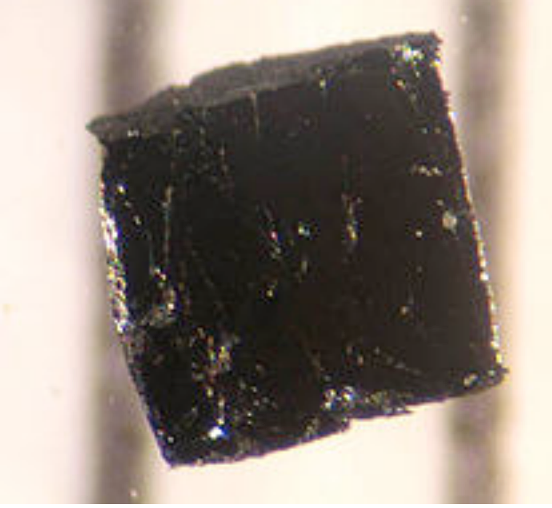
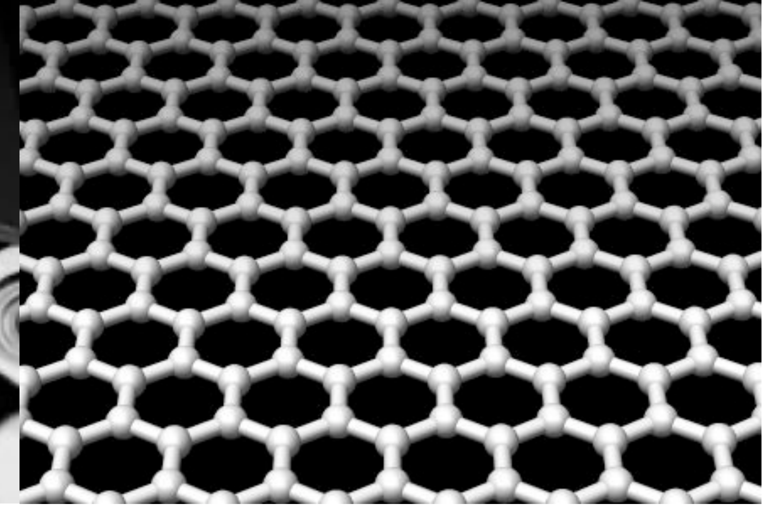
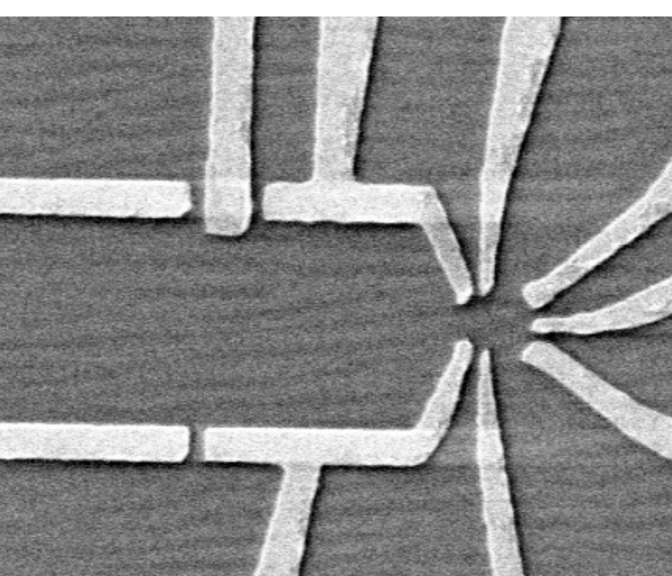




Quantum Science

Bruno Uchoa
Department of Physics and Astronomy
University of Oklahoma

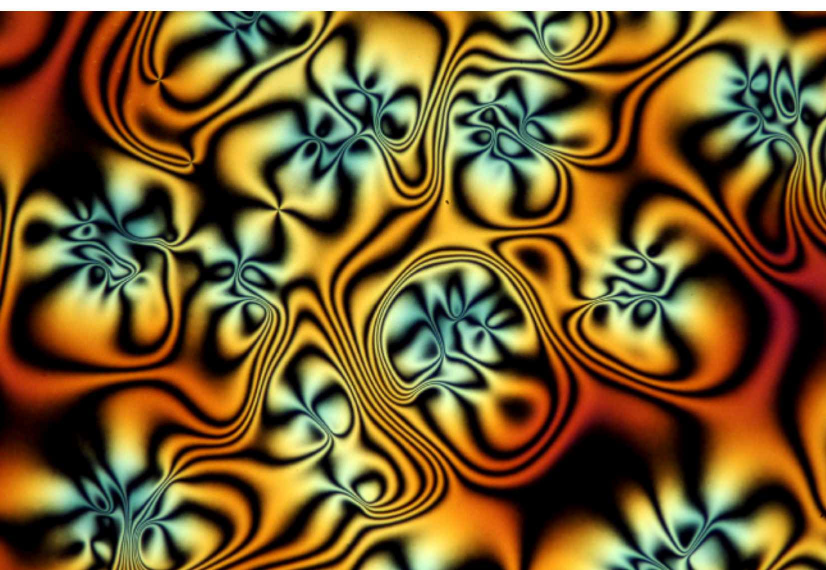
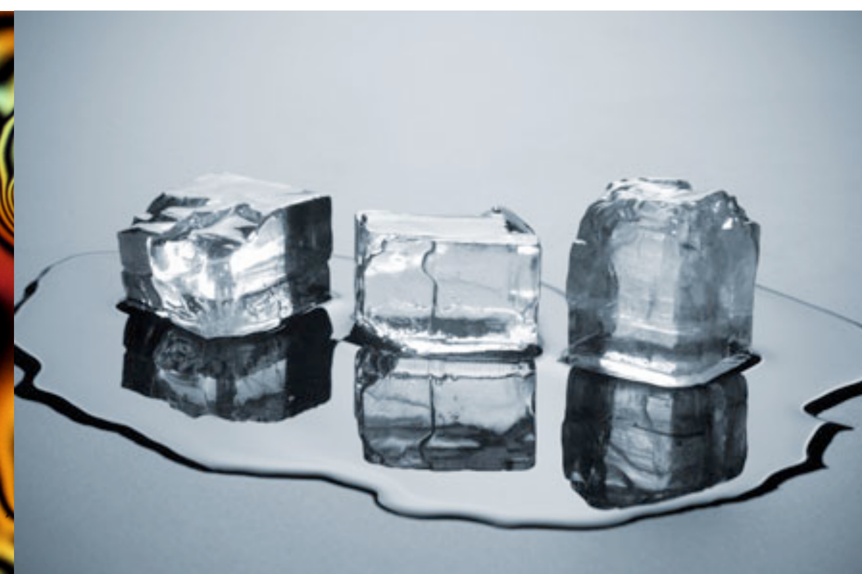



Lecture 3

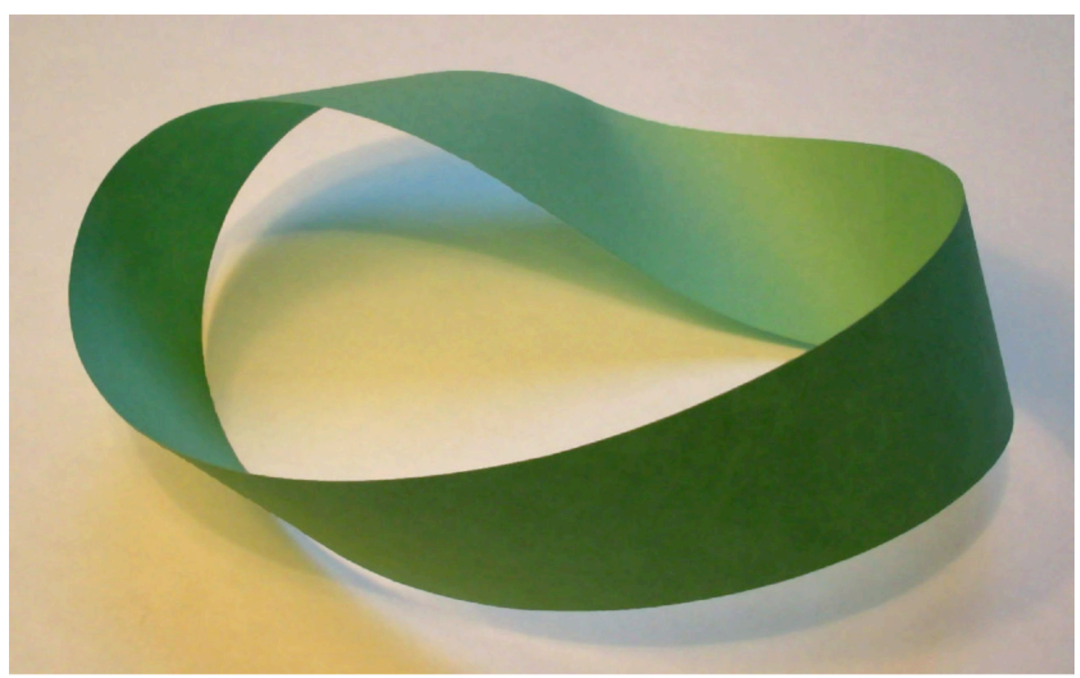
Topology and quantum mechanics

Symmetries in nature

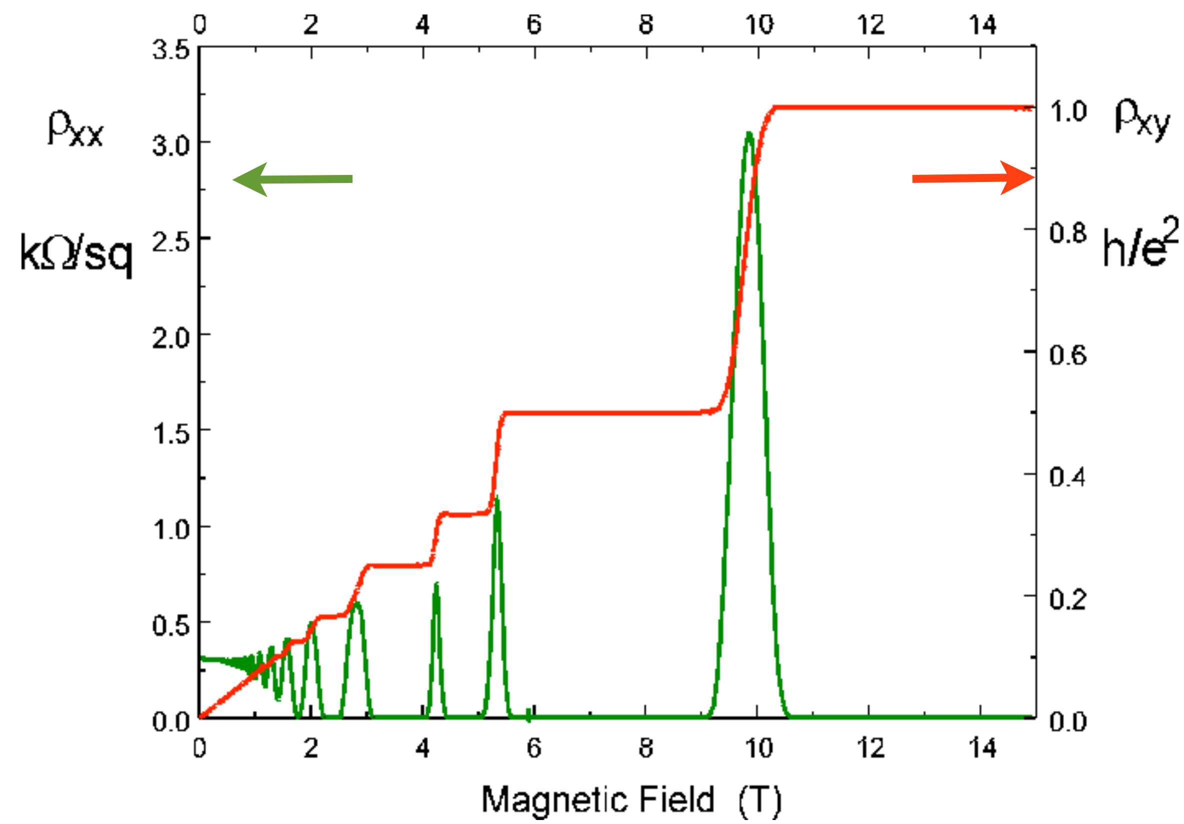
There is a well established concept of order in the way that nature spontaneously breaks symmetries.


More recently, physicists have gone beyond the paradigm of broken symmetries to find new states of matter with non-trivial topology



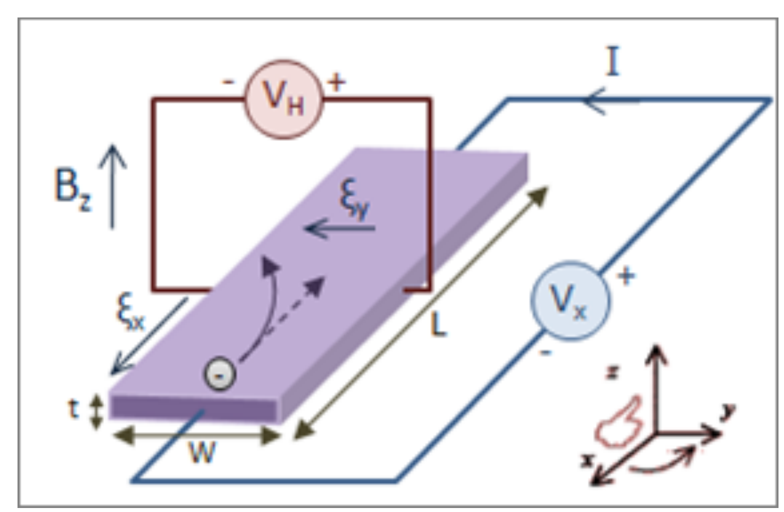
In 1982, Von Klitzing discovered that the Hall conductivity in bad metals is exactly quantized in 2D



The Nobel Prize in Physics 1985 was awarded to Klaus von Klitzing *"for the discovery of the quantized Hall effect".*

$$\sigma_{xy} = n\frac{e^2}{h}$$

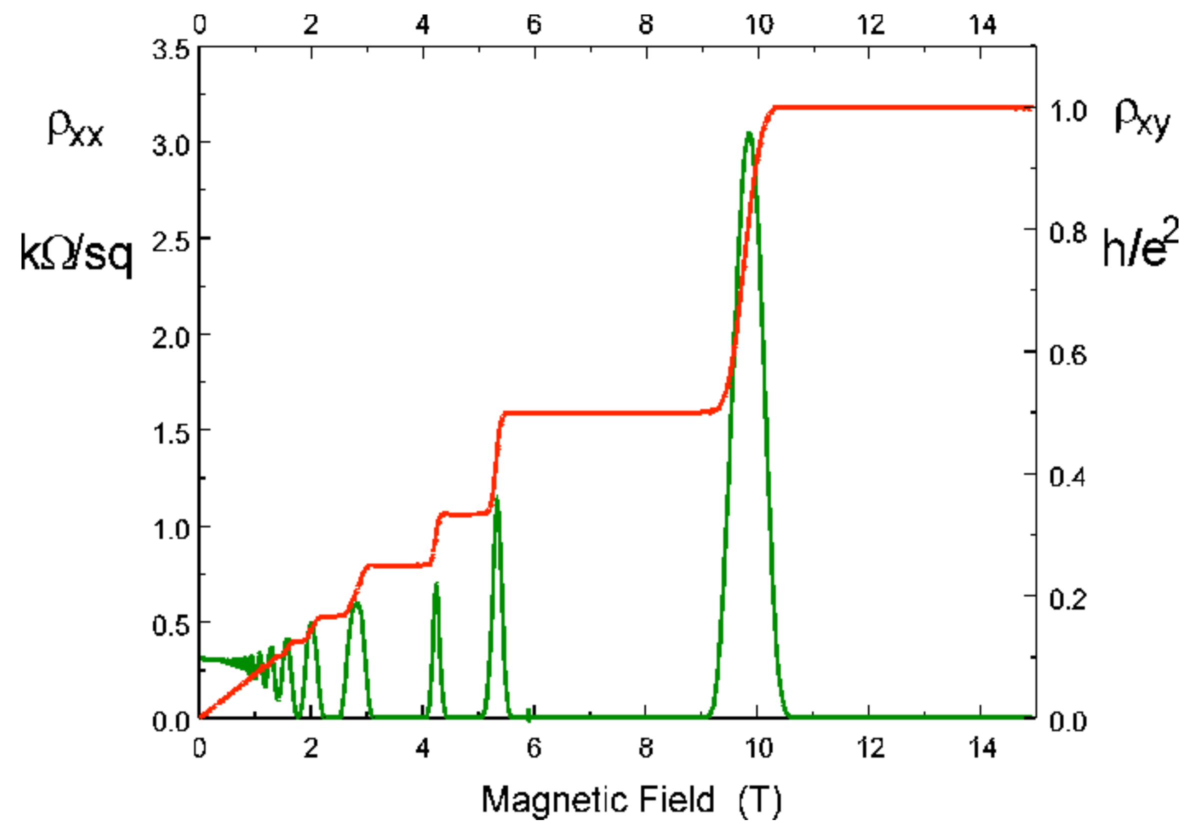
Integer



$$\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2}$$

In 1982, Von Klitzing discovered that the Hall conductivity in bad metals is exactly quantized in 2D

The Nobel Prize in Physics 1985 was awarded to Klaus von Klitzing *"for the discovery of the quantized Hall effect".*



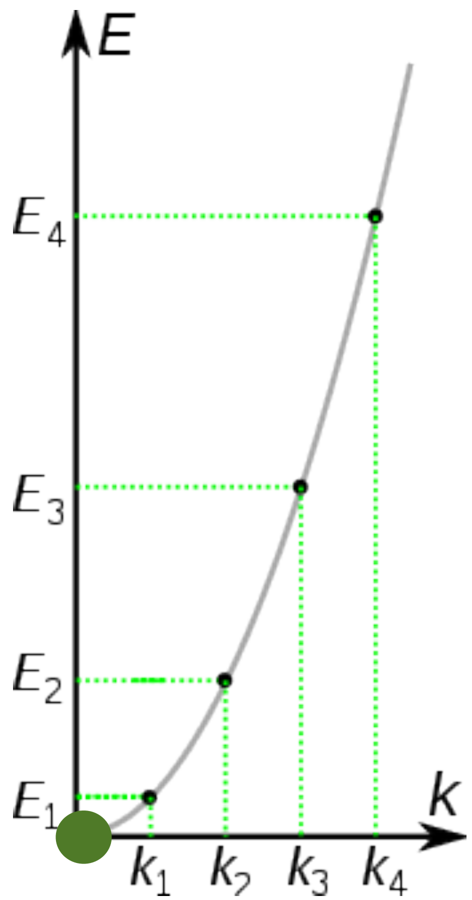
$$\sigma_{xy} = n\frac{e^2}{h}$$

Integer

Subsequent measurements confirmed the quantization in steps of e^2/h with an accuracy of 10^{-9} !

