Problem Set II, Due Monday January 31st (extra credit if three problems turned in on Friday.)

7.1, 7.3, 7.8, 7.10, 7.13, 7.16, 7.17, 7.24, 7.39, 7.42

1 Ohm's Law, the Electromotive Force, and Faraday's law of Induction, and mutual inductance

1.1 Ohm's law

Background: When a marble drops in molasses, the drag it experiences is proportional to the rate it falls (by a drag law due to Stokes.) For this case, the net force on the marble is given by

$$\begin{aligned} \vec{F}_{net} &= m\vec{g} + \vec{F}_d && (1) \\ &= m\vec{g} - \alpha\vec{v} && (2) \end{aligned}$$

terminal velocity is reached very quickly as the marble drops. At terminal velocity, the acceleration is zero and

$$\begin{aligned} \vec{F}_{net} &= \vec{0} && (3) \\ \vec{v} &= \frac{1}{\alpha} m\vec{g} && (4) \end{aligned}$$

The important points of this situation are that (1) a terminal velocity is reached very quickly, (2) the final velocity is proportional to the applied force, (3) the work done by the drag force must be of opposite sign as the work done by the applied force. The power dissipated per unit time is therefore $\vec{P} = vmg$.

The situation for charges (say electrons) moving in matter is similar to marbles in molasses. Free charges in a conductor move by bouncing into hot nuclei doing a kind of random walk through the material. If we are able to apply a small additional force, then the random walk is modified ever so slightly. The collisions with hot nuclei act as a drag force proportional to the drift velocity of the electrons. We have the situation that the drift velocity of the electrons $\vec{v}_e$ is directly proportional to the applied force $\vec{F}_e$:

$$\vec{v}_e = \alpha\vec{F}_e \qquad (5)$$

Actually, 5 is a bit too simple minded. In E&M we write

$$\vec{J} = \sigma\vec{f} \qquad (6)$$

Here $\vec{J}$ is the local velocity of charge carriers times the local charge density. $\vec{f}$ is the force per unit charge. Finally, σ is the conductivity of the material. We will consider forces per unit charge $\vec{f}$ that are electrochemical, magnetic, and electrical. The physics of 6 has little to do with the nature of the force applied and everything to with a drag on the electrons proportional to the drift velocity.

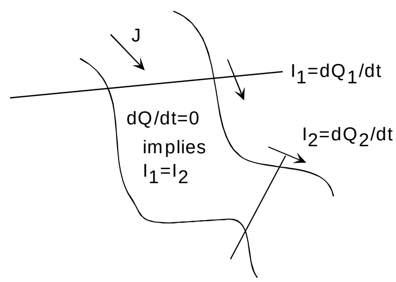
If the force per unit charge is caused by an electric field, then 6 becomes

$$\vec{J} = \sigma \vec{E} \tag{7}$$

which is a less familiar form of ohm's Law.

1.2 Current density and current

To get to the more familiar form of ohms law, we need to introduce the concept of current. Current can replace the more exact current density $\vec{J}$ in a conductor because of the fact that, in most conductors, there is no accumulation of charge. Not every device behaves in this way (capacitors for instance, accumulate charge.) However, even rather poor conductors do not like to build up charge anywhere. This implies that charge per unit time passing any cross section of a conductor is always the same:



(8)

This is extremely handy, for it lets us describe what might be an enormously complex current density $\vec{J}$ with a single quantity I.

Now suppose the only force acting on a charge is electrostatic, so ohm's law holds

$$\vec{J} = \sigma \vec{E} \tag{9}$$

In this case the potential difference between two points is given by

$$V = -\int_b^a \vec{E} \cdot d\vec{\ell} \tag{10}$$

$$= -\frac{1}{\sigma} \int_b^a \vec{J} \cdot d\vec{\ell} \tag{11}$$

As it turns out, the integral of 11 is always proportional to the current flowing from point a to b. The constant of proportionality depends on the geometry of the conductor and is given the name resistance. Thus we have the familiar form of Ohm's law:

$$V = IR \tag{12}$$

SAMPLE PROBLEM: What is the resistance between two cocentric spheres of radius a and $b > a$ if the volume in between is filled with material of conductance σ?

