Chapter 17

Galilean Transformations

Something needs to be understood a bit better. Go back to the (dimensionless) q, p variables, which satisfy

$$\frac{1}{i}[q,p] = 1. \tag{17.1}$$

The finite version of the infinestimal translation of q considered above is [see Problem 1 of Homework Assignment 2]

$$e^{-iq'p} q e^{iq'p} = q - q', \tag{17.2a}$$
$$e^{-iq'p} p e^{iq'p} = p, \tag{17.2b}$$

or with $q \to p, p \to -q$,

$$e^{ip'q} p e^{-ip'q} = p - p', \tag{17.3a}$$
$$e^{ip'q} q e^{-ip'q} = q, \tag{17.3b}$$

Suppose we do both these transformations in succession,

$$U_1 = e^{iq'p} e^{-ip'q}, \tag{17.4a}$$
$$U_2 = e^{-ip'q} e^{iq'p}, \tag{17.4b}$$

which describes the same successive transformations in two different orders. Now

$$U_1^{-1} q U_1 = e^{ip'q} e^{-iq'p} q e^{iq'p} e^{-ip'q} = e^{ip'q}(q - q')e^{-ip'q} = q - q', \tag{17.5a}$$
$$U_1^{-1} p U_1 = e^{ip'q} e^{-iq'p} p e^{iq'p} e^{-ip'q} = e^{ip'q} p e^{-ip'q} = p - p', \tag{17.5b}$$

so we see that U_1 displaces both q and p. So does U_2:

$$U_2^{-1} q U_2 = e^{-iq'p} e^{ip'q} q e^{-ip'q} e^{iq'p} = e^{-iq'p} q e^{iq'p} = q - q', \tag{17.6a}$$
$$U_2^{-1} p U_2 = e^{-iq'p} e^{ip'q} p e^{-ip'q} e^{iq'p} = e^{-iq'p}(p - p')e^{iq'p} = p - p'. \tag{17.6b}$$

The effects of U_1, U_2 are identical. Does that mean that $U_1 = U_2$? No! This is because q and p do not commute:

$$U_2 U_1^{-1} = e^{-ip'q} e^{iq'p} e^{ip'q} e^{-iq'p} = e^{-ip'q} e^{ip'(q+q')} = e^{ip'q'}. \tag{17.7}$$

That is,

$$U_2 = e^{ip'q'} U_1, \tag{17.8}$$

the two transformations differ by a phase. That is why U_2 and U_1 generate the same displacements.

Note that, in general, U and $e^{i\alpha}U$ are equivalent unitary operators. Thus if

$$U^\dagger U = 1 \to \left(e^{i\alpha}U\right)^\dagger \left(e^{i\alpha}U\right) = e^{-i\alpha} U^\dagger e^{i\alpha} U = 1. \tag{17.9}$$

We could observe that $e^{i\alpha}$ is a unitary transformation in its own right. Further,

$$\bar{X} = U^{-1} X U \to e^{-i\alpha} U^{-1} X U e^{i\alpha} = \bar{X}. \tag{17.10}$$

As far as the response on vectors,

$$\overline{\langle\ |} = \langle\ | U \to \overline{\langle\ |} = (\langle\ | U)\, e^{i\alpha}, \tag{17.11}$$

where we see the phase ambiguity of state vectors. Left and right vectors are incomplete objects, complete ones are the measurement symbols and numbers, corresponding to outer and inner products respectively,

$$|a'\rangle\langle a''|, \quad \langle a'|a''\rangle. \tag{17.12}$$

If $|a'\rangle \to e^{-i\alpha}|a'\rangle$ for all vectors, that is α is the same for all vectors, then

$$|a'\rangle\langle a''| \to e^{-i\alpha}|a'\rangle\langle a'| e^{i\alpha} = |a'\rangle\langle a''|, \tag{17.13a}$$

or

$$\langle a'|a''\rangle \to e^{i\alpha}\langle a'|a''\rangle e^{-i\alpha} = \langle a'|a''\rangle. \tag{17.13b}$$

The phase ambiguity disappears in these symbols describing a complete process. Thus, we conclude that nothing physical changes if we multiply a unitary operator by an arbitrary phase.

For an infinitesimal change, we have an infinitesimal unitary transformation,

$$e^{i\alpha} U \to (1 + i\delta\alpha)\left(1 + \frac{i}{\hbar} G\right) = 1 + \frac{i}{\hbar}(G + \hbar\delta\alpha 1). \tag{17.14}$$

Where we use a generator G, we could always add an arbitrary multiple of a unit operator.

