

0.1 Unit 3 Electromagnetic Radiation and Plane waves: When Maxwell's equations become linear

0.1.1 Recognizing linear differential equations

Consider some set of variables x_1, x_2, \dots, x_n and some arbitrary list of functions $f(x_1, x_2, \dots, x_n), g(x_1, x_2, \dots, x_n), h(x_1, x_2, \dots, x_n), \dots$. As you know from your math and physics classes, one often is interested in f, g, h, \dots that satisfy a differential equation. One class of differential equations that often comes up in engineering and math is the class of linear differential equations. Here the differential equations are just linear combinations of derivatives of the functions with respect to the variables. Here is an example of a linear differential equation:

$$\frac{\partial f}{\partial x_1} + \frac{\partial^2 g}{\partial x_1 \partial x_2} = h \quad (1)$$

and here is one that is not linear:

$$g \frac{\partial f}{\partial x_2} = 5 \quad (2)$$

Here is another non-linear differential equation:

$$x_2 \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_1} \quad (3)$$

(Here we might be able to choose the variable $y = \ln(x_2)$ to make the equation linear, but it is not a linear differential equation in the variables x_1 and x_2 .)

Linear equations are so nice to have because they always allow the separable solution

$$f = f_o e^{ik_1 x_1} e^{ik_2 x_2} \dots e^{ik_n x_n} \quad (4)$$

With this substitution, finding a solution for the differential equation becomes simply a problem in algebra. For example, consider a solution to Eq 1. We have

$$\begin{aligned} f &= f_o e^{ik_1 x_1} e^{ik_2 x_2} \\ g &= g_o e^{ik_1 x_1} e^{ik_2 x_2} \\ h &= h_o e^{ik_1 x_1} e^{ik_2 x_2} \end{aligned}$$

Here f_o, g_o , and h_o, k_1 , and k_2 are constants. These constants are not completely arbitrary, but instead must satisfy

$$ik_1 f_o - k_1 k_2 g_o = h_o$$

in order to satisfy Eq 1. Linear differential equations always can be converted to a simple algebraic equation. Thus linear equations can be solved quite easily by both numerical and analytic methods. One very important case of

this is the case of linear circuits (circuits consisting of resistors, capacitors, and inductors.) The entire concept of complex impedance is built around the fact that the time-dependent voltage and current in these devices can be described by linear differential equations. In this chapter we are concerned with four cases for which Maxwell's equations become a linear set of differential equations. They are

- (1) Maxwell's equations in vacuum
- (2) Maxwell's equations in a uniform linear dielectric
- (3) Maxwell's equations in an ohmic material
- (4) Maxwell's equations in a uniform linear dielectric with complex permittivity.

In the following sections we will go over each case.

0.1.2 Linear differential equation #1: Maxwell's equations in vacuum

Through most of this chapter we will be finding approximate solutions to Maxwell's equations, albeit very good ones. For light traveling in a vacuum, however, Maxwell's equations really are just a linear set of differential equations. Here we have

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad (5)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (6)$$

$$\vec{\nabla} \cdot \vec{E} = 0 \quad (7)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (8)$$

where we have let $\mu_o \epsilon_o = 1/c^2$. We previously showed this leads to the three-dimensional wave equation:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} \quad (9)$$

$$= -\nabla^2 \vec{E} \quad (10)$$

$$= -\frac{\partial \vec{\nabla} \times \vec{B}}{\partial t} \quad (11)$$

$$= -\mu_o \epsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}. \quad (12)$$

It is no coincidence that $1/(\epsilon_o \mu_o)$ is exactly the speed of light squared. Thus Maxwell's equations in a vacuum give us

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2 \nabla^2 \vec{E}. \quad (13)$$

