Chapter 31
Magnetic fields in Matter I: Origin of magnetization

31.1 Review of magnetic fields in vacuum

Q: What is the force on a current carrying wire in a magnetic field?

A: The force on a straight wire is simply $I\vec{L}\times\vec{B}$. On a wire we have

$$\vec{F} = -I\int \vec{B}\times d\vec{\ell} \tag{1029}$$

Q: State the Biot-Savart Law

A: The Biot-Savart Law gives the magnetic field in terms of a current density $\vec{J}$, a surface current $\vec{K}$, or a current I :

$$\begin{aligned}\vec{B} &= \frac{\mu_o}{4\pi}\int\frac{\vec{J}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}d\tau' && (31.1030)\\ &= \frac{\mu_o}{4\pi}\int\frac{\vec{K}'\times(\vec{r}-\vec{r}')\times\vec{J}'}{|\vec{r}-\vec{r}'|^3}da' && (31.1031)\\ &= -\frac{\mu_o I}{4\pi}\int\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\times d\vec{\ell}' && (31.1032)\end{aligned}$$

Q: What is the field from a vector potential $\vec{A}$?

A: $\vec{B} = \vec{\nabla}\times\vec{A}$

Q: What is the vector potential from a current density $\vec{J}$ and current loop I?

A: The vector potential $\vec{A}$ is found in much the same way as the potential V:

$$\begin{aligned}\vec{A} &= \frac{\mu_o}{4\pi}\int\frac{\vec{J}}{|\vec{r}-\vec{r}'|}d\tau' && (31.1033)\\ &= \frac{\mu_o I}{4\pi}\oint\frac{1}{|\vec{r}-\vec{r}'|}d\vec{\ell}' && (31.1034)\end{aligned}$$

Q: What is the vector potential far from a current loop?

$$\begin{aligned}\vec{A} &\approx \frac{\mu_o I}{4\pi}\oint(\frac{1}{r}+\frac{\vec{r}\cdot\vec{r}'}{r^3})d\vec{\ell}' && (31.1035)\\ &= \frac{\mu_o I}{4\pi r^3}\oint\vec{r}\cdot\vec{r}'\,d\vec{\ell}' && (31.1036)\\ &= \frac{\mu_o}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} && (31.1037)\end{aligned}$$

where

$$\vec{m} = I \int d\vec{a}' \quad (31.1038)$$

$$= \text{the magnetic dipole moment of a current loop} \quad (31.1039)$$

Here we have used the relation (proved earlier)

$$\oint \vec{C} \cdot \vec{r}' d\vec{\ell}' = -\vec{C} \times \int d\vec{a}' \quad (1040)$$

we have

$$\vec{A} \approx \vec{A}_{dipole} \quad (31.1041)$$

$$= \frac{\mu_o}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} \quad (31.1042)$$

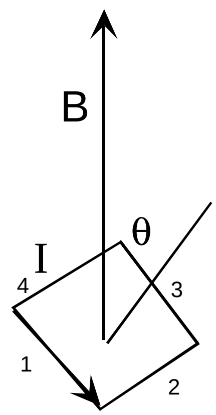
31.2 Effect of field on magnetic dipole of an atom: paramagnetism, diamagnetism

When a magnetic field is applied to a bit of matter it can distort the currents that already exist. These new currents in turn create a new magnetic field that can either enhance or oppose the existing field. In paramagnets, the induced currents enhance the magnetic field whereas in diamagnets, the induced currents oppose it. In this section we give a physical explanation of what causes these two effects.

31.3 Torque on a magnetic dipoles

What is the torque on a loop carrying a current I making an angle θ with a magnetic field $\vec{B}$

?



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Let us first assume that the loop is indeed a square. Then sides 1 and 3 do not contribute to the torque and the forces of magnitude $F = ILB$ from sides 2 and 3 contribute a total torque of magnitude $N = IL^2 B \sin\theta$. By letting $\vec{m}$, the magnetic dipole moment, be a vector of magnitude IL^2in the direction of the loop, then $\vec{N} = \vec{m} \times \vec{B}$.

