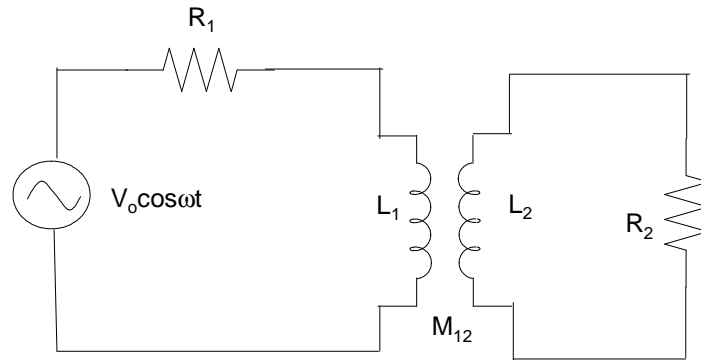


## 1 QUESTION

Dr. Shafer-Ray, most of this section we did touch on last semester, but I've always had a little more trouble understanding inductance. The relationship between mutual inductance (M) and self-inductance (L) is not clear to me. If you had the two loops and the current in loop 1 caused M on loop 2, is the L that the first loop induces on itself the same value as the M that it induces on the other loop? It seems from the reading that self-inductance is typically very small, much smaller seeming than M, although this is never stated. However I don't understand why this would be since it's the same E field created by loop 1.

## 2 ANSWER

There are two issues here. One is whether or not L is bigger or smaller than M. The second is whether the back inductance is important. Let us consider the following circuit:



Suppose we wanted to know the voltage across  $R_2$ . We could start by finding the current  $I_1$  through  $R_1$  using Kirchoff's laws

$$\begin{aligned} V_o \cos \omega t - I_1 R_1 - L_1 \frac{dI_1}{dt} &= 0, \\ \frac{dI_1}{dt} + I_1 \frac{R_1}{L_1} &= \frac{V_o}{L_1} \cos \omega t \end{aligned}$$

The steady-state (particular) solution must be of the form

$$I_1 = A \cos \omega t + B \sin \omega t$$

leading to

$$\begin{aligned} (B \frac{R_1}{L_1} - A\omega) \sin \omega t + (B\omega + A \frac{R_1}{L_1}) \cos \omega t &= \frac{V_o}{L_1} \cos \omega t \\ A &= \frac{BR_1}{L_1\omega}, \\ B\omega \left[ 1 + \left( \frac{R_1}{L_1\omega} \right)^2 \right] \cos \omega t &= \frac{V_o}{L_1} \cos \omega t \\ B &= \frac{V_o}{L_1\omega} \left[ 1 + \left( \frac{R_1}{L_1\omega} \right)^2 \right]^{-1} \\ A &= \frac{V_o}{L_1\omega} \frac{R_1}{L_1\omega} \left[ 1 + \left( \frac{R_1}{L_1\omega} \right)^2 \right]^{-1} \end{aligned}$$

We now simplify using

$$\begin{aligned} I_o &= \sqrt{A^2 + B^2} = \frac{V_o}{L_1\omega} \left[ 1 + \left( \frac{R_1}{L_1\omega} \right)^2 \right]^{-1/2} \\ \phi &= \arctan(B/A) = \arctan\left(\frac{R}{L_1\omega}\right) \\ I_1 &= I_o \cos(\omega t + \phi) \end{aligned}$$

we now analyze the second loop to find

$$\begin{aligned} M_{12} \frac{dI_1}{dt} - V_2 &= 0 \\ V_2 &= V_o \frac{M_{12}}{L_1} \left[ 1 + \left( \frac{R_1}{L_1\omega} \right)^2 \right]^{-1/2} (\cos \omega t + \phi) \end{aligned}$$