Why is the quantization so good?

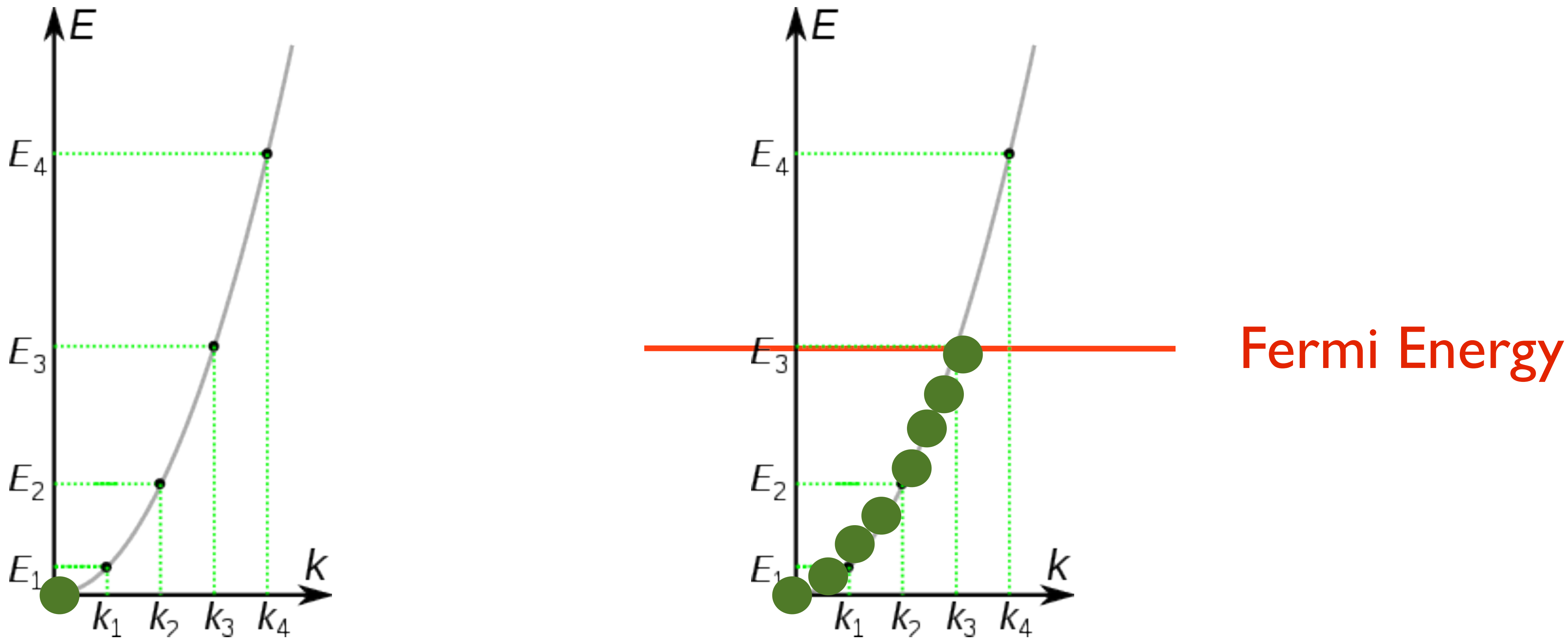
In galilean invariant systems (non-relativistic), the energy of a particle is proportional to the square of the momentum



In the ground state, a single free electron is at rest (k=0)

$$E = \frac{\hbar^2 k^2}{2m}$$

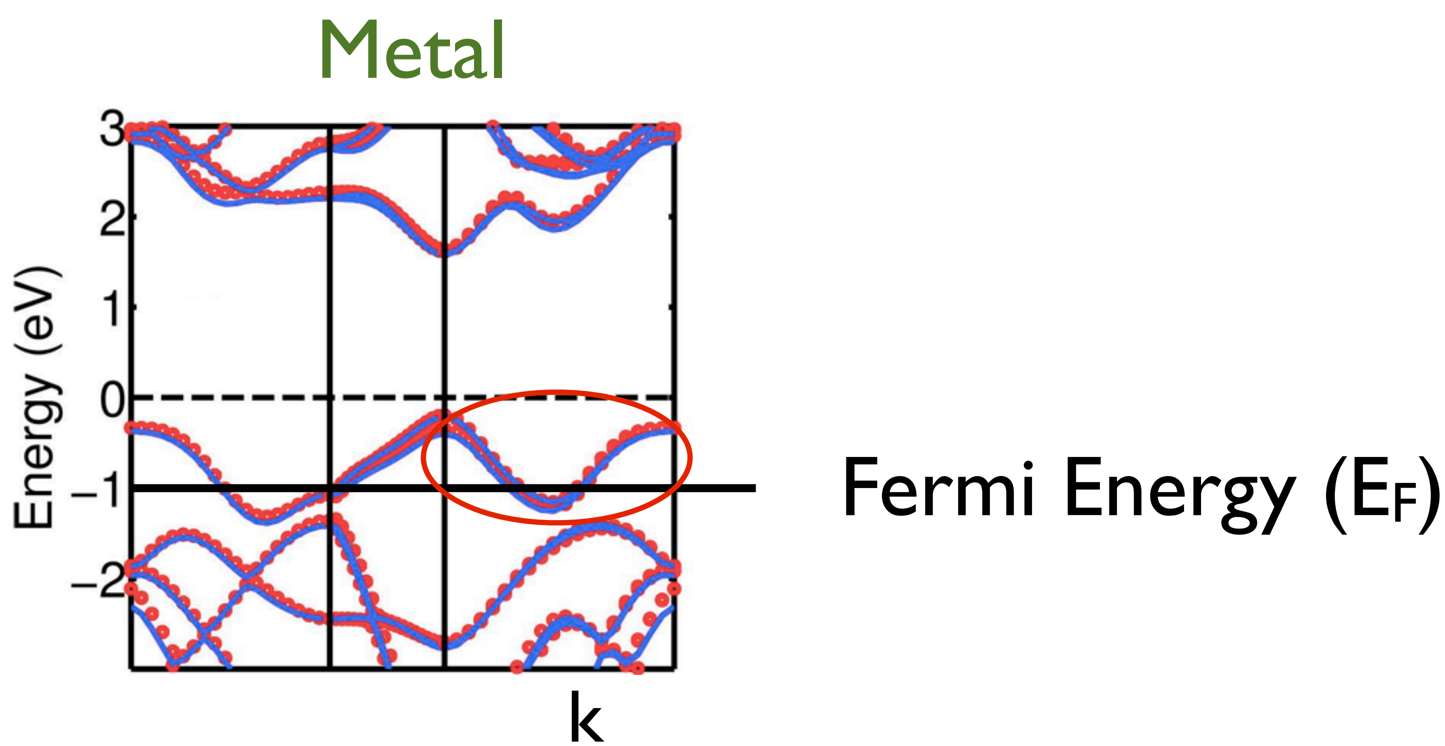
In galilean invariant systems (non-relativistic), the energy of a particle is proportional to the square of the momentum

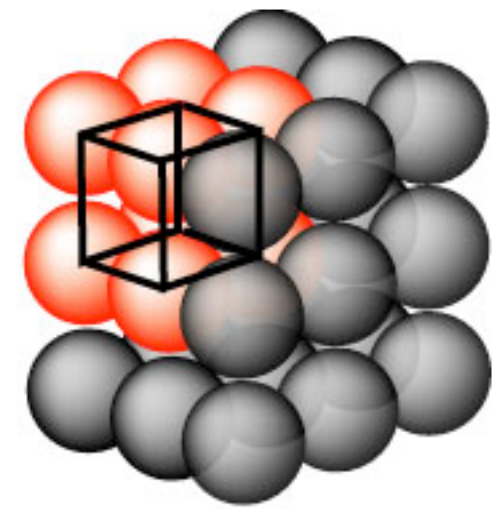


Because of Pauli principle, two electrons cannot occupy the same quantum state.

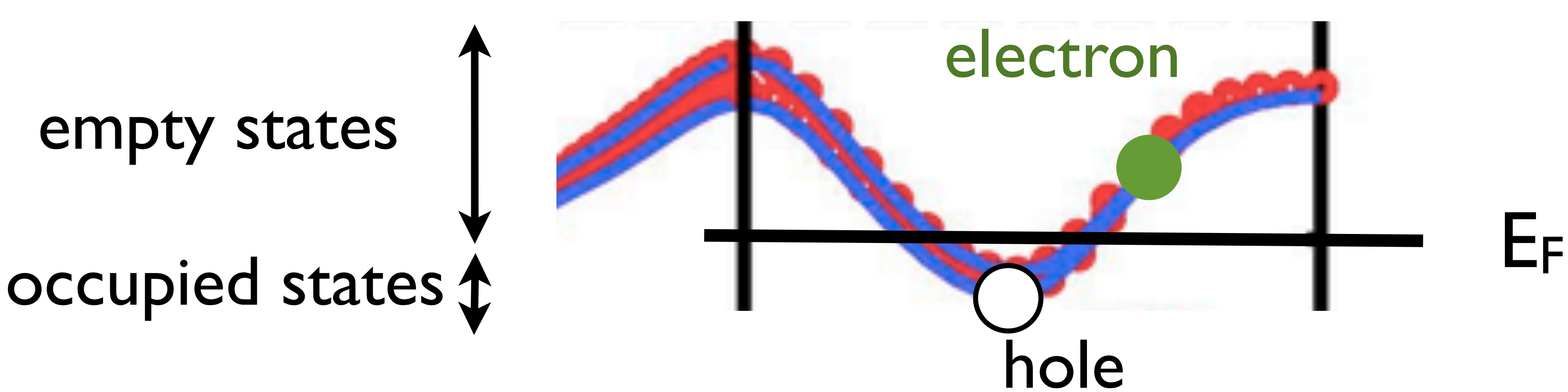
If one adds many free electrons, the states below the Fermi level are occupied and the ones above it are empty.

In solids (periodic potential), the energy spectrum of electrons can be quite complicated



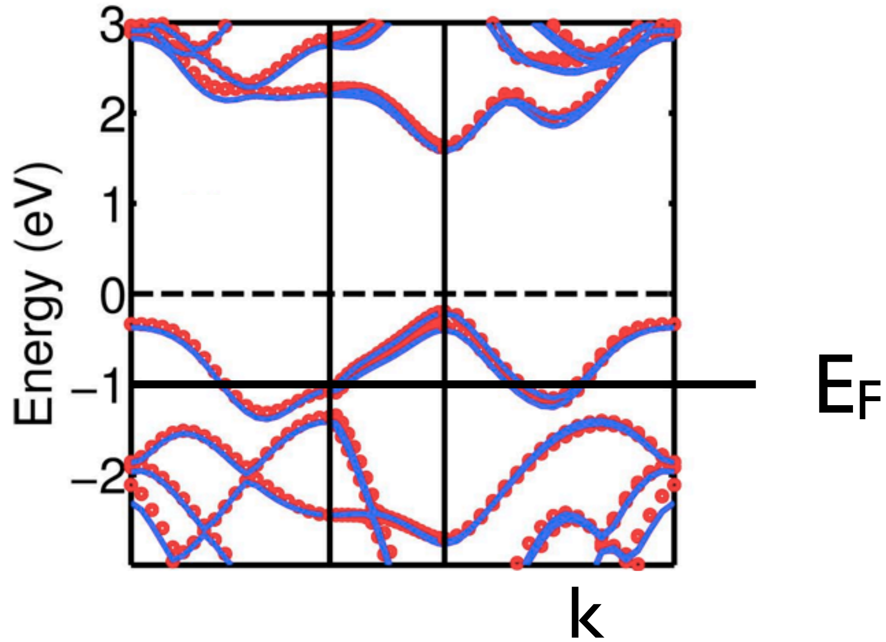


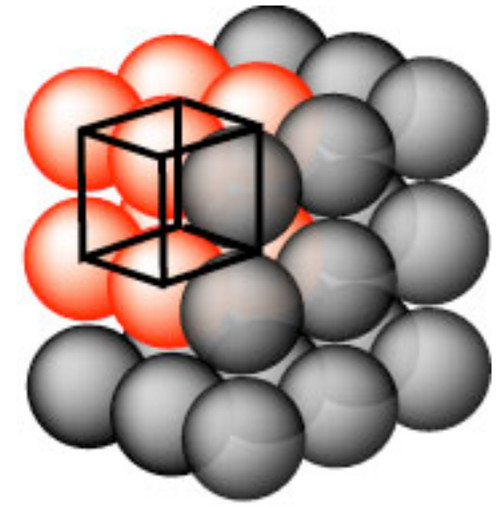
If the Fermi level crosses the electronic bands, the highest occupied state can be easily excited by a bias voltage (metallic state)



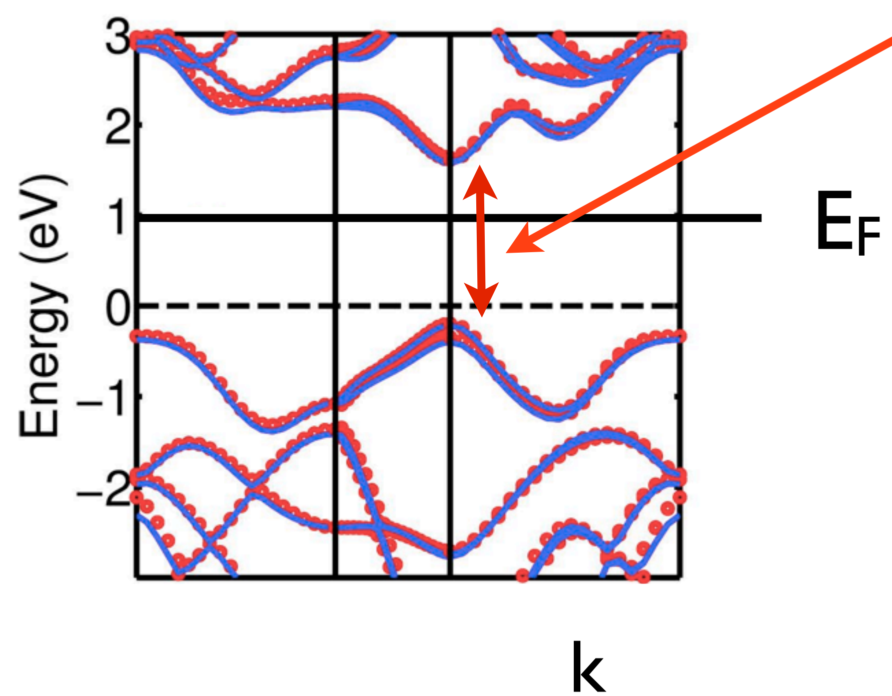
In solids (periodic potential), the energy spectrum of electrons can be quite complicated

Metal



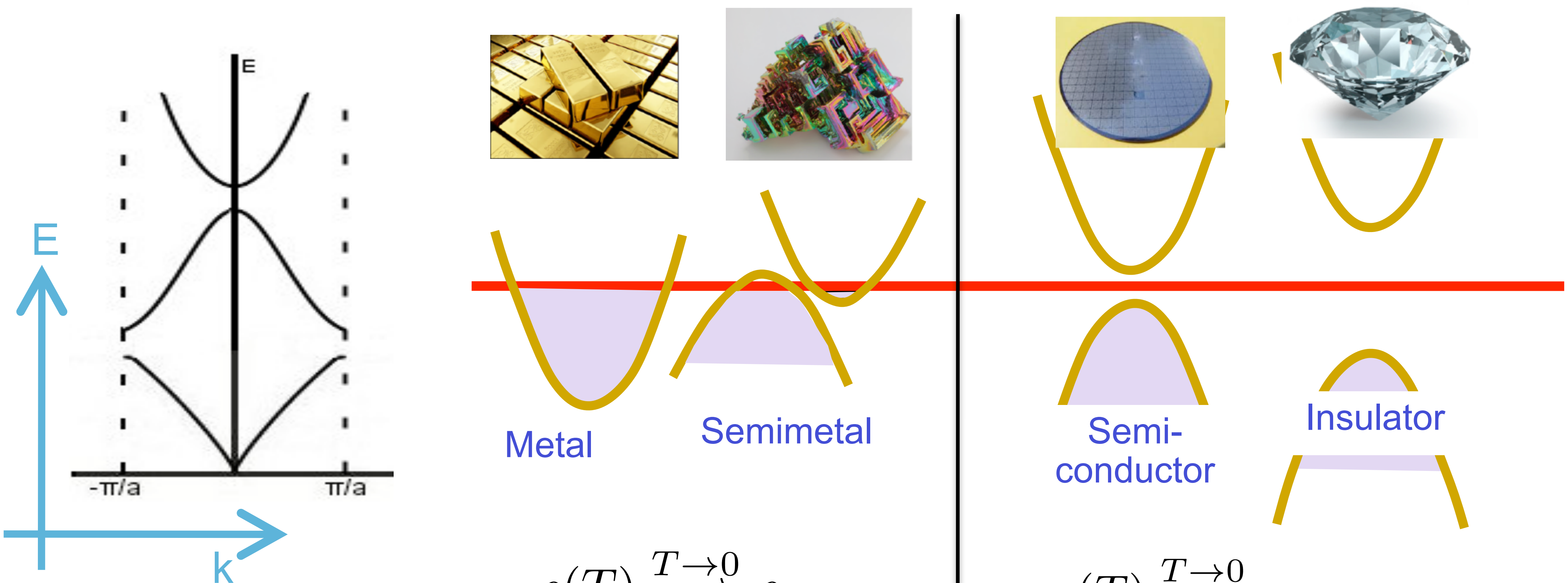


Band Gap



If the Fermi energy lives inside the band gap, the electrons cannot be easily excited with a bias voltage (insulating behavior)

Band Structure: Conductors and Insulators



$$\rho(T) \xrightarrow{T \to 0} \rho_0 \qquad\qquad \rho(T) \xrightarrow{T \to 0} \infty$$

There is a new class of materials that does not fit in this classification: topological materials!

Topology is a field of mathematics concerned with properties that remain invariant under continuous deformations.

