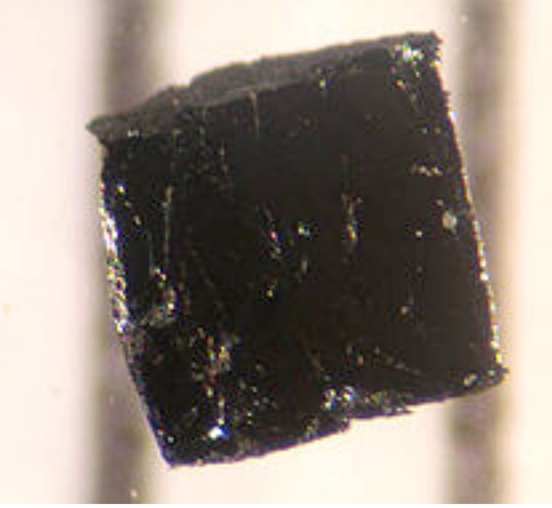
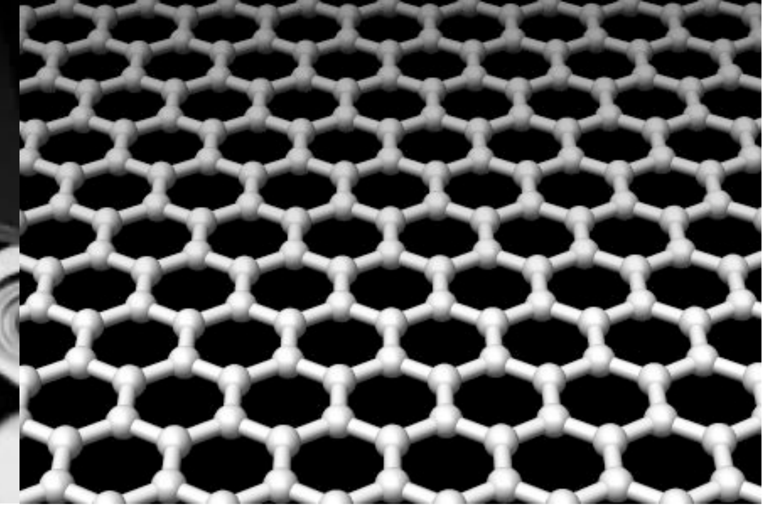
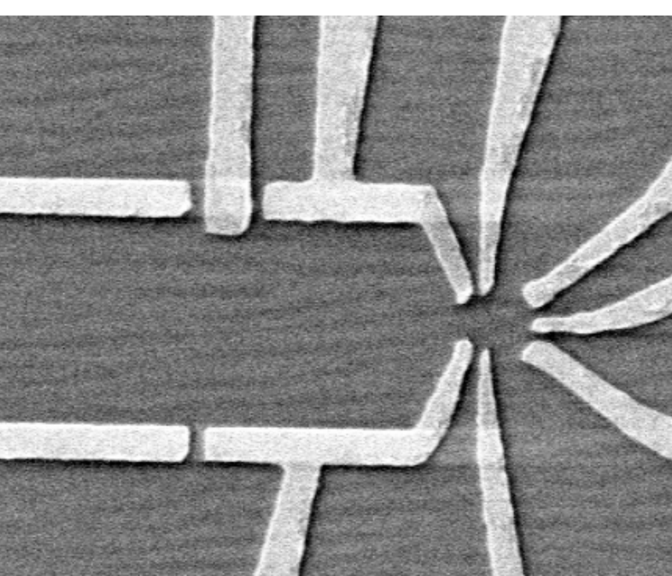




Quantum Science

Bruno Uchoa
Department of Physics and Astronomy
University of Oklahoma



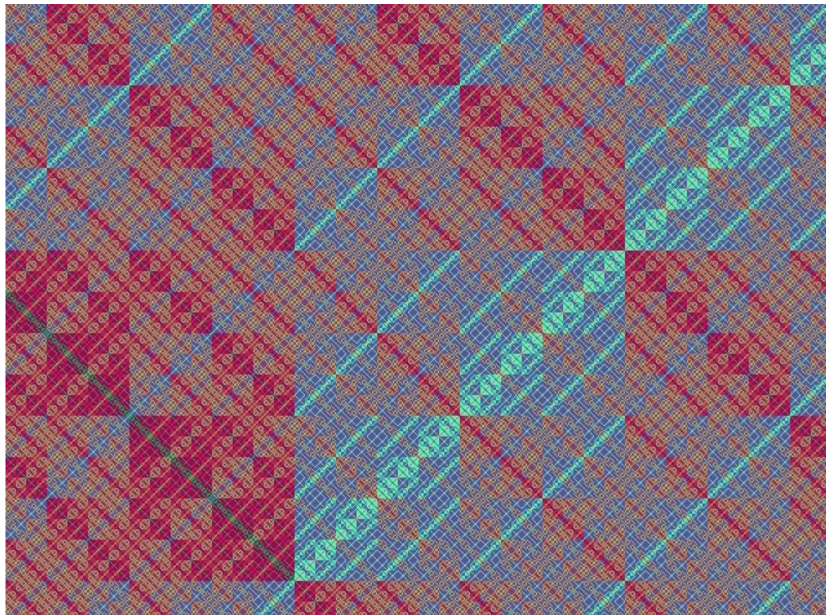

Lecture 4

Topology in quantum computation

The transmission of information in any digital system, including classical computers, relies on error correction.

In coding theory, the simplest way to implement error correction is through redundancy, in which every bit of information is transmitted multiple times.

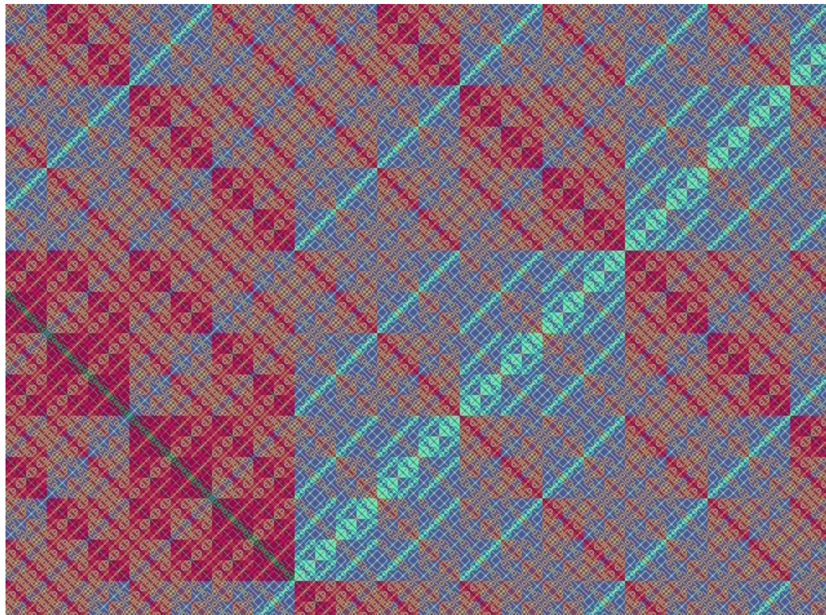
$$101 \to 111000111$$



If Alice would like to send a bit to Bob, she would send 000 for a 0 bit and 111 for a 1 bit.

Alice		**Bob**
0	$\longrightarrow$	000
1	$\longrightarrow$	111

If Bob receives something else, say 101, then an error has occurred.



If the probability of an error is small, then one can assume that the 3 digit string has no more than one error. Therefore,

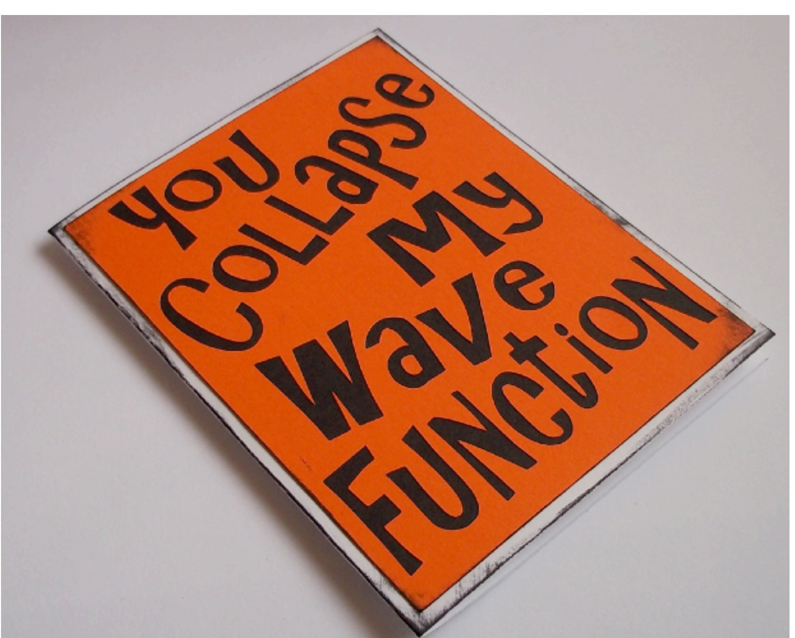
$$101 \xrightarrow[\textbf{Error Correction}]{} 111$$

after error correction.

In quantum computers, one cannot copy qubits because of the non-cloning theorem!

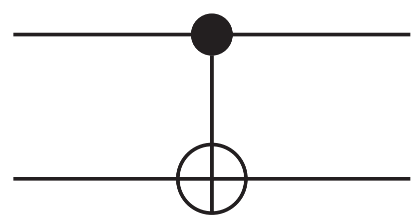
$$|\psi\rangle \nrightarrow |\psi\rangle|\psi\rangle$$

To protect quantum information, any error correction code requires fixing qubits without actually reading them!



In 1998, Peter Shor showed that one could use entanglement of several physical qubits with a logical qubit to correct quantum information.

The logical process of entangling two qubits requires the use of a CNOT gate,

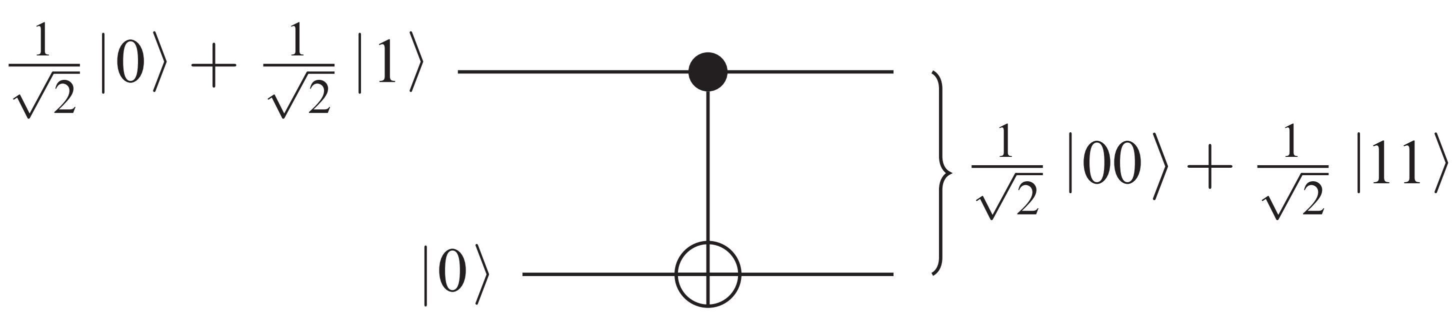


CNOT

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

For example, for an input:

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|0\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$



Output

Quantum bit-flip correction protocol

There are different ways in which qubits can be corrupted.

One type of qubit error is when bits are flipped,

$$a|0\rangle + b|1\rangle \longrightarrow a|1\rangle + b|0\rangle$$

Quantum bit-flip correction protocol

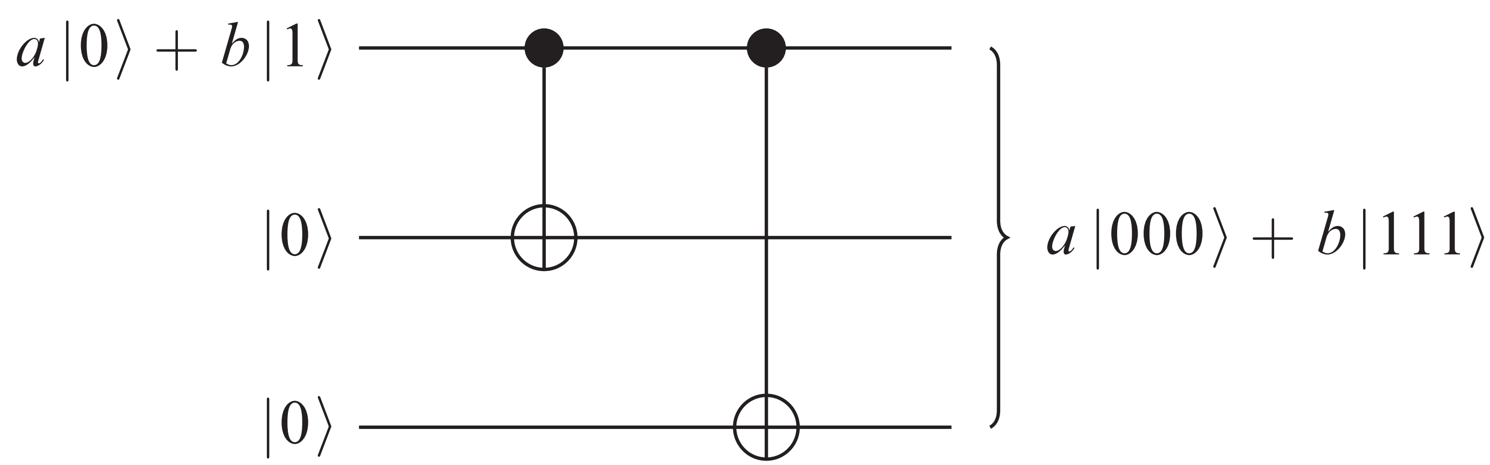
Although Alice would like to send three copies of her qubit

$$a|0\rangle + b|1\rangle$$

to prevent errors, this is not possible because of the non-cloning theorem.