If we assume that  $R_1 \ll L_1\omega$  and  $M_{12}/L_1$  is the "turns ratio" of our transformer, then we get the simple step up or step down of the transformer we might expect. But this is a bit kludge. To do this right, let us consider Kirchhoff's laws again, but this time let's introduce the magnetic flux  $\Phi_1$  through coil 1 and the magnetic flux  $\Phi_2$  through coil 2. Analyzing circuit 1 we have

$$\begin{aligned} V_o \cos \omega t - I_1 R_1 - \frac{d\Phi_1}{dt} &= 0 \\ \frac{-d\Phi_2}{dt} - I_2 R_2 &= 0 \end{aligned}$$

But we also have (by definition of inductance)

$$\begin{aligned} \Phi_1 &= L_1 I_1 - M_{12} I_2 \\ \Phi_2 &= L_2 I_2 - M_{12} I_1 \end{aligned}$$

Which leads to the coupled differential equations

$$\begin{aligned} L_1 \frac{dI_1}{dt} - M_{12} \frac{dI_2}{dt} + I_1 R_1 &= V(t) \\ M_{12} \frac{dI_1}{dt} - L_2 \frac{dI_2}{dt} - I_2 R_2 &= 0 \end{aligned}$$

To solve this, we first solve for the case that  $V(t) = V_+(t) = V_o e^{i\omega t}$ . With the assumption that

$$\begin{aligned} I_1 &= I_1^+ e^{i\omega t} \\ I_2 &= I_2^+ e^{i\omega t} \end{aligned}$$

we get

$$\begin{aligned} (R_1 + i\omega L_1)I_1^+ + (-i\omega M_{12})I_2^+ &= V_o \\ (i\omega M_{12})I_1^+ + (-i\omega L_2 - R_2)I_2^+ &= 0 \end{aligned}$$

which leads immediately to

$$\frac{1}{M_{12}\omega} [(R_1 L_2 + R_2 L_1)\omega - i(\omega^2 M_{12}^2 + R_1 R_2 - \omega^2 L_1 L_2)] I_2^+ = V_o$$

or

$$I_2^+ = \frac{V_o}{R_\omega} e^{i\phi}$$

with

$$\begin{aligned} R_\omega &= \frac{1}{M_{12}\omega} \sqrt{[(R_1 L_2 + R_2 L_1)\omega]^2 + [\omega^2 M_{12}^2 + R_1 R_2 - \omega^2 L_1 L_2]^2} \\ \phi &= \arctan\left(\frac{\omega^2 M_{12}^2 + R_1 R_2 - \omega^2 L_1 L_2}{(R_1 L_2 + R_2 L_1)\omega}\right) \end{aligned}$$

Now let us repeat with  $V(t) = V_-(t) = V_o e^{-i\omega t}$  and  $I_1 = I_1^- e^{-i\omega t}$ ,  $I_2 = I_2^- e^{-i\omega t}$ . Then we have

$$I_2^- = \frac{V_o}{R_\omega} e^{-i\phi}$$

Because we are solving a linear set of differential equations and

$$V_o \cos \omega t = \frac{V_+(t) + V_-(t)}{2}$$

the solution we require is

$$\begin{aligned} I_2(t) &= I_2^+ e^{i\omega t} + I_2^- e^{-i\omega t} \\ &= \frac{V_o}{R_\omega} \cos(\omega t + \phi) \end{aligned}$$

and

$$\begin{aligned} V_2(t) &= R_2 I_2(t) \\ &= \frac{R_2}{R_\omega} V_o \cos(\omega t + \phi) \end{aligned}$$

Let us compare the amplitudes of the two cases

$$\begin{aligned} V_2 &\approx V_o \frac{M_{12}}{L_1} \left[ 1 + \left( \frac{R_1}{L_1 \omega} \right)^2 \right]^{-1/2} \\ V_2 &= V_o \frac{M_{12}}{L_1} \left[ \frac{R_2^2 L_1^2 \omega^2}{[(R_1 L_2 + R_2 L_1) \omega]^2 + [\omega^2 M_{12}^2 + R_1 R_2 - \omega^2 L_1 L_2]^2} \right]^{1/2} \end{aligned}$$

