Final Exam
4183, Spring 2005

Problem 1-c6

An infinitely long cylinder of radius R carries a frozen-in magnetization parallel to the axis

$$\vec{M} = ks\hat{z}$$

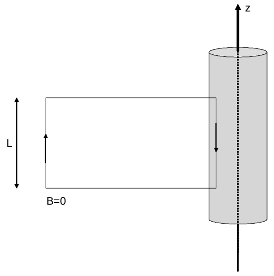
where k is a constant and s is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:

(a) Locate all bound currents and calculate the $\vec{B}$ field they produce.

$\vec{K} = \vec{M} \times \hat{n} = kR\hat{\phi}$

$$\begin{aligned}
\vec{J}_b &= \vec{\nabla} \times \vec{M} \\
&= \det\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & k(x^2+y^2)^{1/2} \end{pmatrix} \\
&= k(\frac{y}{(x^2+y^2)^{1/2}}\hat{x} - \frac{x}{(x^2+y^2)^{1/2}}\hat{y}) \\
&= -k(-\sin\phi\hat{x} + \cos\phi\hat{y}) \\
&= -k\hat{\phi}
\end{aligned}$$

Now we can use Ampere's law:



Far away from the loop, the $\vec{B}$ field must go to zero. (This is because from far away the current density on the right hand side looks as though it is vanishingly close to the current on the left hand side.) Since the cylinder is very long, by symmetry, the top and bottom of the amperian loop do not contribute to the line integral. Thus we have

$$\oint \vec{B} \cdot d\vec{\ell} = B_{in}L$$

$$= \mu_o I_{enc}$$

$$= \mu_o kL(R - s) - \mu_o kLR$$

$$\vec{B}_{in} = \mu_o ks\hat{z}$$

$$\vec{B}_{out} = \vec{0}$$

(b) Use Ampere's law in the form

$$\vec{\nabla} \times \vec{H} = \vec{J}_f$$

and the relationship

$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}.$$

Here we have $\vec{J}_f = \vec{0}$ so $\vec{H}$ is constant.

$$\vec{B}_{in} = \mu_o \vec{H} + \vec{M}$$

$$= \mu_o \vec{H} + \mu_o ks\hat{z}$$

$$\vec{B}_{out} = \mu_o \vec{H}$$

To find the value of the constant $\vec{H}$, we note that far away (by symmetry considerations) the $\vec{B}$ field should go to zero. Thus $\vec{H}$ must be zero and

$$\vec{B}_{in} = \mu_o ks\hat{z}$$

$$\vec{B}_{out} = \vec{0}$$

Problem 2-c7

An alternating current $I_o \cos\omega t$ (amplitude 0.5 A, frequency 60 Hz) flows down a straight wire, which runs along the axis of a toroidal coil with rectangular cross section (inner radius 1 cm, outer radius 2 cm, height 1 cm, 1000 turns). The coil is connected to a 500Ω resistor.

(a) In the quasi-static approximation, what is the emf induced in the toroid? Find the current, $I_r(t)$, in the resistor.

$$\Phi_B = N\int \vec{B} \cdot d\vec{a}$$

$$= Nh\int_a^b \frac{\mu_o I}{2\pi s} ds$$

$$= \frac{\mu_{oN}}{2\pi} h \ln(\frac{b}{a}) I$$

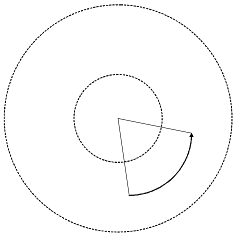
$$= MI$$

with

$$
\begin{aligned}
M &= \frac{\mu_o N}{2\pi} h \ln(\frac{b}{a}) \\
&= \frac{4\pi}{2\pi} 10^{-7} \frac{\mathrm{H}}{\mathrm{m}} (1000)(0.01\mathrm{m}) \ln(2) \\
&= 1.39\mu H = 1.39\mu\Omega \bullet s
\end{aligned}
$$

$$
\begin{aligned}
\xi &= -\frac{d\Phi_B}{dt} \\
&= -M\frac{dI}{dt} \\
&= (1.39\mu\Omega \bullet s)(-2\pi 60 s^{-1})(0.5A)\sin 2\pi f t \\
&= 0.26mV \sin 2\pi f t \\
I &= \frac{\xi}{R} \\
&= 0.52\mu A \sin 2\pi f t
\end{aligned}
$$

(b) Calculate the back emf in the coil due to the current $I_r(t)$. What is the ratio of the amplitudes of this back emf and the "direct" emf in (a)?