Instead, she sends three qubits, one logical and two physical qubits, and entangle them through a CNOT gate,

Alice



Quantum bit-flip correction protocol

Because the communication channel is noisy, Bob may either receive the right qubit or a corrupted one,

Alice

$$a|000\rangle + b|111\rangle$$

Bob

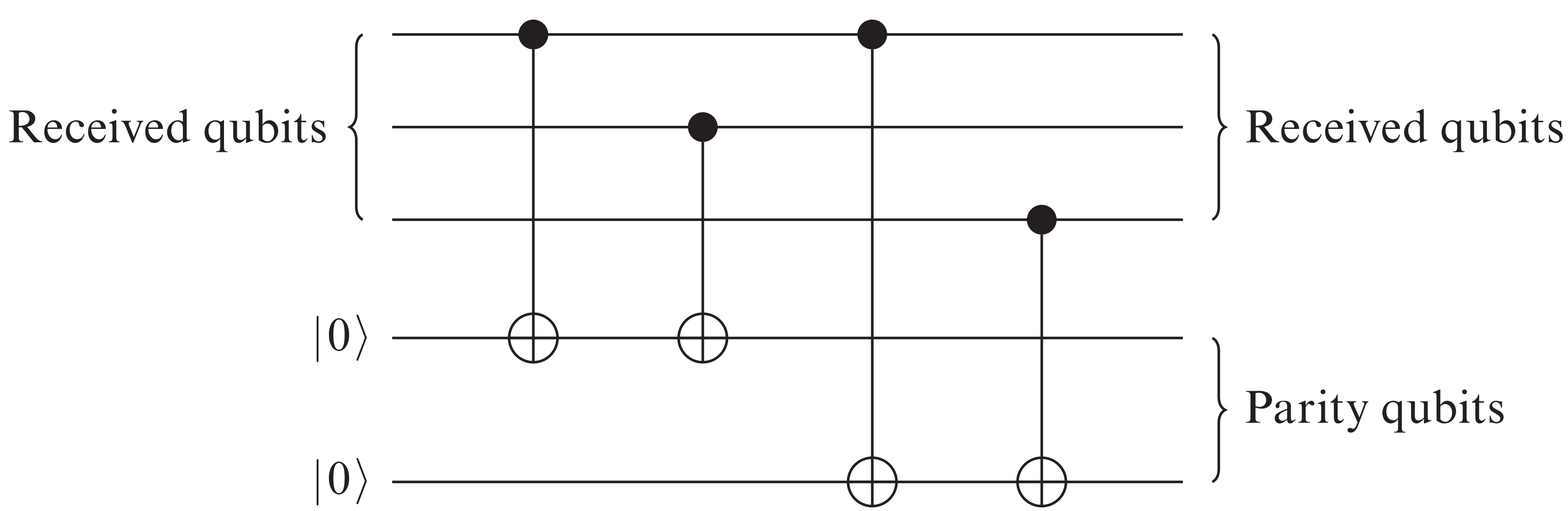
$$a|000\rangle + b|111\rangle \ \checkmark$$

$$\left.\begin{array}{l} a|100\rangle + b|011\rangle \\ a|010\rangle + b|101\rangle \\ a|001\rangle + b|110\rangle \end{array}\right\} \text{Corrupted}$$

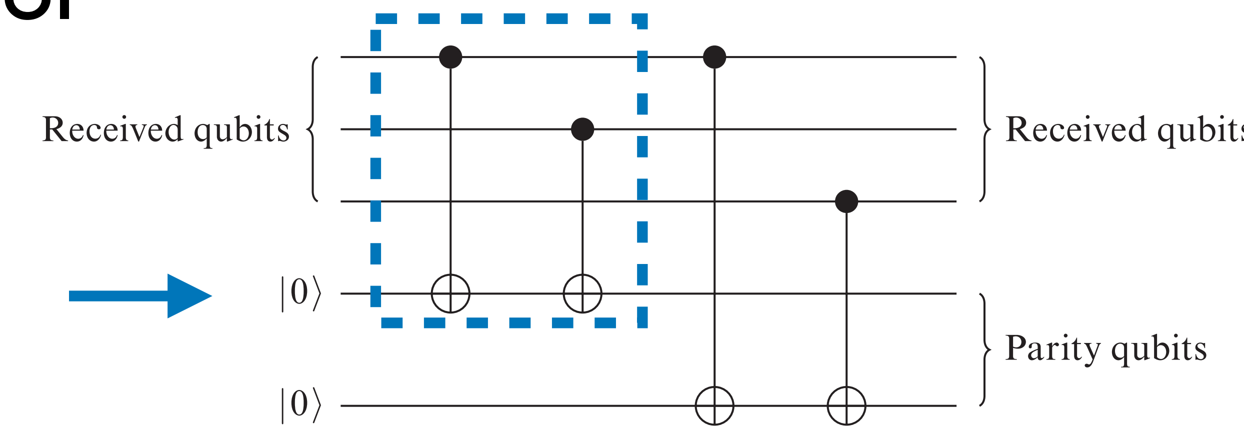
Now Bob has to both figure out if his qubit was corrupted or not and fix it (if necessary) without reading the qubit!

Quantum bit-flip correction protocol

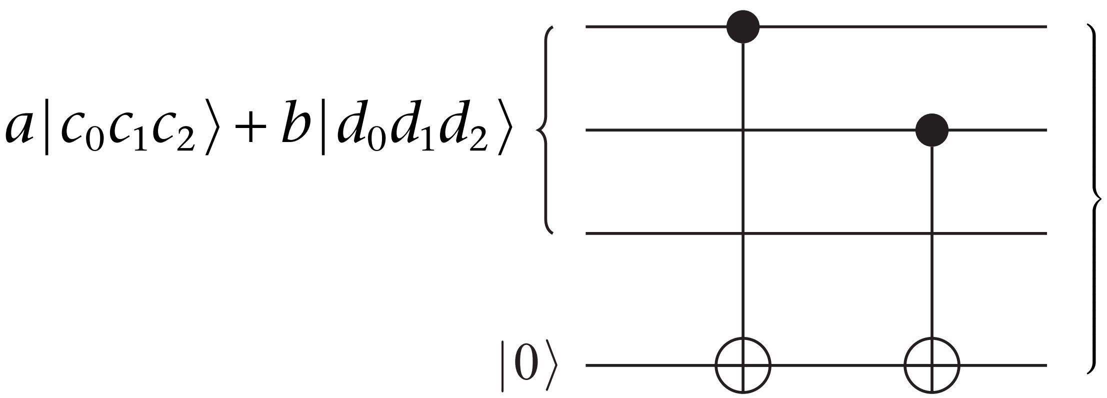
To check if any bits need to be corrected, Bob will add two more physical qubits to the received ones and perform 2 CNOT operations for each extra qubit:



Quantum bit-flip correction protocol



If Bob receives $a|c_0c_1c_2\rangle + b|d_0d_1d_2\rangle$ then the first two CNOT operations with the fourth qubit give

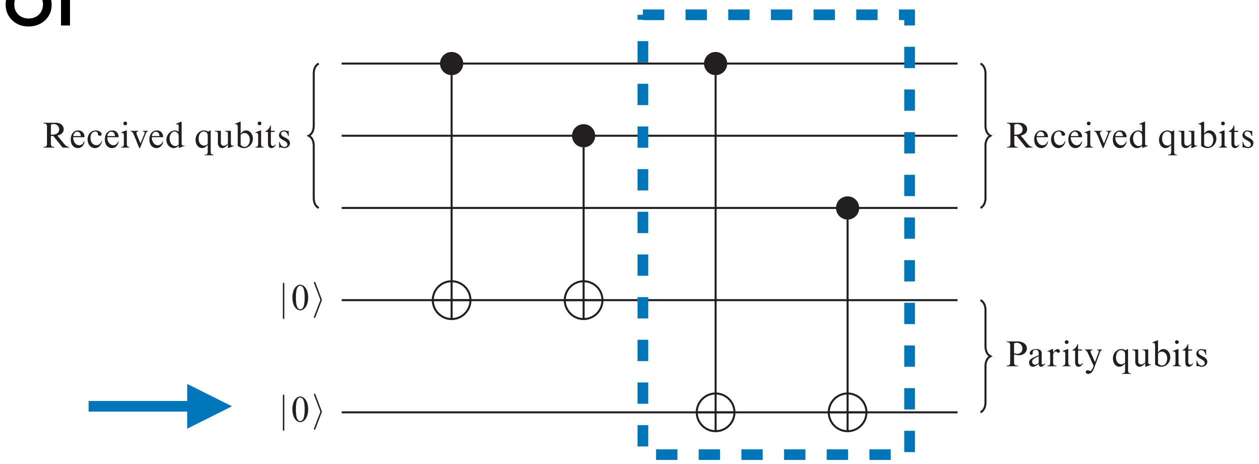


$$(a|c_0c_1c_2\rangle + b|d_0d_1d_2\rangle)|0\rangle \quad \text{if } c_0 \oplus c_1 = d_0 \oplus d_1 = 0$$

$$(a|c_0c_1c_2\rangle + b|d_0d_1d_2\rangle)|1\rangle \quad \text{if } c_0 \oplus c_1 = d_0 \oplus d_1 = 1$$

In either case, the fourth qubit is not entangled with the first three.

Quantum bit-flip correction protocol

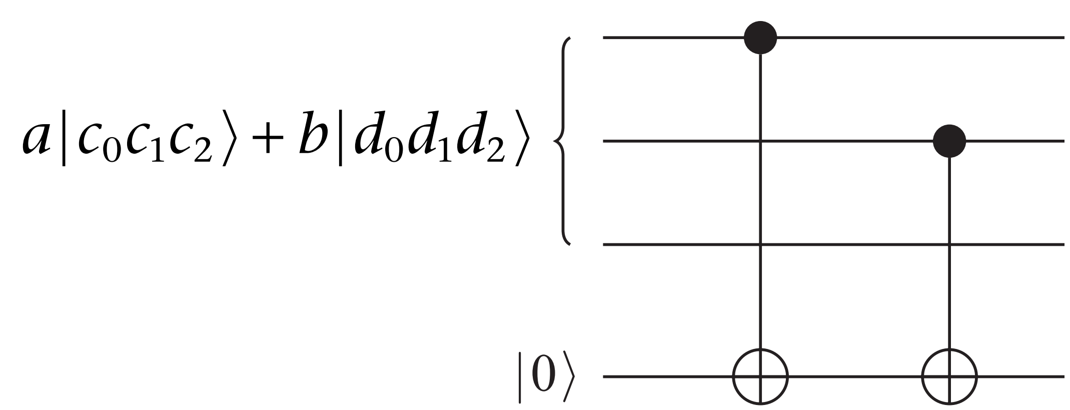


A similar result occurs in the fifth qubit, which is:

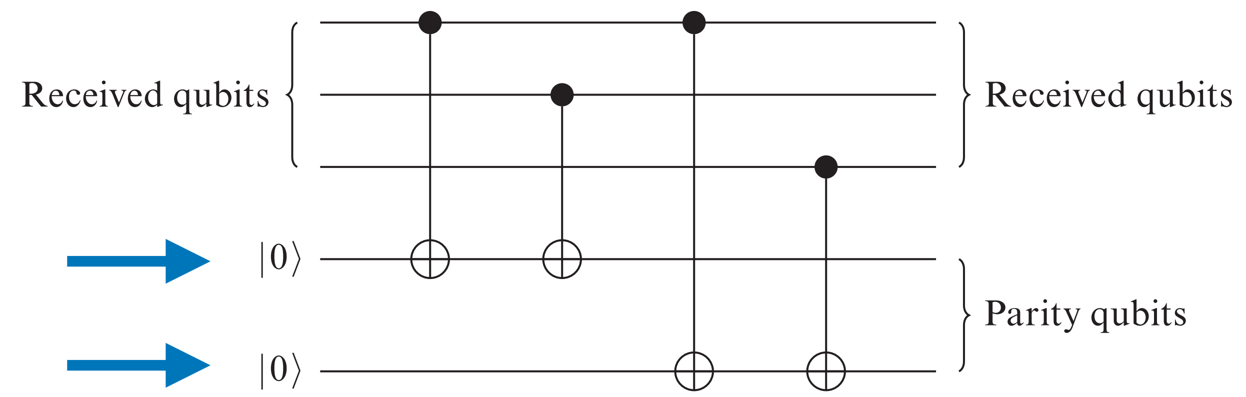
$$|0\rangle \qquad \text{if } c_0 \oplus c_2 = d_0 \oplus d_2 = 0$$

$$|1\rangle \qquad \text{if } c_0 \oplus c_2 = d_0 \oplus d_2 = 1$$

and is not entangled with the first three either.