What if  $R_2 \gg R_1, \omega L_1, \omega M_{12}$ , and  $\omega L_2$ ? Then we need only keep terms of order  $R_2^2$  in the numerator. This leads to

$$\begin{aligned} V_2 &\approx V_o \frac{M_{12}}{L_1} \left[ \frac{R_2^2 L_1^2 \omega^2}{R_2^2 L_1^2 \omega^2 + R_1^2 R_2^2} \right]^{1/2} \\ &= V_o \frac{M_{12}}{L_1} \left[ 1 + \left( \frac{R_1}{L_1 \omega} \right)^2 \right]^{-1/2} \end{aligned}$$

Here we see that we do not need to worry about the back emf in the transformer in the case that  $R_2$  is pretty big. Once we start loading up the secondary, the back emf is important and we better treat this problem correctly.

### 3 QUESTION

In Example 7.3 the ends of a cylindrical resistor of constant cross-sectional area are held at constant potential such that a potential difference exists between the two ends. Griffiths proves that the electric field inside the resistor is uniform and is directed along the length of the cylinder. He states that the “charge arranges itself over the surface of the wire in just such a way as to produce a nice uniform field within”. I’ve been trying to figure out what that means. The potential difference between the two ends creates an electric field. Without the wire, the field between the end plates would not be uniformly directed along the length between the plates. Instead it would be as in Figure 7.3. With the wire in place, the interaction between the field and the surface charge on the wire results in a surface charge distribution that produces “a nice uniform field within”. Is that correct? Is that what Griffiths means? If so, then how is it that the charge arranges itself in “just such a way”?

We are creating a steady state solution with  $\vec{J} = \sigma \vec{E}$ . But we know that no current is leaking out, so  $\vec{J} \cdot \hat{n} = 0$  at the boundary. Thus  $\vec{E} \cdot \hat{n} = 0$  at

the boundary. On the opposite ends,  $V$  is specified to be a constant. But, provided  $\vec{E} = -\vec{\nabla}V$  and  $\nabla^2 V = 0$ , this is sufficient for determining  $\vec{E}$  (by the second uniqueness theorem and Gauss' law.) But this is not very satisfying. How do we know that  $\nabla^2 V = -\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot \vec{J}/\sigma = 0$ . Here's how:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = 0 \text{ for DC circuits}$$

Let's prove this from Maxwell's equations.

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho/\epsilon_o \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_o \vec{J} + \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}$$

We now write

$$\begin{aligned} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) &= \mu_o \vec{\nabla} \cdot \vec{J} + \mu_o \epsilon_o \frac{\partial \vec{\nabla} \cdot \vec{E}}{\partial t} \\ &= 0 \\ &= \mu_o \left( \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right) \text{ (this equation is called the continuity equation.)} \end{aligned}$$

## 4 QUESTION

(1) my first question is over the motional emf. I hope it counts

since it is essentially the same question you we were grappling with in class. Why can we ignore the resistor in when we integrate over the loop in section 7.1.3, figure 7.10. Furthermore, why do we even need the resistor? Won't the current flow anyway? I don't understand why it needs to be in the example at all, much less why it is ignored when integrating. Is it because we are integrating the work (figure 7.12-b) and the resistive force is analogous to drag?

