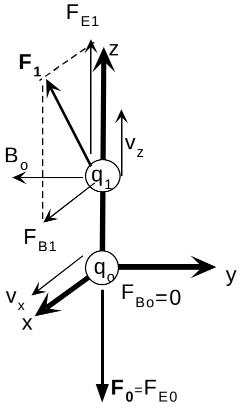
0.1 Conservation of Momentum and the Maxwell Stress Tensor

In the last lecture we started from the fact that the mechanical energy transferred from an E&M field to a particle per unit time is given by

$$\frac{dU_i}{dt} = \vec{F}_i \cdot \vec{v}_i \tag{1}$$

in order to develop a theory of conservation of energy.

Now we wish to develop a theory of conservation of momentum in a similar way. You may ask what makes us suspect that the electromagnetic field carries any momentum? We know from flash cameras that the electric field does indeed store energy, but it not as common to see the effect of the momentum of an electromagnetic field. A simple argument given on page 350 of your book tells us that an electromagnetic field must carry momentum. Here is my version of this argument: Consider a charge $q_o$passing the origin moving with a velocity v_x and another charge q_1 moving with a velocity v_z passing through the z axis at a point $(0,0,z_1)$ as shown:



(2)

If we just consider the forces due to the electric field, we have $\vec{F}_{E1} = q_1\vec{E}_o = -q_o\vec{E}_1 = -\vec{F}_{E0}$. This is a happy situation - we are obeying Newton's third law stating that the force particle 0 exerts on 1 is equal in magnitude but opposed to the force 1 exerts on 0. As the speeds of the particles increase, however, we observe this balance to be lost. We can see this by analyzing the $\vec{B}$ field of the moving charge. The $\vec{B}$ created from the moving charge q_o at q_1 is in the $-y$ direction whereas the $\vec{B}$ field created from the moving charge q_1 is zero at the origin. (We will derive this result when we study special relativity.) Thus the $\vec{B}$ field creates a force on q_1 that is not equal but opposite to the force on q_o. (In addition to the unequal magnetic forces, the electric forces stop balancing as the speed of the particles increase.) We can only save the third law if we associate a momentum with the electromagnetic field. Indeed this is the case.

Of course we need three equations in this case because we have conservation of momentum in the x, y, and z directions. Thus things may get a bit ugly, but let us plow ahead. From Freshman physics we have

$$\begin{aligned}
\vec{F}_i &= \frac{d\vec{p}_i}{dt} \quad (3)\\
&= q_i\left(\vec{E}+\vec{v}_i\times\vec{B}\right) \quad (4)
\end{aligned}$$

The force $\Delta\vec{F}$ on a tiny volume $\Delta\tau$ is found by substituting $\rho\Delta\tau$ for q_i:

$$\begin{aligned}
\Delta\vec{F} &= \left(\rho\vec{E}+\rho\vec{v}\times\vec{B}\right)\Delta\tau \quad (5)\\
&= \left(\rho\vec{E}+\vec{J}\times\vec{B}\right)\Delta\tau \quad (6)
\end{aligned}$$

Now we define $\vec{f}$ as the force per unit volume exerted by the electromagnetic fields:

$$\begin{aligned}
\vec{f} &= \rho\vec{E}+\vec{J}\times\vec{B} \quad (7)\\
&= \varepsilon_o(\vec{\nabla}\cdot\vec{E})\vec{E}+\left(\frac{1}{\mu_o}\vec{\nabla}\times\vec{B}-\varepsilon_o\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B} \quad (8)
\end{aligned}$$

Next we note that

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B}=\frac{\partial\left(\vec{E}\times\vec{B}\right)}{\partial t}-\vec{E}\times\frac{\partial\vec{B}}{\partial t} \quad (9)$$

We can then use the fact that

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad (10)$$

to make the only time derivative to act on the Poynting vector:

$$\begin{aligned}
\frac{\partial\vec{E}}{\partial t}\times\vec{B} &= \frac{\partial\left(\vec{E}\times\vec{B}\right)}{\partial t}+\vec{E}\times(\vec{\nabla}\times\vec{E}) \quad (11)\\
&= \mu_o\frac{\partial\vec{S}}{\partial t}+\vec{E}\times(\vec{\nabla}\times\vec{E}) \quad (12)
\end{aligned}$$

