1 Lecture notes UNIT #4: Time-dependent potentials

By now we have all know by heart the four equations of Maxwell.

$$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_o \tag{1}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} \tag{4}$$

For the steady state case of these equations, we found that we could enforce 2 and 3 by introducing the potentials V and $\vec{A}$ and setting

$$\vec{E} = -\vec{\nabla} \cdot V \tag{5}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{6}$$

For the time-dependent case, we must modify 5. If we keep 6 then the curl of $\vec{E}$ may be written as

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{7}$$

$$= -\frac{\partial \vec{\nabla} \times \vec{A}}{\partial t} \tag{8}$$

so that

$$\vec{\nabla} \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = \vec{0} \tag{9}$$

By Helmholtz's theorem, we must be able to express $\vec{E} + \frac{\partial \vec{A}}{\partial t}$ as the gradient of a scalar potential which we call $-V$:

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V \tag{10}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V \tag{11}$$

Together with 6, Maxwell's equations can now be written

$$\frac{\partial \vec{\nabla} \cdot \vec{A}}{\partial t} + \nabla^2 V = -\rho/\varepsilon_o \tag{12}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \tag{13}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_o \vec{J} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \frac{1}{c^2} \vec{\nabla} \frac{\partial V}{\partial t} \tag{14}$$

$$= -\nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) \tag{15}$$

The last equation leads to

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} - \frac{1}{c^2}\vec{\nabla}\left[c^2\vec{\nabla}\cdot\vec{A} + \frac{\partial V}{\partial t}\right] = -\mu_o\vec{J} \tag{16}$$

We now define the d'Alembertian $\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$. With this definition, Maxwell's equations have been reduced to

$$\begin{aligned} \Box^2\vec{A} - \frac{1}{c^2}\vec{\nabla}\left[c^2\vec{\nabla}\cdot\vec{A} + \frac{\partial V}{\partial t}\right] &= -\mu_o\vec{J} && (17)\\ \frac{\partial\vec{\nabla}\cdot\vec{A}}{\partial t} + \nabla^2 V &= -\rho/\varepsilon_o && (18)\\ &= \Box^2 V + \frac{1}{c^2}\frac{\partial}{\partial t}\left[c^2\vec{\nabla}\cdot\vec{A} + \frac{\partial V}{\partial t}\right] && (19) \end{aligned}$$

where

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{20}$$

and

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \tag{21}$$

What have we accomplished here? We have replaced Maxwell's eight (two vector and two scalar) equations for the six components of $\vec{B}$ and $\vec{E}$ for four (one vector and one scalar) equations for $\vec{A}$ and V. After $\vec{A}$ and V are determined, we may find $\vec{B}$ and $\vec{E}$ by differentiation. Still, the equations we are left with are ugly and difficult to solve. There is one more card up our sleeve, however, and this exploitation of the gauge freedom of the problem.

1.1 Gauge symmetry

We must remember that the electric and magnetic fields are the physical quantities, and that V and $\vec{A}$ are simple mathematical constructions used to determine the fields. This allows us some freedom in our choice of $\vec{A}$ and V. For example, adding any vector to $\vec{A}$ with zero curl will not change $\vec{B}$. A vector with zero curl is of course the gradient of a scalar:

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda \tag{22}$$

then

$$\begin{aligned} \vec{\nabla}\times\vec{A}' &= \vec{\nabla}\times(\vec{A} + \vec{\nabla}\lambda) && (23)\\ &= \vec{\nabla}\times\vec{B} && (24) \end{aligned}$$

But if we modify only $\vec{A}$, we will change the value of $\vec{E}$. Thus we must modify V as well:

$$\begin{aligned}\vec{E} &= -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} && (25)\\ &= -\vec{\nabla}V' - \frac{\partial \vec{A}'}{\partial t} && (26)\\ &= -\vec{\nabla}V' - \frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\lambda) && (27)\\ &= -\vec{\nabla}(V' + \frac{\partial \lambda}{\partial t}) - \frac{\partial}{\partial t}\vec{A} && (28)\end{aligned}$$

So we must change V in such a way that

$$V' = V - \frac{\partial \lambda}{\partial t} \tag{29}$$

Thus we have the freedom to perform the Gauge transformation

$$\begin{aligned}\vec{A}' &= \vec{A} + \vec{\nabla}\lambda && (30)\\ V' &= V - \frac{\partial \lambda}{\partial t} && (31)\end{aligned}$$

for any λ without effecting the electromagnetic field.