I leave it to you to show that Maxwell's equations also imply

$$\frac{\partial^2 \vec{B}}{\partial t^2} = c^2 \nabla^2 \vec{B}. \quad (14)$$

Now let us consider the method suggested to solve a linear differential equation in the variables x, y, z , and t . The first step is to introduce the parameters k_x, k_y, k_z and ω . We choose $\omega = -k_t$. A new symbol is used to emphasize the fact that the units are $1/t$ and not $1/d$. We introduce a sign change so that the vector $\vec{k} = (k_x, k_y, k_z)$ will be in the direction of light propagation. However the formal step of Eq 4 is not changed by this symbol and sign convention. A general solution to our linear differential equation can be found with substitutions of the form

$$\begin{aligned} E_x &= \tilde{E}_{ox} e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t} \\ &= \tilde{E}_{ox} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ E_y &= \tilde{E}_{oy} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ E_z &= \tilde{E}_{oz} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \end{aligned} \quad (15)$$

Here the tilda's indicate that the fields might be complex. Writing this in vector form, and adding the similar equations for the magnetic field, we have

$$\vec{E} = \tilde{\tilde{E}}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad (16)$$

$$\vec{B} = \tilde{\tilde{B}}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (17)$$

with $\tilde{\tilde{E}}_o$ and $\tilde{\tilde{B}}_o$ are constant but perhaps complex vectors. Of course these solutions are hardly physical as they have imaginary components. The point is that if \vec{E} is a solution to a linear differential equation, so is \vec{E}^* and so is $\frac{1}{2}(\vec{E} + \vec{E}^*) = \text{Re}[\vec{E}]$. Thus we start with the complex fields of Eqs 16 and 17, to obtain a set of algebraic equations in $\tilde{\tilde{E}}_o, \tilde{\tilde{B}}_o, \vec{k}$, and ω . In the end we know if \vec{E} is a solution to Maxwell's equations, then $\text{Re}[\vec{E}]$ will be as well.

Now let us see if this assumption will work. Specifically, let us see the conditions on $\tilde{\tilde{E}}_o, \tilde{\tilde{B}}_o, \vec{k}$ and ω that will allow all of Maxwell's equations to be satisfied. Before getting started, let us note that

$$\frac{\partial \vec{E}}{\partial t} = -i\omega \tilde{\tilde{E}}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (18)$$

$$\frac{\partial E_x}{\partial x} = ik_x \tilde{\tilde{E}}_{o,x} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (19)$$

$$\frac{\partial E_i}{\partial x_j} = ik_{x_j} \tilde{\tilde{E}}_{o,i} e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad (20)$$

The last line implies

$$\vec{\nabla} \cdot \vec{E} = (i\vec{k} \cdot \tilde{\tilde{E}}_o) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (21)$$

$$\vec{\nabla} \times \vec{E} = (i\vec{k} \times \tilde{\tilde{E}}_o) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (22)$$

When working to solve Maxwell's equations or the wave equation, every term will have an $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ which can be divided out. Thus, with the assumptions of Eqs 16 and 17, Maxwell's equations become a set of algebraic equations in $\vec{E}_o, \vec{B}_o, \vec{k}$ and ω with $\vec{\nabla} \rightarrow i\vec{k}, \partial_t \rightarrow -i\omega$:

$$i\vec{k} \times \vec{B}_o = -\frac{i\omega}{c^2}\vec{E}_o \quad (23)$$

$$i\vec{k} \times \vec{E}_o = i\omega\vec{B}_o \quad (24)$$

$$i\vec{k} \cdot \vec{E}_o = 0 \quad (25)$$

$$i\vec{k} \cdot \vec{B}_o = 0 \quad (26)$$

And, the wave equations become

$$-\omega^2\vec{E}_o = -k^2c^2\vec{E}_o, \quad -\omega^2\vec{B}_o = -k^2c^2\vec{B}_o \quad (27)$$

Let us work on the constraints on $\vec{E}_o, \vec{B}_o, \vec{k}$ and ω so that all equations are satisfied. First note that the second of these equations gives immediately

$$\vec{B}_o = \frac{1}{\omega}\vec{k} \times \vec{E}_o \quad (28)$$

implying that \vec{E} and \vec{B} fields remain perpendicular. The wave equation implies

$$\omega = \pm kc \quad (29)$$

We only consider the $+\omega$ solution. (We will assume the $-\omega$ solution when we take the real part of the fields in the end.) Taking $\omega = kc$, we have

$$\vec{E} = \vec{E}_o e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad (30)$$

$$\vec{B} = \frac{1}{\omega}(\vec{k} \times \vec{E}_o) e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \frac{1}{c}(\vec{k} \times \vec{E}_o) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad (31)$$

subject to the constraint that

$$\vec{k} \cdot \vec{E}_o = 0 \quad (32)$$

$$\omega = kc \quad (33)$$

we must remember that \vec{k} is arbitrary and \vec{E}_o is an arbitrary complex vector lying in the plane that \vec{k} is normal to. This represents seven degrees of freedom for the real and imaginary parts of the two non-zero components of \vec{E}_o and the three components of \vec{k} . Choose these parameters and you have found a solution to Maxwell's equations.