Topological index and invariants

$$\chi = 2(1-g)$$

# of handles

$$\chi = \frac{1}{2\pi}\iint K \mathrm{d}S$$

Gaussian curvature

Integer

Gaussian curvature

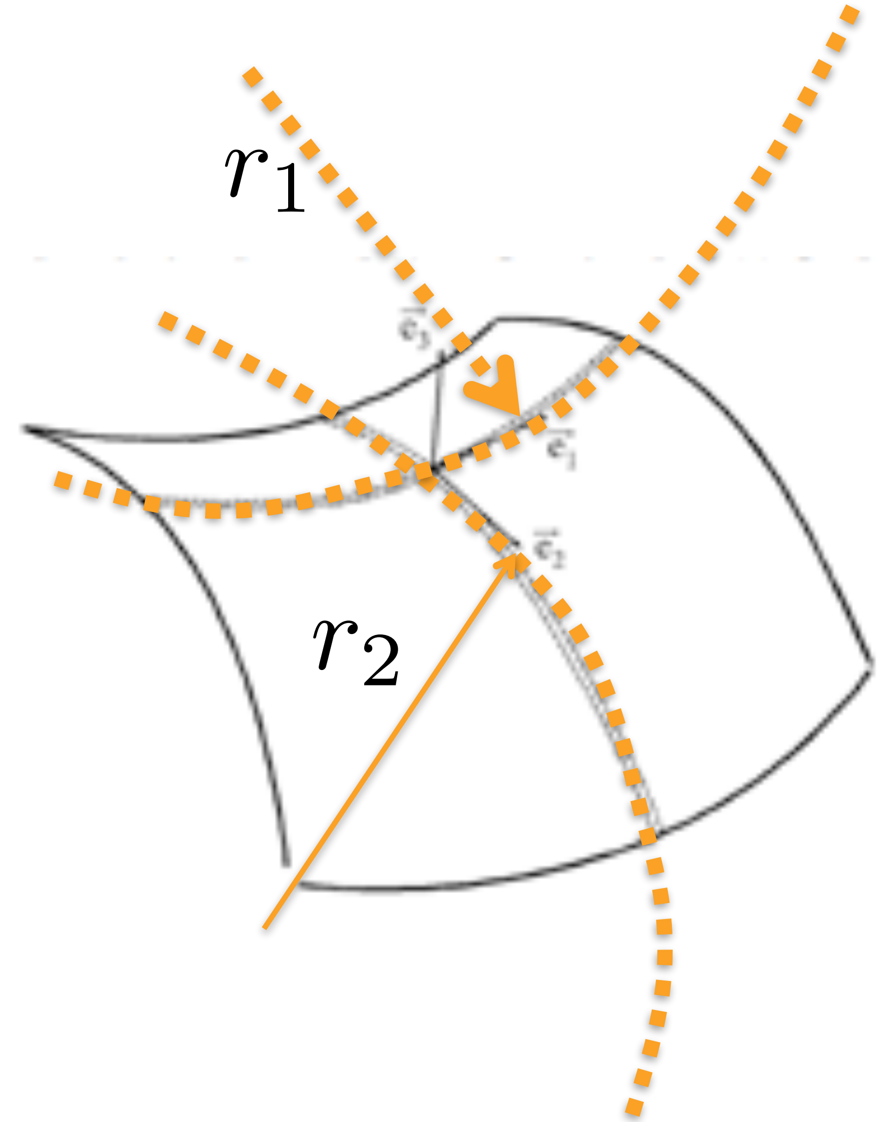
$$K = \kappa_1 \kappa_2$$

maximum curvature

minimum curvature

$$\kappa_{1,2} = \frac{1}{r_{1,2}}$$

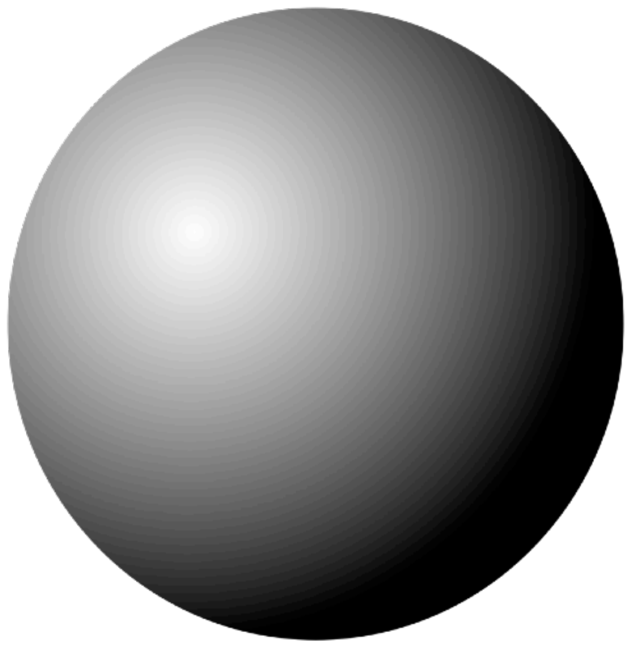
radius of curvature



$$\chi = \frac{1}{2\pi} \iint K \mathrm{d}S$$

Topological invariant

Gaussian curvature (sphere)



$$K = \kappa_1 \kappa_2$$

maximum curvature

minimum curvature

$$Area = 4\pi r^2$$

$$K = \frac{1}{r^2}$$

$$\kappa_{1,2} = \frac{1}{r_{1,2}}$$

radius of curvature

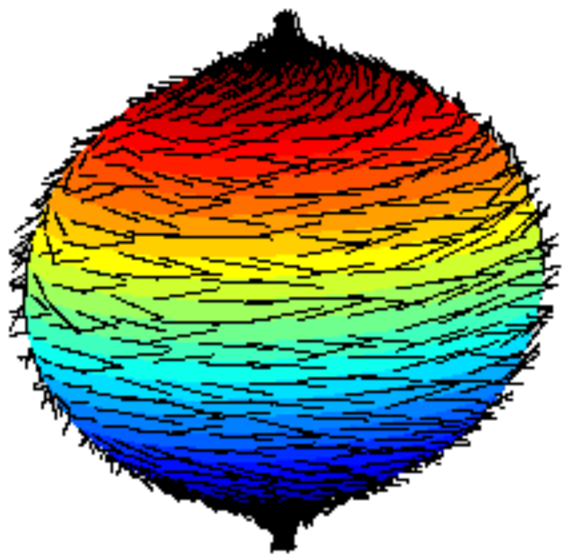
$$\chi = \frac{1}{2\pi} \iint K \mathrm{d}S$$

Topological invariant

$$\chi = \frac{1}{2\pi r^2} \int \mathrm{d}(Area) = \frac{4\pi r^2}{2\pi r^2} = 2 = 2(1-g)$$

Hairy ball theorem

You can't comb a hairy ball without creating a cowlick.



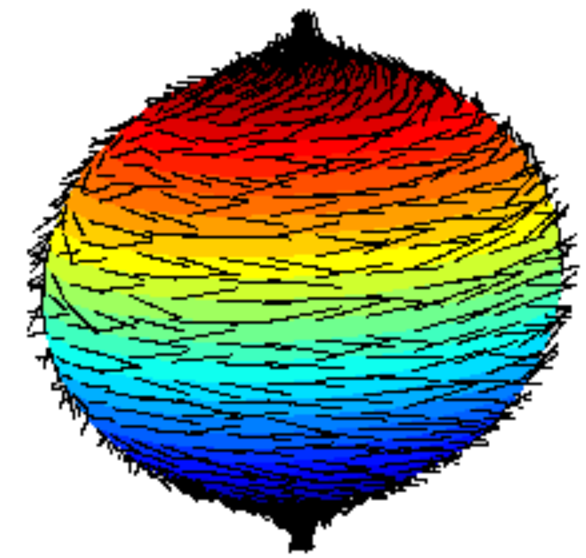
Every zero of the tangential vector field has an index. The sum of the indexes is equal to the Euler topological number

$$\chi = 2(1-g) = 2$$

Therefore a sphere has a least one zero!

Hairy ball theorem

You can't comb a hairy ball without creating a cowlick.

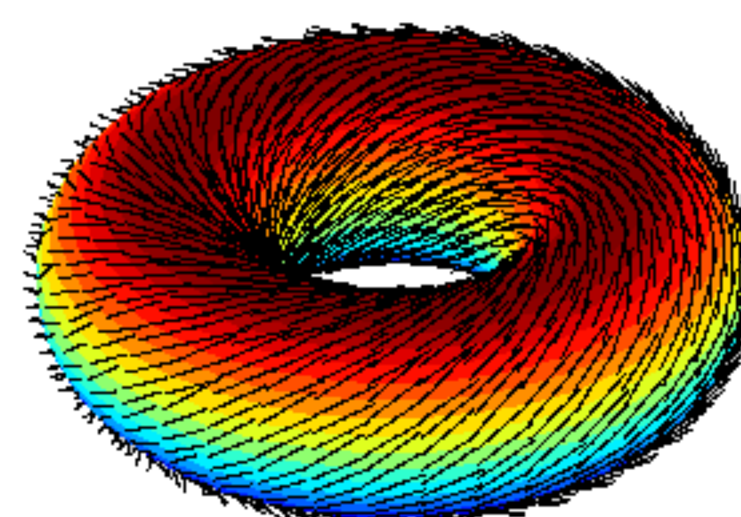


Every zero of the tangential vector field has an index. The sum of the indexes is equal to the Euler topological number.

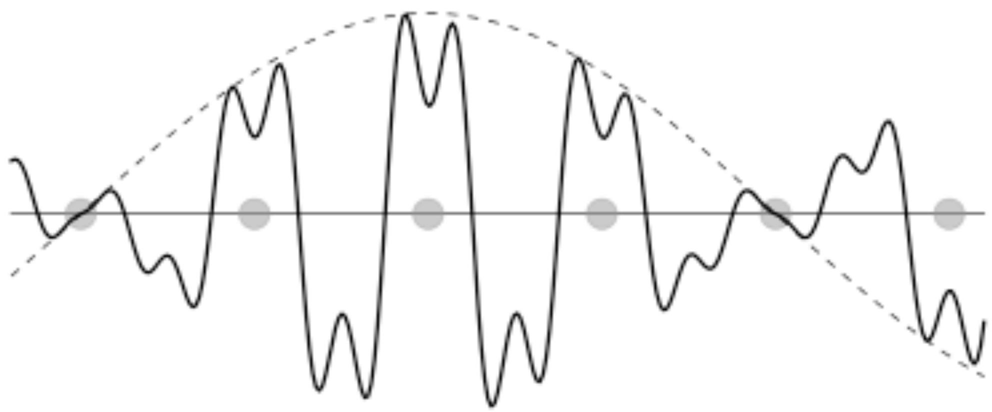
$$\chi = 2(1-g)$$

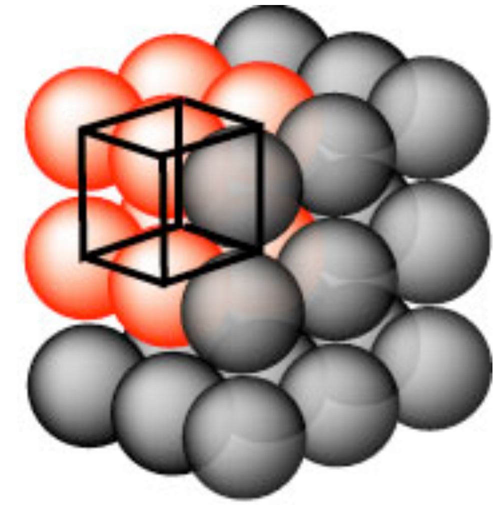
One can comb a torus without any zeros in it.

$$\chi = 2(1-g) = 0$$



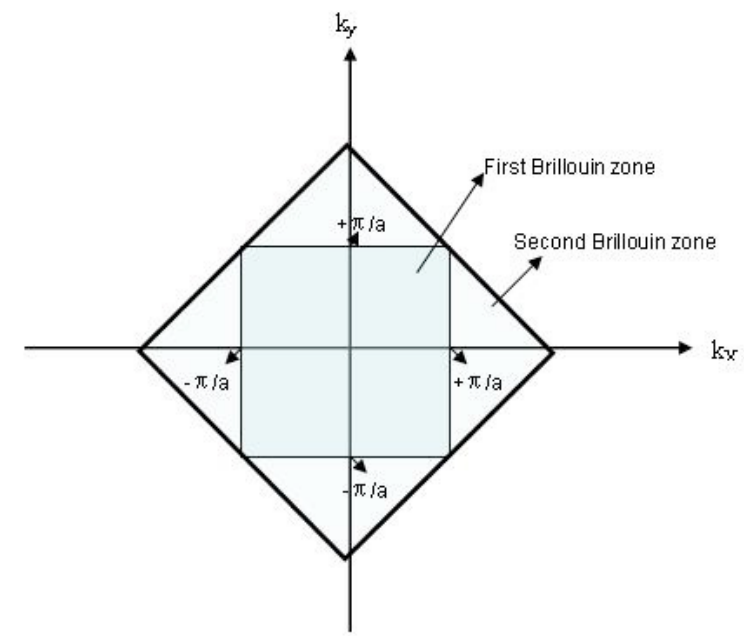
In crystals, the wavefunctions of the electrons are periodic.





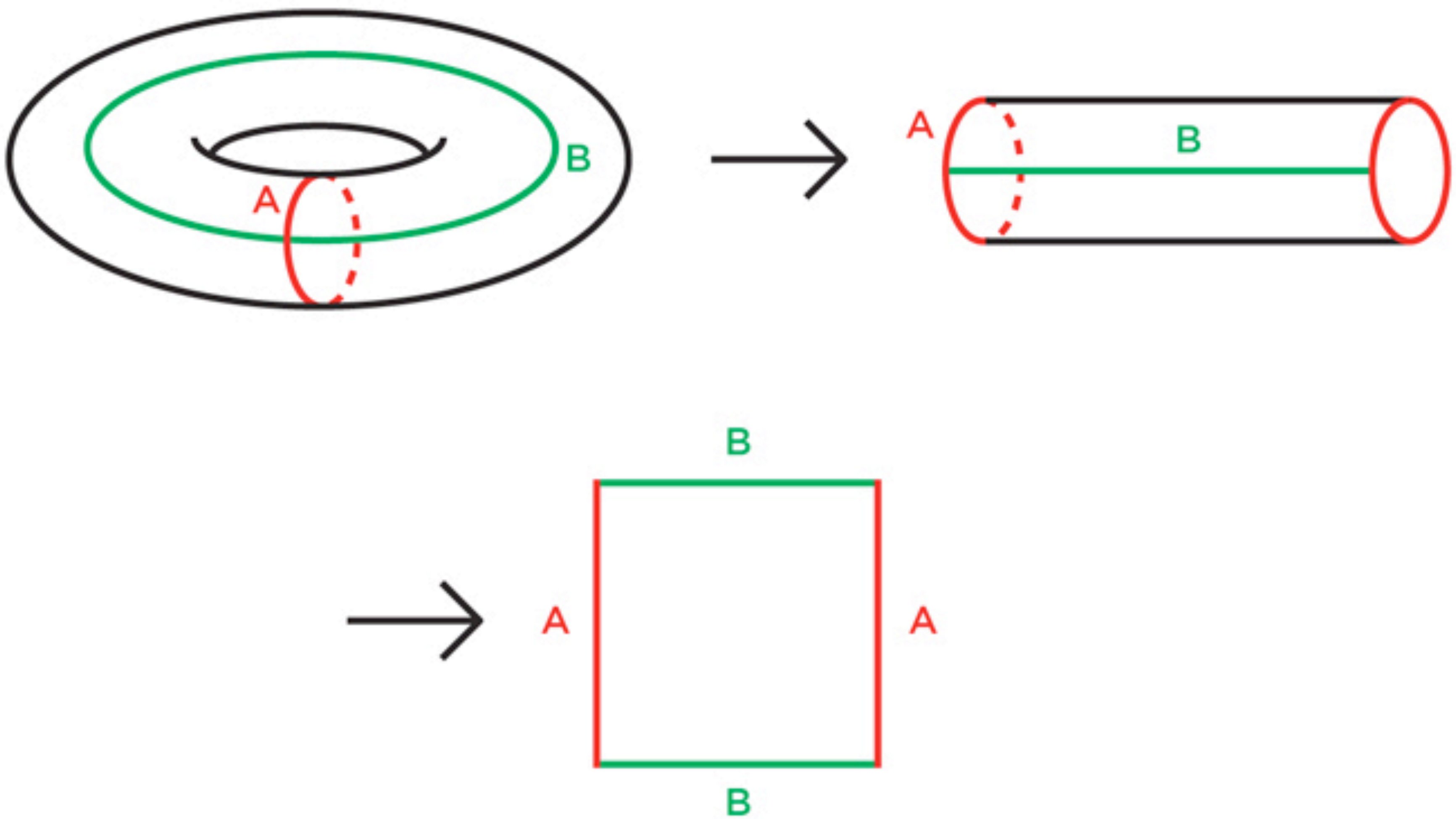
The de Broglie wavelength of the electrons must be larger than the lattice constant of the crystal.

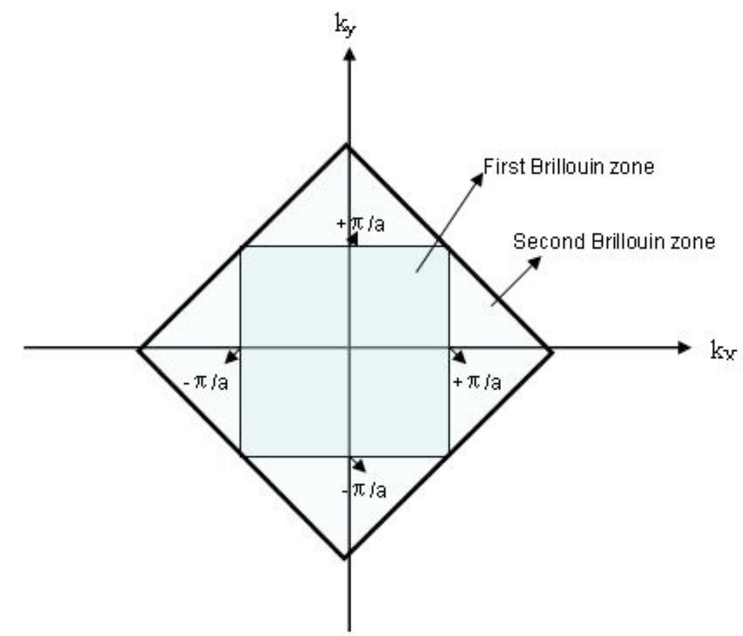
In a square lattice in 2D, the momentum of the electrons is bounded inside a square of size 1/(lattice constant)!

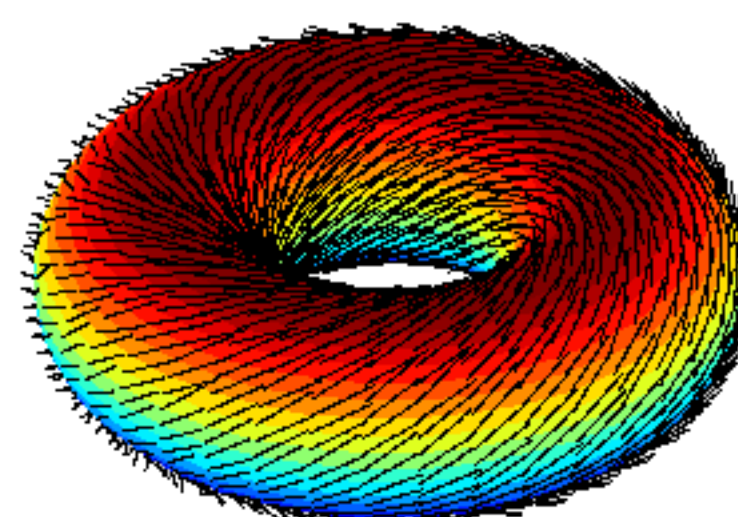


In crystals, the wavefunctions of the electrons are periodic.