I think I answered it. When calculating EMF between any two points, one writes

$$\xi_{ab} = \int_a^b \vec{f} \cdot d\vec{\ell}$$

with

$$\vec{f} = \text{the sum of all forces/charge except resistive forces.} \quad (1)$$

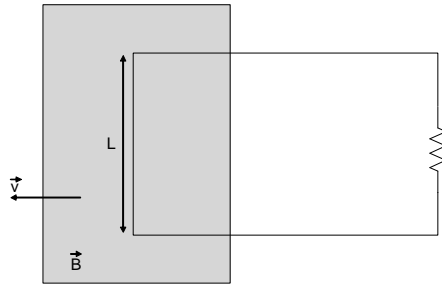
this is because the resistive forces have already been included in our "law of motion" (Ohm's law):

$$\vec{J} = \sigma \vec{f}$$

## 5 QUESTION

student 1: 2. My next question is over the discussion on pages 301-2. In experiment 1 a magnetic field produces the emf. In experiment 2 an electric field produced by a changing magnetic field sets up the emf. The only difference between the two experiments is the coordinate system. How is it that the type of field generating the emf depends on the frame of reference? In the book they say these considerations motivated special relativity, but I don't recall reading anything in our special relativity texts about the types of fields present in a system being relative to a reference frame.

student 2: (2) At the beginning of 7.2.1, Griffiths describes 3 experiments (figure 7.20). He makes a distinction between the first experiment and the second two saying that the first is not an example of Faraday's law because reserves the term Faraday's law for cases where electric fields are induced by changing magnetic fields. Isn't this distinction somewhat arbitrary since it is the relative motion of the charge and the magnetic field that really matter. is there anything special about the lab frame of reference that should make it important in describing the physics involved?



For the first case of moving the loop, there is nothing too strange. But BOTH the second and third case bring up new physics. Let's look at them

(2) Suppose we move the magnetic field (rather than the loop.) In this case the charge in the wire is not moving and the magnetic field on the left hand wire segment is in a constant  $\vec{B}$  field. So why then is

$$\xi = \oint \vec{f} \cdot d\vec{\ell} = vBL = -\frac{d\Phi_B}{dt}$$

Shouldn't it be zero? The answer is that a moving magnetic field creates an electric field. That is

$$\begin{aligned}\vec{E}' &\approx \vec{E} + \vec{v} \times \vec{B} \\ \vec{B}' &\approx \vec{B} - \frac{1}{c^2} \vec{v} \times \vec{E}\end{aligned}$$

Actually these statements are true only for the case that either  $\vec{E} = \vec{0}$  or  $\vec{B} = \vec{0}$ . For the case that  $\vec{E}$  and  $\vec{B}$  are both not zero, these equations work, but only for the case that  $v$  is much less than  $c$ . So let us think about this case (2) solution. In the rest frame of the moving magnet, the field is simply  $\vec{B}$  but in the rest frame of the loop, there is an electric field  $\vec{E}' = \vec{v} \times \vec{B}$ . This upward field acts in the direction of the wire for a distance  $L$ , giving an *EMF*

$$\begin{aligned}\xi &= \oint \vec{f} \cdot d\vec{\ell} \\ &= \oint \vec{E}' \cdot d\vec{\ell} \\ &= vBL = -\frac{d\Phi_B}{dt}\end{aligned}$$

It is in fact different physics than case (1) of a moving loop. In case 1, the  $\vec{v} \times \vec{B}$  force was responsible. In case (2) an electric force is responsible. This transformation is of great historical importance. Suppose we start from Maxwell's equations:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \rho/\epsilon_o \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_o \vec{J} + \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}\end{aligned}$$

We now let

$$\begin{aligned}\vec{E}' &= \vec{E} + \vec{v} \times \vec{B} \\ \vec{B}' &= \vec{B} - \frac{1}{c^2} \vec{v} \times \vec{E}\end{aligned}$$

and

$$\vec{r}' = \vec{r} - \vec{v}t$$

Then we can ask the question does

$$\begin{aligned}\vec{\nabla}' \cdot \vec{E}' &= \rho(\vec{r}')/\epsilon_o \\ \vec{\nabla}' \cdot \vec{B}' &= 0 \\ \vec{\nabla}' \times \vec{B}' &= \mu_o \vec{J}'(\vec{r}') + \mu_o \epsilon_o \frac{\partial \vec{E}'}{\partial t} \\ \vec{\nabla}' \times \vec{E}' &= -\frac{\partial \vec{B}'}{\partial t}\end{aligned}$$

The answer is no. More troubling still is the fact that there is no transformation of fields that will allow  $\vec{r}' = \vec{r} - \vec{v}t$  to lead to the same set of Maxwell's equations. Thus the laws of physics seem to depend on the Galilean frame of reference. The answer is of course that Galilean symmetry is not physically correct.