First we find the field in the toroid. By symmetry, $\vec{B}_r = B\hat{\phi}$ and

$$\oint \vec{B}_r \bullet d\vec{\ell} = B_r \Delta\phi s = \mu_o \frac{\Delta\phi}{2\pi} N I_r$$

$$B_r = \frac{\mu_o N I_r}{2\pi s}$$

The flux is then

$$\Phi_{B_r} = \left(\frac{\mu_o N^2}{2\pi} h \ln \frac{b}{a} \right) I_r$$

so we have

$$
\begin{aligned}
\Phi_{B_r} &= L I_r \\
L &= \left(\frac{\mu_o N^2}{2\pi} h \ln \frac{b}{a} \right) = MN \text{ (for this special case only)}
\end{aligned}
$$

and

$$\begin{aligned}
\xi_r &= -L\frac{dI_r}{dt} = -L\frac{d}{dt}\left(-\frac{M}{R}\frac{dI}{dt}\right) \\
&= \frac{LM}{R}\frac{d^2I}{dt^2} \\
&= \frac{NM^2}{R}\frac{d^2I}{dt^2} = \frac{1000(1.39\times10^{-6}\Omega\bullet s)^2}{500\Omega}(2\pi 60s^{-1})^2(0.5A)\cos 2\pi ft \\
&= (0.27\mu V)\cos 2\pi ft \\
\frac{<\xi_r>}{<\xi>} &= \frac{.27}{.26}\times10^{-3} = 10^{-3}
\end{aligned}$$

Problem 3-c11

An insulating circular ring (radius b) lies in the x-y plane centered at the origin. It carries a linear charge density $\lambda = \lambda_o \sin\phi$, where λ_o is constant and ϕ is the usual azimuthal angle. The ring is now set spinning at a constant angular velocity ω about the z axis. Calculate the power radiated. You may use

$$P = \frac{\mu_o \ddot{p}^2}{6\pi c}.$$

solution:

$$\begin{aligned}
p &= \int(\lambda_o\sin\phi)a\sin\phi a d\phi = \lambda_o\pi a^2 \\
\vec{p} &= \lambda_o\pi a^2(\cos\omega t\hat{x} + \sin\omega t\hat{y}) \\
\frac{d^2\vec{p}}{dt^2} &= -\omega^2\vec{p} \\
\ddot{p} &= \omega^2\lambda_o\pi a^2 \\
P &= \mu_o\frac{\omega^4\lambda_o^2\pi a^2}{6c} \\
&= \frac{\lambda_o^2}{\varepsilon_o}\frac{\omega^4\pi a^4}{6c^3}
\end{aligned}$$

Recall q^2/ε_o has units of energy times distance so $\lambda_o^2/\varepsilon_o$ has units of energy per distance. Thus this entire expression has the proper units of energy / time.

Problem 4-c8 and c12

This problem was not one of the given problems. You may choose to work this problem with your notes, but you must first turn in problems 1,2 and 3.

A long coaxial cable of length ℓ consists of an inner conductor (radius a) and an outer conductor (radius b). It is connected to a battery at one end and a resistor at the other. The inner conductor carries a uniform charge per unit length λ and a steady current I to the right. The outer conductor has the opposite charge and current.

(a) What is the electric field everywhere?

By Gauss' law, $\vec{E} = \frac{\lambda}{2\pi\varepsilon_o s}\hat{s}$ for $a < s < b$ and $\vec{0}$ everywhere else.

(b) What is the magnetic field everywhere?

By Ampere's law $\vec{B} = \frac{\mu_o I}{2\pi s}\hat{\phi}$ for $a < s < b$ and $\vec{0}$ everywhere else.

(c) Show that the electromagnetic momentum stored in the fields is given by $\frac{\lambda I\ell}{2\pi c^2\varepsilon_o}\ln(\frac{b}{a})\hat{z}$ where $\hat{z}$ is the direction of the inner current.

$$\vec{S} = \frac{1}{\mu_o}\vec{E}\times\vec{B}$$

$$\vec{p}_{EM} = \frac{1}{c^2}\vec{S} = \varepsilon_o\vec{E}\times\vec{B}$$

$$\begin{aligned}\vec{p}_{EM} &= \varepsilon_o\vec{E}\times\vec{B}\\ &= \mu_o\frac{I\lambda}{4\pi^2 s^2}\hat{z}\\ \vec{P}_{EM} &= \int\vec{p}_{EM}d\tau\\ &= \hat{z}\ell\mu_o\frac{I\lambda}{4\pi^2}2\pi\int_a^b\frac{1}{s^2}sds\\ &= \ell\mu_o\frac{I\lambda}{2\pi}\ln(\frac{b}{a})\hat{z}\\ &= \frac{\lambda I\ell}{2\pi c^2\varepsilon_o}\ln(\frac{b}{a})\hat{z}\end{aligned}$$