Linearly polarized light Now we treat a special case of linearly polarized light. Let us choose our z axis so that $\vec{k} = k\hat{z}$. Linear polarized light can be written as

$$\vec{E} = \tilde{\tilde{E}}_o e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad (34)$$

$$\tilde{\tilde{E}}_o = \tilde{E}_o(\cos\alpha\hat{x} + \sin\alpha\hat{y}) = E_o e^{i\delta}(\cos\alpha\hat{x} + \sin\alpha\hat{y}) \quad (35)$$

Note that $\tilde{\tilde{E}}_o$ does not have to be real to create linearly polarized light. Instead we only require that the phase be the same for the x and y components of \vec{E} . The real part of \vec{E} (representing physical solutions to the wave equation) are then

$$\text{Re}[\vec{E}] = E_o(\cos\alpha\hat{x} + \sin\alpha\hat{y})(\cos(kz - \omega t - \delta))$$

Here the angle α gives the plane in which \vec{E}_o , the electric field of the light, lies.

Circularly polarized light Now let us assume that \hat{y} is pure imaginary and \hat{x} is real. That is

$$\tilde{\tilde{E}}_o = E_o e^{i\delta}(\hat{x} \pm i\hat{y}) \quad (36)$$

Then we have

$$\text{Re}[\vec{E}] = E_o(\cos(\omega t - (kz + \delta))\hat{x} \pm \sin(\omega t - (kz + \delta))\hat{y}) \quad (37)$$

Notice that now there is no set plane of the electric field. Instead the electric field at any point in space rotates in either the clock-wise or counter-clockwise sense.

0.1.3 Linear Equation #2: Maxwell's equations in a linear dielectric

Here we consider such things as the propagation of light in glass or some other medium in which absorption by the material is minimal. In the absence of free charges or currents, we have

$$\vec{\nabla} \cdot \vec{D} = 0 \quad (38)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (39)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (40)$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad (41)$$

given $\vec{D} = \varepsilon \vec{E}$ and $\vec{H} = \frac{1}{\mu} \vec{B}$ this becomes

$$\vec{\nabla} \cdot \varepsilon \vec{E} = 0 \quad (42)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (43)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (44)$$

$$\vec{\nabla} \times \frac{1}{\mu} \vec{B} = \frac{\partial \varepsilon \vec{D}}{\partial t} \quad (45)$$

Normally these equations are very difficult to solve because μ and ε have values that can depend on space and even on time. If we are concerned with light traveling in a uniform media, ε and μ may be considered constants and we have

$$\vec{\nabla} \cdot \vec{E} = 0 \quad (46)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (47)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (48)$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{v^2} \frac{\partial \vec{E}}{\partial t} \quad (49)$$

with $v = 1/\sqrt{\varepsilon\mu}$. This is exactly the result of Maxwell's equations in a vacuum, but with c replaced by v . Thus the solution is the same, but this time $\omega = \pm kv$:

A solution to Maxwell's equation in a linear dielectric media with $v = 1/\sqrt{\varepsilon\mu}$ is given by

$$\vec{E} = \vec{\tilde{E}}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (50)$$

$$\vec{B} = \frac{1}{v} (\hat{k} \times \vec{\tilde{E}}_o) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (51)$$

subject to the constraint that

$$\vec{k} \cdot \vec{\tilde{E}}_o = 0 \quad (52)$$

$$\omega = kv \quad (53)$$

0.1.4 Linear Equation #3: Maxwell's equations in ohmic material

The following example is normally applied to model the electric and magnetic fields of radio to microwave radiation in conductors. In a conductor (ohmic material) we have

$$\vec{J} = \sigma \vec{E} \quad (54)$$

leading to

$$\vec{\nabla} \cdot \vec{E} = \rho_f \quad (55)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (56)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (57)$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \mu\sigma \vec{E} \quad (58)$$