1.3 The free-electron model of conductivity

The free electron model of conductivity (due to Drude) helps us gain a physical insight to the fact that $\vec{J}$ is proportional to $\vec{f}$. In this picture, a conductor consists of a certain number per unit volume n_e of charge carriers that move a very high velocities. With a frequency ν, These electrons undergo collisions with nuclei in the matter with a period τ. Each collision completely scrambles the electrons original velocity. If a force per charge $\vec{f}$ is applied, the electrons of charge $-e$ are accelerated, but only during the time between collisions. Thus the average drift of the electrons due to the field is given by

$$\vec{v}_d = \frac{-e\tau}{2m}\vec{f} \tag{13}$$

The current density $\vec{J}$ is given by the current density $\rho = -e\,n_e$ multiplied by this drift speed:

$$\vec{J} = \frac{(-e)^2 n_e \tau}{2m}\vec{f} \tag{14}$$

If we assume λ is the average distance between nuclei in the conductor and $v_{thermal}$ is the average speed of the electrons, we have

$$\vec{J} = \frac{e^2 n_e \lambda}{2m v_{thermal}}\vec{f} \tag{15}$$

Thus, according to the free-electron model, the conductivity of a material of charge density n_e and charge $-e$ is given by

$$\sigma = \frac{e^2 n_e \lambda}{2m v_{thermal}} = \frac{e^2 n_e \tau}{2m} \tag{16}$$

Notice that the conductivity of a piece of material is independent of the sign of the charge carrier. In a conductor, the conductivity tends to go down with increasing temperature (because $v_{thermal}$ increases.) In a semiconductor or insulator, conductivity tends to increase with increasing temperature because the number of charge carriers per unit volume (n_e) increases.

1.4 Electromotive Force

In practice there are two types of forces on charge carriers: the first is source forces which we designate with a force per unit charge $\vec{f_s}$. These forces might be chemical, mechanical, a temperature gradient, or magnetic. The second type of force per charge are electrostatic forces $\vec{E}$. The electrostatic forces prevent charge from accumulating in any conductor and transfer charges from the source to distance parts of the circuit. In any case, the total source per unit charge is given by

$$\vec{f} = \vec{f_s} + \vec{E} \tag{17}$$

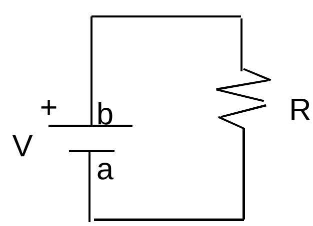
The electromotive force in a circuit is not a force. Instead it is the force per unit charge integrated about a closed loop:

$$\begin{aligned} \xi &= \oint \vec{f} \cdot d\vec{\ell} & (18) \\ &= \oint \vec{f}_s \cdot d\vec{\ell} + \oint \vec{E} \cdot d\vec{\ell} & (19) \\ &= \oint \vec{f}_s \cdot d\vec{\ell} & (20) \end{aligned}$$

There is a very important point in this definition of ξ that must not be left out. $\vec{f}$ is not the net force per unit charge. Instead it is the sum of electric forces and those forces that do positive work on the electron. It neglects resistive forces! For a steady-state circuit, if resistive forces were not neglected, $\xi = 0$ or the electrons could not go at a constant velocity.

1.4.1 Case 1: the ideal battery.

Consider an ideal battery that supplies power with no internal resistance (i.e., no heat loss) to a resistor in a simple circuit as shown below:



(21)

In ideal battery supplies charge without resistance. Thus σ may be taken as infinite, which implies for finite $\vec{J}$, $\vec{f} = \vec{0}$. Given $\vec{f} = \vec{0}$, the source force and electrostatic forces in the battery must cancel ($\vec{f}_s = -\vec{E}$.) Thus, if we call one terminal of the battery b and the other a, the potential at b with respect to a is given by

$$\begin{aligned} V &= -\int_a^b \vec{E} \cdot d\vec{\ell} & (22) \\ &= \int_a^b \vec{f}_s \cdot d\vec{\ell} & (23) \end{aligned}$$

On the other hand, there are no source forces in the resistor, so

$$\int_b^a \vec{f}_s \cdot d\vec{\ell} = 0 \qquad (24)$$

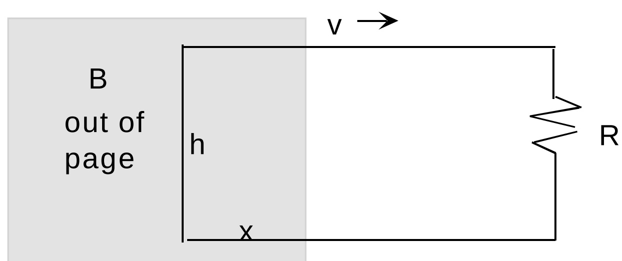
finally we have

$$\xi = \int_a^b \vec{f}_s \cdot d\vec{\ell} + \int_b^a \vec{f} \cdot d\vec{\ell} = V \qquad (25)$$

1.4.2 CASE 2: Motional emf

You might be thinking the last case was a bit cloudy. We got to $\xi = V$ by assuming σ is infinite in the battery instead of by examining the nature of the chemical forces that actually move the charges about. The case of motional ξ, the source of energy is somebody actually pulling a wire and the forces are much easier to see. The case of motional ξ is also actually more common: it is the way in which power generators produce the electricity that is delivered to our houses.