If the loop is not square, this result does not change. The force $d\vec{F}$ from a small segment of wire is given by

$$d\vec{F} = -I\vec{B} \times d\vec{\ell} \tag{1044}$$

The torque on this section of wire (with respect to an arbitrary origin) is given by

$$\begin{aligned} d\vec{N} &= \vec{r} \times (-I\vec{B} \times d\vec{\ell}) && (31.1045) \\ &= -I\vec{B}(\vec{r} \cdot d\vec{\ell}) + I(\vec{B} \cdot \vec{r})d\vec{\ell} && (31.1046) \end{aligned}$$

and

$$\vec{N} = -I\vec{B} \oint (\vec{r} \cdot d\vec{\ell}) + I \oint (\vec{B} \cdot \vec{r})d\vec{\ell} \tag{1047}$$

Defining the center of the loop so that

$$\oint (\vec{r} \cdot d\vec{\ell}) = 0 \tag{1048}$$

we have

$$\vec{N} = I \oint (\vec{B} \cdot \vec{r})d\vec{\ell} \tag{1049}$$

Finally, we use

$$\oint (\vec{C} \cdot \vec{r})d\vec{\ell} = -\vec{C} \times \int d\vec{a} \tag{1050}$$

to write

$$\vec{N} = \vec{m} \times \vec{B} \tag{1051}$$

where

$$\vec{m} = I \int d\vec{a}' \tag{31.1052}$$
$$= \text{the magnetic dipole moment of a current loop.} \tag{31.1053}$$

Notice that although we were first introduced to $\vec{m}$ in the context of an approximate theory, here we have obtained an exact expression for the torque on a loop in terms of $\vec{m}$.

We know from our experience with dipoles ($\vec{N} = \vec{p} \times \vec{E}$) leads to $U = -\vec{p} \cdot \vec{E}$. It stands to reason that

$$U = -\vec{m} \cdot \vec{B}. \tag{1054}$$

Indeed, if we recheck our derivation of U for an electric dipole, it indeed uses only mechanics and the statement $\vec{N} = \vec{p} \times \vec{E}$. Thus U is indeed given by $-\vec{m} \cdot \vec{E}$We must be very careful with these analogies however! for example, the force on an magnetic dipole in a non-uniform field is given by

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B}) \tag{1055}$$

whereas the force on an electric dipole is

$$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E} \tag{1056}$$

Small difference such as this will continue to creep up, so be alert when solving problems!

With 31.1054 we immediately have the fact that magnetic dipoles in matter will tend to line up with a magnetic field. The degree to which they line up will depend on the temperature of the matter and the strength of the field. The fact that the dipoles line up parallel the field implies the induced current will enhance the magnetic field. Thus matter made up of dipoles that are free to move tend to be paramagnets and the phenomena is known as paramagnetism.

The most likely microscopic dipole to align with a magnetic field is the electron itself, which carries a tiny permanent magnetic moment. It may seem that, since electrons are so plentiful, all mater should display paramagnetism. However, because of relativistic quantum mechanics, electrons like to pair up in matter, with one electron carrying the opposite spin (and hence opposite magnetic moment) as the other. This does not change in the presence of a magnetic field (provided the field is not too strong.) Thus, when one of the paired electrons starts to point its magnetic moment in the direction of the field, the other tends to point its moment in the opposite direction, thus canceling any net magnetization. For this reason, paramagnetism due to electrons is far weaker than one might expect from arguments that invoke only classical physics. One might expect paramagnets tend to be atoms or molecules with an odd number of electrons. From the table below, we see that other factors also are at play:

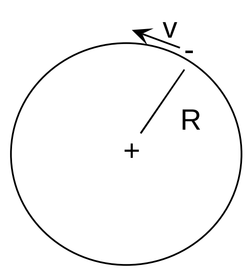
material	At No	susceptibility	type
Bismuth	83	-1.6×10^{-4}	diamagnetic
Gold	79	-3.4×10^{-5}	diamagentic
Silver	47	-2.4×10^{-5}	diamagentic
Copper	29	-9.7×10^{-6}	diamagnetic
Water	10	-9.0×10^{-6}	diamagnetic
Carbon Dioxide	22	-1.2×10^{-8}	diamagnetic
Hydrogen	1	-2.2×10^{-9}	diamagnetic
Oxygen	8	1.9×10^{-6}	paramagnetic
Sodium	11	8.5×10^{-6}	paramagnetic
Aluminum	13	2.1×10^{-5}	paramagnetic
Tungsten	74	7.8×10^{-5}	paramagnetic
Liquid Oxygen	8	3.9×10^{-3}	paramagnetic
Gadolinium	64	4.8×10^{-1}	paramagnetic

(1057)

31.4 distortion of a atomic orbitals and diamagnets

When introducing the distortion of an atomic orbital due to an electric field, we admitted that the proper treatment required quantum mechanics (namely consideration of the Stark effect) but an approximate picture could be gained from simple arguments. Here again we seek a simple picture of a more complicated quantum effect (this time the Zeeman effect.)

We first consider an electron orbiting a core in a circular orbit.



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Given the centripetal force is provided by the electric field, we have

$$\frac{mv^2}{R} = \frac{e^2}{4\pi\varepsilon_o R^2} \tag{1059}$$

If we add a magnetic field out of the page and assume the effect is to change the speed of the electron (but not the shape of the orbital) we have:

$$\frac{m_e v'^2}{R} = \frac{e^2}{4\pi\varepsilon_o R^2} + ev'B \qquad (31.1060)$$

$$= \frac{m_e v^2}{R} + ev'B \qquad (31.1061)$$

$$ev'B = \frac{m_e}{R}(v'-v)(v'+v) \qquad (31.1062)$$

Now we assume $v' \approx v$. (If this is not the case, the field will do much more than a small distortion.) With this we have

$$evB \approx \frac{2m_e v}{R}\Delta v, \qquad (31.1063)$$

$$\Delta v = \frac{eR}{2m_e}B \qquad (31.1064)$$

The magnet dipole moment of the loop is $m = Ia = \frac{e}{T}a = \frac{e}{2\pi R/v}\pi R^2 = \frac{evR}{2}$. Thus the change in the dipole moment is given by

$$\Delta m = \frac{e\Delta v R}{2} = \frac{e^2 R^2}{4m_e}B \qquad (1065)$$

Because the new stable orbit has a slower moving electron, the direction of the change in dipole moment is to oppose the field:

$$\Delta\vec{m} = -\frac{e^2 R^2}{4m_e}\vec{B} \qquad (1066)$$

We emphasize that this is a simple classical model of an quantum model. It is not worth refining details such as asking if the speed or radius changes because the answer will be wrong unless quantum mechanics is used. The answer does give the proper order-of magnitude of the effect and indicates a trend that magnetic fields tend to alter electronic orbits in such a fashion that a new dipole moment is created to oppose the field, i.e., the distortion of atomic orbitals tends to lead to diamagnetism.

31.5 The Magnetization vector $\vec{M}$

As in our discussion of electric fields in matter, we now forget for the time being the microscopic picture of how an magnetic field influences the field in matter and instead turn to the macroscopic picture. For now, let us forget how it happens, but suppose we have a piece of matter with a magnetization $\vec{M}$. By this we mean that any sufficiently small cube of volume $d\tau$ will have a dipole moment given by $d\vec{m} = \vec{M}d\tau$. Thus $\vec{M}$ is the dipole moment per unit volume. We ask the question what is the vector potential $\vec{A}$? To answer this question, we recall that the vector potential $d\vec{A}$ from a single dipole is given by

$$d\vec{A} = \frac{\mu_o}{4\pi}\frac{d\vec{m} \times \vec{r}}{r^3} \tag{1067}$$

So from the entire distribution of dipoles we must have

$$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{d\vec{m}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \tag{31.1068}$$

$$= \frac{\mu_o}{4\pi}\int \frac{\vec{M}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\tau' \tag{31.1069}$$