(d) Let us assume that the current is accomplished by the motion of electrons. Let us let λ_m be the mass of these electrons per unit distance of conductor. Then we have

$$\begin{aligned}\lambda_m^{inner}\vec{v}_m^{inner} &= \frac{mI}{-e}\hat{z}\\ \lambda_m^{outer}\vec{v}_m^{outer} &= -\frac{mI}{-e}\hat{z}\end{aligned}$$

(Here e is a positive constant.) Classically, this implies that there is no mechanical momentum:

$$\vec{P}_{mech,classical} = \ell\lambda_m^{inner}\vec{v}_m^{inner} + \ell\lambda_m^{outer}\vec{v}_m^{outer} = 0$$

However, while classical momentum is given by $p = mv$ and classical kinetic energy is $\frac{1}{2}mv^2$, the relativistic momentum is given by $p = mv/(1-v^2/c^2)^{1/2} = \gamma mv$ and relativistic kinetic energy is $\gamma mc^2 - mc^2$. Use the fact that the total momentum of the system is zero (electromagnetic + mechanical) to find the difference in the total energy of an inner drift electrons and an outer drift electrons. That is find

$$\Delta U = [(-eV_{out}) + (\gamma_{out}mc^2 - mc^2)] - [(-eV_{in}) + (\gamma_{in}mc^2 - mc^2)]$$

$$\begin{aligned}
\Delta U &= [(-eV_{out}) + (\gamma_{out}mc^2 - mc^2)] - [(-eV_{in}) + (\gamma_{in}mc^2 - mc^2)] \\
&= -e(V_{out} - V_{in}) + (\gamma_{out} - \gamma_{in})mc^2 \\
\vec{P}_{mech} + \vec{P}_{EM} &= \vec{0} \\
\vec{P}_{mech} &= (\gamma_{in}\lambda_m^{inner}\ell\vec{v}_m^{inner} + \gamma_{out}\lambda_m^{outer}\ell v_m^{outer})\hat{z} \\
&= -\frac{m}{e}I(\gamma_{in} - \gamma_{out})\hat{z} \\
&= -\vec{P}_{EM} \\
&= -\frac{\lambda I\ell}{2\pi c^2\varepsilon_o}\ln(\frac{b}{a})\hat{z} \\
(\gamma_{in} - \gamma_{out})mc^2 &= \frac{ec^2}{I}\frac{\lambda I\ell}{2\pi c^2\varepsilon_o}\ln(\frac{b}{a}) \\
&= \frac{e\lambda}{2\pi\varepsilon_o}\ln(\frac{b}{a})
\end{aligned}$$

But

$$\begin{aligned}
V &= \frac{-\lambda}{2\pi\varepsilon_o}\ln r \\
V_{out} - V_{in} &= \frac{-\lambda}{2\pi\varepsilon_o}\ln\frac{b}{a}
\end{aligned}$$

so we have

$$(\gamma_{in} - \gamma_{out})mc^2 = (-e)(V_{out} - V_{in})$$

and the remarkable result that

$$\Delta U = 0$$

Problem 5-c9

This problem was not one of the given problems. You may choose to work this problem with your notes, but you must first turn in problems 1,2 and 3.

Consider light traveling in a media which obeyed the modified Maxwell equations

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= 0 \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial\vec{B}}{\partial t} - \frac{1}{\tau}\vec{B} \\
\vec{\nabla} \times \vec{B} &= \frac{1}{v^2}\frac{\partial\vec{E}}{\partial t}
\end{aligned}$$

In this problem we find the response of this media to a perturbation with time dependence $e^{-i\omega t}$, that is we assume the electric and magnetic fields are of the form

$$\vec{E} = \vec{E}_o e^{i\vec{k}\cdot\vec{r}} e^{-i\omega t}$$
$$\vec{B} = \vec{B}_o e^{i\vec{k}\cdot\vec{r}} e^{-i\omega t}$$

and determine the constraints are on $\vec{E}_o, \vec{B}_o, \vec{k}$ so that these equations are solved.