Putting this all together we have

$$\vec{f}=\varepsilon_o(\vec{\nabla}\cdot\vec{E})\vec{E}+\frac{1}{\mu_o}(\vec{\nabla}\times\vec{B})\times\vec{B}+\varepsilon_o(\vec{\nabla}\times\vec{E})\times\vec{E}-\mu_o\varepsilon_o\frac{\partial\vec{S}}{\partial t} \quad (13)$$

Now we simplify the cross product of the curl of a vector with itself:

$$\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}=\varepsilon_{uiv}\left(\varepsilon_{ijk}\partial_j B_k\right)B_v\widehat{x}_u \quad (14)$$

Here ε_{ijk} is the anti-symmetric tensor and summation over repeating indices is assumed. Now we use

$$\begin{aligned}
\varepsilon_{uiv}\varepsilon_{ijk} &= -\varepsilon_{iuv}\varepsilon_{ijk} \quad (15)\\
&= -(\delta_{uj}\delta_{vk}-\delta_{uk}\delta_{vj}) \quad (16)
\end{aligned}$$

So that

$$\begin{aligned}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B} &= (\delta_{uk}\delta_{vj}-\delta_{uj}\delta_{vk})(\partial_j B_k)B_v\widehat{x}_u && (17)\\ &= (\vec{B}\cdot\vec{\nabla})\vec{B}-\frac{1}{2}\vec{\nabla}B^2 && (18)\end{aligned}$$

Substituting 17 into 13 we have

$$\begin{aligned}\vec{f} &= \varepsilon_o\left[\vec{E}(\vec{\nabla}\cdot\vec{E})+\left(\vec{E}\cdot\vec{\nabla}\right)\vec{E}\right]\\ &\quad +\frac{1}{\mu_o}\left[\vec{B}(\vec{\nabla}\cdot\vec{B})+\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}\right]\\ &\quad -\vec{\nabla}\left(\frac{\varepsilon_o}{2}E^2+\frac{1}{2\mu_o}B^2\right)-\frac{1}{c^2}\frac{\partial\vec{S}}{\partial t} && (19)\end{aligned}$$

(We have thrown in a term proportional to $\vec{\nabla}\cdot\vec{B}=0$ just to make this look more symmetric.) The last two terms of this expression are perhaps easier to rationalize. The second-to-last term is the gradient of the potential energy per unit volume-exactly what we expect for a force per unit volume. We will soon see that the last term is proportional the time-dependent change in momentum of the E&M field. I wish I could tell you the first two terms vanish-but they do not. They occur because of the vector nature of the electromagnetic radiation.

What does one due when one has a complex expression in physics? Invent new notation to make it look more simple. Here we invent the Maxwell Stress Tensor $\overleftrightarrow{T}$ where

$$\overleftrightarrow{T}_{ij}=\varepsilon_o(E_iE_j-\frac{1}{2}\delta_{ij}E^2)+\frac{1}{\mu_o}(B_iB_j-\frac{1}{2}\delta_{ij}B^2) \qquad (20)$$

Question: What are $\overleftrightarrow{T}_{xy}$ and $\overleftrightarrow{T}_{zz}$?.

Question: What is $\overleftrightarrow{T}_{xx}+\overleftrightarrow{T}_{yy}+\overleftrightarrow{T}_{zz}$?

Question: What is $\vec{a}\cdot\overleftrightarrow{T}$?
Answer:

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{3}a_i\overleftrightarrow{T}_{ij}\widehat{x}_j &= \varepsilon_o\left((\vec{a}\cdot\vec{E})\vec{E}-\frac{1}{2}\vec{a}(E^2)\right) && (21)\\ &\quad +\frac{1}{\mu_o}\left((\vec{a}\cdot\vec{B})\vec{B}-\frac{1}{2}\vec{a}(B^2)\right) && (22)\end{aligned}$$