Quantum bit-flip correction protocol



Bob can now measure the last two qubits, without interfering with the first three sent by Alice and make parity checks.

Alice

$$a|000\rangle + b|111\rangle$$

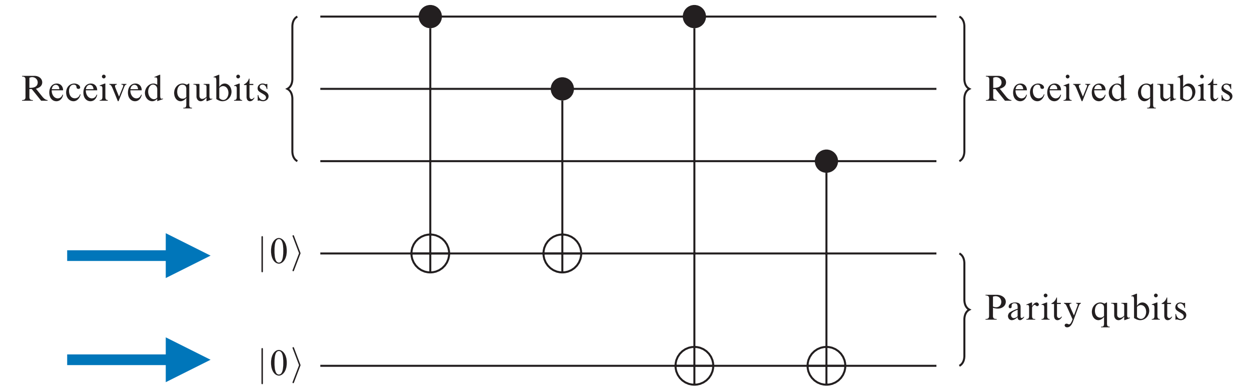
Bob

$$a|c_0c_1c_2\rangle + b|d_0d_1d_2\rangle$$

If the result is $|00\rangle$ then all c bits and d bits are the same.

Nothing to correct!

Quantum bit-flip correction protocol



Bob can now measure the last two qubits, without interfering with the first three sent by Alice.

Alice

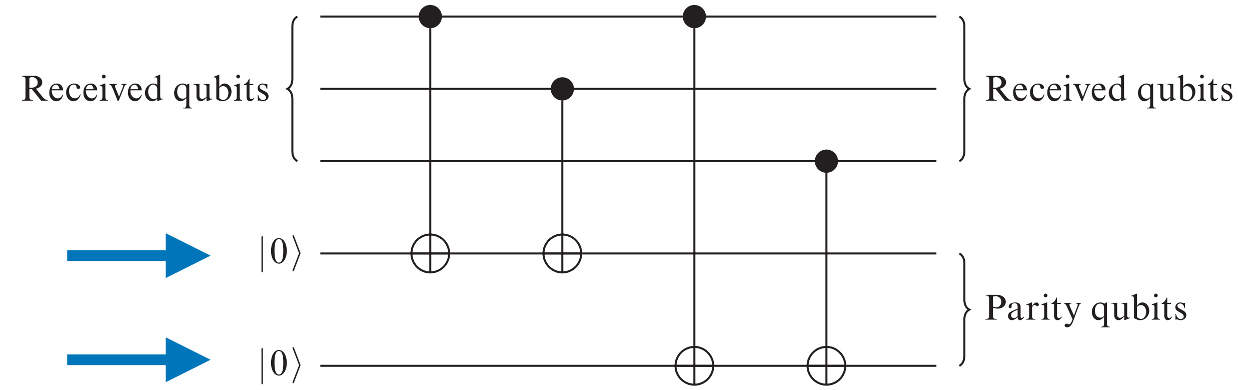
$$a|000\rangle + b|111\rangle$$

Bob

$$a|c_0c_1c_2\rangle + b|d_0d_1d_2\rangle$$

If the result is $|10\rangle$ then the c_1 and d_1 bits are the different ones. They need to be flipped with an X quantum gate!

Quantum bit-flip correction protocol



Bob can now measure the last two qubits, without interfering with the first three sent by Alice.

Alice

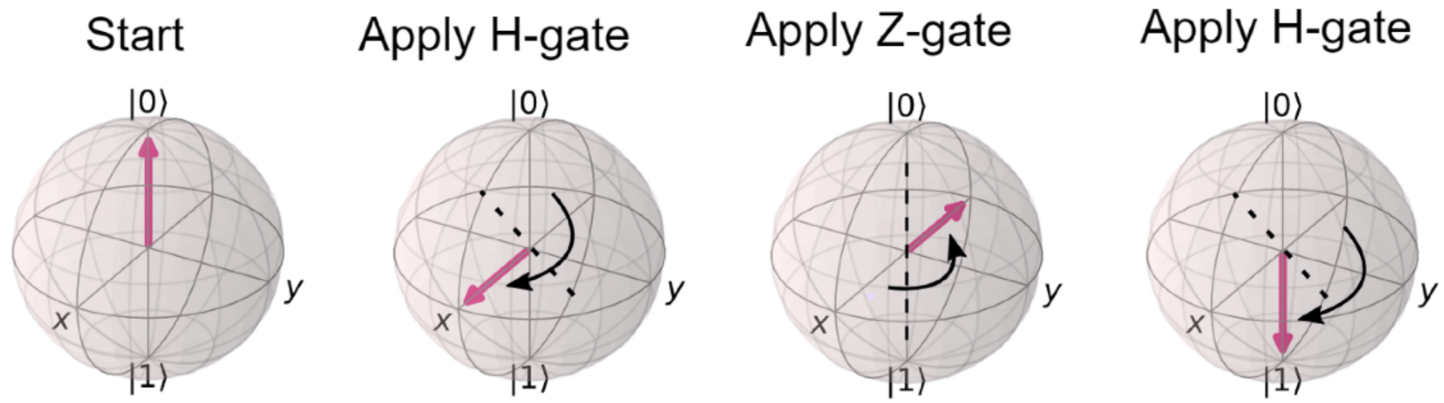
Bob

$$a|000\rangle + b|111\rangle$$

$$a|c_0c_1c_2\rangle + b|d_0d_1d_2\rangle$$

If the result is $|01\rangle$ then the c_2 and d_2 bits are the different ones.
If Bob measures $|11\rangle$ then the c_0 and d_0 bits need to be flipped!

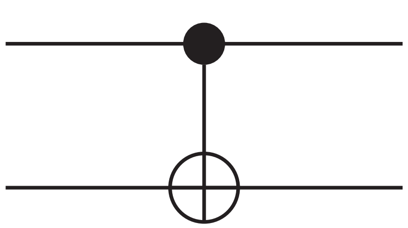
Quantum gates



All logical operations in a quantum computer can be performed with five gates acting in one bit plus the CNOT gate, which acts on two.

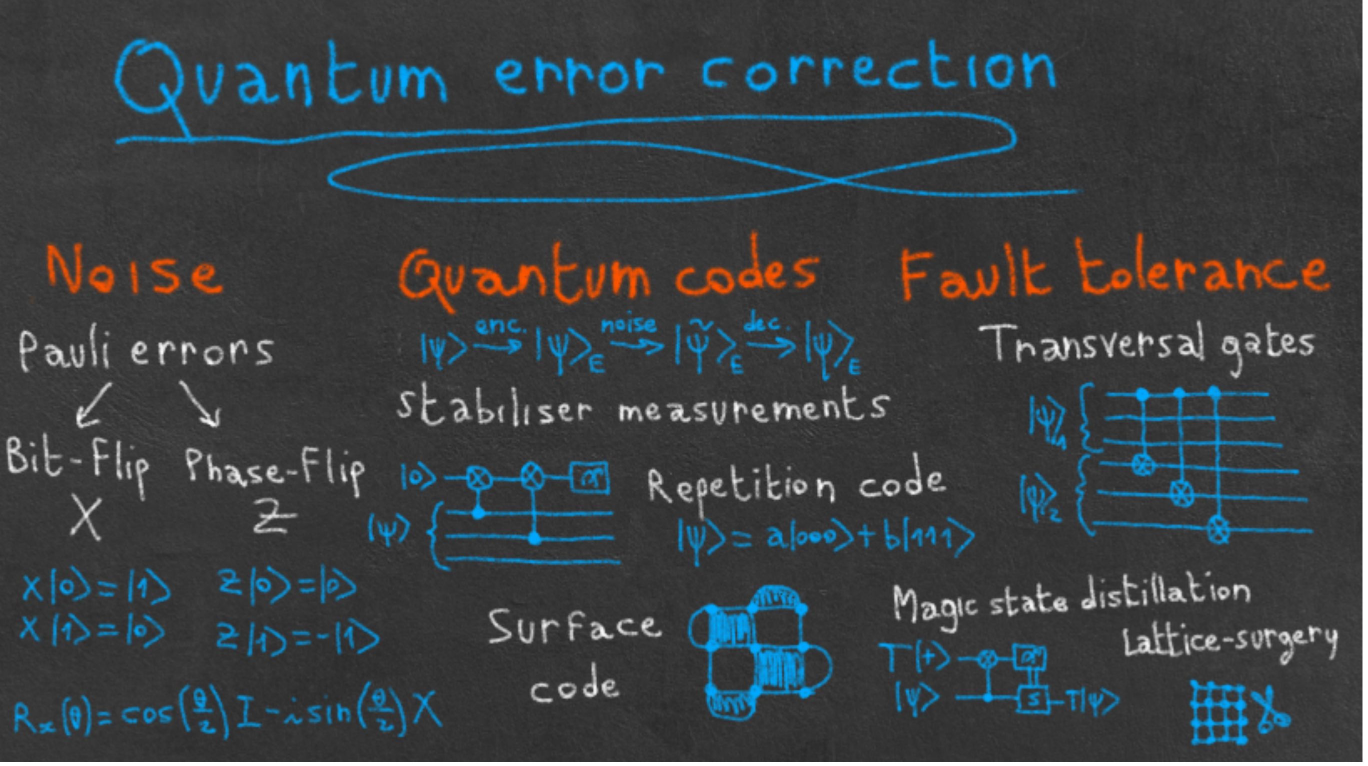
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

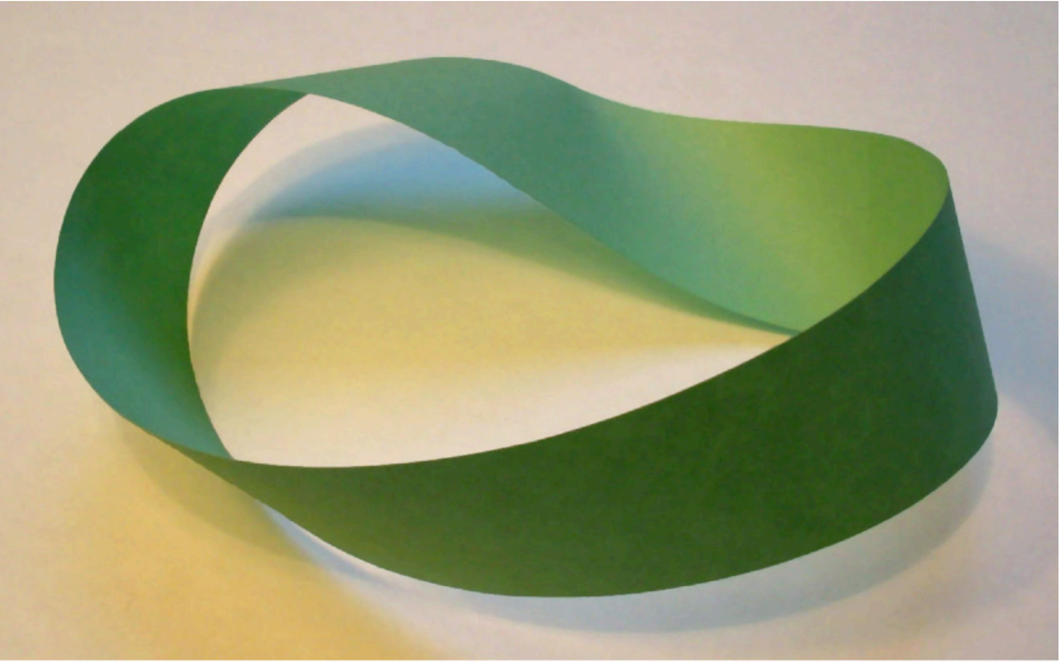
CNOT



To be useful, quantum computers need to protect qubits from the environment.