If $\rho_f = 0$, we get a linear set of differential equations. Let us see when this is ok by solving for ρ_f :

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla} \cdot \vec{J} \text{ (by conservation of charge)} \quad (59)$$

$$= -\sigma \vec{\nabla} \cdot \vec{E} \quad (60)$$

$$= -\frac{\sigma}{\epsilon} \rho_f \quad (61)$$

Assume

$$\rho_f = \rho(\vec{r})f(t) \quad (62)$$

then

$$\begin{aligned} \frac{df}{dt} &= -\frac{\sigma}{\epsilon} f \\ \frac{1}{f} df &= \frac{-\sigma}{\epsilon} dt \\ \int \frac{1}{f} df &= \frac{-\sigma}{\epsilon} \int dt \\ \ln f &= \frac{-\sigma}{\epsilon} t + C \\ f &= e^C e^{-\sigma t/\epsilon} \\ &= f_0 e^{-\sigma t/\epsilon} \end{aligned}$$

So that

$$\rho_f(t) = \rho_0 e^{-\sigma t/\epsilon}. \quad (63)$$

What does this (homogeneous) solution imply? It implies that if we set up some non-zero charge distribution in an ohmic material, we expect it to disappear (go to the surfaces of the material) in a time roughly given by ϵ/σ . We are interested in driven solutions. That is, we assume somebody somewhere is shaking some charges, creating a plane wave of electromagnetic radiation. We want to know the nature of this radiation as it passes through (or is reflected by) a conductor. If the frequency of this shaking is much smaller than σ/ϵ , we can expect the system to constantly approach the steady state conditions of

$\rho_f = 0$. In practice, the approximation that $\rho_f = 0$ breaks down long before the frequency of the radiation reaches σ/ε because of a breakdown in Ohm's law. In actuality the time to redistribute is longer than ε/σ for a good conductor. In any case, for a wide range of useful frequencies, we may assume $\rho_f = 0$ and reap the benefits of a linear set of differential equations. We have

$$\vec{\nabla} \cdot \vec{E} = \rho_f \quad (64)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (65)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (66)$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{v^2} \frac{\partial \vec{E}}{\partial t} + \mu\sigma \vec{E} \quad (67)$$

In the usual way we let

$$\vec{E} = \vec{\tilde{E}}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad (68)$$

$$\vec{B} = \vec{\tilde{B}}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (69)$$

and with $\vec{\nabla} \rightarrow i\vec{k}$ and $\partial_t \rightarrow -i\omega$, we have

$$\vec{k} \cdot \vec{\tilde{E}}_o = 0 \quad (70)$$

$$\vec{k} \times \vec{\tilde{E}}_o = \omega \vec{\tilde{B}}_o \quad (71)$$

$$\vec{k} \cdot \vec{\tilde{B}}_o = 0 \quad (72)$$

$$\vec{k} \times \vec{\tilde{B}}_o = -\left(\frac{\omega}{v^2} + i\mu\sigma\right) \vec{\tilde{E}}_o \quad (73)$$

Note I have assumed that \vec{k} may be complex. Indeed we find this must be the case. Now let us take $\vec{k} \times$ both sides of Eq 71 we find

$$\begin{aligned} \vec{k}^2 \vec{\tilde{E}}_o &= \left(\frac{\omega^2}{v^2} + i\mu\omega\sigma\right) \vec{\tilde{E}}_o \\ &= \frac{1}{v^2} (\omega^2 + i\omega\gamma) \vec{\tilde{E}}_o \end{aligned}$$

