Chapter 3

Elementary Transcendental Functions

3.1 Exponential Function

Define, for all complex z, the exponential function by

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n. \tag{3.1}$$

By the ratio test,

$$\frac{n!}{(n+1)!}|z| = \frac{|z|}{n+1} \to 0 \quad \forall z, \tag{3.2}$$

the series converges everywhere. By the theorem of the Sec. 2.7, that means that the series converges uniformily in any finite closed region.

Note that the following property holds:

$$\begin{aligned}
\exp(z_1 + z_2) &= \sum_{n=0}^{\infty} \frac{1}{n!}(z_1 + z_2)^n \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{n!} \frac{n!}{m!\,(n-m)!} z_1^m z_2^{n-m} \\
&= \left(\sum_{k=0}^{\infty} \frac{1}{k!} z_1^k\right) \left(\sum_{l=0}^{\infty} \frac{1}{l!} z_2^l\right) \\
&= \exp(z_1)\exp(z_2).
\end{aligned} \tag{3.3}$$

Then, by induction

$$(e^z)^n = e^{nz}, \tag{3.4}$$

where n is any positive integer.

Hyperbolic and *trigonometric* functions are defined in terms of the exponential function:

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \cosh z = \frac{e^z + e^{-z}}{2}, \tag{3.5a}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tag{3.5b}$$

so that

$$i \sin z = \sinh iz, \tag{3.6a}$$

$$\cos z = \cosh iz, \tag{3.6b}$$

for all complex z.

Note that

$$e^{iz} = \cos z + i \sin z. \tag{3.7}$$

Therefore, the polar representation of a complex number,

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta}, \end{aligned} \tag{3.8}$$

becomes a most useful and compact representation. In particular,

$$z^n = r^n e^{in\theta} \tag{3.9}$$

implies De Moivre's formula,

$$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n. \tag{3.10}$$

3.1.1 Definition of π

There exists a positive number π such that

1. $$e^{\pi i/2} = i, \quad \text{and} \tag{3.11a}$$

2. $$e^z = 1 \quad \text{if and only if} \quad z = 2\pi i n, \tag{3.11b}$$

   where n is an integer.

Hence $\exp(z)$ is periodic with period $2\pi i$,

$$\exp(z + 2\pi i) = \exp(z)\exp(2\pi i) = \exp(z). \tag{3.12}$$

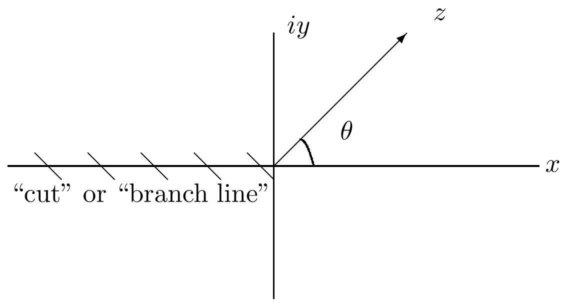

Figure 3.1: Cut plane for defining the logarithm.

3.2 The Natural Logarithm

If $z = re^{i\theta}$, we define

$$\ln z \equiv \log z \equiv \ln r + i\theta, \tag{3.13}$$

where $\ln r$ is defined as the inverse of the exponential function for real positive r,

$$r = e^{\ln r}. \tag{3.14}$$

Thus we have

$$z = e^{\zeta} \quad \text{where} \quad \zeta = \ln r + i\theta = \log z. \tag{3.15}$$

Recall that $\theta = \arg z$ is a multivalued function, because θ is only defined up to an arbitrary multiple of 2π. [This is just the periodic property (3.12).] Recall further that we defined the principal value of the argument as that which satisfied

$$-\pi < \arg z \leq \pi. \tag{3.16}$$

Correspondingly, we say that the single-valued logarithm function (also denoted $\log z$) is defined in the *cut plane* shown in Fig. 3.1. In measuring θ from the $+x$ axis, one is not allowed to cross the cut along the $-x$ axis. (Where the cut is placed is an arbitrary convention.) The correspondingly defined single-valued functions $\arg z$ and

$$\log z = \log |z| + i \arg z, \tag{3.17}$$

or

$$-\pi < \text{Im} \log z \leq \pi, \tag{3.18}$$

are also referred to as the principal values of the argument and logarithm, respectively.