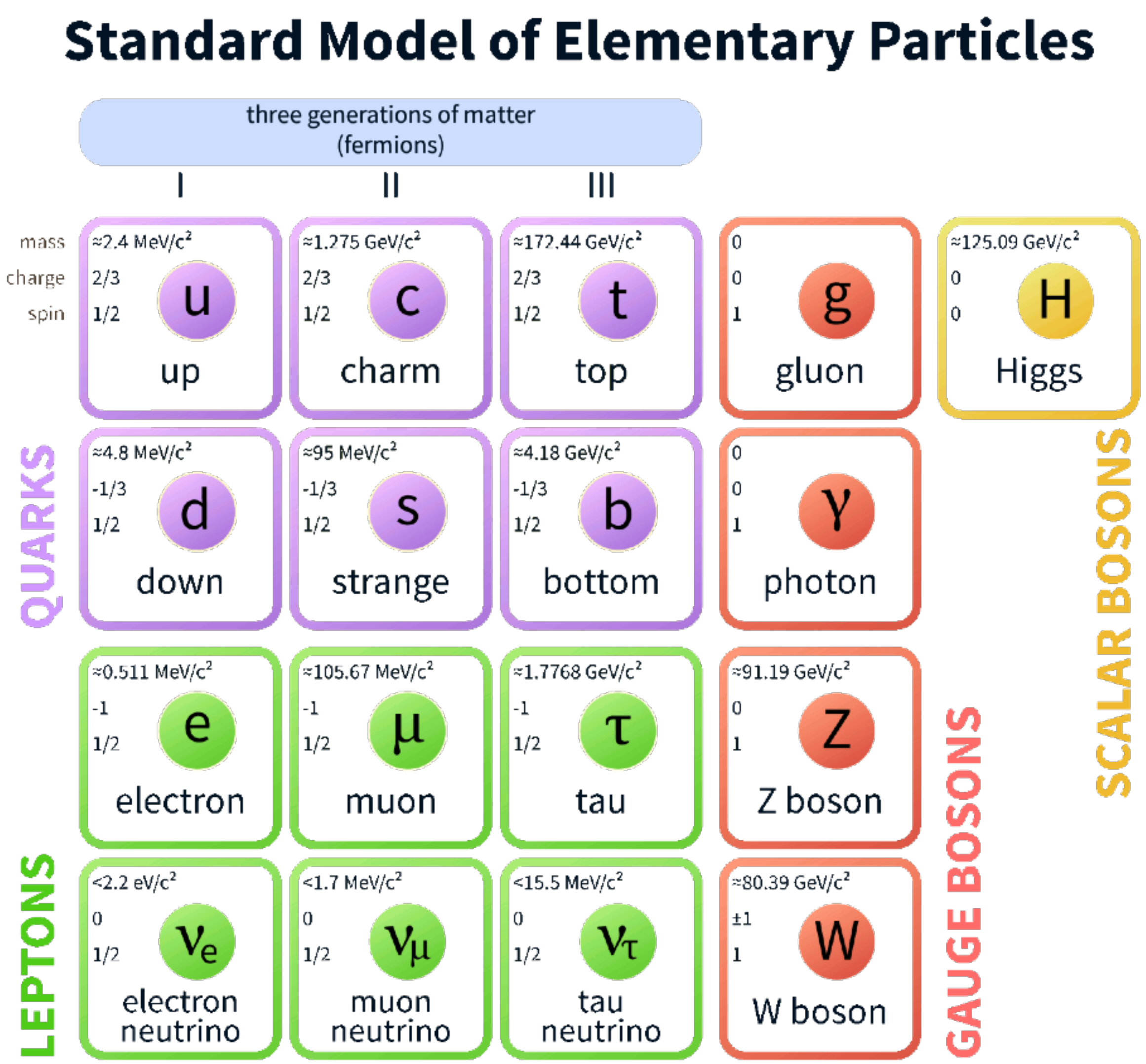
To correct all possible kinds of errors it may be necessary to entangle each logical qubit with dozens of physical qubits!

That could make quantum computing very hard!



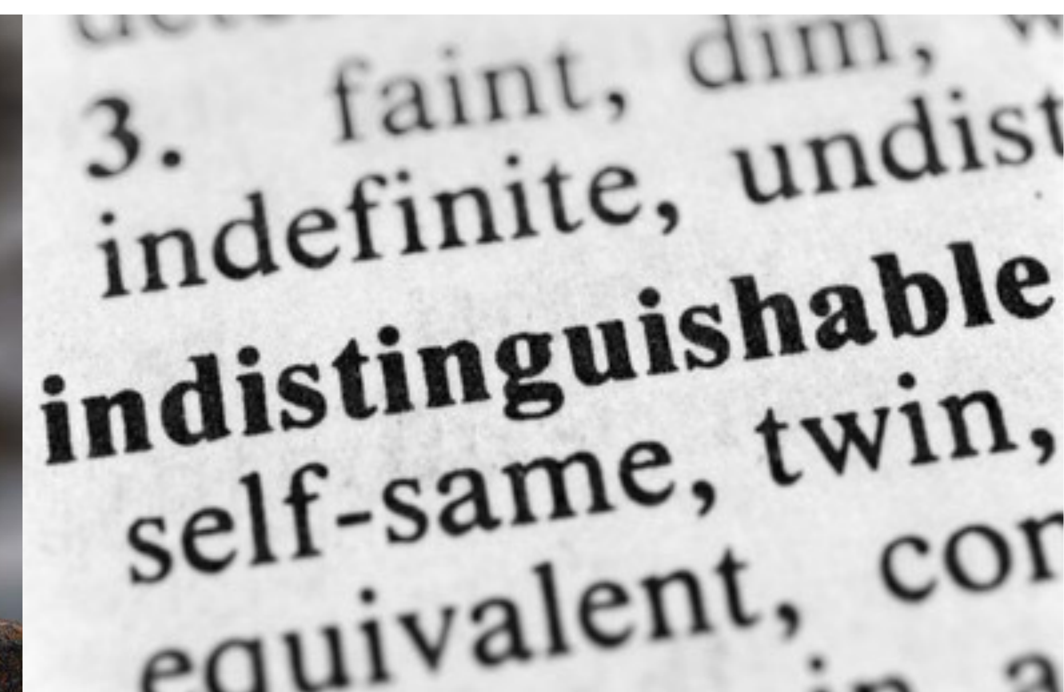
Could topology be used for fault-tolerant quantum computation?

Quantum particles can be classified as bosons or fermions depending on their having integer of half-integer spin.



Quantum particles can be classified as bosons or fermions
depending on their having integer of half-integer spin.

Identical particles are indistinguishable from each other.
Therefore, one cannot tell which individual particle
lives in a given quantum state!



Quantum particles can be classified as bosons or fermions depending on their having integer of half-integer spin.

A system with two identical bosons is described by an entangled state that is symmetric under the exchange of the two particles,

$$|\psi_1, \psi_2\rangle = \frac{1}{\sqrt{2}} \left(|\psi_1\rangle|\psi_2\rangle + |\psi_2\rangle|\psi_1\rangle \right)$$

Two identical fermions are described by an entangled state that is anti-symmetric under the exchange of the two particles,

$$|\psi_1, \psi_2\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle|\psi_2\rangle - |\psi_2\rangle|\psi_1\rangle\right)$$

Two identical fermions cannot occupy the same quantum state!

$$|\psi_1, \psi_1\rangle = 0$$

Without loss of generality, whenever two identical particles are swapped, the total wave function picks up a phase.

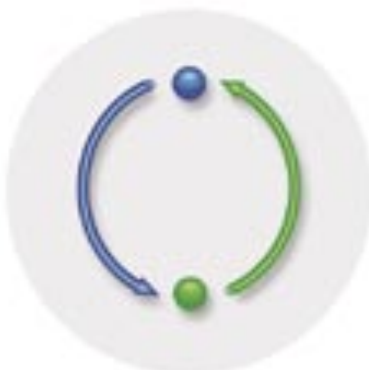
$$|\psi_1, \psi_2\rangle \rightarrow \mathrm{e}^{i\theta}|\psi_2, \psi_1\rangle$$

where $\mathrm{e}^{i\theta} = +1$ for bosons and -1 for fermions.

PHASE

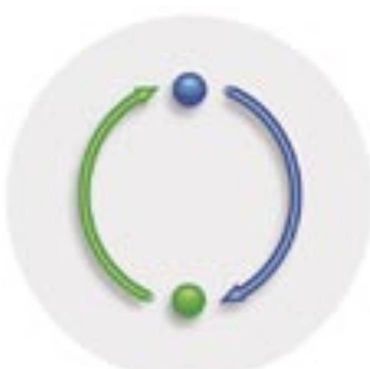
In 2D, it is mathematically possible to have particles with fractional statistics (anyons)!

Counterclockwise



$$|\psi_1, \psi_2\rangle \to \mathrm{e}^{i\theta}|\psi_2, \psi_1\rangle \qquad \theta \neq n\pi, \quad n \in \mathbb{Z}$$