What about the final case (3). Here the  $\vec{B}$  field is not moving, but just decreasing in magnitude. Now we might expect the the emf really should be zero. However (and this was a really profound addition by Faraday) a changing magnetic field creates an electric field with

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Integrating to find the emf in a loop gives us

$$\oint \vec{E} \cdot d\vec{\ell} = \int \int \vec{\nabla} \times \vec{E} \cdot d\vec{A} = -\frac{\partial}{\partial t} \int \int \vec{B} \cdot d\vec{A} = -\frac{d}{dt} \Phi_B$$

Notice that the case of the transformation of the moving  $\vec{B}$  field into an electric field  $\vec{E}$  is a special case of Faraday's law, but Faraday's law is far more general. I would say the first experiment really is different, but case (2) and (3) are closely related.

## 6 Question

1. I'm having some difficulty understanding how the emf is calculated in problems such as Example 7.4. I understand that the emf can usually be calculated using either equation 7.9 or equation 7.13. Griffiths explains that in Ex 7.4 the flux rule (7.13) cannot be applied because "it assumes that the current flows along a well-defined path" and "in this example the current spreads out over the whole disk." So, equation 7.9 must be used instead. However, it seems to me that equation 7.9 also assumes that the current flows along a well defined path. The equation requires a line integral taken along this path. I don't know what this path should be in example 7.4. I don't understand why when Griffiths solves the problem he only integrates along the radius of the disk. The emf should be calculated by the line integral of the force per charge around the circuit, but I don't know what the path of the circuit is in a problem such as this.

The point is it does not matter the path taken, you choose it. However, you must choose a path for which the integral is easy to solve. For the case of the spinning disk, we see that the source force is always radially outward, thus we can integrate easily with a bath that is radially outward.

## 7 Question

1. On page 293 in the book the electromotive force is defined as "the integral of a force per unit charge." I am having difficulty picturing this description. For example, today in class we discussed the analogy that current works like a water hose. Is there any such description that could help me understand the physical meaning of the electromotive force?

What makes this so confusing is that we are using an empirical law ( $\vec{J} = \sigma \vec{f}$ ) rather than  $\vec{F} = m\vec{a}$ . This forces us to think about other quantities that are a bit less intuitive. Here the quantity is the integral over the sum of the electrical force and all other forces used to overcome the drag (resistive) forces and move charges in a circuit.

2. In example 7.6 the direction of the flux is given as upward. I do not understand where this was derived from. What is the general rule for finding the direction of flux? (I also have difficulty understanding how to find the direction of the induced electric field, as in example 7.7.)

This is sloppy. The direction of the magnetic field is upward (as determined by the right hand rule.) Flux is a scalar. It does not have a direction. However the sign is determined by the right hand rule.

3. On pages 318-319, there is a discussion which says we should not worry about whether energy is stored in the magnetic field or in the current distribution, if it is even located anywhere. Why is this information disregarded?

The reason it is not important is that we have found this from Maxwell's time independent equation (Ampere's law)

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$

thus is it in  $\vec{J}$  or in  $\vec{B}$ ? A physical picture of energy stored in the field does not come until quantum electrodynamics. I will say my preference: Indeed even when  $\vec{J} = \vec{0}$ , we have an energy per volume of  $\frac{1}{2\mu_o} B^2$ , so I think it more general to think of the energy as stored in the magnetic field. But the classical picture does not distinguish the two points of view for steady state situations.