In order to keep track of units we have introduced a new frequency $\gamma = \sigma/\varepsilon$. Taking $\vec{k} \times$ both sides of Eq 73 we find

$$\vec{k}^2 \vec{\tilde{B}}_o = \frac{1}{v^2} (\omega^2 + i\omega\gamma) \vec{\tilde{B}}_o \quad (74)$$

apparently we have found that \vec{k} is complex. Let us consider propagation in the \hat{z} direction with

$$\vec{k} = \tilde{k} \hat{z} \quad (75)$$

and

$$\tilde{k}^2 = \frac{1}{v^2}(\omega^2 + i\omega\gamma). \quad (76)$$

Setting

$$\tilde{k} = k + i\kappa \quad (77)$$

we have

$$\tilde{k}^2 = k^2 - \kappa^2 + 2ik\kappa \quad (78)$$

leading to an algebraic equation of the form

$$\begin{aligned} k^2 - \kappa^2 &= k_1^2 \\ k\kappa &= \frac{1}{2}k_2^2 \end{aligned} \quad (79)$$

In the case at hand we have

$$k_1 = \frac{\omega}{v}, \quad k_2 = \frac{\sqrt{\omega\gamma}}{v} \quad (80)$$

but we will soon solve this problem again with k_1 and k_2 having a different definition. Solving for k and κ is done by solving a quadratic equation in k^2 to find

$$k = k_1 \left(\frac{\sqrt{1 + \left(\frac{k_2}{k_1}\right)^4} + 1}{2} \right)^{1/2} \quad (81)$$

$$\kappa = k_1 \left(\frac{\sqrt{1 + \left(\frac{k_2}{k_1}\right)^4} - 1}{2} \right) \quad (82)$$

which, upon substitution, reduces to

$$k = \frac{\omega}{v} \left(\frac{\sqrt{1 + \left(\frac{\gamma}{\omega}\right)^2} + 1}{2} \right)^{1/2} \quad (83)$$

$$\kappa = \frac{\omega}{v} \left(\frac{\sqrt{1 + \left(\frac{\gamma}{\omega}\right)^2} - 1}{2} \right)^{1/2} \quad (84)$$

Notice if $\gamma = \sigma/\varepsilon = 0$, we have $k = \omega/v$ and $\kappa = 0$, just as we would have for the case of a linear dielectric. But we find now

$$\vec{E} = \vec{E}_o e^{i(kz - \omega t)} e^{-\kappa z}. \quad (85)$$

Thus as the beam propagates, it decays. The energy goes into heating the conductor. Interestingly, we still have

$$\vec{E}_o \cdot \vec{k} = 0 \quad (86)$$

and

$$\vec{\tilde{B}}_o = \frac{1}{\omega} \vec{\tilde{k}} \times \vec{\tilde{E}}_o \quad (87)$$

But now that $\vec{\tilde{k}}$ is complex there is a phase difference between the real part of $\vec{\tilde{B}}_o$ and $\vec{\tilde{E}}$. For simplicity, assume $\vec{\tilde{E}}_o = E_o \hat{x}$. Then

$$\vec{\tilde{B}}_o = \frac{E_o}{\omega} (k + i\kappa) \hat{y} \quad (88)$$

$$= \frac{E_o}{v} (1 + (\frac{\gamma}{\omega})^2)^{1/4} e^{i\phi} \quad (89)$$

$$\phi = \arctan \frac{\kappa}{k} = \arctan \left(\sqrt{\frac{\sqrt{1 + (\frac{\gamma}{\omega})^2} - 1}{\sqrt{1 + (\frac{\gamma}{\omega})^2} + 1}} \right) \quad (90)$$

Taking the real part of the magnetic and electric fields, we find

$$\vec{E} = E_o \cos(kz - \omega t) e^{-\kappa z} \hat{x} \quad (91)$$

$$\vec{B} = \frac{E_o}{v} (1 + (\frac{\gamma}{\omega})^2)^{1/4} \cos(kz - \omega t + \phi) e^{-\kappa z} \quad (92)$$

0.1.5 Linear Equation #4: Maxwell's equations in complex dielectric material

A simple model of complex permittivity Here we consider the case for which $\mu \approx \mu_o$ and $\tilde{\epsilon} = \epsilon + i\xi$. As we shall see, the index of refraction and absorption of light can be directly related to a complex permittivity $\tilde{\epsilon}$. With a lot of work, one can also determine $\tilde{\epsilon}$ reasonably well from first principles. Here I present a very simple model of complex permittivity that has the right flavor of a realistic model and introduces many of the issues one must consider when finding how $\tilde{\epsilon}$ depends on ω . Keep in mind, however, that an accurate model of a complex permittivity requires statistical mechanics and quantum mechanics and may be both quantitatively and qualitatively different than the model I will present here.