This notation is really powerful and saves a lot of writing once you get used to it. Eventually we will get to an even slicker notation that will allow us to omit the summation symbols. If your confused, consider the same multiplication in matrix form for the case $\vec{B} = 0$:

$$\begin{aligned}
&\vec{a} \cdot \overleftrightarrow{T} \qquad \qquad (23) \\
&= (a_x, a_y, a_z)\varepsilon_o \begin{matrix} \frac{(E_x^2 - E_y^2 - E_z^2)}{2} & E_xE_y & E_xE_z \\ E_yE_x & \frac{(E_y^2 - E_x^2 - E_z^2)}{2} & E_yE_z \\ E_zE_x & E_zE_y & \frac{(E_z^2 - E_x^2 - E_y^2)}{2}) \end{matrix} \qquad (24) \\
&= \left(a_x \frac{(E_x^2 - E_y^2 - E_z^2)}{2} + E_yE_xa_y + E_zE_xa_z, \; ... \right. \qquad (25) \\
&= \left(a_x \frac{(-E_x^2 - E_y^2 - E_z^2)}{2} + E_yE_xa_y + E_zE_xa_z + E_xE_xa_x, \; ... \right. \qquad (26) \\
&= \left(\frac{-a_xE^2}{2} + (\vec{a} \cdot \vec{E})E, \; ... \right. \qquad (27) \\
&= \left(\frac{-a_xE^2}{2} + (\vec{a} \cdot \vec{E})E_x, \frac{-a_yE^2}{2} + (\vec{a} \cdot \vec{E})E_y, \frac{-a_zE^2}{2} + (\vec{a} \cdot \vec{E})E_z \right) \qquad (28) \\
&= -\frac{E^2}{2}(a_x, a_y, a_z) + \vec{a} \cdot \vec{E}(E_x, E_y, E_z) \qquad (29) \\
&= \left(\vec{a} \cdot \vec{E} \right) \vec{E} - \frac{1}{2}E^2\vec{a} \qquad (30)
\end{aligned}$$

This might be a bit backwards from your usual way of matrix multiplication. But this order makes it easier for us to consider the divergence of $\overleftrightarrow{T}$:

$$\begin{aligned}
\vec{\nabla} \cdot \overleftrightarrow{T} &= \sum_{i=1}^{3}\sum_{j=1}^{3} \partial_i \overleftrightarrow{T}_{ij}\widehat{x}_j \qquad (31) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \partial_i \left(\varepsilon_o(E_iE_j - \frac{1}{2}\delta_{ij}E^2) + \frac{1}{\mu_o}(B_iB_j - \frac{1}{2}\delta_{ij}B^2) \right) \widehat{x}_j \qquad (32) \\
&= \varepsilon_o \left((\vec{\nabla} \cdot \vec{E})\vec{E} + (\vec{E} \cdot \vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}(E^2) \right) \qquad (33) \\
&\quad + \frac{1}{\mu_o} \left((\vec{\nabla} \cdot \vec{B})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}(B^2) \right) \qquad (34)
\end{aligned}$$

We are used to the divergence of a vector field giving us a scalar. Here the divergence of a tensor field gives us a vector. So it goes. Let us take another look for our expression for the force per unit volume exerted by a electromagnetic field:

$$\begin{aligned}
\vec{f} &= \varepsilon_o \left[\vec{E}(\vec{\nabla} \cdot \vec{E}) + \left(\vec{E} \cdot \vec{\nabla} \right) \vec{E} \right] \\
&\quad + \frac{1}{\mu_o} \left[\vec{B}(\vec{\nabla} \cdot \vec{B}) + \left(\vec{B} \cdot \vec{\nabla} \right) \vec{B} \right] \\
&\quad - \vec{\nabla} \left(\frac{\varepsilon_o}{2}E^2 + \frac{1}{2\mu_o}B^2 \right) - \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} \qquad (35)
\end{aligned}$$

The divergence of the Maxwell-stress tensor is exactly all the terms in this expression except the term proportional to the time derivative of $\vec{S}$:

$$\vec{f} = \vec{\nabla} \cdot \overleftrightarrow{T} - \frac{1}{c^2} \frac{\partial \vec{S}}{\partial t} \tag{36}$$

A force per unit area is a pressure, thus $\vec{T}$ has units of pressure. Indeed the diagonal elements of $\overleftrightarrow{T}$ correspond to the pressure acting on a differential area element. Unlike forces due to the pressure of an ideal gas, an area element in the electromagnetic field also feels a force in a direction that is not normal to the element. This shear (rather than pressure) is given by the off-diagonal elements of $\vec{T}$.