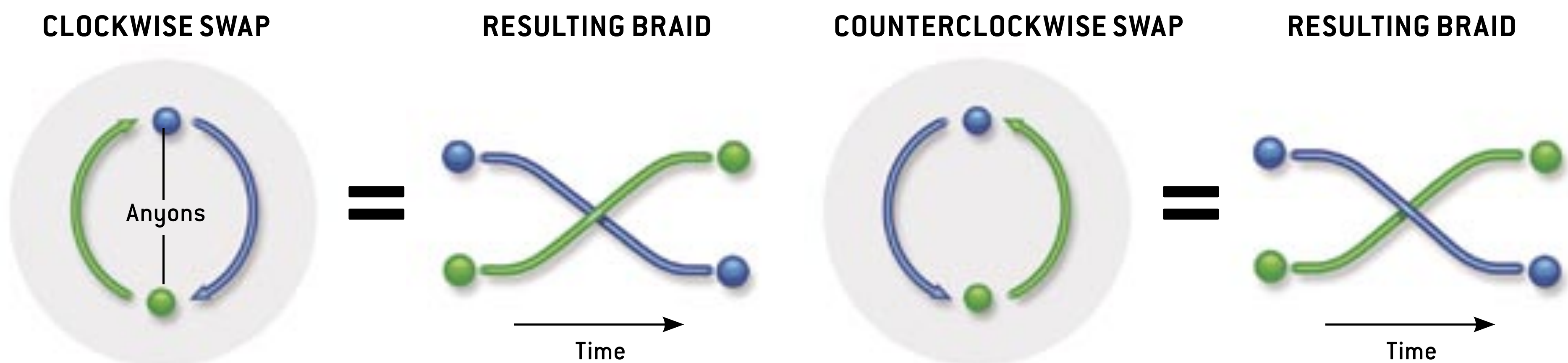
Clockwise

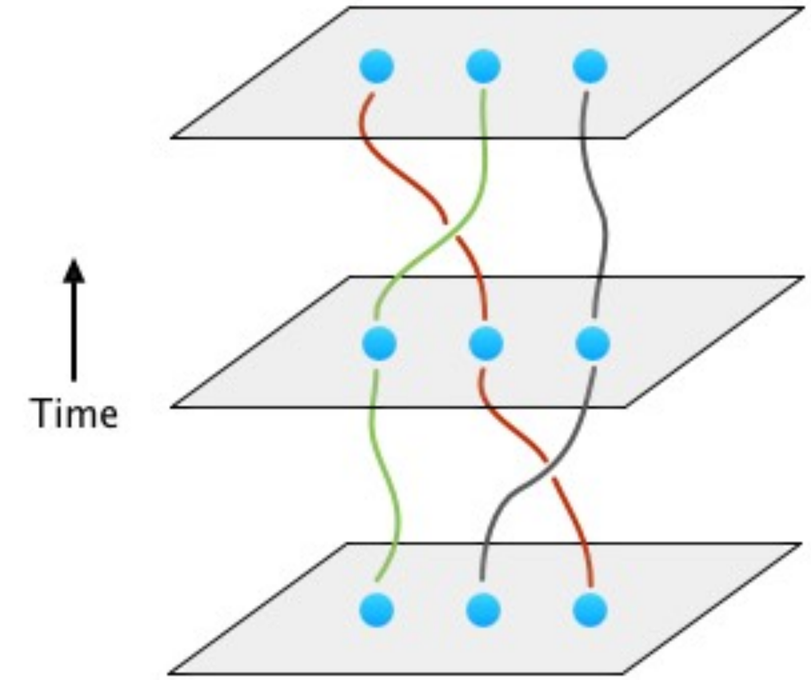


$$|\psi_1, \psi_2\rangle \to \mathrm{e}^{-i\theta}|\psi_2, \psi_1\rangle$$

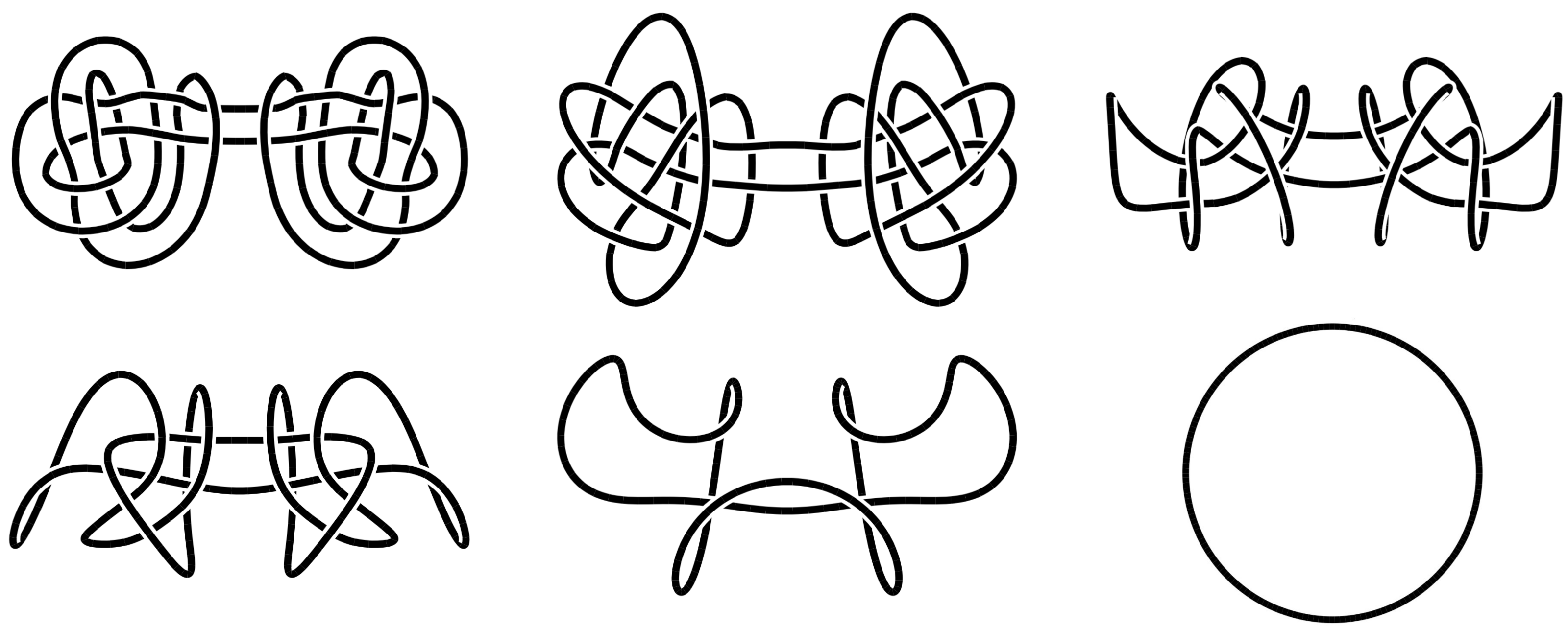
In that case it matters if the particles are swapped in the clockwise or counterclockwise direction!

The act of exchanging two anyons corresponds to two distinct braiding operations, depending on how they are swapped!

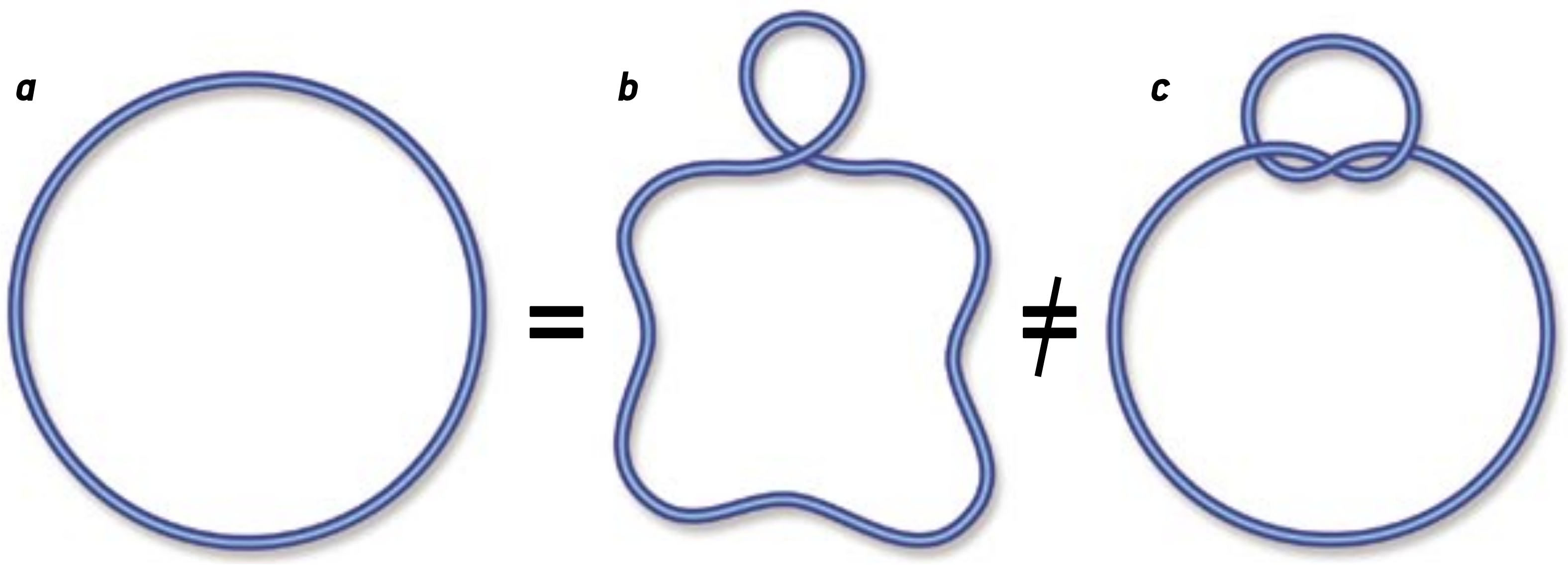




World line



Non-trivial topology can also appear in the form of knots of world lines!



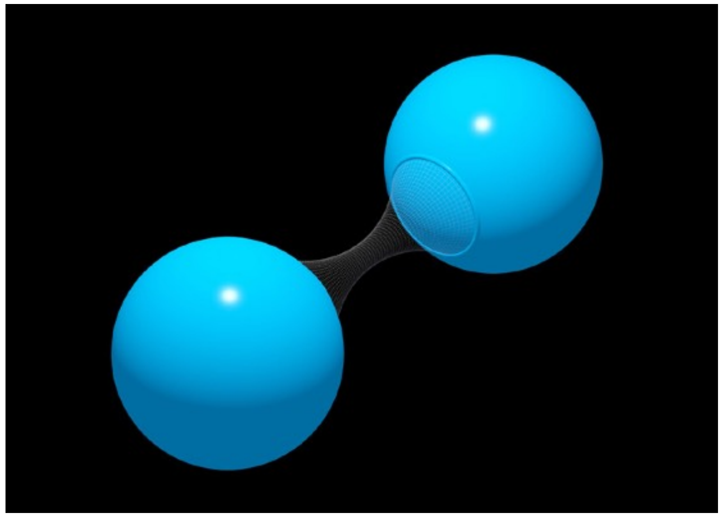
One can separate world lines in different topological classes based on the number of knots.

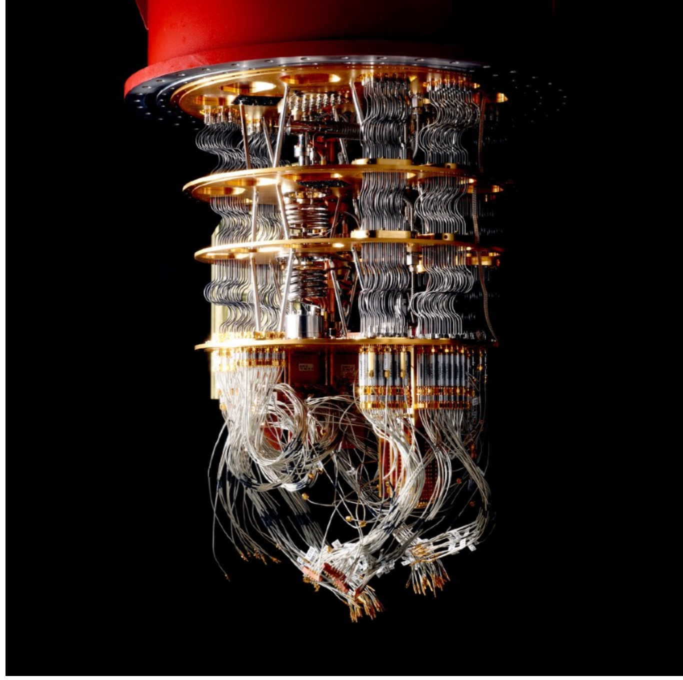
In 1997, Alexei Kitaev suggested that building qubits with anyons would permit creating quantum computers that are immune to decoherence!

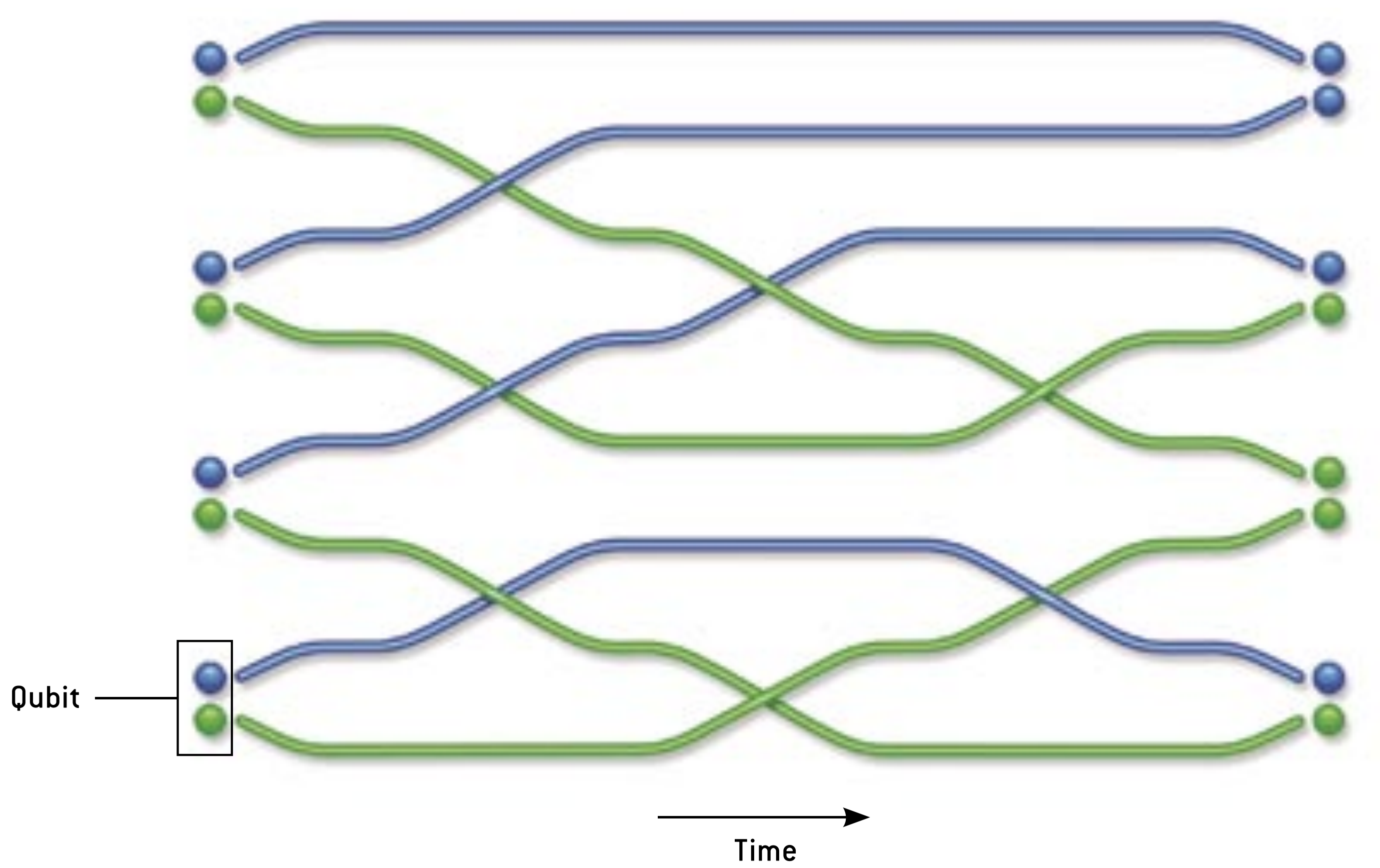
Anyons cannot exist in isolation and must be entangled with other anyons!

They are hard to observe because they are insensitive to local measurements, and can only be destroyed when merged with other anyons.

On the other hand, their qubits are hard to decohere!