The Gauge freedom allows us to pick any function for the diveregence of $\vec{A}$. You may ask, "how is this possible?" After all $\vec{\nabla} \cdot \vec{A} = f(\vec{r}, t)$ is not what we said we could do. We simply said we could add $\vec{\nabla}\lambda$ to $\vec{A}$ and subtract $\frac{\partial \lambda}{\partial t}$ from V. Here is how it works. Suppose we have found $\vec{A}'$ that satisfies 17 and 18. We can always choose λ such that

$$\vec{A} = \vec{A}' + \vec{\nabla}\lambda \tag{32}$$

and

$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}' + \vec{\nabla}^2\lambda \tag{33}$$

Now suppose we choose λ to satisfy Poisson's equation

$$\vec{\nabla}^2\lambda = -\vec{\nabla} \cdot \vec{A}' + f(\vec{r}, t) \tag{34}$$

then we have choosen the divergence of $\vec{A}$.

The choice of $\vec{\nabla} \cdot \vec{A} = 0$ is the Coulomb gauge. Here 18 becomes

$$\vec{\nabla} \cdot V = -\rho/\varepsilon_o \tag{35}$$

and, as in the steady-state case

$$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d\tau' \tag{36}$$

This potential has a very unphysical nature. Suppose the charge density changes at a point $\vec{r}'$ in space. According to this equation, the potential at $\vec{r}$ changes immediately, violating the idea that information can only move at the speed of light. However, we must remember that the potential is only a mathematical construction. It is not a problem if $V(\vec{r}, t)$ is unphysical, but only if $\vec{E}$ is. Recall that

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \tag{37}$$

Apparantly $\vec{A}$ some how cancels out any information passing that might occur at the speed of light. It is no suprise that although V is very easy to calculate in the Coulomb gauge, it is not terribly useful because we need $\vec{A}$ to do physics and $\vec{A}$ is difficult to calculate. Instead we use the choice of the Lorentz gauge where

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \tag{38}$$

This leads to the beautifully symmetric form of Maxwell's equations

$$\Box^2 V = -\rho/\varepsilon_o \tag{39}$$

$$\Box^2 \vec{A} = -\mu_o \vec{J} \tag{40}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{41}$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \tag{42}$$

1.2 The Retarded Potential

It is interesting to try to fix 36 in order to gain a more physical potential function. If we let

$$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\vec{r}', t')}{|\vec{r} - \vec{r}'|} d\tau' \tag{43}$$

where $t' = t - \frac{|\vec{r} - \vec{r}'|}{c}$ then this potential does not respond to the field in a way that is unphysically quick. As it turns out, this potential is precisely the form of the potential in the Lorentz gauge. To see this, we note

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_o} \int \nabla^2 (\frac{\rho(\vec{r}', t')}{|\vec{r} - \vec{r}'|}) d\tau' \tag{44}$$

Now we have to be careful taking the Laplacian because both the $1/|\vec{r} - \vec{r}'|$ term and t' now depend on $\vec{r}$. Taking this into consideration, have

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_o} \int \rho(\vec{r}', t') \nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} d\tau' \tag{45}$$

$$+ \frac{1}{4\pi\varepsilon_o} \int \frac{\nabla^2 \rho(\vec{r}', t')}{|\vec{r} - \vec{r}'|} d\tau' \tag{46}$$

$$+ \frac{1}{2\pi\varepsilon_o} \int \left(\vec{\nabla} \rho(\vec{r}', t')\right) \cdot \left(\vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|}\right) d\tau' \tag{47}$$