We consider the simple case of a loop of wire being pulled through a region of nonzero magnetic field, as shown in the figure below:



(26)

Motion through the magnetic field causes the charges to feel a force per charge of magnitude

$$f_s = vB \tag{27}$$

that is directed from the top of the page to the bottom. The emf is therefore

$$\xi = \oint \vec{f}_s \cdot d\vec{\ell} \tag{28}$$

$$= vBh \tag{29}$$

QUESTION: What is the force required to move the loop at the constant velocity $\vec{v}$?

ANSWER: The power dissipated by the resistor is $\xi^2/R = (vBh)^2/R$ which is in turn given by $\vec{F}_{applied} \cdot \vec{v}$. Thus $\vec{F}_{applied} = \frac{B^2h^2}{R}\vec{v}$.

Notice that

$$\xi = vBh \tag{30}$$

$$= \frac{d\Phi_B}{dt} \tag{31}$$

where

$$\Phi_B = \oint \vec{B} \cdot d\vec{a} \text{ through the loop} \tag{32}$$

$$= Bhx. \tag{33}$$

The virtue of 31 is that it is true for any loop moving in an arbitrary direction through an arbitrary field. This is proved in your text.

1.5 Faraday's law of induction

The concept of relativity is so ingrained in our teaching that you might not be as inspired by the following experiment as the likes of Faraday and Einstein. If you look at the situation too casually, you can almost miss the point.

Faraday did the following three experiments:

(1) He observed a current flow through a circuit because of motional emf by pulling a wire loop through a magnetic field as pictured in the last section. This was not an unexpected result. The force of the magnetic field on moving charges was well understood at the time and the current flowing was induced simply by this force.

(2) The next thing he did was to move the magnetic field rather that the charges. AN INCREDIBLE THING HAPPENED! He observed a current in the loop. BUT WHY? the charges in the loop in this case are stationary. Magnetic fields do not put forces on stationary charges.

(3) The final thing he did was to reduce the value of the magnetic field (using an electromagnet.) Again a current was induced in the loop. BUT WHY? Magnetic fields to not put forces on stationary charges.

What Faraday concluded was the beginning of the fusion of electricity and magnetism. He concluded that a changing magnetic field somehow creates an electric field such that

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} \tag{34}$$

$$= -\oint \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \tag{35}$$

Using Stokes' law, this can be written

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{36}$$

It is important to realize what an incredible difference there is between the first experiment and the second two. The first experiment involved no great departure from previous theories of electricity and magnetism: A magnetic field was known to put forces on moving charges. The second two experiments, however, were truly new results. The force in on the charges that create the emf in this case are electric, not magnetic, in nature and are created because of 36.

In the end, the fact that the mechanisms for inducing the emf are all very different does not change the simple result

$$\xi = -\frac{d\Phi_B}{dt} \tag{37}$$

This result is known as Faraday's law of induction. The fact that this exact expression holds regardless if magnetic or electric forces are ultimately responsible for the motion of the charges inspired Einstein's theory of special relativity.

1.6 Energy dissipated in a circuit

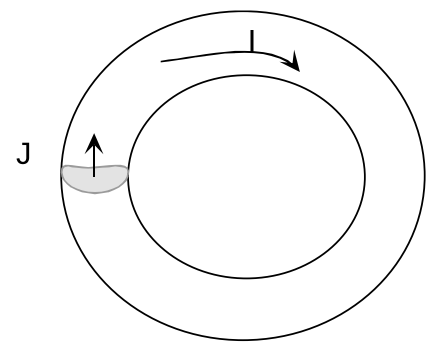
Earlier we concluded that the velocity of charges in a circuit is proportional to the force applied because of an inherent drag force. Of course there is heating associated with this drag force that corresponds to the power spent per charge to keep it from accelerating. Specifically, the power per charge is given by $\vec{f} \cdot \vec{v}$ so the power per volume must be

$$\frac{dP}{dV} = \vec{f} \cdot \rho\vec{v} \tag{38}$$

$$= \vec{f} \cdot \vec{J} = fJ \tag{39}$$

$$P = \int fJdV \tag{40}$$

(The dot product can be dropped because $\vec{J}$ is proportional to $\vec{f}$.) Now let us consider the total power dissipated in a current loop. To do this, we must evaluate the integral of 40. We keep matters simple by considering a neat donut shaped loop with $\vec{J}$ (and $\vec{f} = \frac{1}{\sigma}\vec{J}$) uniform. We then divide the volume of the loop it into cross sections for which $\vec{f}$ (and $\vec{J}$) is constant and normal to the cross section.