The momentum space of a periodic crystal (Brillouin zone) is homeomorphic to a torus



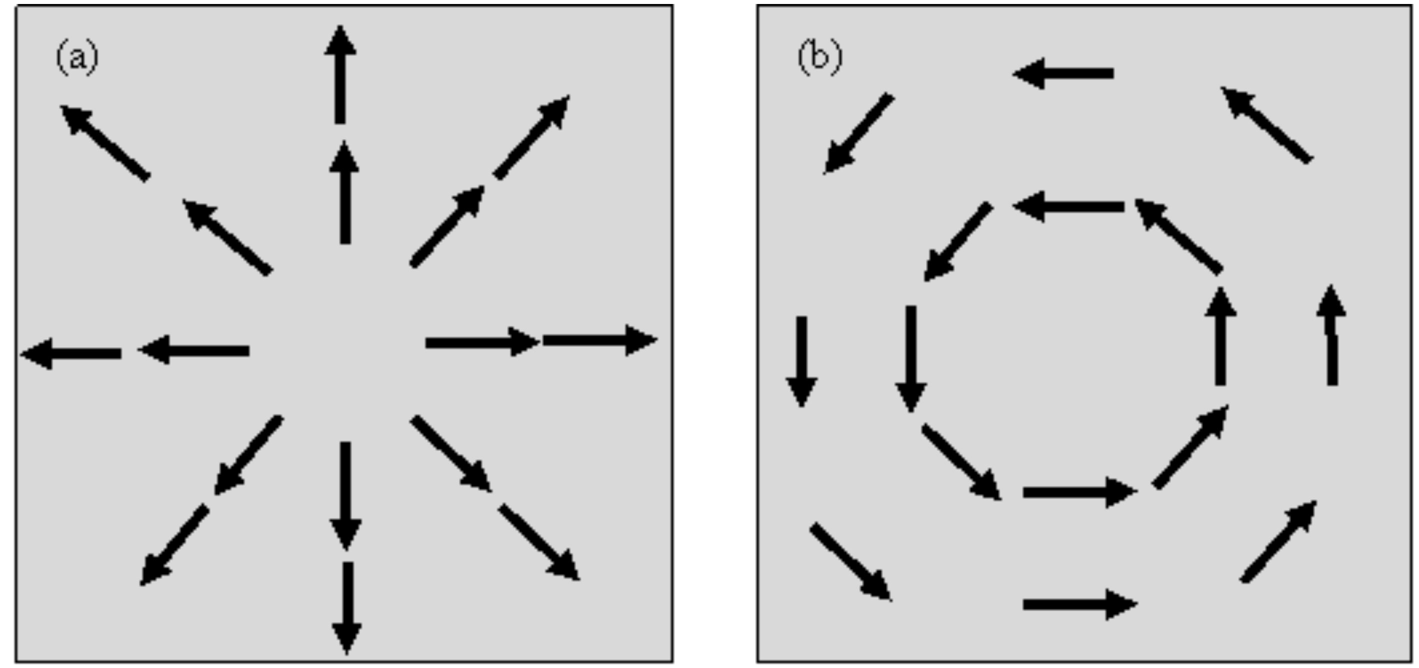


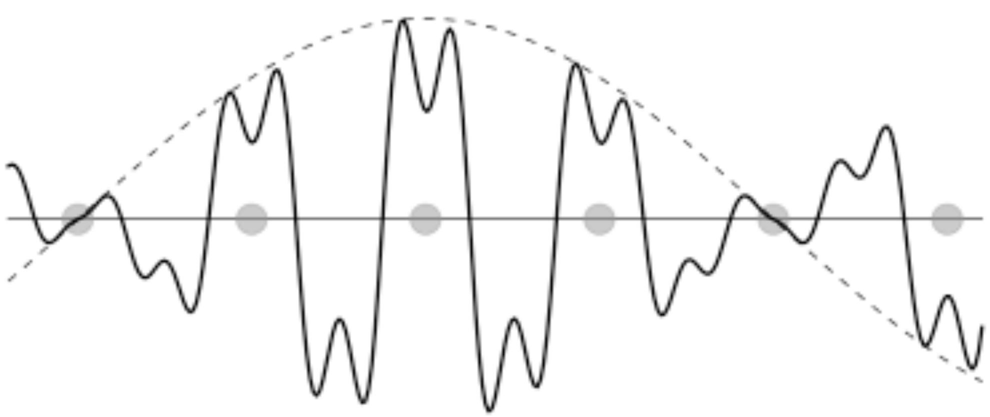


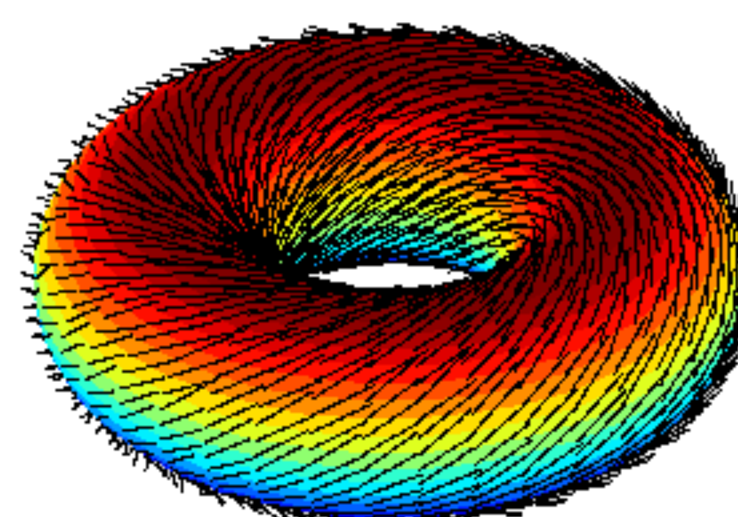
Bloch bands

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = u_{n,\mathbf{k}}(\mathbf{r})\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$$
$$= \langle \mathbf{r} | \psi_{n,\mathbf{k}} \rangle$$

Bloch wavefunction





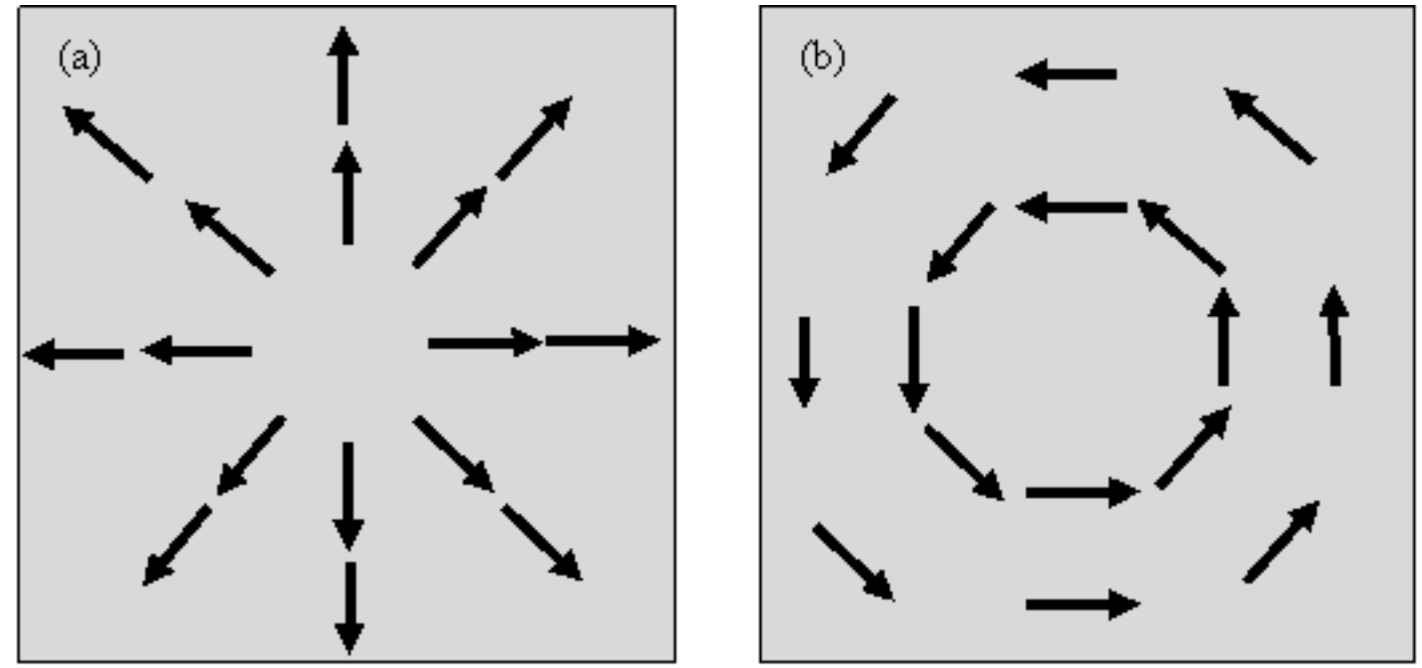


Bloch wavefunctions describe the wavefunction of the electrons in a periodic crystal.

Bloch bands

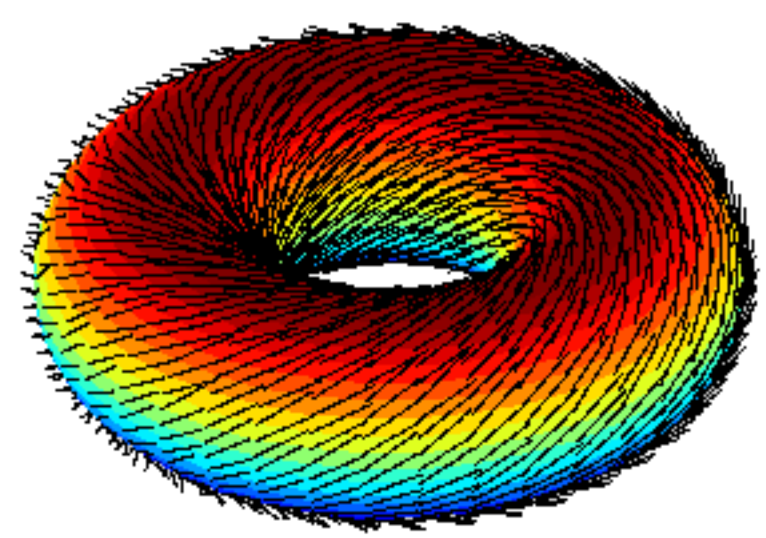
$$\psi_{n,\mathbf{k}}(\mathbf{r}) = u_{n,\mathbf{k}}(\mathbf{r})\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$$
$$= \langle \mathbf{r} | \psi_{n,\mathbf{k}} \rangle$$

Bloch wavefunction

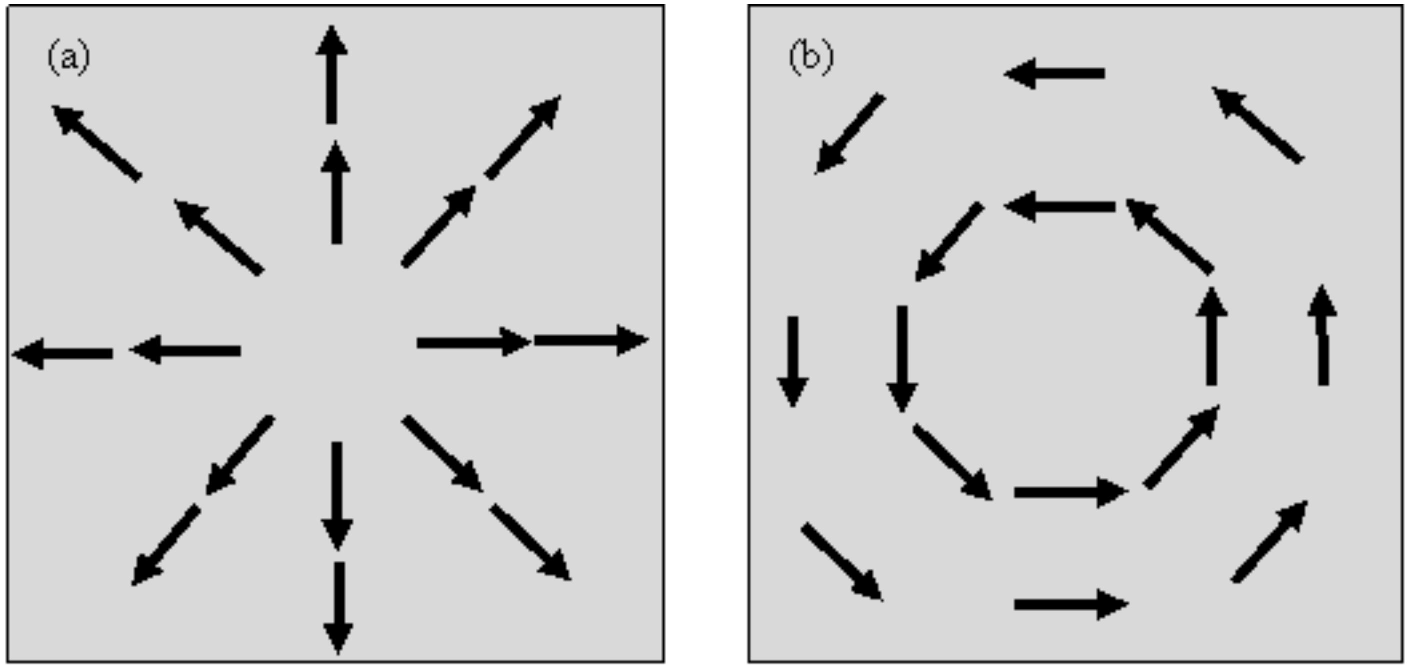


The Berry connection of the Bloch band kets, $\overbrace{i\langle \psi_{n,\mathbf{k}} | \frac{\partial}{\partial \mathbf{k}} | \psi_{n,\mathbf{k}} \rangle}^{\mathbf{A}}$ behaves as a vector field in the Brillouin zone (torus).

Loopholes or hedgehogs of the vector field are topological obstructions with a topological index and add up to a non-zero Euler topological number in the hairy torus!



Bloch bands



$$\psi_{n,\mathbf{k}}(\mathbf{r}) = u_{n,\mathbf{k}}(\mathbf{r})\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$$
$$= \langle \mathbf{r} | \psi_{n,\mathbf{k}} \rangle$$

Bloch wavefunction

$$\nu_n = \iint \mathrm{d}\mathbf{S} \cdot \boxed{\nabla \times \overbrace{i \langle \psi_{n,\mathbf{k}} | \frac{\partial}{\partial \mathbf{k}} | \psi_{n,\mathbf{k}} \rangle}^{\mathbf{A}}} \longrightarrow \text{Berry curvature}$$

Chern number (integer)

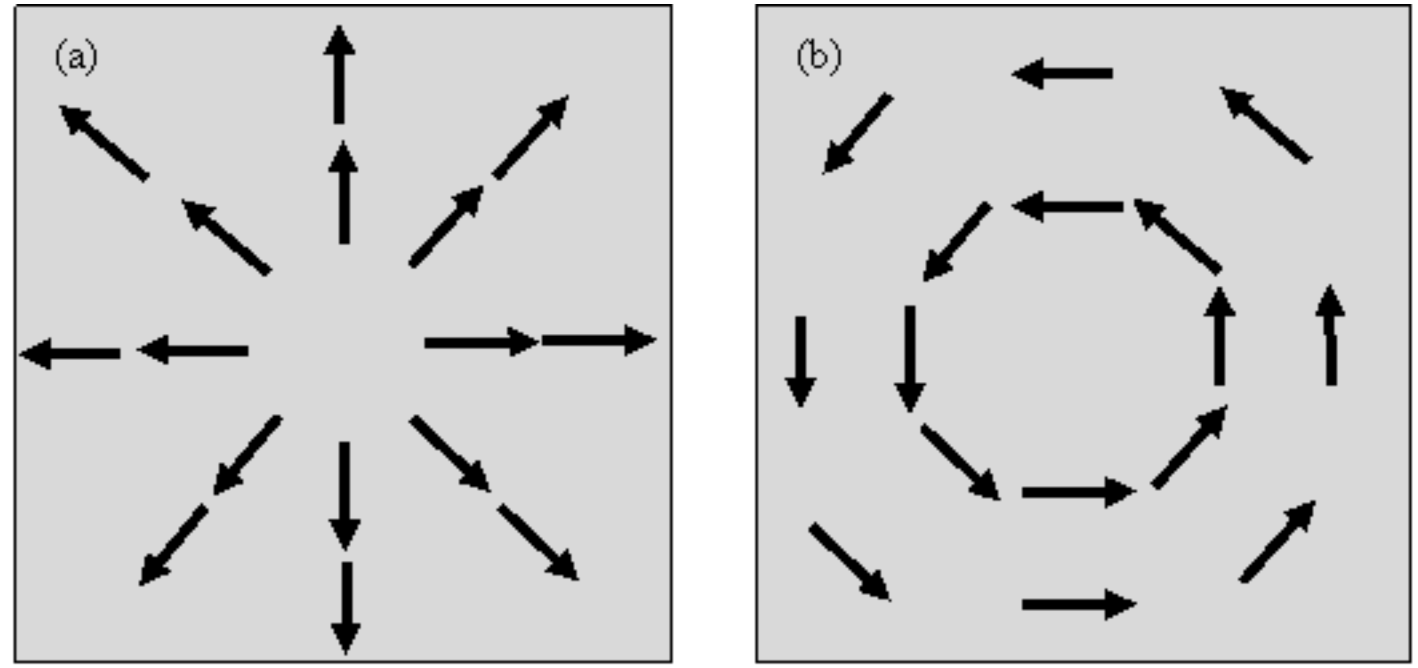
The Chern number is a topological invariant of Bloch bands

$$\chi = \frac{1}{2\pi} \iint K \mathrm{d}S$$

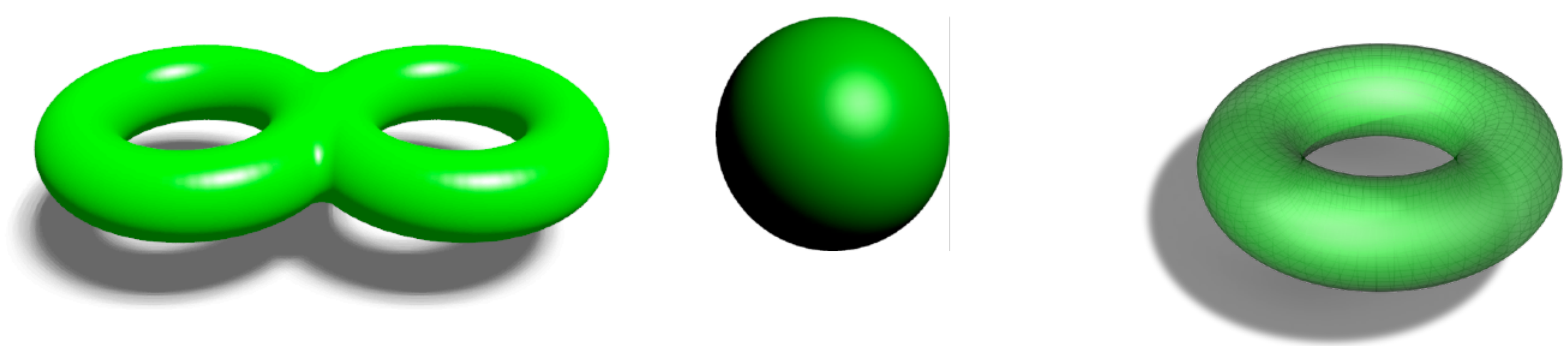
Bloch bands

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = u_{n,\mathbf{k}}(\mathbf{r})\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$$