The fundamental unit of information (qubit) in a topological quantum computer is a linear superposition of two anyons, which can be braided with other qubits!

PRL **95,** 140503 (2005) PHYSICAL REVIEW LETTERS week ending 30 SEPTEMBER 2005

Braid Topologies for Quantum Computation

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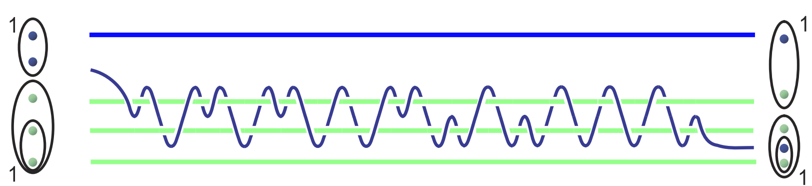
(Received 23 May 2005; published 29 September 2005)

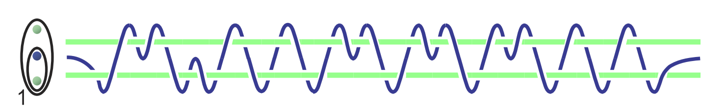
In topological quantum computation, quantum information is stored in states which are intrinsically protected from decoherence, and quantum gates are carried out by dragging particlelike excitations (quasiparticles) around one another in two space dimensions. The resulting quasiparticle trajectories define world lines in three-dimensional space-time, and the corresponding quantum gates depend only on the topology of the braids formed by these world lines. We show how to find braids that yield a universal set of quantum gates for qubits encoded using a specific kind of quasiparticle which is particularly promising for experimental realization.

(a)



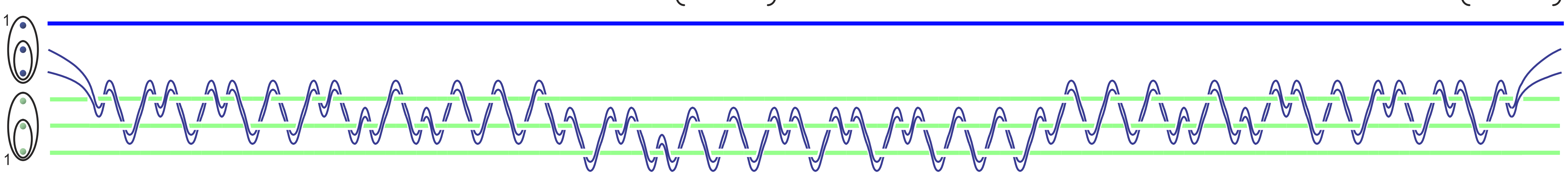
$$\sigma_2^{3}\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{-4}\sigma_2^{-2}\sigma_1^{4}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{3} \approx \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

(b)

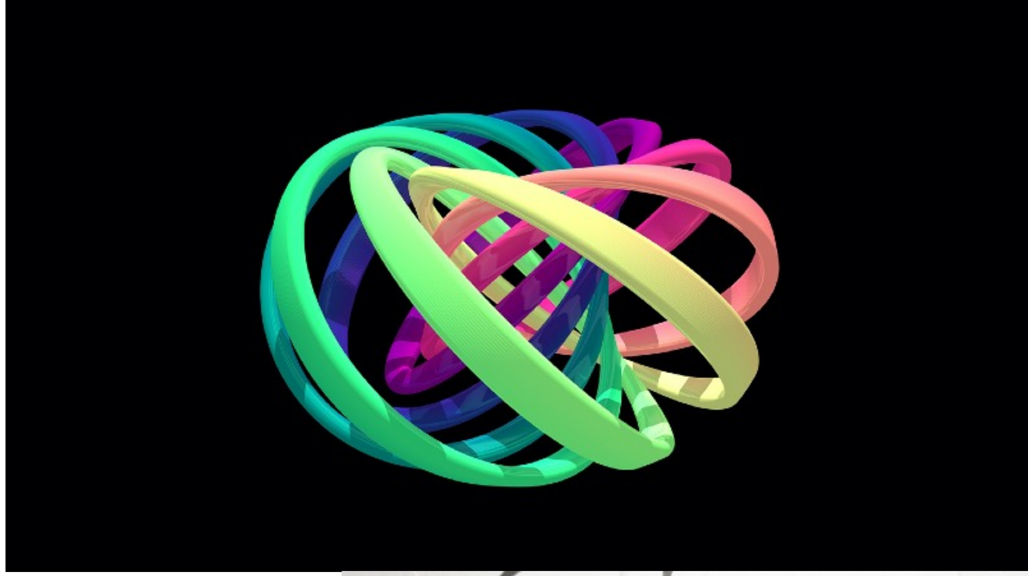


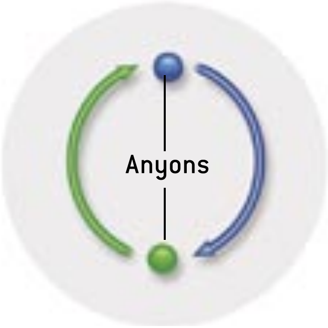
$$\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{4}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2} \approx \left(\begin{array}{cc|c} 0 & i & 0 \\ i & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

(c)



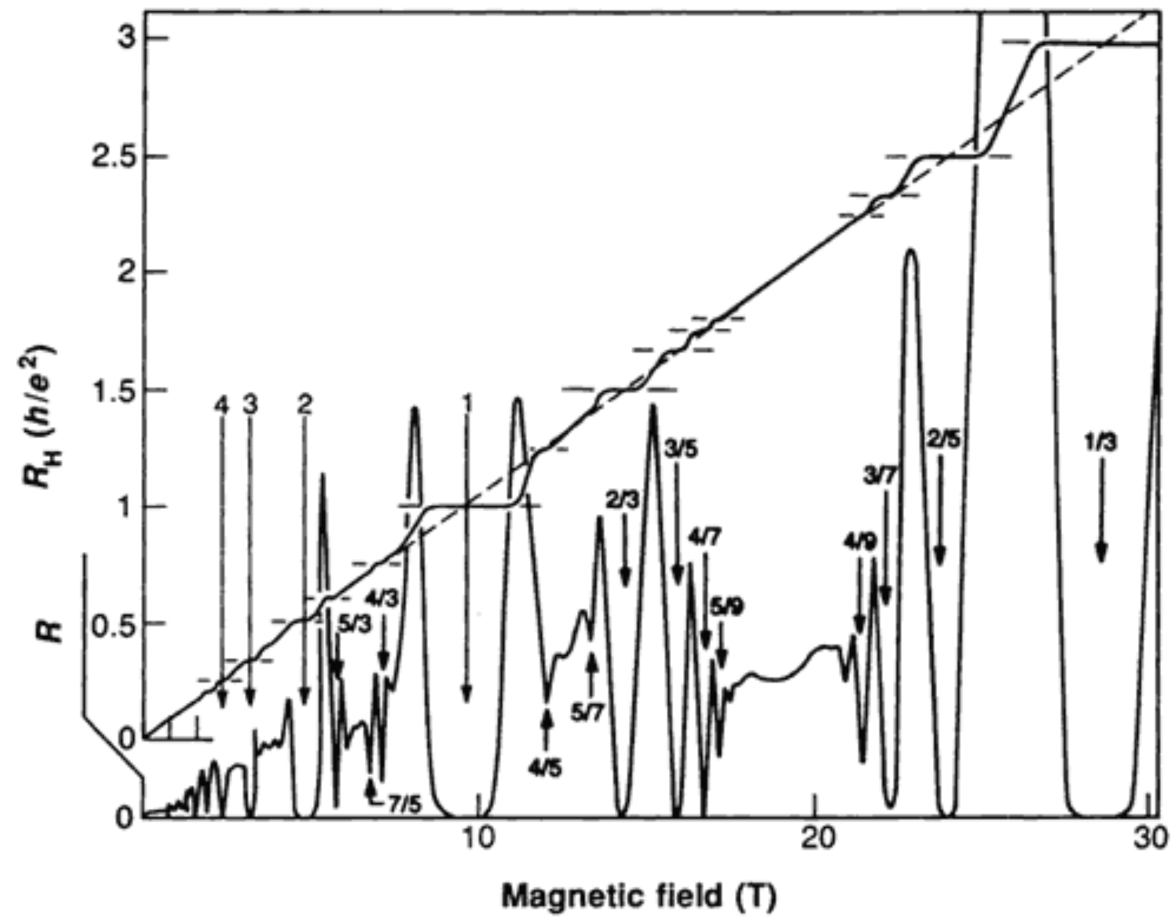
In 2005, it was explicitly shown (theoretically) how to perform logical operations by braiding world lines of topological quantum particles!





Do anyons exist?

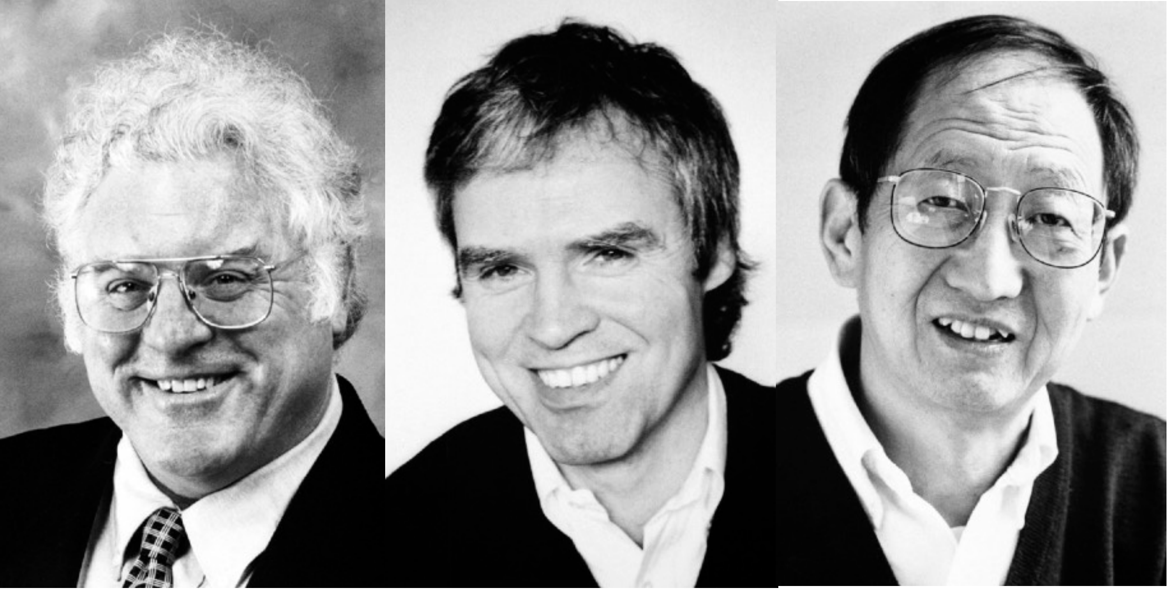
In 1982, right after the discovery of the integer quantum Hall effect, Tsui and Stomer discovered the appearance of quantum Hall plateaus at some rational fractions of magnetic flux quanta per unit cell.



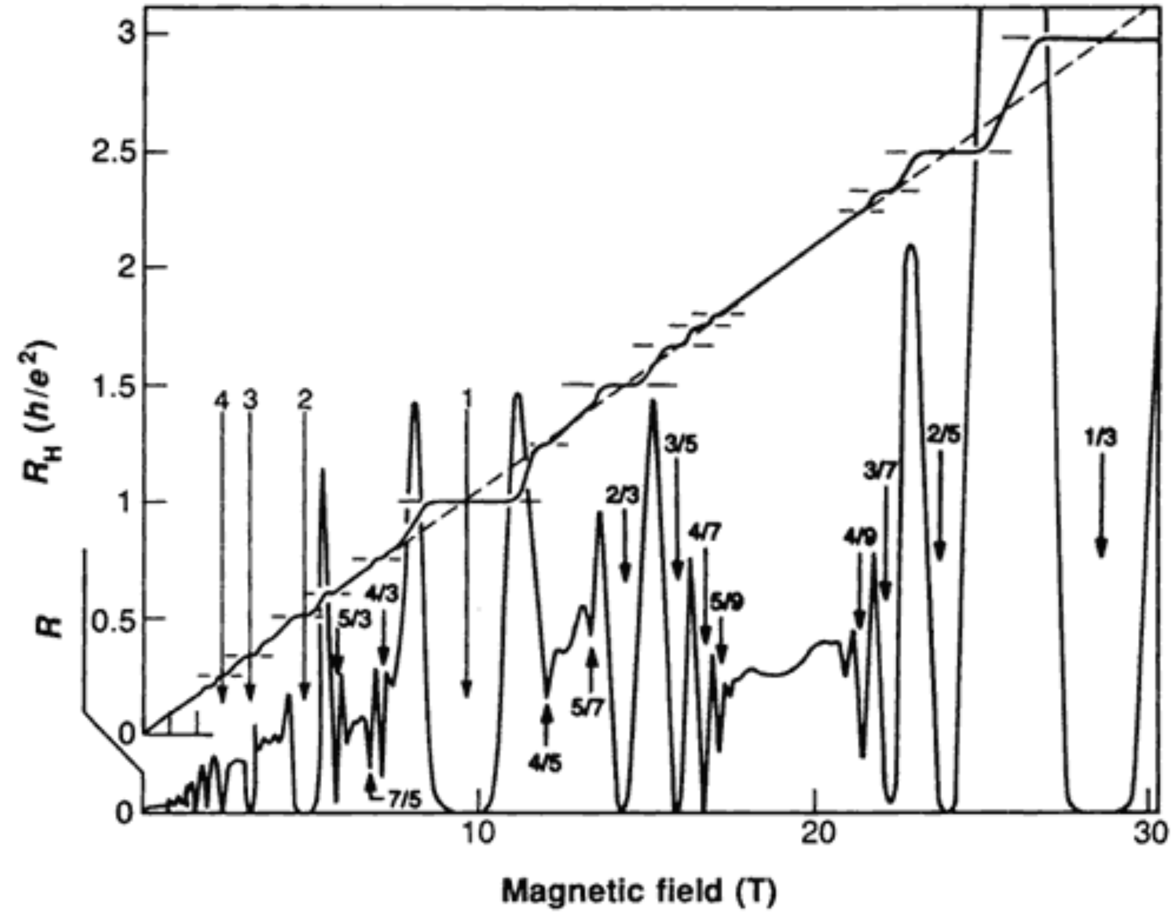
Flux quanta $\nu = \frac{p}{q} = \frac{1}{3}, \frac{3}{5}, \frac{4}{7}, \ldots$

One year later, Bob Laughlin showed the experiment can be explained by the existence of particles with fractional charge!