The total force on a volume is given by integrating the force per unit volume:

$$\vec{F} = \int \left(\vec{\nabla} \cdot \overleftrightarrow{T} - \frac{1}{c^2} \frac{\partial \vec{S}}{\partial t} \right) d\tau \tag{37}$$

For each j^{th} element of $\overleftrightarrow{T}_{ij}$ we can apply the divergence theorem allowing us to write

$$\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a} - \frac{1}{c^2} \frac{d}{dt} \int \vec{S} d\tau \tag{38}$$

The first term is the integrated shear and pressure at the surface of the object. The second will become clear

Let us try to give physical meaning to 36. Recall that the energy leaving a differential area per unit time is given by $\vec{S}$. Let us suppose that this energy was carried away by little particles (say photons) of velocity $\vec{c}$ and energy E. Then we would expect the energy escaping the volume would be given by

$$\vec{S} = \rho_N E \vec{c} \tag{39}$$

Here $\vec{v}$ is the local velocity of the particles, E is the energy per particle, and ρ_N is the number of particles per unit volume.

The momentum of a tiny volume $\Delta\tau$ of space is expected to be the number of particles in that volume ($N = \rho_N \Delta\tau$) multiplied by the momentum of each particle $\vec{p}$. Thus, the momentum per unit volume is simply.

$$\vec{p}_{em} = \rho_N \vec{p} \tag{40}$$

How does 39 relate to 40? That depends how energy and momentum are related. Recall that lovable physics slogan $E = mc^2$. You know that is short for

$$E = \sqrt{m^2 c^4 + p^2 c^2} \tag{41}$$

If $m = 0$, we get

$$E = pc \tag{42}$$

so that

$$E\vec{c} = \vec{p}c^2 \tag{43}$$

and

$$\rho_N E\vec{c} = \rho_N \vec{p} c^2 \tag{44}$$

But $\vec{S} = \rho_N E\vec{c}$ and $\vec{p}_{em} = \rho_N \vec{p} = \rho_N E\vec{c}/c^2 =$. Thus we have (very informally) justified the expression

$$\vec{p}_{em} = \frac{1}{c^2}\vec{S} \tag{45}$$

This states that the momentum per unit volume associated with the electromagnetic field is given by $\vec{S}/c^2$. The force acting by the field per unit volume is therefore given by

$$\vec{f} = \vec{\nabla} \cdot \overleftrightarrow{T} - \frac{\partial \vec{p}_{em}}{\partial t} \tag{46}$$

What is this force per unit volume doing? It is moving around charged particles and, by doing so, changing the total amount of mechanical energy in the system. Thus

$$\frac{\partial \vec{p}_{mech}}{\partial t} = \vec{\nabla} \cdot \overleftrightarrow{T} - \frac{\partial \vec{p}_{em}}{\partial t} \tag{47}$$

This can be written as a conservation law by grouping the time-dependent terms together:

$$\frac{\partial (\vec{p}_{mech} + \vec{p}_{em})}{\partial t} = \vec{\nabla} \cdot \overleftrightarrow{T} \tag{48}$$

Thus $\overleftrightarrow{T}$ represents the time rate of change of electromagnetic momentum out of a differential volume volume. It has two contributions. One is the change of mechanical momentum of the system and the second is the change of particle momentum in the system.

Notice the very different role that $\vec{S}$ plays in conservation of energy (Poynting's theorem) and in conservation of momentum. In Poynting's theorem, $\vec{S}$ is a flux that is integrated over an area to find the energy escaping a given volume. For conservation of momentum $\vec{p} = \vec{S}/c^2$ is a momentum per unit volume to be integrated if one wants to determine the total electromagnetic momentum in a given volume.