Now we use

$$\vec{\nabla}\rho = \frac{\partial\rho}{\partial t}\vec{\nabla}(\frac{|\vec{r}-\vec{r}'|}{c}) \tag{48}$$

$$= \frac{1}{c}\frac{\partial\rho}{\partial t}(\vec{r}-\vec{r}') \tag{49}$$

and

$$\nabla^2\frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\delta(\vec{r}-\vec{r}') \tag{50}$$

to show that

$$\Box^2 V = -\rho/\varepsilon_o \tag{51}$$

It can be proven in the same way that

$$\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r}',t-|\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|}d\tau' \tag{52}$$

satisfies

$$\Box^2\vec{A} = -\mu_o\vec{J} \tag{53}$$

1.3 Jefimenko's Equations

In the last section we found

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int\frac{\rho(\vec{r}',t-|\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|}d\tau' \tag{54}$$

and

$$\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r}',t-|\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|}d\tau' \tag{55}$$

satisfy Maxwell's equations in the Lorentz gauge:

$$\Box^2 V = -\rho/\varepsilon_o \tag{56}$$

$$\Box^2\vec{A} = -\mu_o\vec{J} \tag{57}$$

where

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \tag{58}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{59}$$

Now we find $\vec{E}$ and $\vec{B}$ explicitly:

$$4\pi\vec{E} = -\vec{\nabla}\int\frac{\rho(\vec{r}',t-|\vec{r}-\vec{r}'|/c)}{\varepsilon_o|\vec{r}-\vec{r}'|}d\tau' \tag{60}$$

$$-\frac{\partial}{\partial t}\int\frac{\mu_o\vec{J}(\vec{r}',t-|\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|}d\tau' \tag{61}$$

We use

$$\vec{\nabla}\frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} = \rho\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}+\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\rho}{\partial t}\vec{\nabla}(\frac{|\vec{r}-\vec{r}'|}{c}) \quad (62)$$

$$= \rho\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}+\frac{\dot{\rho}}{|\vec{r}-\vec{r}'|c} \quad (63)$$

and the fact that $\mu_o = 1/\varepsilon_o c^2$ to find $\vec{E}$:

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\int\left[\frac{(\vec{r}-\vec{r}')\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|^3}+\frac{\frac{\partial}{\partial t_r}\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|c}-\frac{\frac{\partial}{\partial t_r}\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|c^2}\right]d\tau' \quad (64)$$

The curl of $\vec{A}$ gives $\vec{B}$:

$$\vec{\nabla}\times\vec{A} = \vec{\nabla}\times\frac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r}',t-|\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|}d\tau' \quad (65)$$

We note that

$$\vec{\nabla}\times\frac{\vec{J}}{|\vec{r}-\vec{r}'|} = \frac{1}{|\vec{r}-\vec{r}'|}\vec{\nabla}\times\vec{J}-\vec{J}\times\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} \quad (66)$$

$$= \frac{\vec{\nabla}\times\vec{J}}{|\vec{r}-\vec{r}'|}+\frac{\vec{J}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \quad (67)$$

Using

$$\vec{\nabla}\times\vec{J}(\vec{r}',t-|\vec{r}-\vec{r}'|/c) = \frac{\partial}{\partial t_r}\vec{J}\times(\vec{r}-\vec{r}')/c \quad (68)$$

we have

$$\vec{B} = \frac{\mu_o}{4\pi}\int\left[\frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}-\frac{\frac{\partial}{\partial t_r}\vec{J}(\vec{r}',t_r)}{c|\vec{r}-\vec{r}'|}\right]\times(\vec{r}-\vec{r}')d\tau' \quad (69)$$

1.4 The Lienard-Wiechert Potentials

Let us find the potential due to a point charge moving with a constant velocity $\vec{v}$. It's position is given by

$$\vec{r}_t = \vec{r}_o+\vec{v}_o t \quad (70)$$

so it's potential must be given by

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int\frac{q\delta(\vec{r}'-\vec{r}_o-\vec{v}_o(t-|\vec{r}-\vec{r}'|/c))}{|\vec{r}-\vec{r}'|}d\tau' \quad (71)$$