Bloch bands

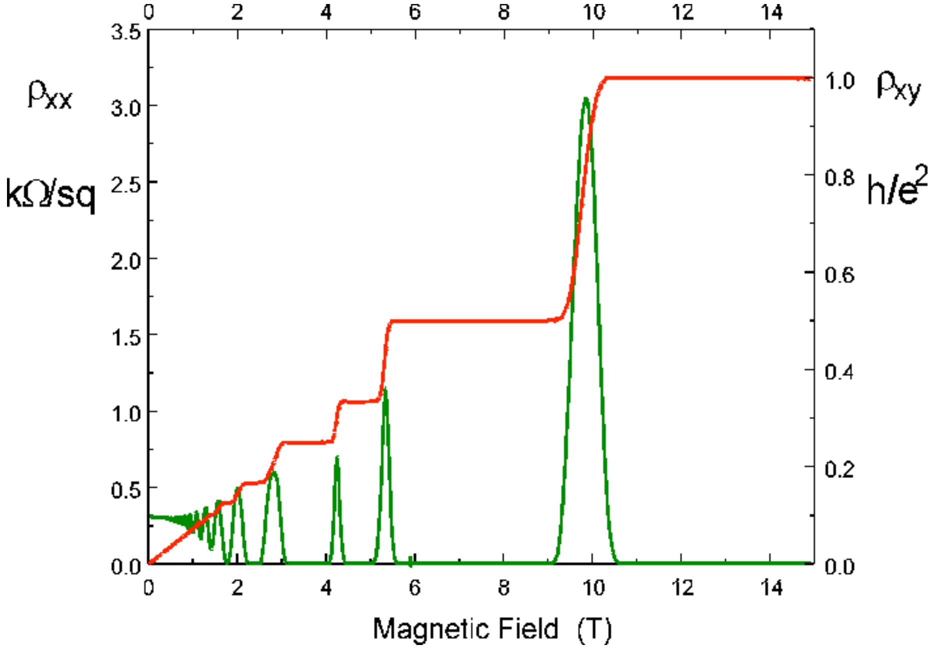


$$\nu_n = \iint \mathrm{d}\mathbf{S} \cdot \boxed{\nabla \times i\langle\psi_{n,\mathbf{k}}|\frac{\partial}{\partial\mathbf{k}}|\psi_{n,\mathbf{k}}\rangle} \longrightarrow$$ Berry curvature



$$\chi = \frac{1}{2\pi}\iint K \mathrm{d}S$$

Cowlicks in the Berry curvature change the topological class of the hairy torus (Brillouin zone)!



Why is the Hall conductivity quantized?

VOLUME 49, NUMBER 6 PHYSICAL REVIEW LETTERS 9 AUGUST 1982

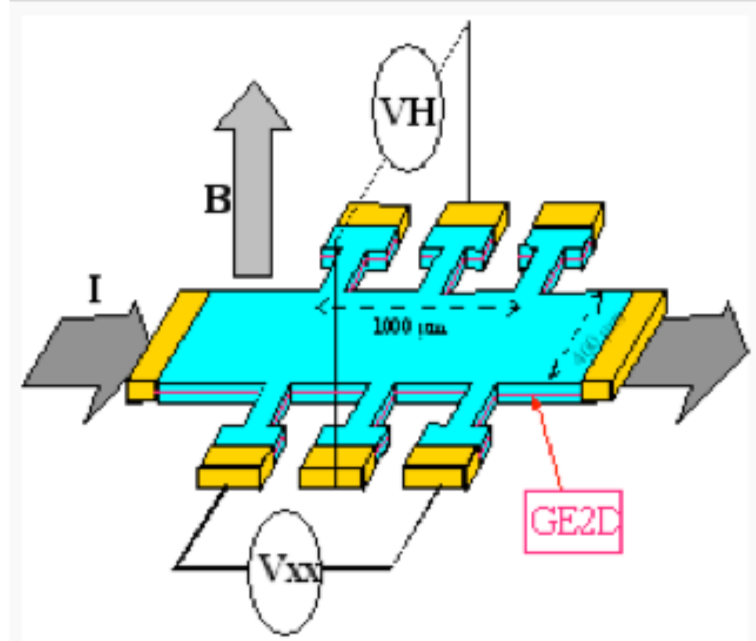
Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

The Hall conductance of a two-dimensional electron gas has been studied in a uniform magnetic field and a periodic substrate potential U. The Kubo formula is written in a form that makes apparent the quantization when the Fermi energy lies in a gap. Explicit expressions have been obtained for the Hall conductance for both large and small $U/\hbar\omega_c$.

$$\nu_n = \iint \mathrm{d}\mathbf{S} \cdot \nabla \times i\langle \psi_{n,\mathbf{k}} | \frac{\partial}{\partial \mathbf{k}} | \psi_{n,\mathbf{k}} \rangle$$

$$\sigma_{xy} = \frac{e^2}{h} \sum_n \nu_n$$

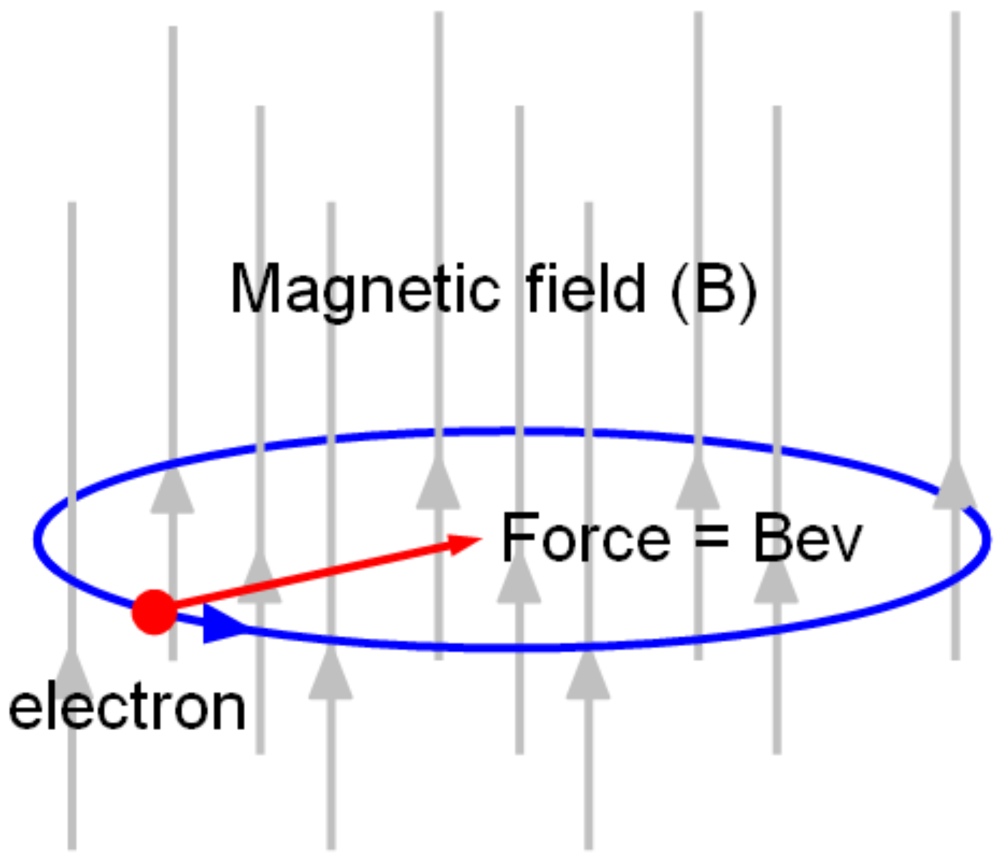


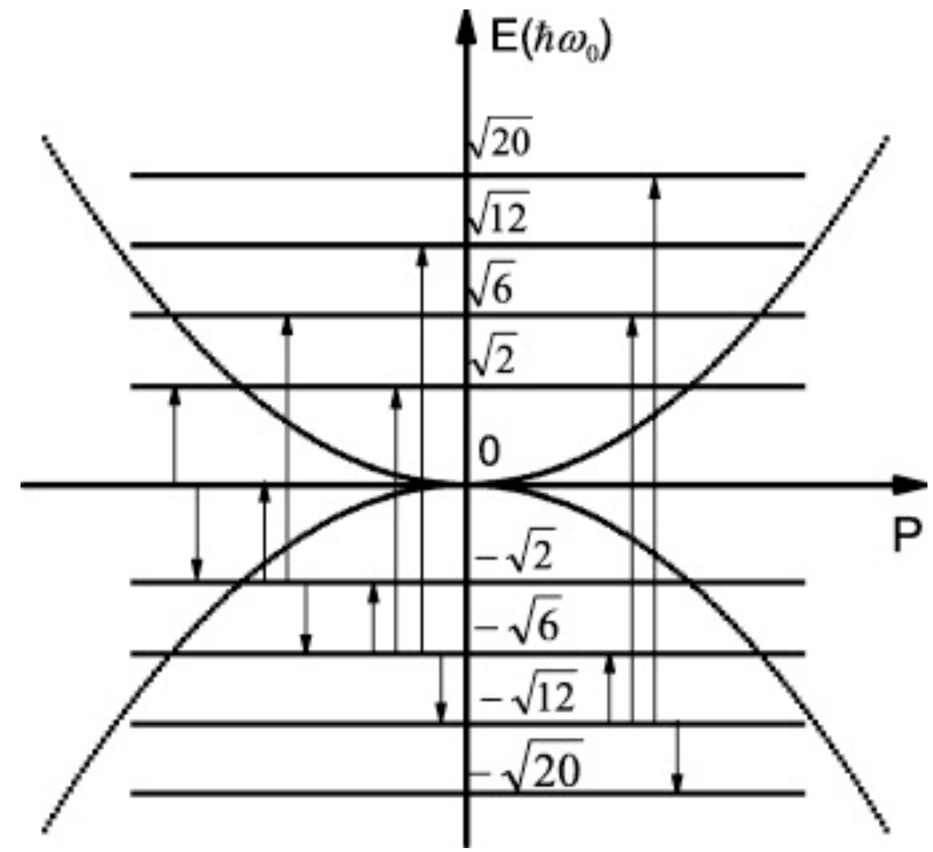
two-dimensional electron gas
with Hall bar geometry

Quantum hall conductivity is quantized by the Chern number!

Topological matter

In 2D, electrons move in cyclotron orbits in the presence of a magnetic field

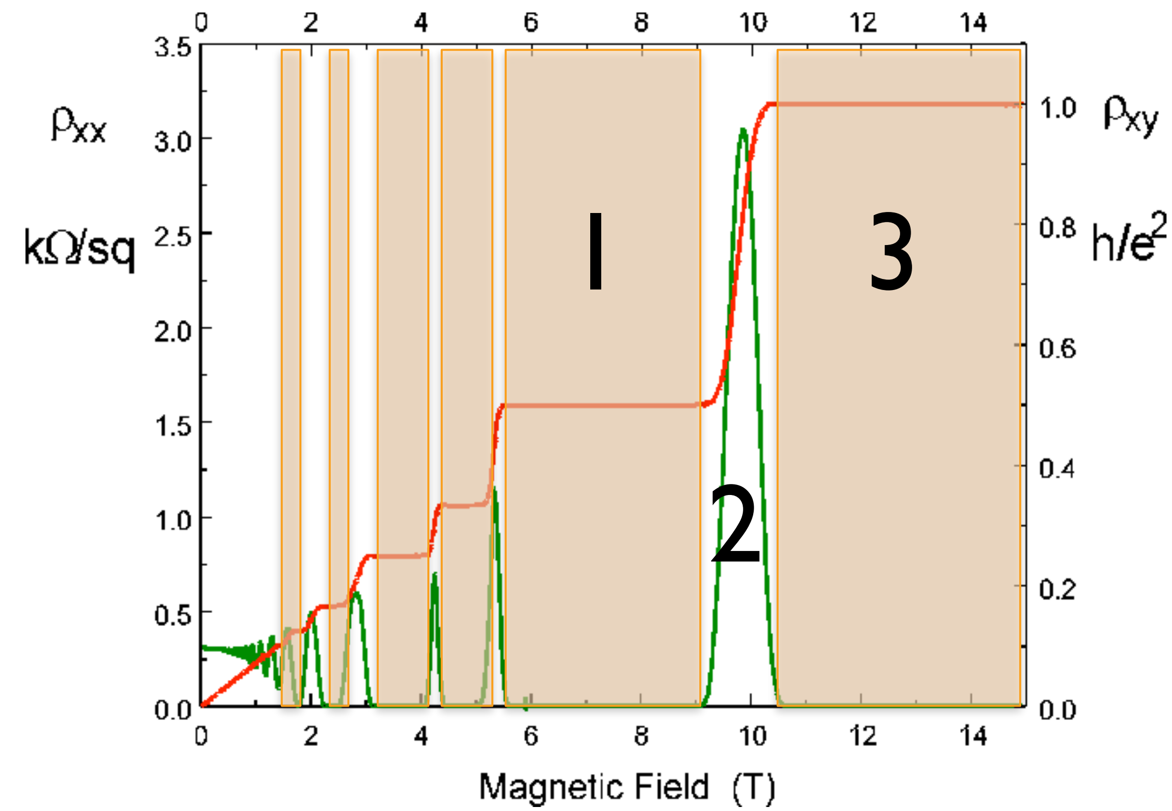




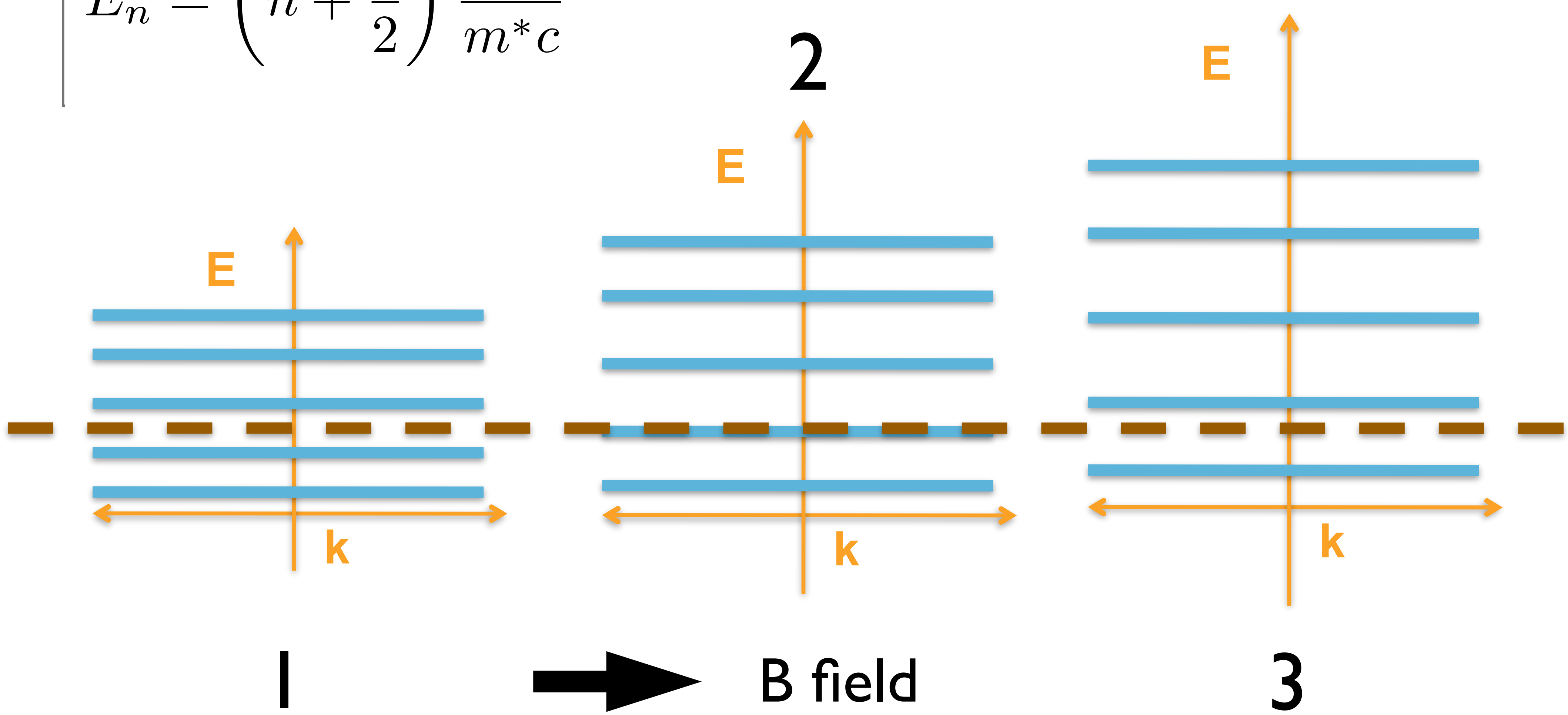
Due to quantum interference, only a discrete number of wavelengths are allowed (Landau energy levels)