Fractional quantum Hall effect!



The Nobel Prize in Physics 1998 was awarded jointly to Robert B. Laughlin, Horst L. Störmer and Daniel C. Tsui "for their discovery of a new form of quantum fluid with fractionally charged excitations."



Are particles with fractionalized charge anyons?

VOLUME 53, NUMBER 7 PHYSICAL REVIEW LETTERS 13 AUGUST 1984

Fractional Statistics and the Quantum Hall Effect

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and

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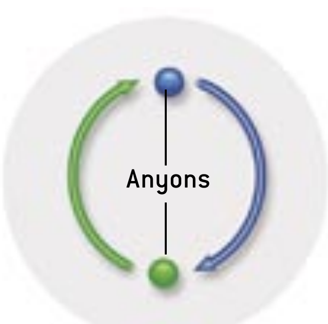
(Received 18 May 1984)

The statistics of quasiparticles entering the quantum Hall effect are deduced from the adiabatic theorem. These excitations are found to obey fractional statistics, a result closely related to their fractional charge.

In 1984, Arovas, Schrieffer and Wilczek showed (theoretically) that particles with fractional charge $e^* = e/q$ in the FQH effect correspond to anyons with phase



$$\theta = \frac{\pi}{q}$$

$$|\psi_1, \psi_2\rangle \to \mathrm{e}^{i\theta} |\psi_2, \psi_1\rangle$$

MESOSCOPIC PHYSICS

Fractional statistics in anyon collisions

H. Bartolomei[1]*, M. Kumar[1]*†, R. Bisognin[1], A. Marguerite[1]‡, J.-M. Berroir[1], E. Bocquillon[1], B. Plaçais[1], A. Cavanna[2], Q. Dong[2], U. Gennser[2], Y. Jin[2], G. Fève[1]§

Bartolomei *et al.*, *Science* **368**, 173–177 (2020)   10 April 2020

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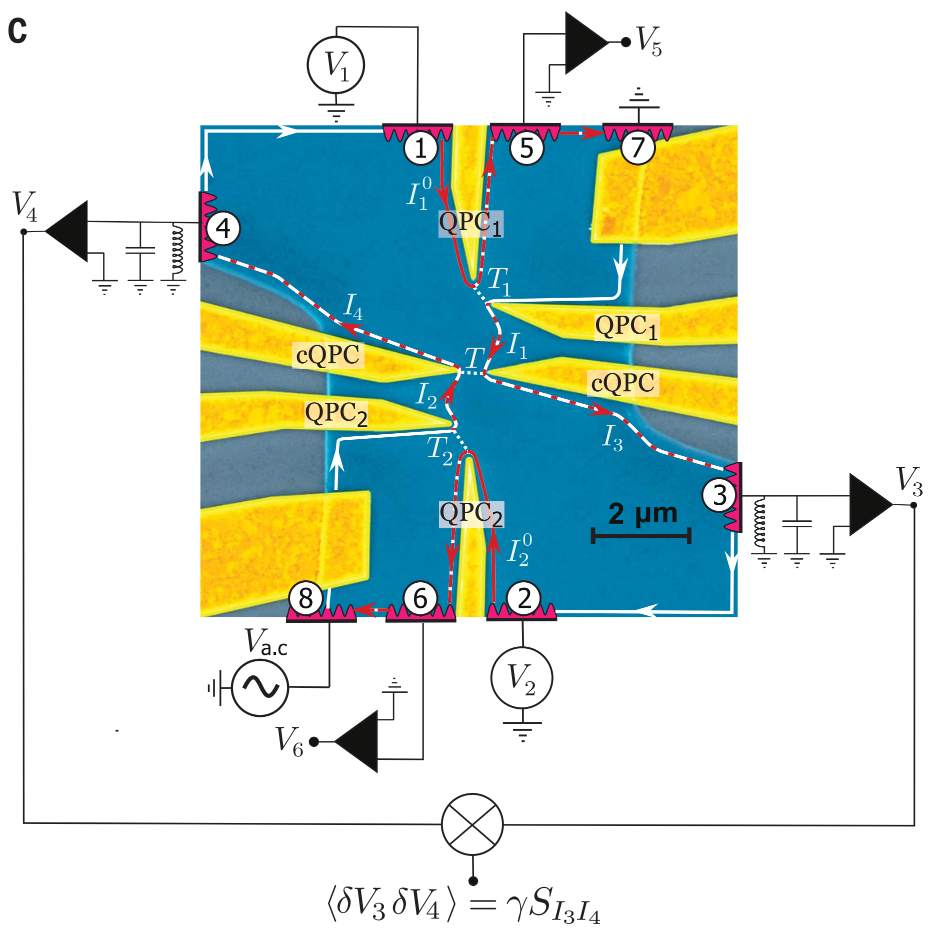
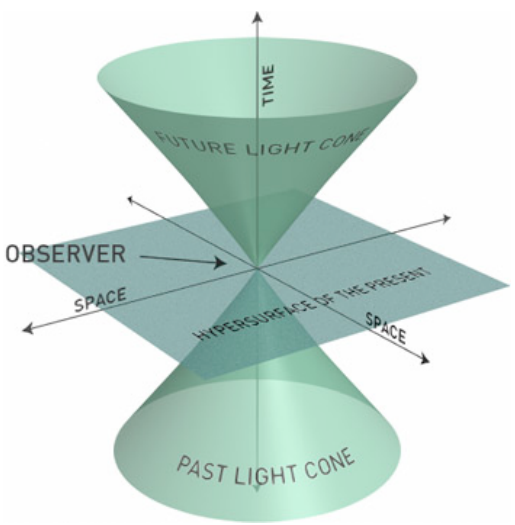


Experimental evidence that they exist!

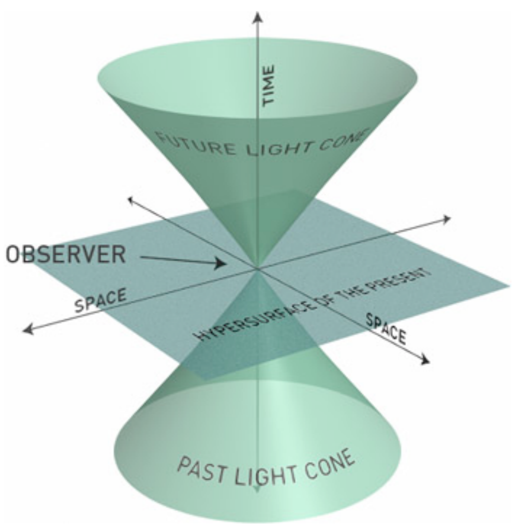
In 1928, Paul Dirac proposed a relativistic wave equation for spin 1/2 particles, such as electrons, neutrinos and quarks,

$$(i\partial\!\!\!/ - m)\psi = 0\,.$$



In 1928, Paul Dirac proposed a relativistic wave equation for spin 1/2 particles, such as electrons, neutrinos and quarks,

$$(i\partial\!\!\!/ - m)\psi = 0\,.$$



Etore Majorana proposed in 1937 a variation of the Dirac wave equation for neutral particles with spin 1/2.

He showed that their wave functions are real, and that an electron can be fractionalized in two entangled Majorana fermions, which are charge neutral,

$$|e\rangle = \frac{1}{\sqrt{2}}\left(|\gamma_1\rangle + i|\gamma_2\rangle\right)$$

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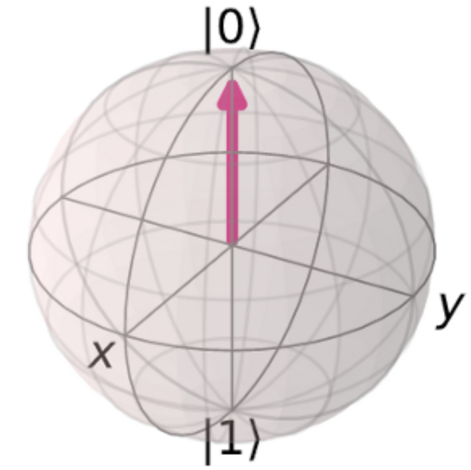

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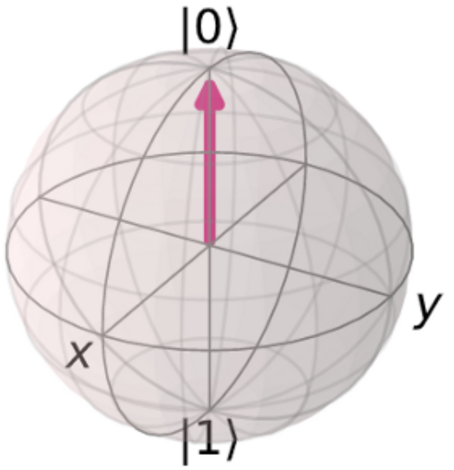
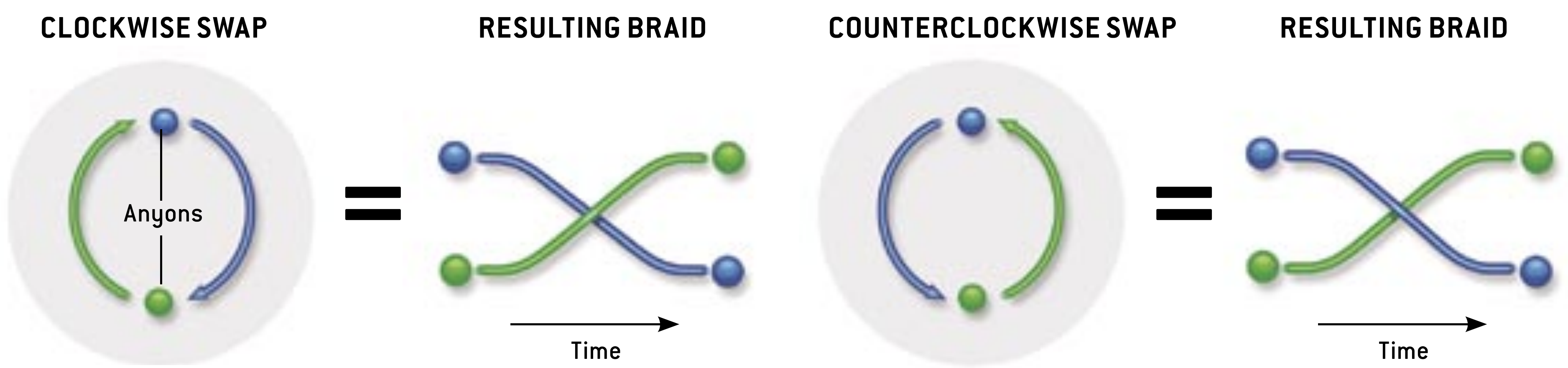
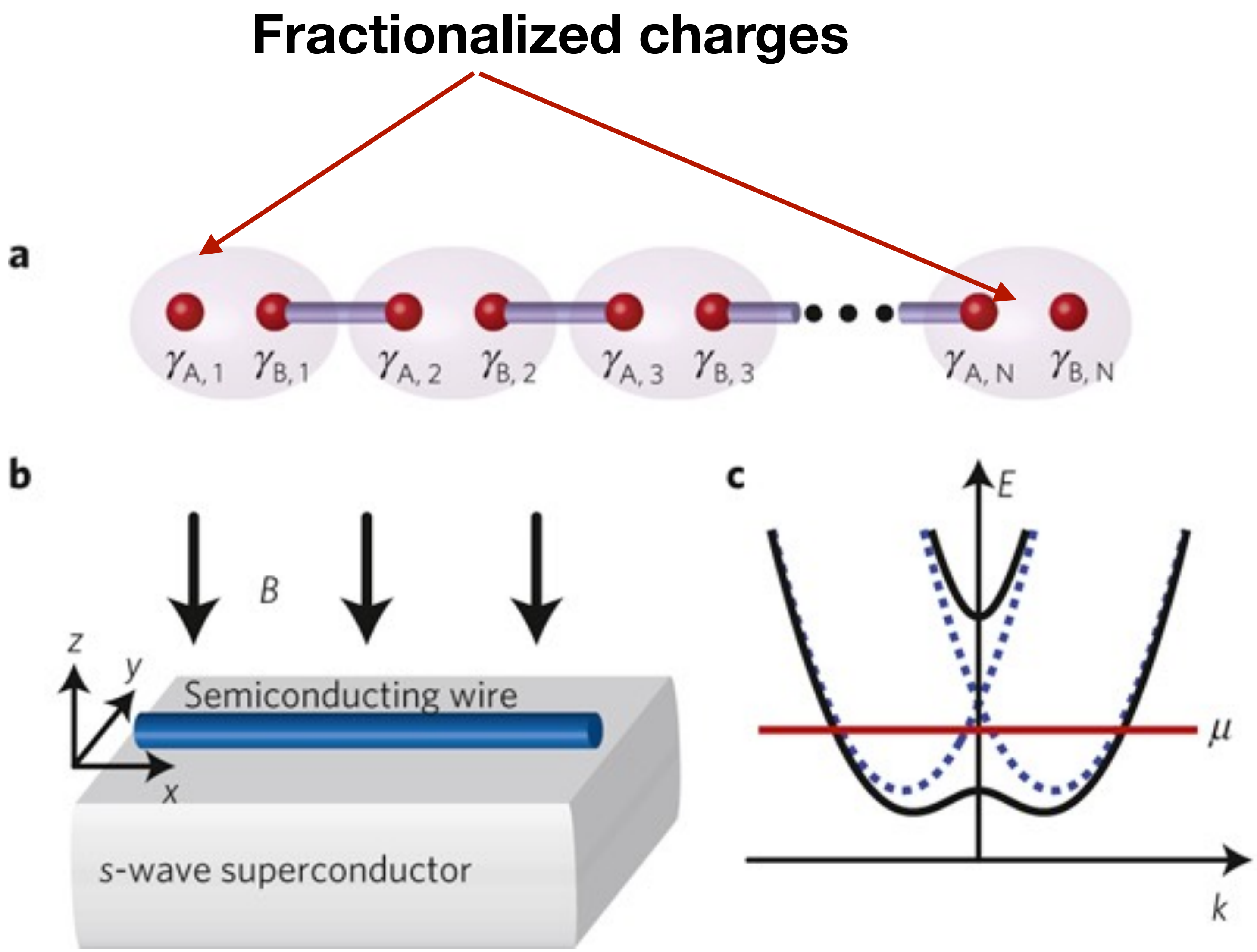
Qubit?