We begin by considering how electrons in an insulator move when subject to the oscillating electric field of a light wave. We assume the electrons are bound by a harmonic force and experience a drag:

$$m\ddot{x} = -m\omega_j^2 x - m\gamma_j \dot{x}$$

If we drive this system with an electric field polarized in the x direction and oscillating according to

$$\vec{E} = E_o e^{-i\omega t} \hat{x}$$

Then we find

$$\vec{r} = x \hat{x} = \frac{q}{m(\omega_j^2 - \omega^2 - i\gamma_j \omega)} E_o \hat{x} e^{-i\omega t}$$

Note that \vec{r} is complex only because we are considering driving the electron with a complex electric field. Of course, we don't create imaginary E -fields in the laboratory. However, because the equation of motion of the particle is itself linear, if we drive the system with the real part of $\vec{E} = \text{Re}[\vec{E}]$, we will find the position of the particle with a function of time will be given by the real part of $x = \text{Re}[x]$.

The time-dependent polarization is due to a population of N electrons per volume each with a probability f_j of being in a location of the solid such that the force is given by ω_j and damping γ_j is then given by

$$\vec{P} = N \sum_j q f_j \vec{r}_j \quad (93)$$

$$= \sum_j \frac{q^2 N f_j}{m(\omega_j^2 - \omega^2 - i\gamma_j \omega)} E_o \hat{x} e^{-i\omega t} \quad (94)$$

$$= \sum_j \frac{q^2 N f_j}{m(\omega_j^2 - \omega^2 - i\gamma_j \omega)} \vec{E} \quad (95)$$

$$= (\tilde{\epsilon} - \epsilon_o) \vec{E} \quad (96)$$

$$\tilde{\epsilon} = \epsilon_o + \sum_j \frac{q^2 N f_j}{m(\omega_j^2 - \omega^2 - i\gamma_j \omega)} \quad (97)$$

By a quick manipulation of our expression for $\tilde{\epsilon}$, we find the result of this model:

$$\tilde{\epsilon} = \epsilon + i\xi \quad (98)$$

$$\epsilon = \epsilon_o + \frac{q^2 N}{m} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2} \quad (99)$$

$$\xi = \frac{q^2 N}{m} \sum_j \frac{f_j \gamma_j \omega}{(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2} \quad (100)$$

It still might not rest easy with you that we have made a physical property of a material complex. But again note that the permittivity is complex only because we are assuming that we are driving the system with a complex electric field. In the end, we really drive the system with the real part of the complex field and we only consider the real part of resulting electric or magnetic fields.

Maxwell's equation in a dielectric with a complex permittivity Now that we have a model for why it makes sense to introduce a complex permittivity. Lets take a bit of time to say what the implications are. To solve Maxwell's equations for a linear dielectric we have

$$\tilde{k}_o^2 = \tilde{\epsilon} \mu_o \omega^2 \vec{E}_o$$

which implies

$$\begin{aligned} \tilde{k}^2 &= \epsilon \mu_o \omega^2 + i\xi \mu_o \omega^2 \\ &= k_1^2 + i k_2^2 \end{aligned}$$

We did this algebra in the last section, so we need only define

$$\begin{aligned} k_1^2 &= \varepsilon\mu_o\omega^2 \\ k_2^2 &= \xi\mu_o\omega^2 \end{aligned}$$

Leading to

$$\begin{aligned} \tilde{k} &= k + i\kappa \\ k &= \sqrt{\varepsilon\mu_o}\omega \left(\frac{\sqrt{1 + (\xi/\varepsilon)^2} + 1}{2} \right)^{1/2} \\ &\approx \sqrt{\frac{\varepsilon}{\varepsilon_o}} \frac{\omega}{c} \\ \kappa &= \sqrt{\varepsilon\mu_o}\omega \left(\frac{\sqrt{1 + (\xi/\varepsilon)^2} - 1}{2} \right)^{1/2} \\ &\approx \frac{\xi}{\varepsilon} \sqrt{\frac{\varepsilon}{\varepsilon_o}} \frac{\omega}{2c} \end{aligned}$$