Integer quantum Hall effect

$$\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2}$$

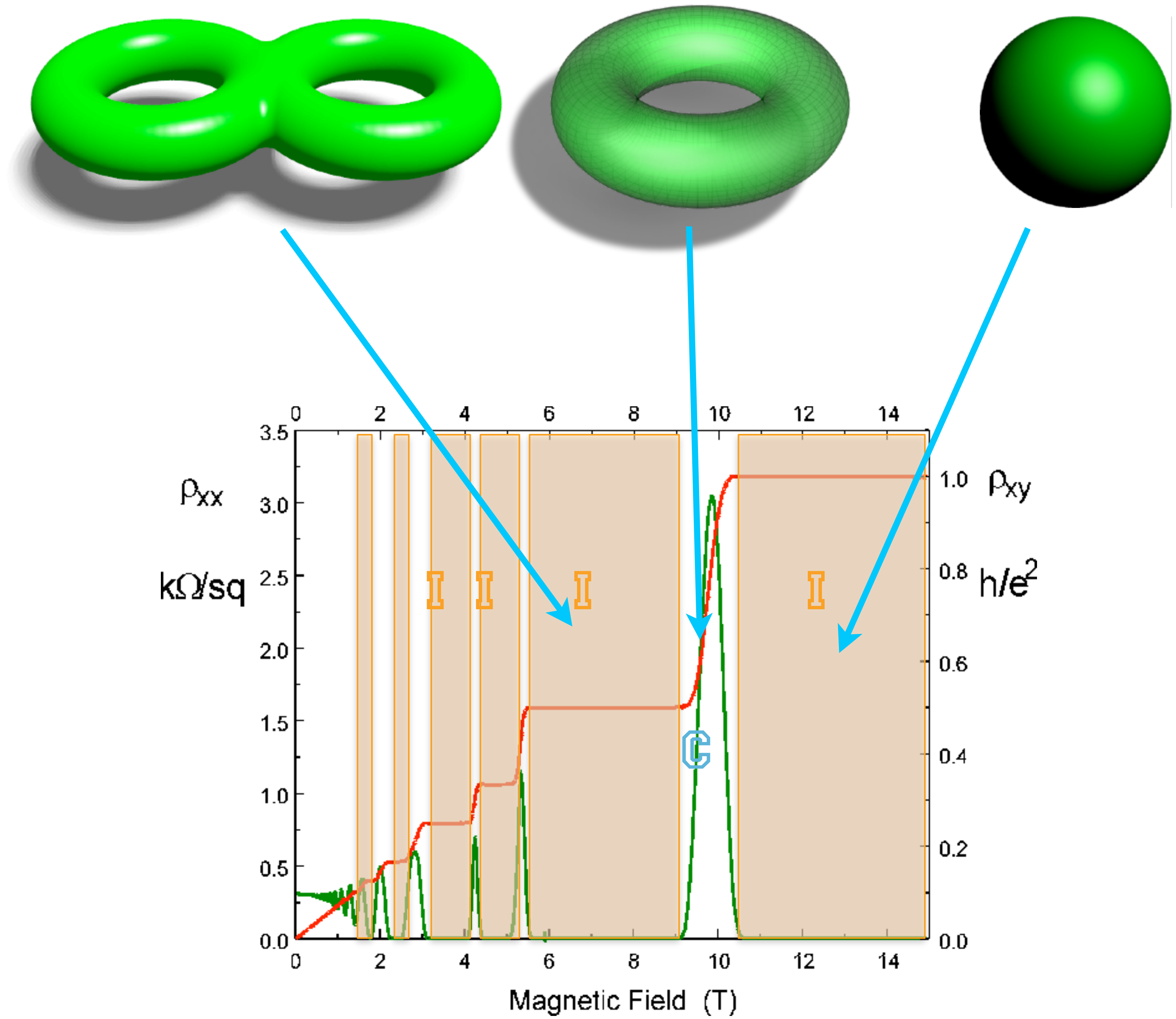


$$E_n = \left(n + \frac{1}{2}\right)\frac{heB}{m^*c}$$



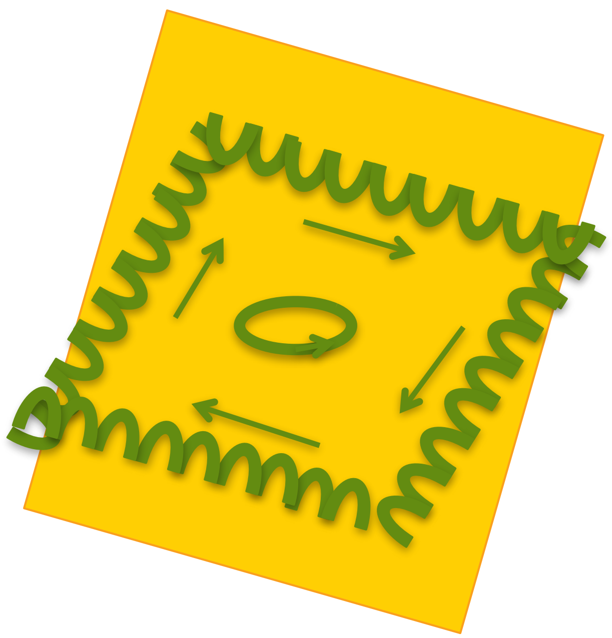
Quantum Hall conductivity is quantized by the Chern number

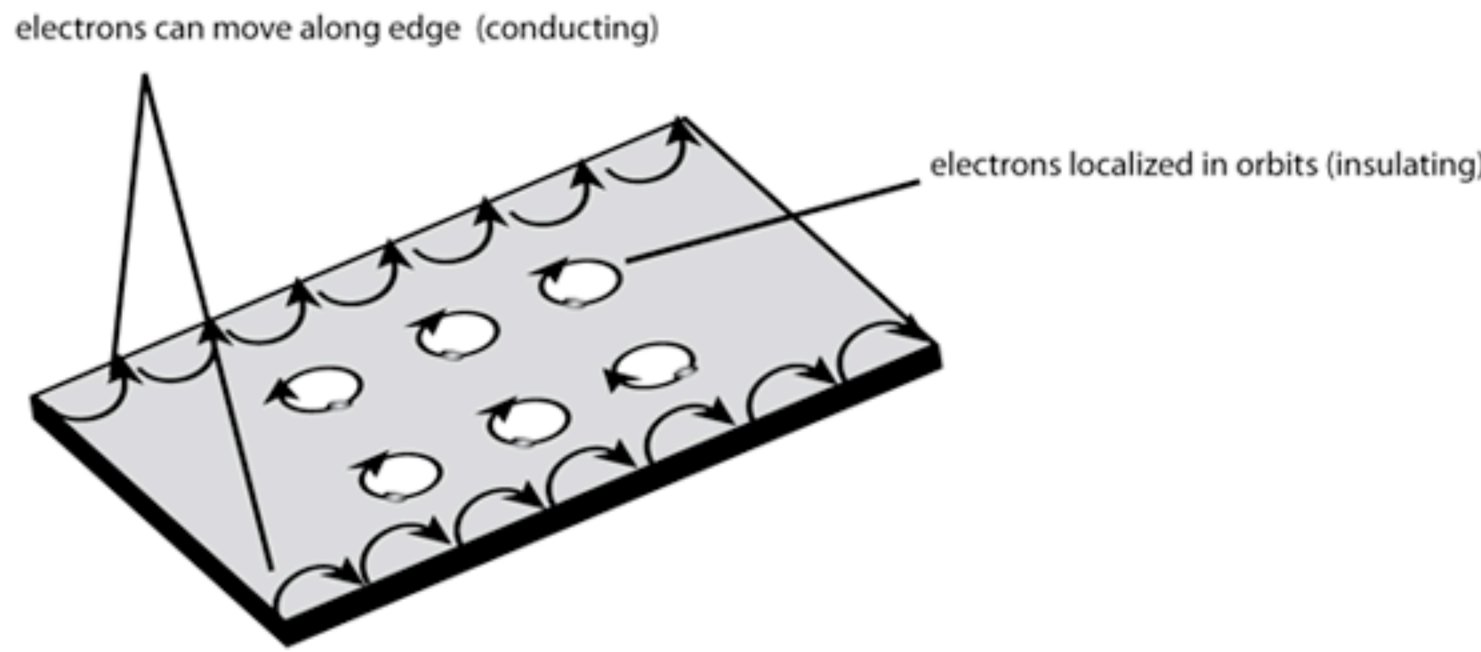
$$\sigma_{xy} = \frac{e^2}{h} \sum_n \nu_n$$



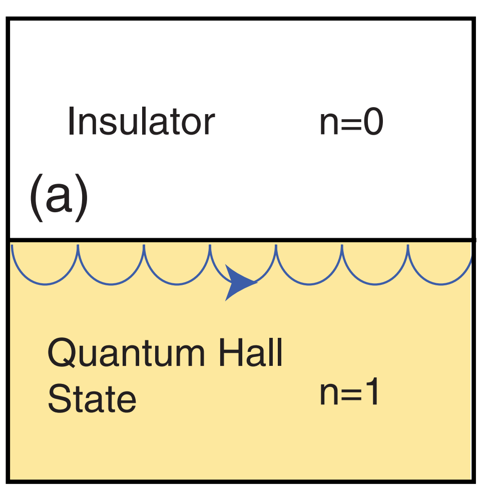
Topologically distinct insulating phases

Bulk-Boundary Correspondence



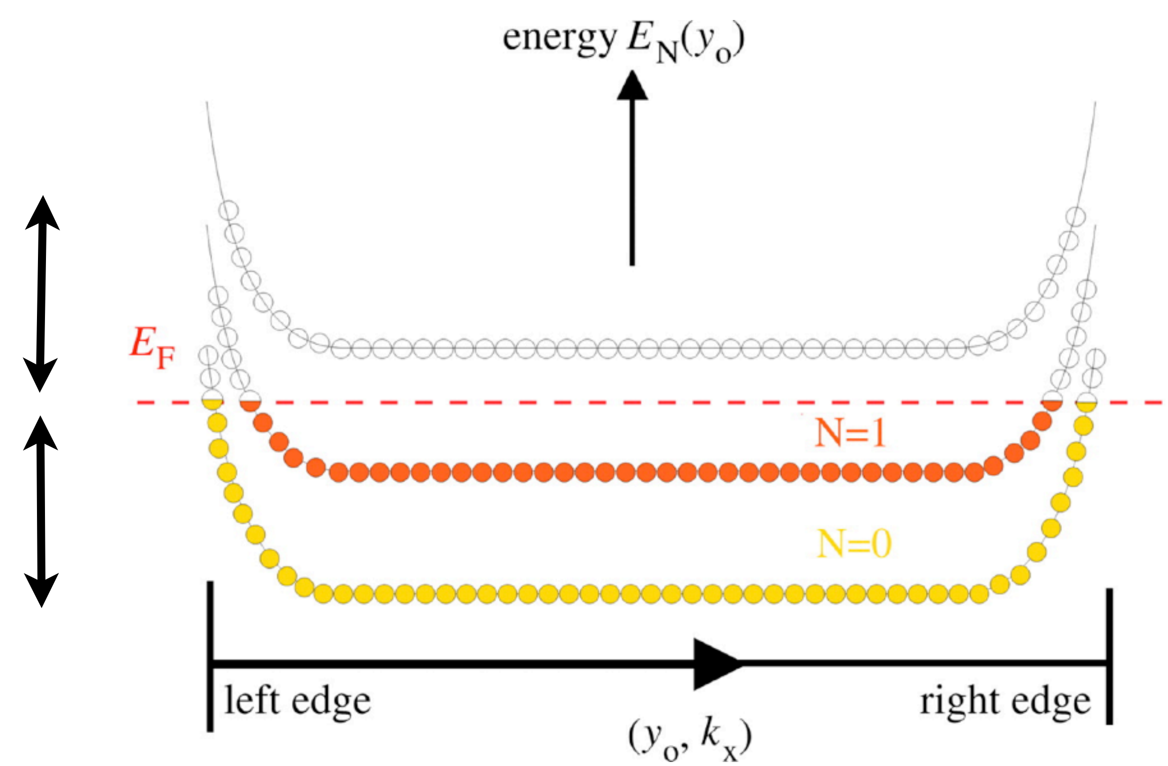


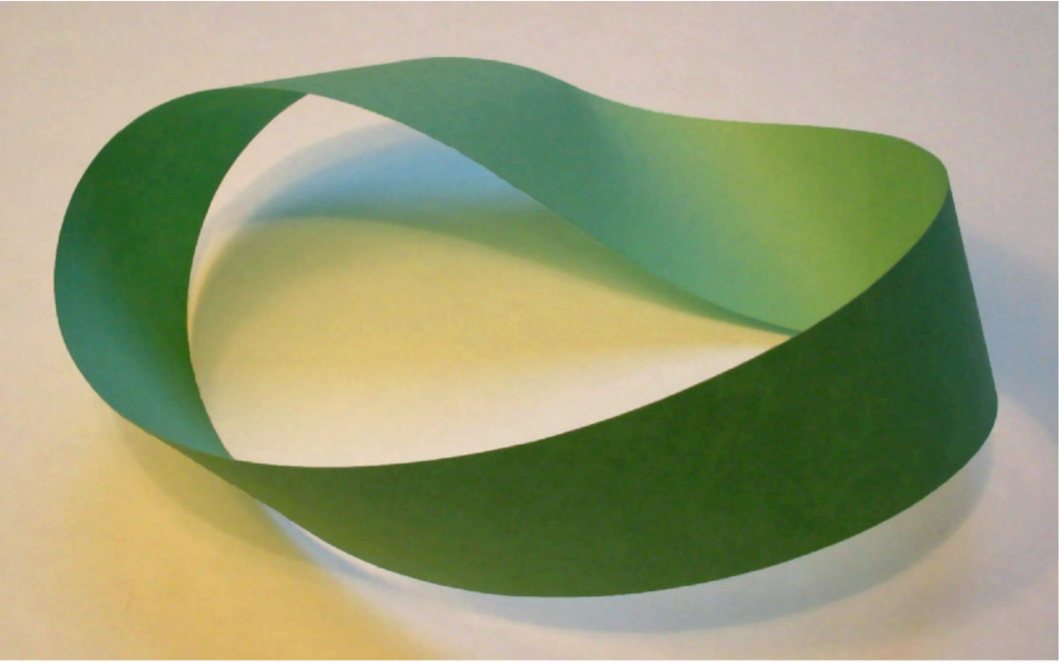
Hall conductivity is quantized by the number of 1D channels at the edge!



empty states

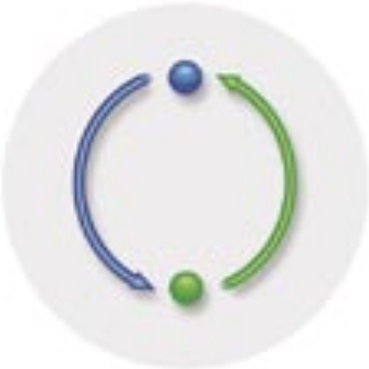
occupied states





Can topology be used for quantum computation?

In quantum mechanics, whenever two identical particles are swapped, the total wave function picks up a phase.



$$|\psi_1, \psi_2\rangle \rightarrow \mathrm{e}^{i\theta}|\psi_2, \psi_1\rangle$$

where $\mathrm{e}^{i\theta} = +1$ for bosons and -1 for fermions.

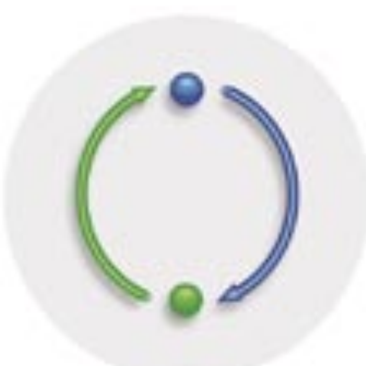
PHASE

In 2D, it is mathematically possible to have particles with fractional statistics (anyons)!

Counterclockwise

$$|\psi_1, \psi_2\rangle \to \mathrm{e}^{i\theta}|\psi_2, \psi_1\rangle \qquad \theta \neq n\pi, \quad n \in \mathbb{Z}$$

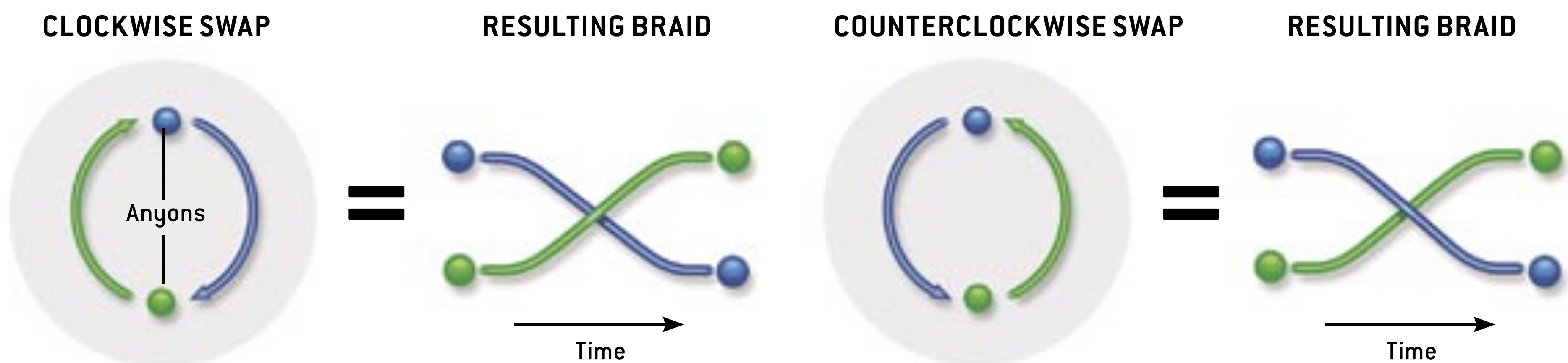
Clockwise

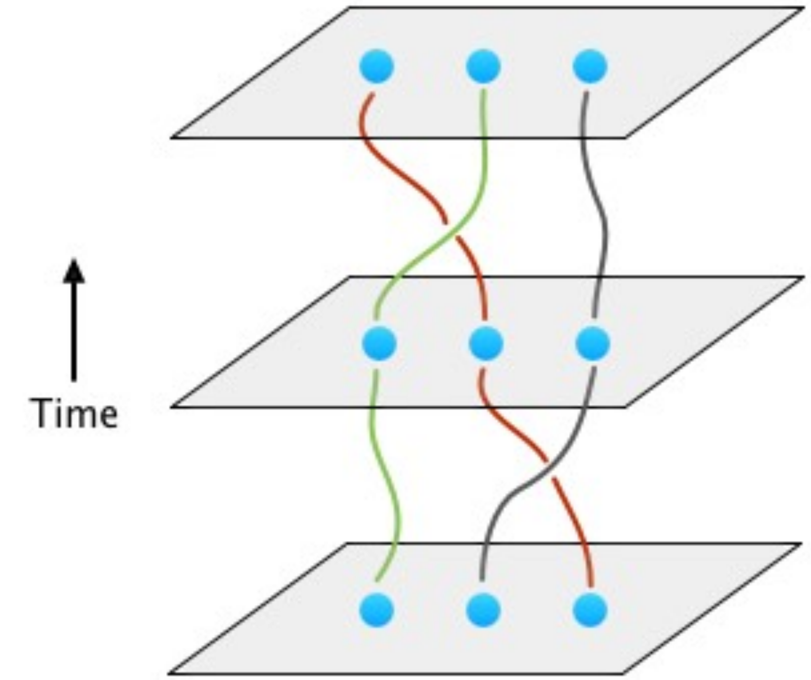


$$|\psi_1, \psi_2\rangle \to \mathrm{e}^{-i\theta}|\psi_2, \psi_1\rangle$$

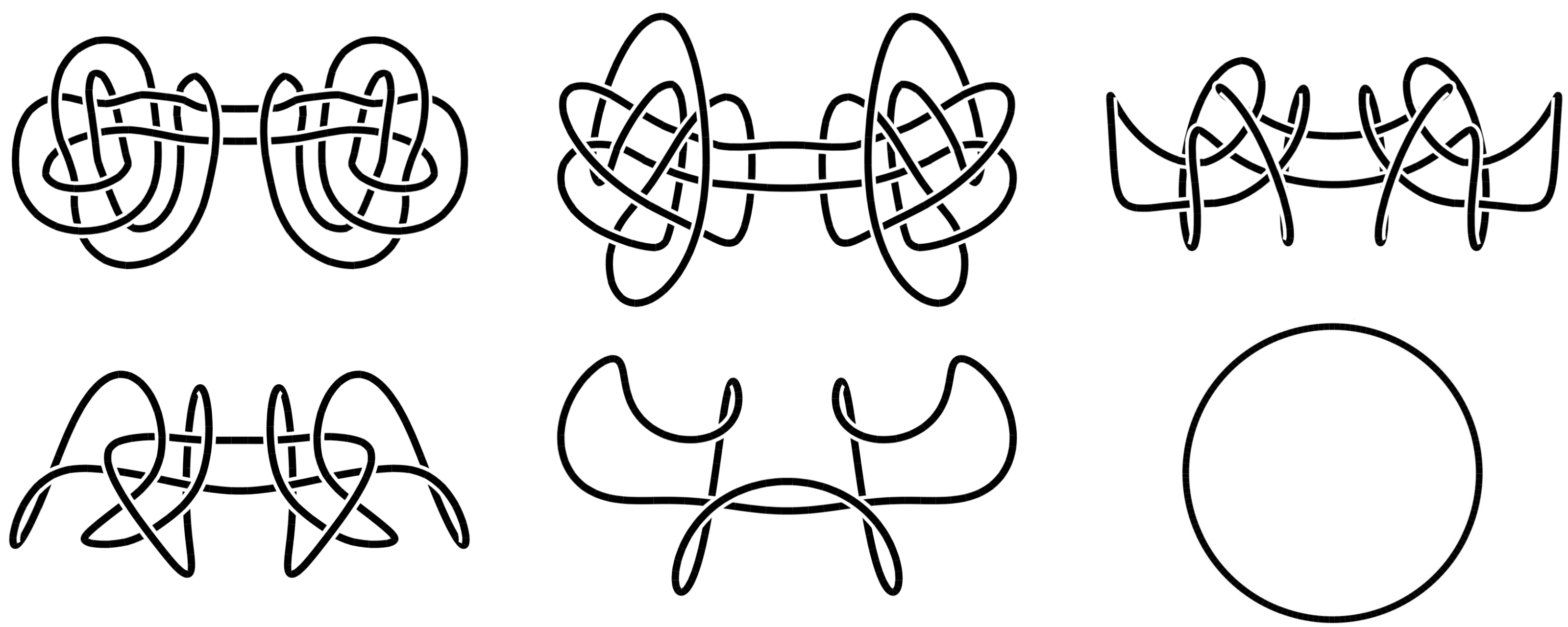
In that case it matters if the particles are swapped in the clockwise or counterclockwise direction!