(41)

We have then

$$P = \int fJdV \tag{42}$$

$$= \oint f \left[\int Jda \right] d\ell \tag{43}$$

$$= I \oint \vec{f} \cdot d\vec{\ell} \tag{44}$$

$$= \xi I \tag{45}$$

I have been a little sloppy with my "proof". Specifically I have not correctly treated the general case. (If any of you would like to offer a better derivation of P, I would welcome it.) In any case, it is important to realize that the source of the emf, be it mechanical, electrical, or magnetic in nature is not what gives us the familiar power law. It is instead the fact that the current density is proportional to the forces applied which requires a drag force that dissipates power as given by $P = \xi I$.

1.7 Inductance

1.7.1 Mutual Inductance

Now we depart from lofty experiments that helped define the course of modern physics to the more practical matter of electronic devices such as choke coils and transformers. When a current passes through a loop 1, it creates a magnetic field $\vec{B}_1$. If a second loop is nearby, it will have a magnetic flux Φ_2 pass through it given by

$$\Phi_2 = \int_{loop2} \vec{B}_1 \cdot d\vec{a} \tag{46}$$

But $\vec{B}_1$ is given by

$$\vec{B}_1 = \frac{\mu_o}{4\pi} I_1 \oint_{loop1} \frac{d\vec{\ell}_1 \times (\vec{r} - \vec{r}_1)}{|\vec{r} - \vec{r}_1|^3} \tag{47}$$

Because $\vec{B}_1$ is proportional to I_1, so is Φ_2. Specifically

$$\Phi_2 = M_{21} I_1 \tag{48}$$

where

$$M_{21} = \frac{\mu_o}{4\pi} \int_{loop2} \left[\oint_{loop1} \frac{d\vec{\ell}_1 \times (\vec{r} - \vec{r}_1)}{|\vec{r} - \vec{r}_1|^3} \right] \cdot d\vec{a} \tag{49}$$

It is actually not so unusual to find the mutual inductance of two loops directly from 49. There is another form of this expression that shows a useful property of the mutual inductance: We get it by expressing $\vec{B}_1$ in terms of the vector potential $\vec{A}$ in the Coulomb gauge:

$$\begin{aligned} \Phi_2 &= \int_{loop2} \vec{B}_1 \cdot d\vec{a} && (50) \\ &= \int_{loop2} (\vec{\nabla} \times \vec{A}_1) \cdot d\vec{a} && (51) \\ &= \oint \vec{A}_1 \cdot d\vec{\ell}_2 && (52) \end{aligned}$$

But, in the Coulomb gauge, $\vec{A}_1$ is given by

$$\vec{A}_1 = \frac{\mu_o I_1}{4\pi} \oint \frac{d\vec{\ell}_1}{|\vec{r} - \vec{r}_1|} \tag{53}$$

This leads to

$$M_{21} = \frac{\mu_o}{4\pi} \oint \oint \frac{d\vec{\ell}_2 \cdot d\vec{\ell}_1}{|\vec{r}_2 - \vec{r}_1|} \tag{54}$$

Perhaps the most important use of this result (called the Neumann formula) is that it shows $M_{21} = M_{12}$. Thus if a current I in loop 1 induces a flux Φ_2 in loop 2, then a current I in loop 2 will induce a flux $\Phi_1 = \Phi_2$ in loop 1. Thus we may drop the index 21 from M and write

$$\Phi_2 = MI_1 \tag{55}$$
$$\Phi_1 = MI_2 \tag{56}$$

This analysis become important when we consider the application of time-dependent currents. If we set a time dependent current in loop 1, we induce as emf in loop 2:

$$\xi_2 = -\frac{d\Phi_2}{dt} \tag{57}$$
$$= -M\frac{dI_1}{dt} \tag{58}$$

This is the basis of the transformer.