(a) By substituting these assumed fields into the modified Maxwell equations, find four vector algebraic equations in terms of $\vec{k}$, ω,, $\vec{E}_o$, and $\vec{B}_o$.

$$\vec{k} \cdot \vec{E}_o = 0 + \tag{I}$$
$$\vec{k} \cdot \vec{B}_o = 0 \tag{II}$$
$$\vec{k} \times \vec{E}_o = (\omega + i\frac{1}{\tau})\vec{B}_o \tag{III}$$
$$\vec{k} \times \vec{B}_o = -\frac{\omega}{v^2}\vec{E}_o \tag{IV}$$

(b) Use your answer to part (a) to explain why it is possible, without loss of generality, to choose a coordinate system with

$$\vec{k} = k\hat{z}$$
$$\vec{B} = B_o\hat{y}$$
$$\vec{E} = E_o\hat{x}$$

with $k, B_o,$ and E_o arbitrary (possibly imaginary) constants.

(c) Find expressions for B_o^2 and k^2 in terms of E_o, ω, τ, and v.

Eq III gives

$$kE_o\hat{y} = (\omega + \frac{i}{\tau})\vec{B}_o$$
$$B_o = \frac{kE_o}{(\omega + i/\tau)}$$

whereas Eq IV gives

$$-kB_o\hat{x} = -\frac{\omega}{v^2}E_o\hat{x}$$
$$kB_o = \frac{\omega}{v^2}E_o$$

Combining these two equations, one has

$$k = \frac{\omega}{v^2 B_o}E_o$$

so that

$$B_o^2 = \frac{\omega}{(\omega + i/\tau)}\frac{E_o^2}{v^2}$$

similar algebra leads to our expression for k^2.

$$\begin{aligned}
\frac{k^2 E_o}{(\omega + i/\tau)} &= \frac{\omega}{v^2} E_o \\
k^2 &= \frac{\omega(\omega + i/\tau)}{v^2} \\
&= k_o^2 (1 + \frac{i}{\omega\tau}), \\
k_o &= \frac{\omega}{v}
\end{aligned}$$

(d) Find expressions for the real and imaginary part of k as a function of ω, v, and τ. That is find k_r and κ so that $k = k_r + \iota\kappa$. (Hint, as a check, when τ goes infinite, κ should go to zero and k_r should approach $k_o = \omega/v$. Check your final answer to make sure this is true.)

$$\begin{aligned}
k^2 &= k_o^2 (1 - \frac{i}{\omega\tau}) \\
k &= k_r + \iota\kappa \\
k^2 &= (k_r^2 - \kappa^2) + 2i\kappa k_r = k_o^2 - i k_o^2/\omega\tau \\
k_r^2 - \kappa^2 &= k_o^2 \\
\kappa k_r &= \frac{k_o^2}{2\omega\tau} \\
k_r^2\kappa^2 - \kappa^4 &= k_o^2\kappa^2 = \frac{k_o^4}{4\omega^2\tau^2} - \kappa^4 \\
\kappa^4 + k_o^2\kappa^2 - \frac{k_o^4}{4\omega^2\tau^2} &= 0 \\
\kappa &= +\frac{1}{2}(-k_o^2 + (k_o^4 + \frac{k_o^4}{\omega^2\tau^2})^{1/2})^{1/2}
\end{aligned}$$

(note roots taken so that κ is real and positive)

$$\begin{aligned}
&= k_o \sqrt{\frac{\sqrt{1 + 1/(\omega^2\tau^2)} - 1}{2}} \\
k_r &= \frac{k_o^2}{2\omega\tau\kappa} \\
&= k_o \frac{1}{\omega\tau\sqrt{2\sqrt{1 + 1/(\omega^2\tau^2)} - 1}} \\
&= k_o \sqrt{\frac{\sqrt{1 + 1/(\omega^2\tau^2)} + 1}{2}} \\
k &= \frac{\omega}{v}\left[\sqrt{\frac{\sqrt{1 + 1/(\omega^2\tau^2)} + 1}{2}} + \iota\sqrt{\frac{\sqrt{1 + 1/(\omega^2\tau^2)} - 1}{2}}\right]
\end{aligned}$$

(e)What is the absorption depth d of the material?

The absorption depth is the distance it will take for the amplitude of E^2 to be reduced by a factor of $1/e$:

$$
\begin{aligned}
d &= 1/(2k_r) \\
&= \frac{v}{2\omega}\left[\;\frac{1}{\sqrt{1+1/(\omega^2\tau^2)}\;-1)}\;\right]
\end{aligned}
$$