0.2 Conservation of angular momentum

We have found that the momentum per unit volume associated with a magnetic field is given by

$$\begin{aligned} \vec{p}_{em} &= \frac{1}{c^2}\vec{S} && (49) \\ &= \frac{1}{\mu_o c^2}\vec{E} \times \vec{B} && (50) \end{aligned}$$

Not surprisingly, if we specify and origin, we can speak of the angular momentum per unit volume with respect to that origin. It is given by

$$\begin{aligned} \vec{\ell}_{em} &= \vec{r} \times \vec{p}_{em} && (51) \\ &= \frac{1}{\mu_o c^2}\vec{r} \times (\vec{E} \times \vec{B}) && (52) \end{aligned}$$

What is really amazing is the fact that even a static field can contain nonzero angular momentum.

As a sample problem, consider the angular momentum due to the interaction of a charge q_e and a hypothetical magnetic charge q_m. Let us put the electric charge at the origin and the magnetic charge at a location $\vec{d}$. Then

$$\begin{aligned} \vec{E} &= \frac{q_e}{4\pi\varepsilon_o}\frac{\vec{r}}{r^3} && (53) \\ \vec{B} &= \frac{q_m\mu_o}{4\pi}\frac{(\vec{r}-\vec{d})}{|\vec{r}-\vec{d}|^3} && (54) \end{aligned}$$

The angular momentum per unit volume is given by

$$\begin{aligned} \vec{\ell}_{em} &= \frac{1}{\mu_o c^2}\vec{r}\times(\vec{E}\times\vec{B}) && (55) \\ &= \frac{\mu_o q_m q_e}{16\pi^2 r^3|\vec{r}-\vec{d}|^3}\vec{r}\times\left(\vec{r}\times(\vec{r}-\vec{d})\right) && (56) \\ &= \frac{\mu_o q_m q_e}{16\pi^2 r^3|\vec{r}-\vec{d}|^3}\vec{r}\times\left(\vec{d}\times\vec{r}\right) && (57) \\ &= \frac{\mu_o q_m q_e}{16\pi^2 r^3|\vec{r}-\vec{d}|^3}\left[r^2\vec{d}-\left(\vec{r}\cdot\vec{d}\right)\vec{r}\right] && (58) \end{aligned}$$

To get the total angular momentum of the system, we choose a coordinate system with $\vec{d}=d\widehat{z}$ so

$$\begin{aligned} \vec{\ell}_{em} &= \frac{\mu_o q_m q_e d}{16\pi^2 r(r^2+d^2-2dr\cos\theta)^{3/2}}\left[\widehat{z}+\widehat{r}\right] && (59) \\ &= \frac{\mu_o q_m q_e d}{16\pi^2 r(r^2+d^2-2dr\cos\theta)^{3/2}}\left[(1-\cos\theta)\widehat{z}-\sin\theta(\cos\phi\widehat{x}+\sin\phi\widehat{y})\right] && (60) \end{aligned}$$

The next step is to integrate this expression over all space:

$$\begin{aligned} \vec{L}_{em} &= \int \vec{\ell}_{em}d\tau && (61) \\ &= \int \vec{\ell}_{em}r^2\sin\theta drd\theta d\phi && (62) \end{aligned}$$

The integration over ϕ will cause the x and y components of $\vec{L}_{em}$ to vanish, leaving

$$L_{em,z} = \frac{\mu_o q_m q_e}{8\pi}\int_o^\infty\int_o^\pi \frac{rd(1-\cos\theta)\sin\theta}{(r^2+d^2-2dr\cos\theta)^{3/2}}d\theta dr \qquad (63)$$

Now we let $R=r/d$ and $\chi=\cos\theta$ so that

$$L_{em,z} = \frac{\mu_o q_m q_e}{8\pi}\int_o^\infty\int_{-1}^1 \frac{R(1-\chi)}{(1+R^2-2R\chi)^{3/2}}d\chi dR \qquad (64)$$

Without even solving this integral we recognize something amazing: The integral is just a constant number happens to be 2. Thus the angular momentum associated with the electric field of a single charge at the origin and a monopole at $\vec{d}$ is given by

$$\vec{L}_{em} = \frac{\mu_o q_m q_e}{4\pi}\hat{d}$$

This answer is independent of d!!! Kim will discuss the implications of this in his guest lecture.