Now we use the trick

$$\vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|} = \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \tag{1070}$$

to write

$$\vec{A} = \frac{\mu_o}{4\pi}\int \vec{M}' \times \vec{\nabla}\frac{1}{|\vec{r} - \vec{r}'|} d\tau' \tag{1071}$$

Finally, we use

$$\vec{\nabla} \times (f\vec{M}) = f\vec{\nabla} \times \vec{M} - \vec{M} \times \vec{\nabla} f \tag{1072}$$

with $f = \frac{1}{|\vec{r} - \vec{r}'|}$ so that

$$\vec{A} = \frac{-\mu_o}{4\pi}\int \vec{\nabla}' \times (\frac{\vec{M}'}{|\vec{r} - \vec{r}'|}) d\tau' + \frac{\mu_o}{4\pi}\int \frac{\vec{\nabla}' \times \vec{M}'}{|\vec{r} - \vec{r}'|} d\tau' \tag{1073}$$

Now we use the following corollary of the fundamental theorems:

$$\int \vec{\nabla} \times \vec{v} d\tau = -\oint \vec{v} \times d\vec{a} \tag{31.1074}$$

$$= -\oint \vec{v} \times \hat{n}\, da \tag{31.1075}$$

to write

$$\vec{A} = \frac{\mu_o}{4\pi}\oint \frac{\vec{M}' \times \hat{n}'}{|\vec{r} - \vec{r}'|} da' + \frac{\mu_o}{4\pi}\int \frac{\vec{\nabla}' \times \vec{M}'}{|\vec{r} - \vec{r}'|} d\tau' \tag{1076}$$

Now we define a bound surface current density

$$\vec{K}_b = \vec{M} \times \hat{n} \tag{1077}$$

and a bound current density

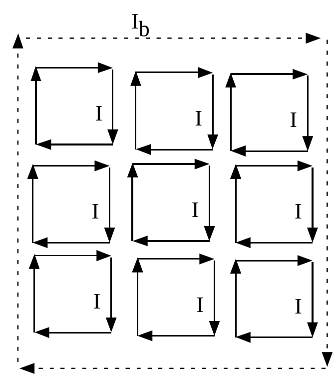
$$\vec{J}_b = \vec{\nabla} \times \vec{M} \tag{1078}$$

so that

$$\vec{A} = \frac{\mu_o}{4\pi} \oint \frac{\vec{K}_b'}{|\vec{r} - \vec{r}'|} da' + \frac{\mu_o}{4\pi} \int \frac{\vec{J}_b'}{|\vec{r} - \vec{r}'|} d\tau' \tag{1079}$$

We have just found something useful: The effect of the magnetism $\vec{M}$ can be determined by simple assuming the existence of a new surface current $\vec{K}_b$ and current density $\vec{J}_b$. Thus, just as in the case of electrostatic fields, we have taken a new problem (find the magnetic field of matter with magnetization $\vec{M}$) and re-formed it in terms of an old problem (find the magnetic field from a current density $\vec{J}_b = \vec{\nabla} \times \vec{M}$ and surface current density $\vec{K}_b = \vec{M} \times \widehat{n}$.

The surface current density has a physical interpretation: it is the current that accumulates at the edge of an array of dipoles because the current is not canceled by an adjacent dipole



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If we have something not quite as abrupt as a sudden change from magnetic dipoles to vacuum, it stands to reason we will still have effective moving charges. Indeed we do, and this moving charge is described by the current density $\vec{J}_b = \vec{\nabla} \times \vec{M}$

Chapter 32
The vector field $\vec{H}$ and linear material

32.1 Review:

Last lecture we discovered that the magnetic field due to a piece of material with magnetization $\vec{M}$ can be by assuming bound currents