Now we define complex powers of complex numbers as follows:

$$\zeta^z \equiv e^{z \log \zeta}, \tag{3.19}$$

where $\log \zeta$ is defined in the cut plane. Then

$$\left(e^{\xi}\right)^z = e^{z \log e^{\xi}} = e^{z(\text{Re}\,\xi + i\text{Im}\,\xi)} = e^{\xi z} \tag{3.20}$$

when

$$\arg e^{\xi} = \operatorname{Im} \xi \tag{3.21}$$

lies between

$$-\pi < \operatorname{Im} \xi \leq \pi. \tag{3.22}$$

If this is not so,

$$\log e^{\xi} = \xi + 2\pi i n \tag{3.23}$$

where n is so chosen that

$$-\pi < \operatorname{Im} (\xi + 2\pi i n) \leq \pi, \tag{3.24}$$

and

$$\left(e^{\xi}\right)^z = e^{z(\xi+2\pi i n)}. \tag{3.25}$$

For example,

$$\sqrt{z} = z^{1/2} = e^{\frac{1}{2}\log z} \tag{3.26}$$

is defined as a single-valued function only in the cut plane

$$-\pi < \arg z \leq \pi. \tag{3.27}$$

3.3 Inverse Hyperbolic and Trigonometric Functions

The inverse hyperbolic and trigonometric functions are defined in terms of the logarithm:

$$\operatorname{arcsinh} z = \log \left[z + (z^2+1)^{1/2}\right], \tag{3.28a}$$

$$\operatorname{arccosh} z = \log \left[z + (z^2-1)^{1/2}\right], \tag{3.28b}$$

$$\operatorname{arctanh} z = \frac{1}{2} \log \frac{1+z}{1-z}, \tag{3.28c}$$

which are defined in the cut planes shown in Fig. 3.2.

$$\begin{aligned} \arcsin z &= -i \operatorname{arcsinh} iz \\ &= -i \log \left[iz + (1-z^2)^{1/2}\right], \end{aligned} \tag{3.29a}$$

$$\begin{aligned} \arccos z &= -i \operatorname{arccosh} z \\ &= -i \log \left[z + (z^2-1)^{1/2}\right], \end{aligned} \tag{3.29b}$$

$$\begin{aligned} \arctan z &= -i \operatorname{arctanh} iz \\ &= \frac{i}{2} \log \frac{1-iz}{1+iz} = \frac{i}{2} \log \frac{i+z}{i-z}, \end{aligned} \tag{3.29c}$$

which are defined in the cut planes shown in Fig. 3.3. Note that the branch

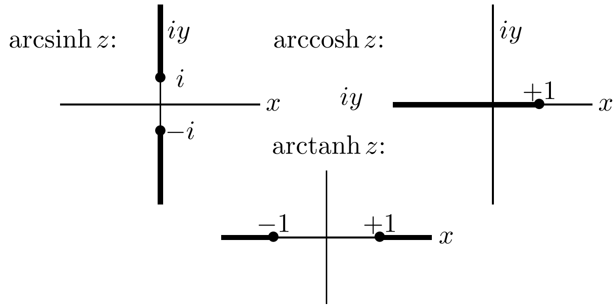

Figure 3.2: Cut planes for defining the inverse hyperbolic functions. The thick lines represent the cuts.

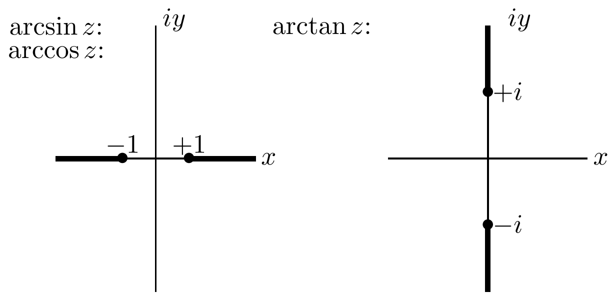

Figure 3.3: Cut plane for defining the inverse trigonometric functions.

lines (cuts) are chosen so as not to cross the region where both the range and the domain of the functions are real, because for real x,

$$\sin x, \quad \cos x \in [-1, 1], \tag{3.30a}$$
$$\tan x \in (-\infty, \infty), \tag{3.30b}$$
$$\sinh x \in (-\infty, \infty), \tag{3.30c}$$
$$\cosh x \in [1, \infty), \tag{3.30d}$$
$$\tanh x \in [-1, 1]. \tag{3.30e}$$

An alternative notation for the inverse functions is provided by the superscript -1, as for example,

$$\operatorname{arcsinh} z = \sinh^{-1} z, \tag{3.31}$$

which does *not* mean $1/\sinh z$.