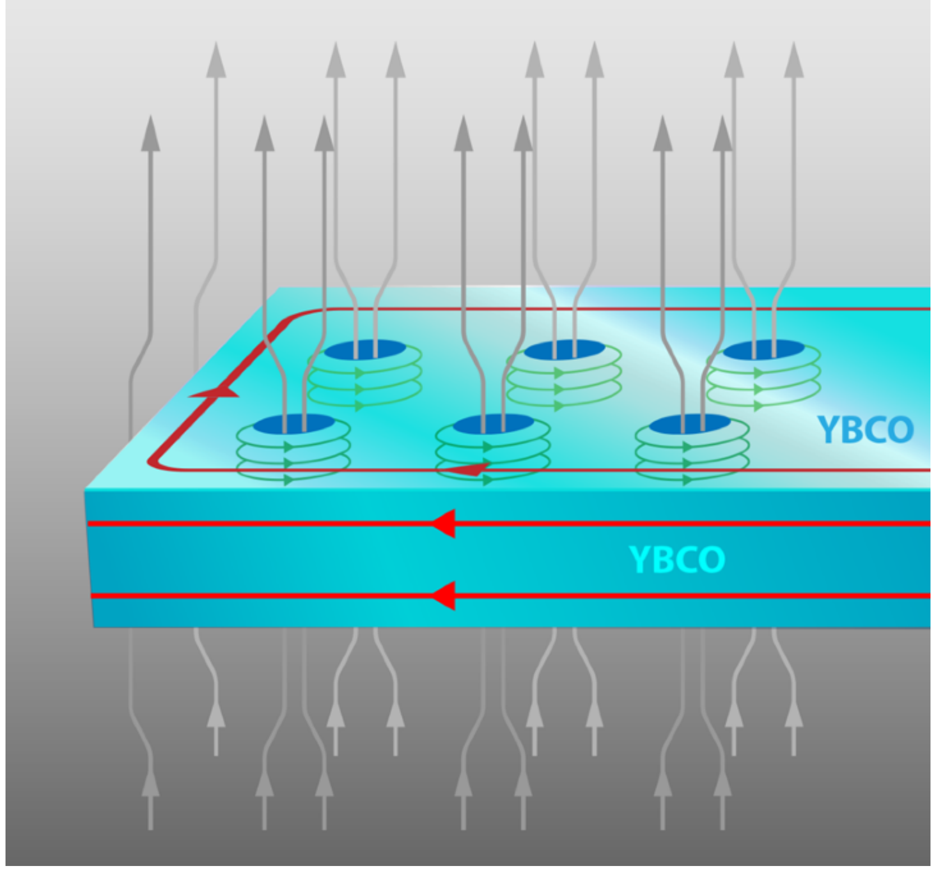
The only way one can destroy a Majorana fermion is by bringing them close to each other and annihilate them.





Whether they recombine into an electron or nothing depends on how they are braided. Otherwise, they are topological excitations and cannot be destroyed by local perturbations!





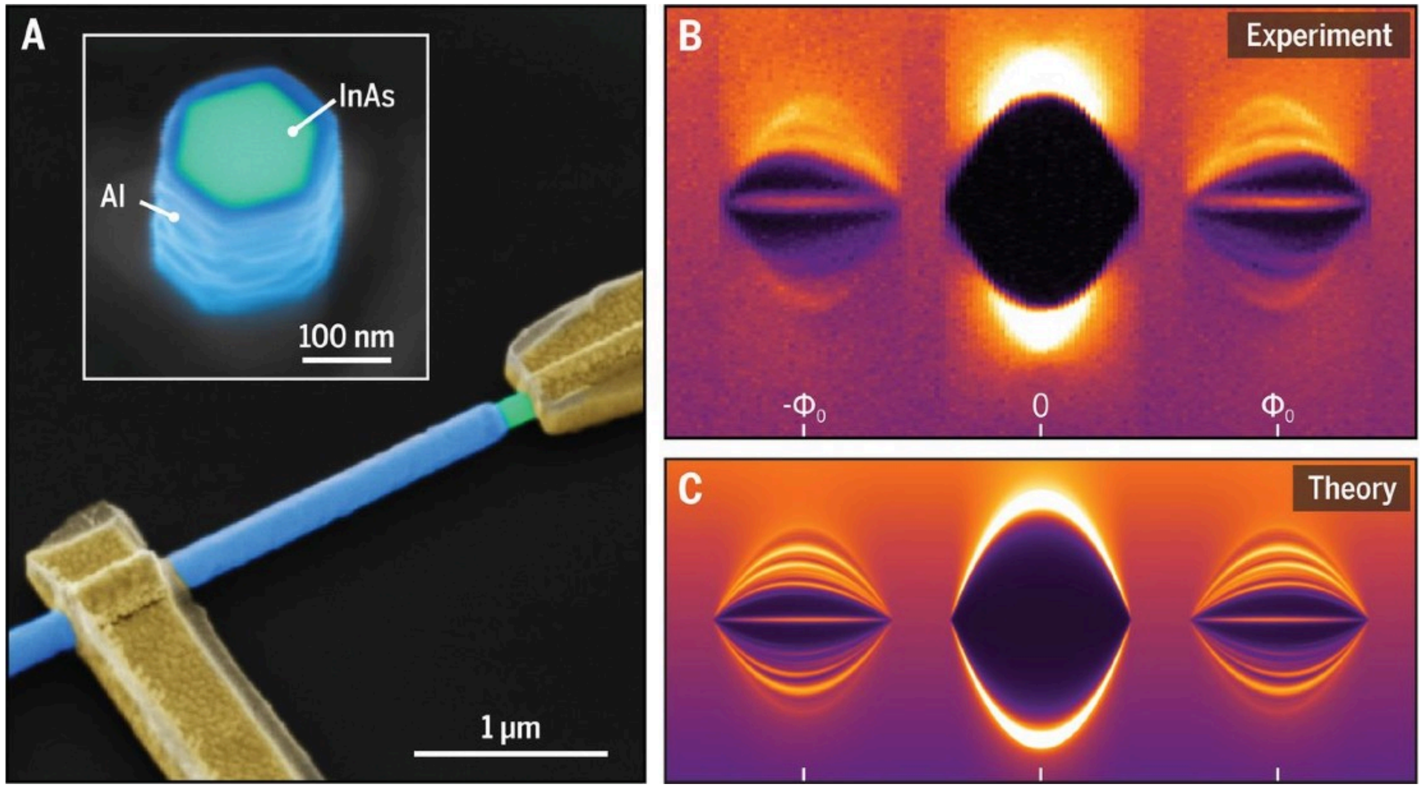
Fractionalized charges may also exist at the two ends of a superconducting wire or inside the core of a superconducting vortex!

RESEARCH ARTICLE

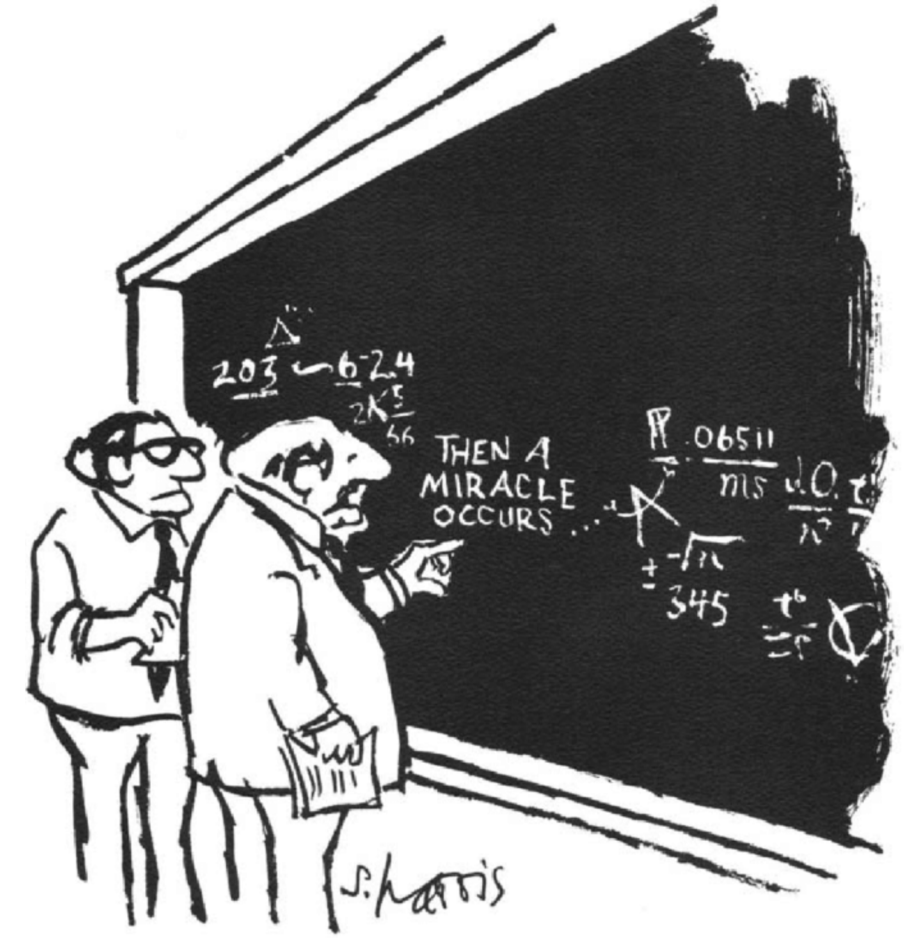
TOPOLOGICAL MATTER

Flux-induced topological superconductivity in full-shell nanowires

S. Vaitiekėnas[1], G. W. Winkler[2], B. van Heck[2], T. Karzig[2], M.-T. Deng[1], K. Flensberg[1], L. I. Glazman[3], C. Nayak[2], P. Krogstrup[1], R. M. Lutchyn[2]*, C. M. Marcus[1]*



Possible experimental observation of signatures of Majorana fermions!



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

It seems extremely difficult to observe Majorana fermions in nature and nobody knows for sure (yet) if they exist.

Quantum computing is a major open challenge!