The act of exchanging two anyons corresponds to two distinct braiding operations, depending on how they are swapped!

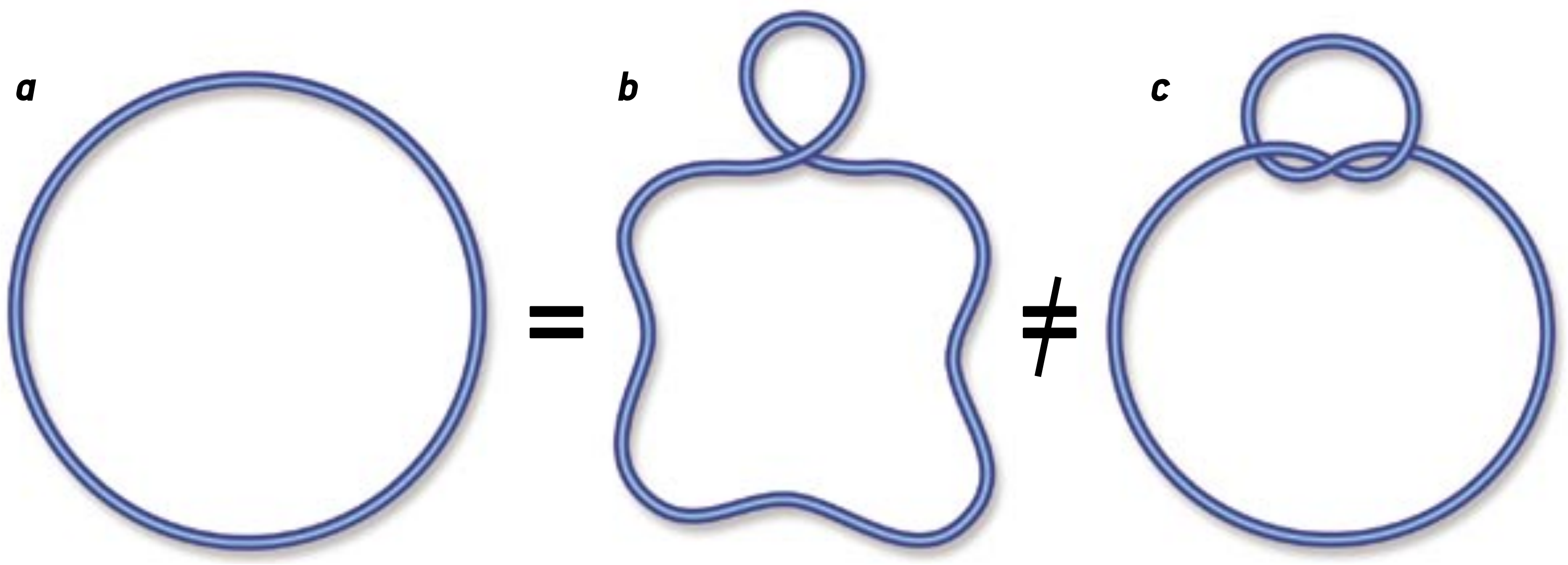




World line



Non-trivial topology can also appear in the form of knots of world lines!



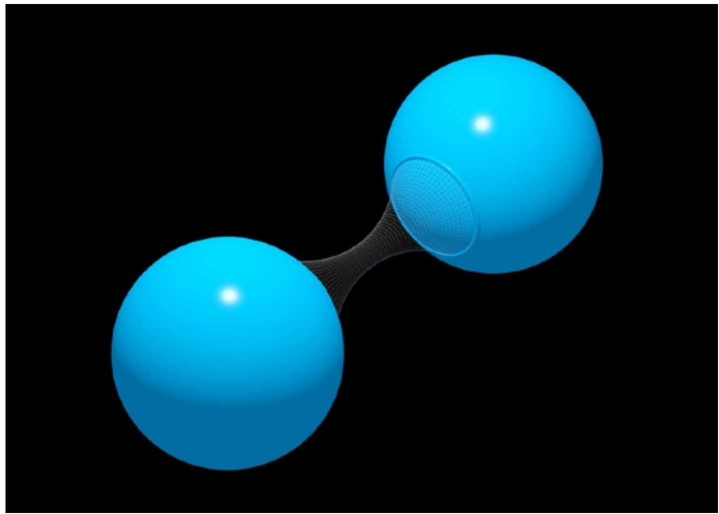
One can separate world lines in different topological classes based on the number of knots.

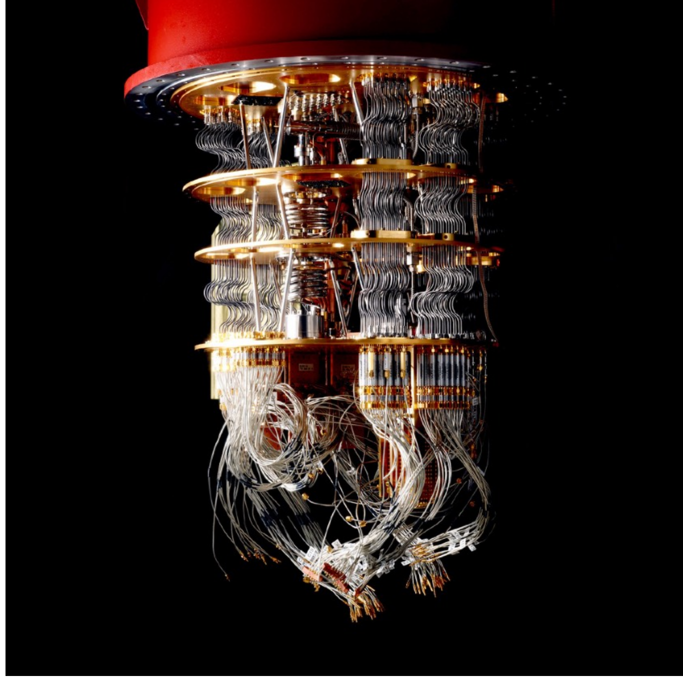
In 1997, Alexei Kitaev suggested that building qubits with anyons would permit creating quantum computers that are immune to phase decoherence!

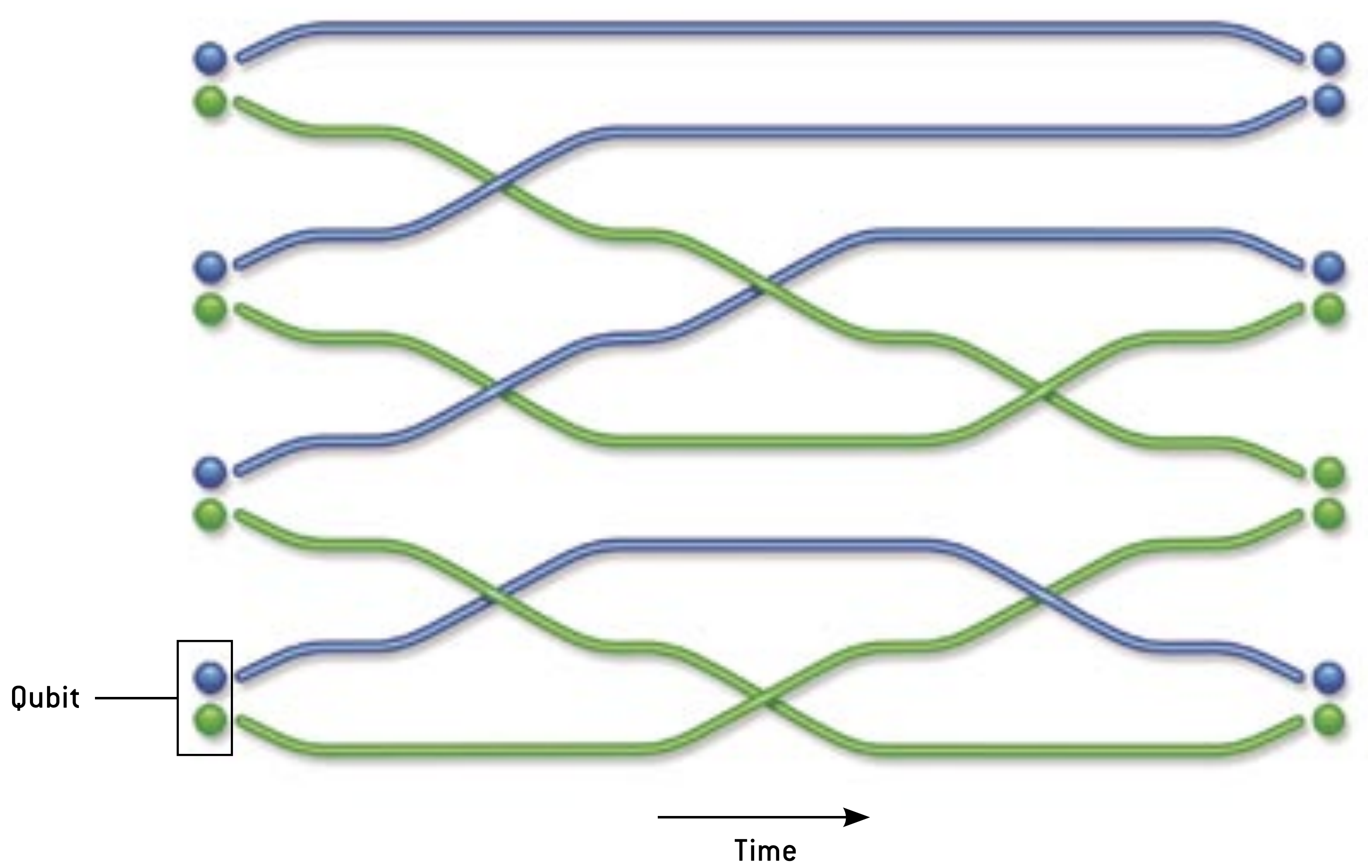
Anyons cannot exist in isolation and must be entangled with other anyons!

They are hard to observe because they are insensitive to local measurements, and can only be destroyed when merged with other anyons.

On the other hand, their qubits are hard to decohere!







The fundamental unit of information (qubit) in a topological quantum computer is a linear superposition of two anyons, which can be braided with other qubits!

Braid Topologies for Quantum Computation

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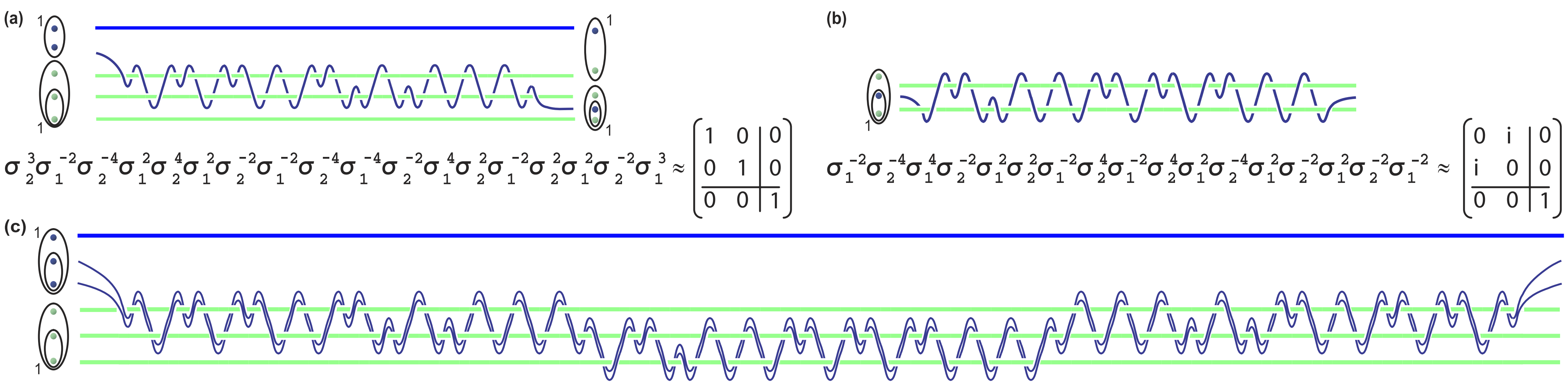
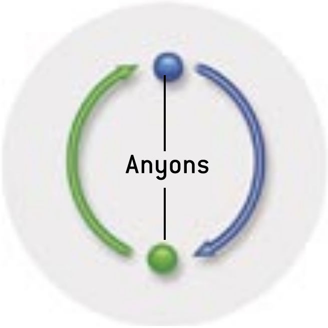
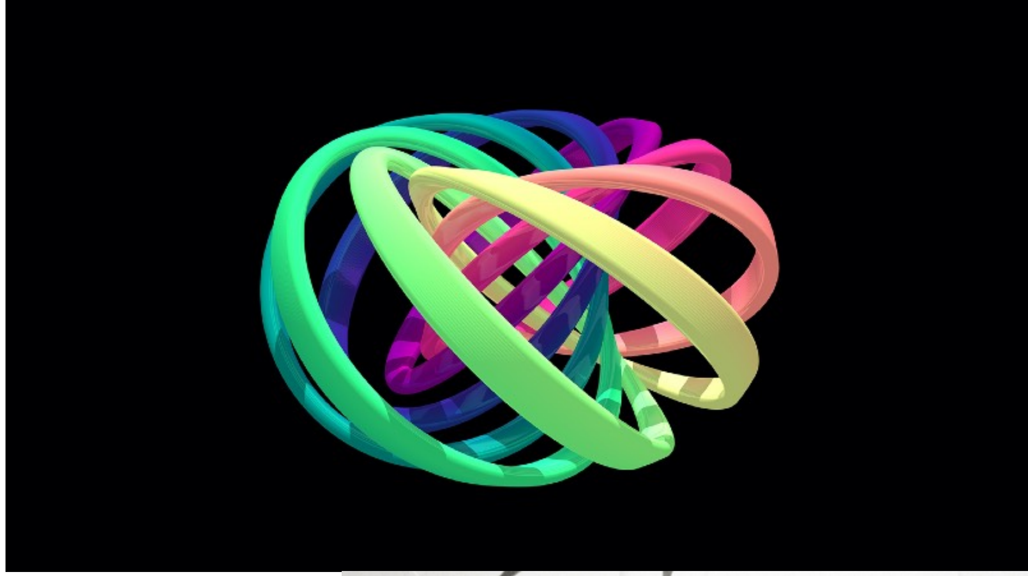
In topological quantum computation, quantum information is stored in states which are intrinsically protected from decoherence, and quantum gates are carried out by dragging particlelike excitations (quasiparticles) around one another in two space dimensions. The resulting quasiparticle trajectories define world lines in three-dimensional space-time, and the corresponding quantum gates depend only on the topology of the braids formed by these world lines. We show how to find braids that yield a universal set of quantum gates for qubits encoded using a specific kind of quasiparticle which is particularly promising for experimental realization.