Without loss of generality, we can choose a coordinate system with $\vec{v}_o = v_o\hat{z}$. In this system we have

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int\frac{q\delta(x'-x_o)\delta(y'-y_o)\delta(z'-z_o-v_o(t-|\vec{r}-\vec{r}'|/c))}{|\vec{r}-\vec{r}'|}dx'dy'dz' \quad (72)$$

The integrals over x' and y' are trivial and lead to

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\int_{-\infty}^{\infty}\frac{\delta(z' - z_o - v_o(t - \frac{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z')^2}}{c}))}{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z')^2}}dz' \quad (73)$$

Now I substitute $\vec{r}_t = \vec{r}_o + \vec{v}_o t$ position of the particle at a time t and $\beta = v_o/c$. This implies that $z_t = z_o + v_o t$ is the $z-$position of the particle. We have

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\int_{-\infty}^{\infty}\frac{\delta(z' - z_t + \beta\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z')^2})}{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z')^2}}dz' \quad (74)$$

Recall the rule for evaluating a delta function of a function:

$$\int g(z')\delta(f(z'))dz' = \sum_i \frac{g(z_i)}{|\frac{df}{dz'}|\Big|_{z_i}} \quad (75)$$

In our case,

$$f(z') = z' - z_t + \beta\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z')^2} \quad (76)$$

so that

$$\frac{df}{dz'} = 1 + \beta\frac{z'-z}{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z')^2}} \quad (77)$$

$$= \frac{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z')^2} + \beta(z'-z)}{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z')^2}} \quad (78)$$

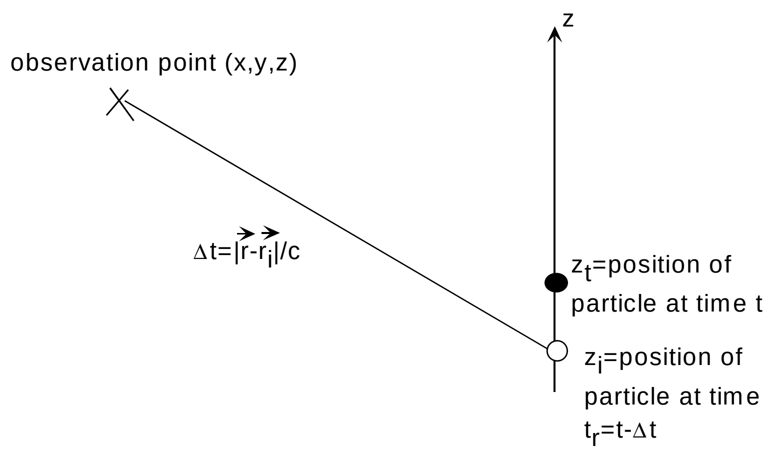
$$\frac{g(z_i)}{|\frac{df}{dz'}|\Big|_{z_i}} = \left|\frac{g(z_i)}{\frac{df}{dz'}\Big|_{z_i}}\right| \quad (79)$$

$$= \frac{1}{|\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_i)^2} - \beta(z-z_i)|} \quad (80)$$

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\sum_i \frac{1}{|\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_i)^2} - \beta(z-z_i)|} \quad (81)$$

We now pause to remember what these variables are. The variables x, y, and z are the point at which we wish to evaluate the potential at a time t. The variables x_o and y_o are (time independent) position of the charge moving along the z-axis. $\beta = v_o/c$ is the speed of the particle in units of the speed of light. z_i is the position of the particle at the retarded time t_r. We find this position when we satisfy

$$z_i - z_t + \beta\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_i)^2} = 0 \quad (82)$$



(83)

Which leads to a quadratic equation. After a great deal of manipulating the solutions to this quadratic, one finds:

$$z_i = z_o + v_o \left(t \pm \left[\frac{\frac{(z-z_t)v_o}{c} + \sqrt{R^2(1-\beta^2) + \left[\frac{(z-z_t)\cdot v_o}{c}\right]^2}}{c(1-\beta^2)} \right] \right) \tag{84}$$

Here I have let $R^2 = (x-x_o)^2+(y-y_o)^2+(z-z_t)^2$. Note that the term in square brackets is always positive. Thus on solution makes sense. It corresponds to the particle at times less than t. The other solution does not make sense as it corresponds to where the particle will be in the future. We throw away the future solution because it is not causal.