1.7.2 Self Inductance

The current in a circuit creates a magnetic field which in turn creates a magnetic flux through not only nearby current loops, but the source loop itself! This flux is given by

$$\Phi = LI \tag{59}$$

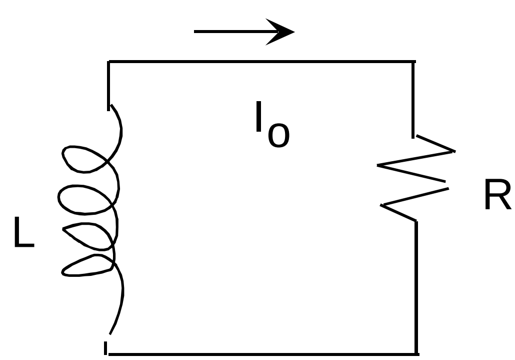
where L is called the self inductance. Specifically, we have

$$\xi = -L\frac{dI}{dt} \tag{60}$$

Fortunately, for many practical circuit applications, the self inductance is small enough to be ignored. Devices consisting of hundreds or thousands of tightly wound loops called inductors or choke coils can be placed in a circuit when significant self inductance is desired. The voltage drop across such a device is then given by 60.

1.8 Energy in magnetic fields

Consider the following circuit:



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An inductor is made of a long cylindrical solenoid of radius a, length h and n turns of wire per unit length. Initially the circuit carries a current I_o.

(a) Find an expression for $I(t)$ in terms of L, I_o, and R.

(b) How much energy is dissipated in the resistor as the current changes from I_o to zero? Where did this energy come from?

(c) Find an expression for the initial magnetic field inside the coil in terms of a, h,n, I_o and μ_o.

(d) Find an expression for the inductance in the coil as a function of a, h, n and μ_o.

(e) Find an expression for the energy per unit volume u_B stored in the coil in terms of a,h, μ_o, and B.

Solution

(a) The voltage across the inductor must be the same as that across the resistor. Thus

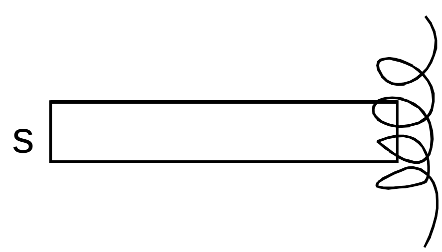
$$-L\frac{dI}{dt} = IR \quad (62)$$
$$\frac{dI}{dt} = -\frac{R}{L}I \quad (63)$$
$$I = I_o e^{-Rt/L} \quad (64)$$

(b) The power is given by $IV = I^2R$ so that

$$\frac{dU}{dt} = I_o^2 R\, e^{-2Rt/L} \quad (65)$$
$$U = I_o^2 R \int_o^\infty e^{-2Rt/L} dt \quad (66)$$
$$= \frac{1}{2} L I_o^2 \quad (67)$$

This energy must have been stored in the inductor.

(c) In brief, we assume the magnetic field is zero everywhere but inside the coil where it is axial. Creating a rectangular amperian loop as shown,



(68)

we find that

$$\oint \vec{B} \cdot d\vec{\ell} = sB \tag{69}$$

$$= \mu_o I_{enc} \tag{70}$$

$$= \mu_o N I_o \tag{71}$$

$$B = \mu_o \frac{N}{s} I_o \tag{72}$$

$$= \mu_o n I_o \tag{73}$$

(d) The flux through the solenoid is given by

$$\Phi_B = NBA \tag{74}$$

where N is the number of turns, B is the strength of the field, and A is the area of the solenoid. So we have

$$\Phi_B = nhB\pi a^2 \tag{75}$$

$$= \mu_o \pi n^2 h a^2 I_o \tag{76}$$

$$= LI \tag{77}$$

$$L = \mu_o \pi n^2 h a^2 \tag{78}$$

(e) The energy in the coil was found in part (b). Here we simply rewrite it:

$$U = \frac{1}{2} L I^2 \tag{79}$$

$$= \frac{1}{2} \mu_o \pi n^2 h a^2 \left(\frac{B}{\mu_o n} \right)^2 \tag{80}$$

$$= \frac{1}{2\mu_o} \pi h a^2 B^2 \tag{81}$$

Notice that this implies that

$$u_B = \frac{U}{Vol} = \frac{1}{2\mu_o} B^2. \tag{82}$$

This result is significant. It is the energy per unit required to create a magnetic field B. We have shown this to be the case for a simple solenoid. In actuality, this result is completely general and complements an previous result for the electric field:

$$u_E = \frac{\varepsilon_o}{2} E^2 \tag{83}$$

$$u_B = \frac{1}{2\mu_o} B^2 \tag{84}$$

$$u = \frac{1}{2} \left[\varepsilon_o E^2 + \frac{1}{\mu_o} B^2 \right] \tag{85}$$