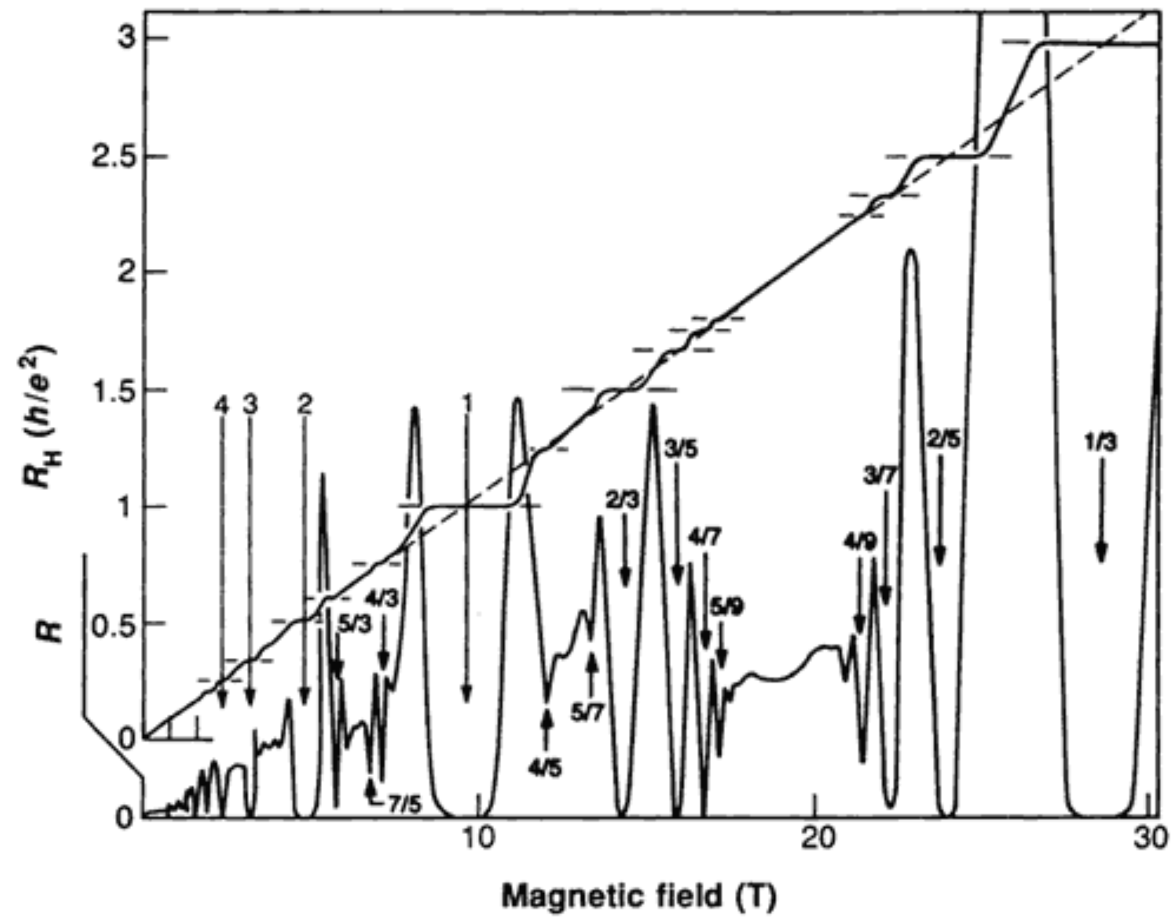
In 2005, it was explicitly shown (theoretically) how to perform logical operations by braiding world lines of topological quantum particles!



Do anyons exist?



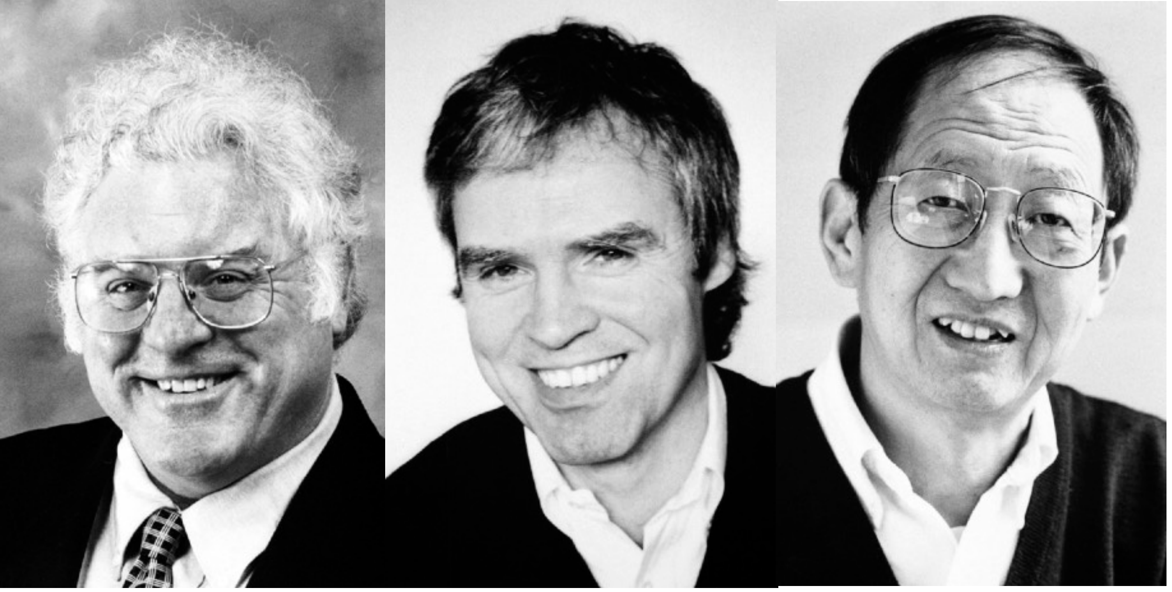
In 1982, right after the discovery of the integer quantum Hall effect, Tsui and Stomer discovered the appearance of quantum Hall plateaus at some rational fractions of magnetic flux quanta per unit cell.



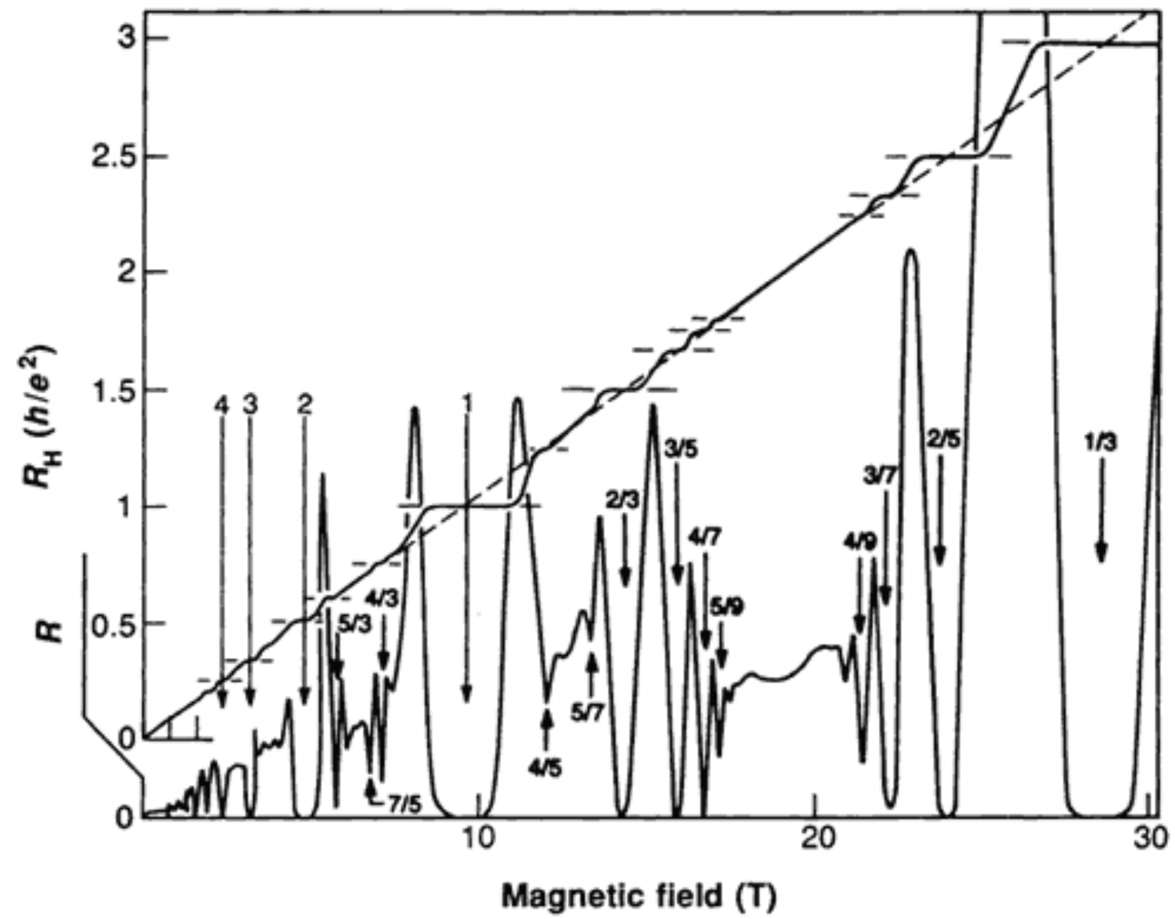
Flux quanta $$\nu = \frac{p}{q} = \frac{1}{3}, \frac{3}{5}, \frac{4}{7}, \ldots$$

One year later, Bob Laughlin showed the experiment can be explained by the existence of particles with fractional charge!

Fractional quantum Hall effect!



The Nobel Prize in Physics 1998 was awarded jointly to Robert B. Laughlin, Horst L. Störmer and Daniel C. Tsui "for their discovery of a new form of quantum fluid with fractionally charged excitations."



Are particles with fractionalized charge anyons?

VOLUME 53, NUMBER 7 PHYSICAL REVIEW LETTERS 13 AUGUST 1984

Fractional Statistics and the Quantum Hall Effect

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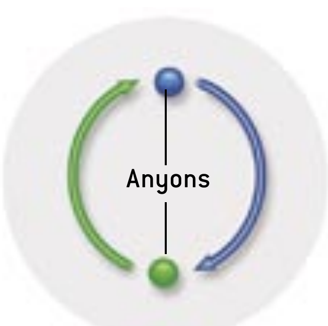
(Received 18 May 1984)

The statistics of quasiparticles entering the quantum Hall effect are deduced from the adiabatic theorem. These excitations are found to obey fractional statistics, a result closely related to their fractional charge.

In 1984, Arovas, Schrieffer and Wilczek showed (theoretically) that particles with fractional charge $e^* = e/q$ in the FQH effect correspond to anyons with phase



$$\theta = \frac{\pi}{q}$$

$$|\psi_1, \psi_2\rangle \rightarrow \mathrm{e}^{i\theta} |\psi_2, \psi_1\rangle$$

MESOSCOPIC PHYSICS

Fractional statistics in anyon collisions

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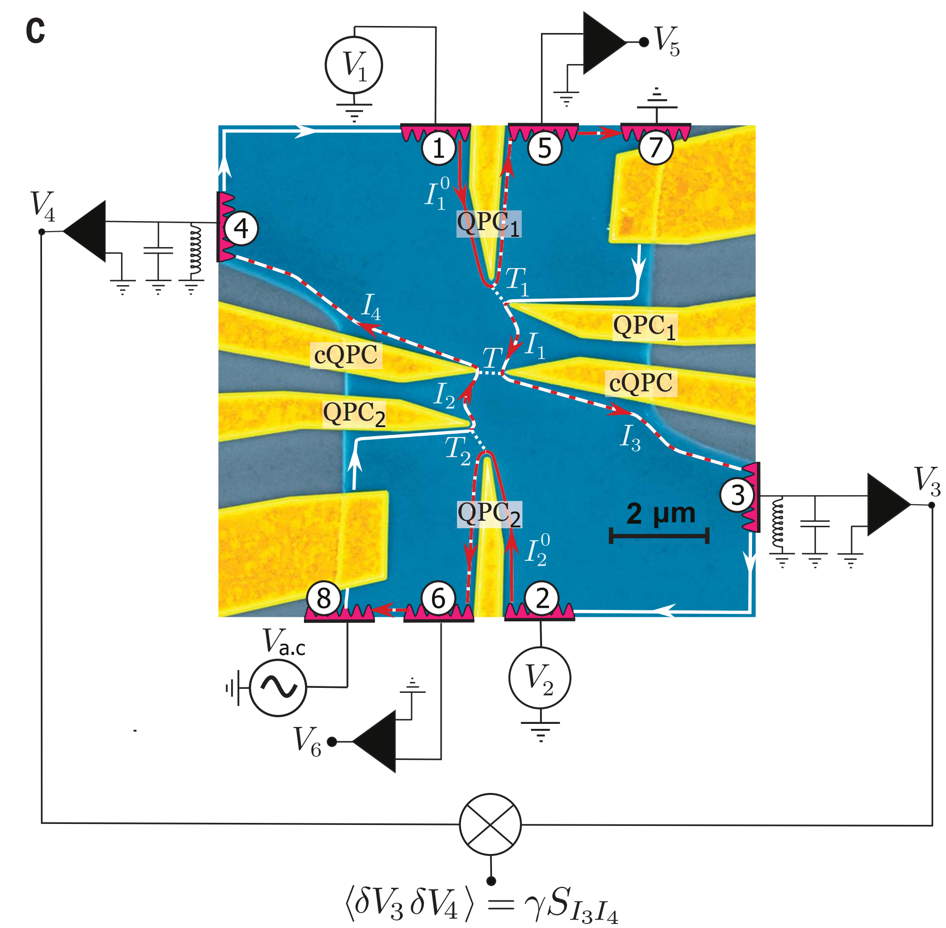
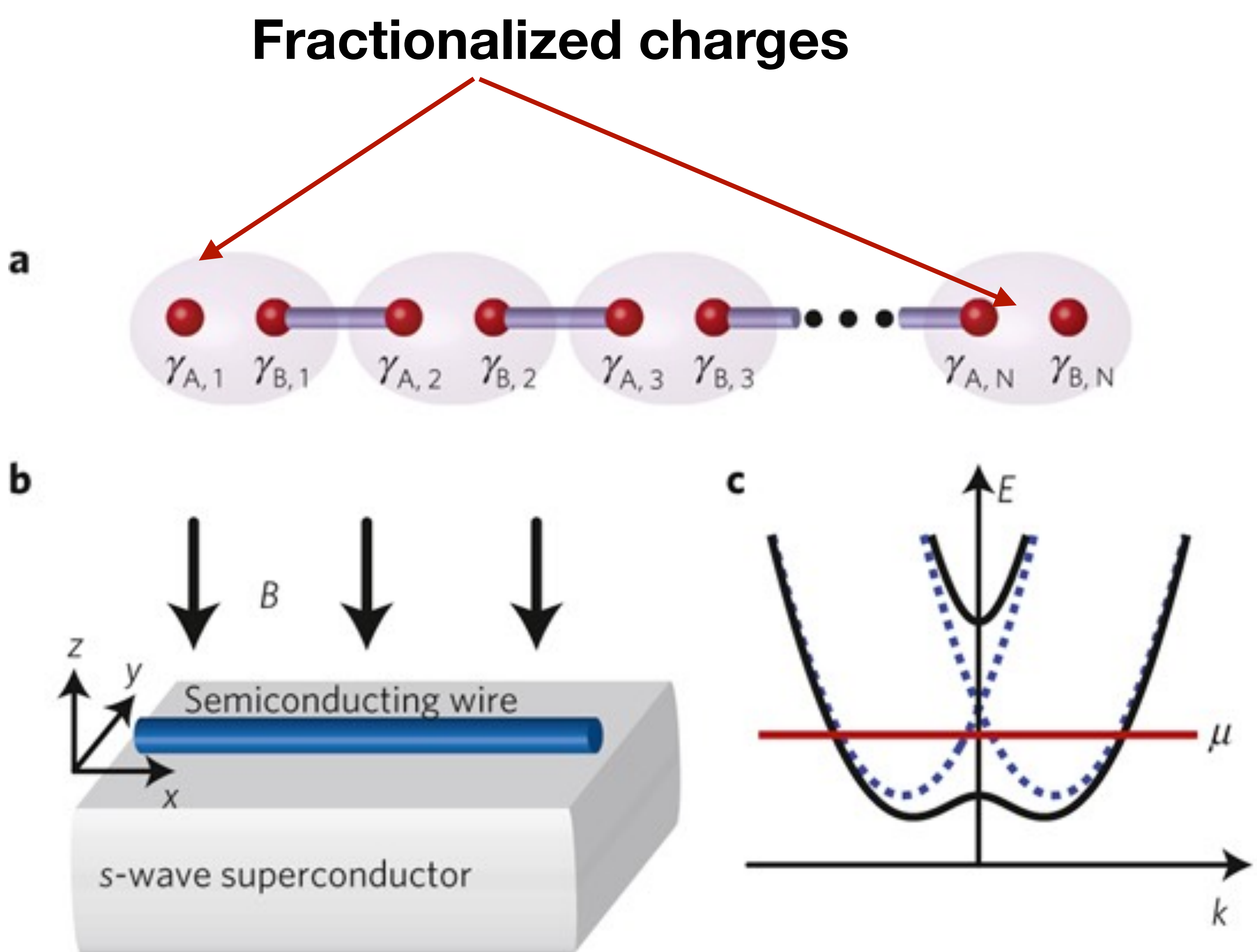
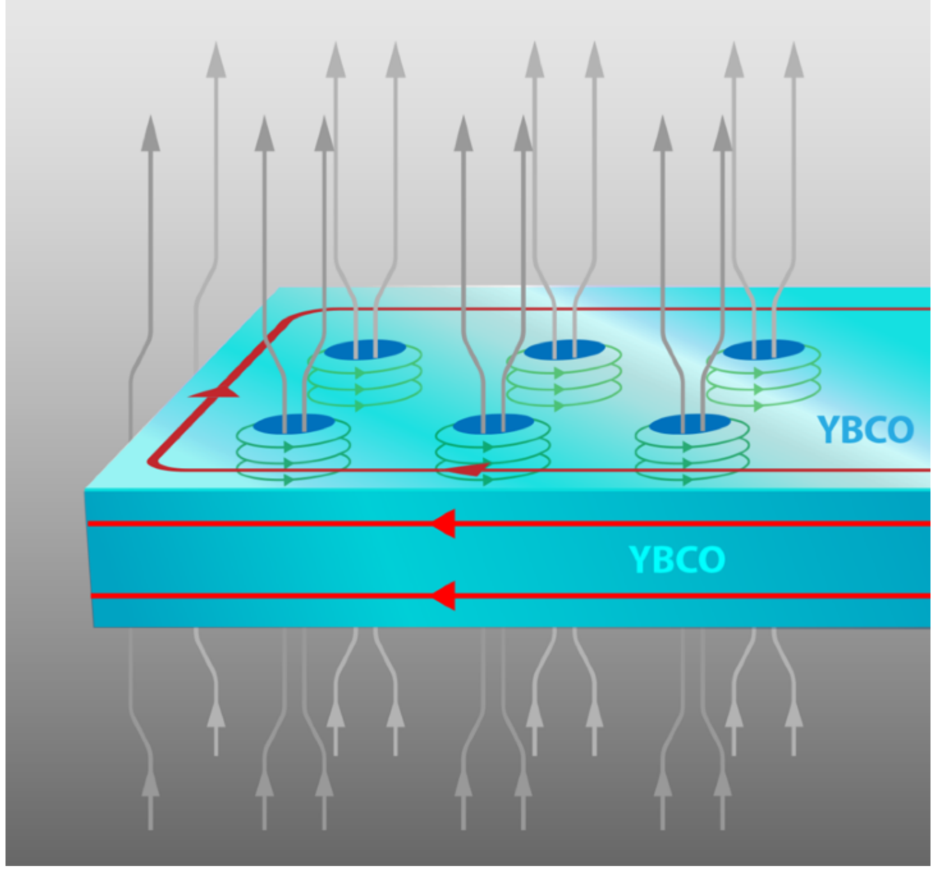
Bartolomei *et al.*, *Science* **368**, 173–177 (2020)   10 April 2020



Experimental evidence that they exist!





Fractionalized charges may also exist at the two ends of a superconducting wire or inside the core of a superconducting vortex!

Some concepts appear to be pure mathematical abstractions until they lead the way to understanding new fundamental discoveries.

Try to learn the language, even if you would like to become an experimentalist!