Here we have simplified by assuming that ξ is much smaller than ε , which is true in practice. (We shall see that $\frac{2\pi}{k} = \lambda$ is the wavelength of the light (typically $10^{-5}m$) and $\frac{1}{2\kappa}$ is the absorption depth, typically $1m$.) The plane wave solution is apparently

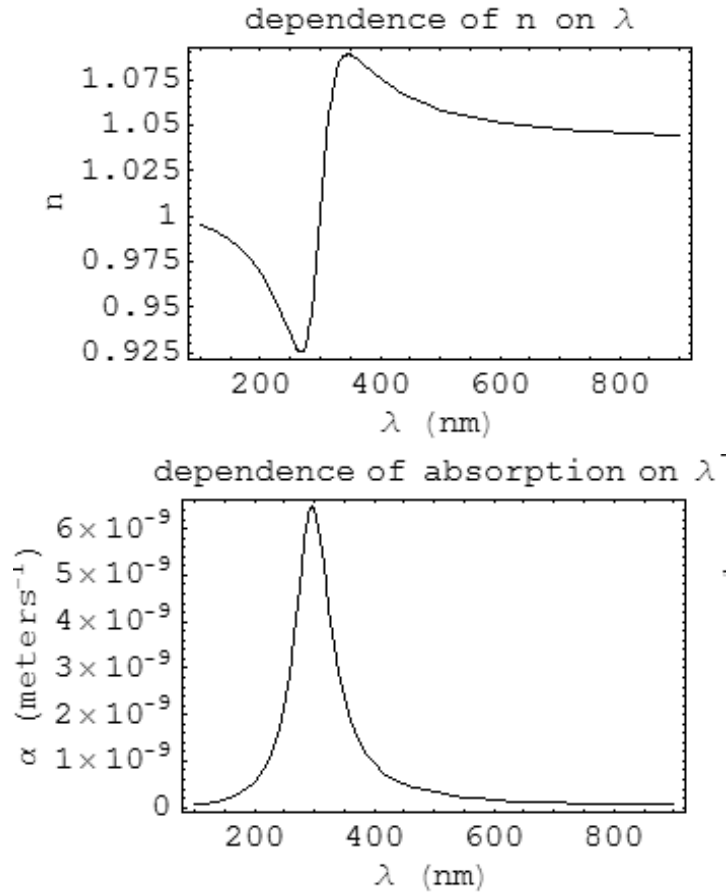
$$\vec{E} = E_o \hat{x} e^{i(kz - \omega t)} e^{-\kappa z}$$

Recall that $\kappa \ll k$ (It takes meters visible light to be absorbed by glass, yet $k \approx 2\pi/500nm$.) Thus the light pretty much behaves like a plane wave for the case that the permittivity is real (case 2.) Thus it makes sense to define an index of refraction

$$\begin{aligned} n &= \frac{kc}{\omega} \\ &\approx \sqrt{\frac{\varepsilon}{\varepsilon_o}} \\ \alpha &= \frac{2\kappa}{\omega} \\ &\approx \frac{\xi}{\varepsilon} \frac{2\pi n}{\lambda} \end{aligned}$$

Here λ is the vacuum wavelength of the radiation, that is $\lambda = 2\pi c/\omega$.

It is useful to go back to the classical model giving ε and ξ to see what n and α look like when plotted against λ . We If we assume just one absorption frequency at $\frac{2\pi c}{\omega_1} = 300nm$ and a damping coefficient $\gamma_1 = 1.5 \text{ sec}^{-1}$, a density of electrons of $10^{27}m^{-3}$, then the index of refraction and α are as shown in the plots below:



Note in our simple model we see a typical feature found in nature; namely a region of strong absorption. Near this regions of strong absorption, $n - 1$ tends to look much like the negative of the derivative of the absorption peak. Away from the absorption, the index of refractions drops slowly with increasing wavelength. One should not take too seriously either the magnitude of n or the magnitude of α . One needs a far more carefully thought out model of the media to come up with a quantitative model of absorption and dispersion.