Now let us assume $\vec{r}_i$ is the causal solution. The potential is then

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_o} \frac{1}{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_i)^2} - \beta(z_i - z)} \tag{85}$$

We can throw out the absolute value sign because β is less than one and $(z_i - z)$ is always less than the square root term. We wish to write this solution in a coordinate system independent way. To do this, we note

$$(z-z_i)\beta = (z-z_i)v_o/c \tag{86}$$
$$= (z-z_i)v_{oz}/c \tag{87}$$
$$= (\vec{r}-\vec{r}_i)\cdot\vec{v}_o/c \tag{88}$$

and that

$$\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_i)^2} = |\vec{r}-\vec{r}_i| \tag{89}$$

so we have

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{|\vec{r}-\vec{r}_i| - (\vec{r}-\vec{r}_i)\cdot\vec{v}_o/c}\right] \tag{90}$$

The current density is given from $\vec{J} = \rho\vec{v}_o$. Following the same arguments with

$$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{J}}{|\vec{r}-\vec{r}'|}d\tau' \tag{91}$$

one finds

$$\begin{aligned}\vec{A} &= \frac{\mu_o}{4\pi}\left[\frac{q\vec{v}_o}{|\vec{r}-\vec{r}_i| - (\vec{r}-\vec{r}_i)\cdot\vec{v}_o/c}\right] && (92)\\ &= \frac{\vec{v}}{c^2}V(\vec{r},t) && (93)\end{aligned}$$

Equations 90 and 92 give the Lienard-Wiechert potentials. Although we have only derrived these expressions for a charge moving at a constant velocity, they are true in general.

For a charge moving at a constant velocity, the point $\vec{r}_i$ is given by 84. This expression may be written in a coordinate independent way by introducing the current displacement vector $\vec{R} = \vec{r} - \vec{r}_t = \vec{r} - \vec{r}_o - \vec{v}_o t$. Here $\vec{R}$ gives the instantaneous displacement of the charge from the observer.

$$\vec{r}_i = \vec{r}_o + \vec{v}_o\left[t - \frac{\left[\frac{\vec{R}\cdot\vec{v}_o}{c}\right] + \sqrt{R^2(1-\beta^2) + \left[\frac{\vec{R}\cdot\vec{v}_o}{c}\right]^2}}{c(1-\beta^2)}\right] \tag{94}$$

Let us return to the potential functions V :

$$\begin{aligned}V(\vec{r},t) &= \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{|\vec{r}-\vec{r}_i| - (\vec{r}-\vec{r}_i)\cdot\vec{v}_o/c}\right] && (95)\\ &= \frac{q}{4\pi\varepsilon_o}\frac{1}{|\vec{r}-\vec{r}_i|}\left[\frac{1}{1-(\vec{r}-\vec{r}_i)\cdot\vec{v}_o/c|\vec{r}-\vec{r}_i|}\right] && (96)\end{aligned}$$

Notice how subtle E&M has become. We know that the potential due to a static point charge is given by

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{|\vec{r}-\vec{r}_i|}\right] \tag{97}$$

and we have been told that the potential due to time-dependent sources is given by

$$V(\vec{r},t) = \int \frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}d\tau' \tag{98}$$

Apparently this does not imply we can simply assume that $\vec{r}_i$ is replaced by the retarded position to find V. This extra factor of

$$\frac{1}{1-(\vec{r}-\vec{r}_i)\cdot\vec{v}_o/c|\vec{r}-\vec{r}_i|} = \frac{1}{1-v_{||}/c} \tag{99}$$

can actually be understood in terms of a simple geometric argument. (See your text, page 431.)

The $\vec{E}$ and $\vec{B}$ fields of a moving charge are given in by equation 10.65 of your text. They are derived from Eq 90, Eq 92, and a lot of work.