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \quad (32.1081)$$

$$\vec{K}_b = \hat{n} \times \vec{M} \quad (32.1082)$$

and then finding $\vec{B}$ and $\vec{A}$ in the usual way:

$$\vec{B} = \frac{\mu_o}{4\pi} \int \frac{(\vec{r} - \vec{r}') \times \vec{J}_b'}{|\vec{r} - \vec{r}'|^3} d\tau' + \frac{\mu_o}{4\pi} \int \frac{(\vec{r} - \vec{r}') \times \vec{K}_b'}{|\vec{r} - \vec{r}'|^3} da' \quad (1083)$$

or

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad (1084)$$

where (in the Coulomb gauge)

$$\vec{A} = \frac{\mu_o}{4\pi} \int \frac{\vec{J}_b'}{|\vec{r} - \vec{r}'|} d\tau' + \frac{\mu_o}{4\pi} \int \frac{\vec{K}_b'}{|\vec{r} - \vec{r}'|} da'. \quad (1085)$$

In either case, we have

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}_b \quad (1086)$$

Now consider a slightly more complex situation: Not only do we have a piece of polarized material with magnetization $\vec{M}$, but we also have real currents actually moving around. In this case we have

$$\vec{B} = \frac{\mu_o}{4\pi} \int \frac{(\vec{r} - \vec{r}') \times (\vec{J}_b' + \vec{J}_f')}{|\vec{r} - \vec{r}'|^3} d\tau' + \frac{\mu_o}{4\pi} \int \frac{(\vec{r} - \vec{r}') \times (\vec{K}_b' + \vec{K}_f')}{|\vec{r} - \vec{r}'|^3} da' \quad (1087)$$

or, more compactly,

$$\vec{\nabla} \times \vec{B} = \mu_o (\vec{J}_b + \vec{J}_f) \quad (1088)$$

(You may be wondering how 32.1088 could possibly be a similar expression to 32.1087. After all, 32.1088 does not even mention $\vec{K}_b$ and $\vec{K}_f$. Recall that $\vec{K}_b$ and $\vec{K}_f$ are the surface currents. If instead of defining $\vec{K}_b$ and $\vec{K}_f$ as currents in an infinitely thin sheet, we define an infinitesimally thin region thickness ε of current density $\vec{J} = \vec{K}/\varepsilon$, then this surface current would be included in 32.1088.)

Just as we defined the displacement vector field $\vec{D} = \varepsilon_o\vec{E} + \vec{P}$ so that $\vec{\nabla} \cdot \vec{D} = \rho_f$, we now define a vector field $\vec{H}$ so that $\vec{\nabla} \times \vec{H} = \vec{J}_f$. To do this, we let

$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M} \tag{1089}$$

This vector field vector is called $\vec{H}$. Some people call $\vec{H}$ the magnetic field and $\vec{B}$ the magnetic induction. These people are as crazed as advocates of killing people for splitting infinitives. Fortunately, unlike the grammatical zealots who have succeeded to effectively kill a perfectly nice construction, the magnetic induction zealots appear to have lost the battle. It is now almost universally taught that $\vec{B}$ is the magnetic field and $\vec{H}$. . ., well $\vec{H}$ is just $\vec{H}$.

Just as it is a no-no to use Coulombs law to find $\vec{D}$ from ρ_f (because $\vec{\nabla} \times \vec{D}$ is not necessarily zero, it is a no-no to use the Biot-Savart law to find $\vec{H}$ from $\vec{J}_f$ because $\vec{\nabla} \cdot \vec{H}$ is not necessarily zero. We can, however, use ampere's law to find $\vec{H}$ from $\vec{J}_f$. We will do this in our problem solving session at the end of today's lecture.

32.2 Linear paramagnetism and diamagnetism

We now consider material which responds in a linear fashion to a applied magnetic field. By this we mean that the magnetization (appearing by aligning magnetic dipoles or by distorting the shape of electronic orbitals) is proportional to the strength of the magnetic field:

$$\vec{M} = \frac{1}{\mu_o}\frac{\chi_m}{1+\chi_m}\vec{B} \tag{1090}$$

Why such a downright bizarre normalization as this? Why can't we write $\vec{M} = const/\mu_o \cdot \vec{B}$ just as we wrote $\vec{P} = \varepsilon_o\chi_e\vec{E}$? I do not know the history of χ_m, but let us write $\vec{M}$ in terms of $\vec{H}$ rather than $\vec{B}$ to see that there is some reason behind this madness:

$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M} \tag{32.1091}$$

$$= \frac{1+\chi_m}{\chi_m}\vec{M} - \vec{M} \tag{32.1092}$$

$$= \frac{1}{\chi_m}\vec{M} \tag{32.1093}$$

Thus we see that $\vec{M}$ is traditionally defined in terms of the $\vec{H}$ field rather than the $\vec{B}$ field:

$$\vec{M} = \chi_m\vec{H}. \tag{1094}$$

With linear media, $\vec{M}$, $\vec{H}$, and $\vec{B}$ are all related by simple constants:

	$\vec{M}$	$\vec{H}$	$\vec{B}$
$\vec{M}$	$\vec{M}$	$\vec{M}=\chi_m\vec{H}$	$\vec{M}=\frac{\chi_m}{(1+\chi_m)\mu_o}\vec{B}$
$\vec{H}$	$\vec{H}=\frac{1}{\chi_m}\vec{M}$	$\vec{H}$	$\vec{H}=\frac{1}{(1+\chi_m)\mu_o}\vec{B}$
$\vec{B}$	$\vec{B}=\frac{(1+\chi_m)\mu_o}{\chi_m}\vec{M}$	$\vec{B}=(1+\chi_m)\mu_o\vec{H}$	$\vec{B}$

(1095)

This implies that once one determines either $\vec{M}$, $\vec{B}$, or $\vec{H}$ for a linear paramagnetic or diamagnetic material, one can find the other two. Notice that sometimes one is not given the magnetic susceptibility χ_m, but instead the permeability of the material. Here $\mu=(1+\chi_m)\mu_o$ so that $\vec{B}=\mu\vec{H}$.

As a review, we write a similar expression for linear dielectrics:

	$\vec{P}$	$\vec{D}$	$\vec{E}$
$\vec{P}$	$\vec{P}$	$\vec{P}=\frac{\chi_e}{1+\chi_e}\vec{D}$	$\vec{P}=\chi_e\varepsilon_o\vec{E}$
$\vec{D}$	$\vec{D}=\frac{1+\chi_e}{\chi_e}\vec{P}$	$\vec{D}$	$\vec{D}=(1+\chi_e)\varepsilon_o\vec{E}$
$\vec{E}$	$\vec{E}=\frac{1}{\chi_e\varepsilon_o}\vec{P}$	$\vec{E}=\frac{1}{(1+\chi_e)\varepsilon_o}\vec{D}$	$\vec{E}$

(1096)

Most materials are linear media, with the electric susceptibility χ_e being relatively large in solids and liquids. (a relatively weak dielectric is Benzene with $\chi_e=1.28$. A very strong dielectric is water, with $\chi_e=79.1$) The magnetic susceptibility of most materials is much smaller. (The magnetic susceptibility of liquid oxygen at $\chi_m=0.039$ is considered very large. The magnetic susceptibility of water is -9.0×10^{-6}.

QUESTION: What does the fact that most media are linear tell us about the scale of forces on atoms and molecules due to external electric and magnetic fields as compared to the microscopic forces that hold matter together?

QUESTION: What does the fact that the electric susceptibility of most materials is much larger than the magnetic susceptibility tell us?

32.3 Words of warning: $\vec{\nabla}\times\vec{D}\neq\vec{0}$ and $\vec{\nabla}\cdot\vec{H}\neq 0$

You may recall a word of warning when we learned that, for dielectrics (both linear and nonlinear), although $\vec{\nabla}\cdot\vec{D}=\rho_f$ we did not have $\vec{\nabla}\times\vec{D}=\vec{0}$. Whereas $\vec{\nabla}\cdot\vec{D}=\rho_f$ buys us Gauss's law:

$$\oint\vec{D}\cdot d\vec{a}=Q_{f,enc} \tag{1097}$$

Because $\vec{\nabla}\times\vec{D}=\vec{\nabla}\times\vec{P}$ which is not necessarily zero, we can not write down a Coulomb's