What difference does this make? Remember we showed [Eq. (16.21)] that if successive rotations are done in different orders we get different results, differing by a rotation about a perpendicular axis:

$$\delta_{[12]}\boldsymbol{\omega} = \delta_1\boldsymbol{\omega} \times \delta_2\boldsymbol{\omega}. \tag{17.15}$$

Here, look at this again by considering the following sequence of rotations: rotation 1, followed by rotation 2, followed by the inverse of rotation 1, followed by the inverse of rotation 2. This measures the extent to which the order of transformation matters:

$$\mathbf{r}_1 = \mathbf{r} - \delta_1\boldsymbol{\omega} \times \mathbf{r}, \tag{17.16a}$$

$$\mathbf{r}_2 = \mathbf{r}_1 - \delta_2\boldsymbol{\omega} \times \mathbf{r}_1, \tag{17.16b}$$

$$\mathbf{r}_{1^{-1}} = \mathbf{r}_2 + \delta_1\boldsymbol{\omega} \times \mathbf{r}_2, \tag{17.16c}$$

$$\mathbf{r}_{2^{-1}} = \mathbf{r}_{1^{-1}} + \delta_2\boldsymbol{\omega} \times \mathbf{r}_{1^{-1}}. \tag{17.16d}$$

Now put these together:

$$\mathbf{r}_2 = \mathbf{r} - \delta_1\boldsymbol{\omega} \times \mathbf{r} - \delta_2\boldsymbol{\omega} \times \mathbf{r} + \delta_2\boldsymbol{\omega} \times (\delta_1\boldsymbol{\omega} \times \mathbf{r}), \tag{17.17a}$$

$$\mathbf{r}_{1^{-1}} = \mathbf{r} - \delta_2\boldsymbol{\omega} \times \mathbf{r} + \delta_2\boldsymbol{\omega} \times (\delta_1\boldsymbol{\omega} \times \mathbf{r}) - \delta_1\boldsymbol{\omega} \times (\delta_2\boldsymbol{\omega} \times \mathbf{r}), \tag{17.17b}$$

$$\mathbf{r}_{2^{-1}} = \mathbf{r} + \delta_2\boldsymbol{\omega} \times (\delta_1\boldsymbol{\omega} \times \mathbf{r}) - \delta_1\boldsymbol{\omega} \times (\delta_2\boldsymbol{\omega} \times \mathbf{r}). \tag{17.17c}$$

Here we have neglected $(\delta_1\boldsymbol{\omega})^2$ and $(\delta_2\boldsymbol{\omega})^2$ terms, which would have cancelled if kept, but are not interesting. That is, we keep only linear and bilinear terms. Now using the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0, \tag{17.18}$$

we find

$$\mathbf{r}_{2^{-1}} = \mathbf{r} - \delta_{[12]}\boldsymbol{\omega} \times \mathbf{r}, \quad \delta_{[12]}\boldsymbol{\omega} = \delta_1\boldsymbol{\omega} \times \delta_2\boldsymbol{\omega}, \tag{17.19}$$

as we saw before. Side by side with this is the sequence of unitary transformations,

$$U_2^{-1}U_1^{-1}U_2U_1 = \left(1 - \frac{i}{\hbar}G_2\right)\left(1 - \frac{i}{\hbar}G_1\right)\left(1 + \frac{i}{\hbar}G_2\right)\left(1 + \frac{i}{\hbar}G_1\right) \to 1 + \frac{i}{\hbar}G_{[12]}, \tag{17.20}$$

where again we have kept only bilinear terms. Here

$$G_{[12]} = \frac{1}{i\hbar}[G_1, G_2]. \tag{17.21}$$

From this we conclude for rotations

$$\frac{1}{i\hbar}[\delta_1\boldsymbol{\omega} \cdot \mathbf{J}, \delta_2\boldsymbol{\omega} \cdot \mathbf{J}] = (\delta_1\boldsymbol{\omega} \times \delta_2\boldsymbol{\omega}) \cdot \mathbf{J}, \tag{17.22}$$

which of course is what we already know.

What about the ambiguity in the generator,

$$G \to G + \hbar\delta\alpha 1? \tag{17.23}$$

We might expect

$$\frac{1}{i\hbar}[\delta_1\boldsymbol{\omega} \cdot \mathbf{J} + \hbar\delta\alpha_1, \delta_2\boldsymbol{\omega} \cdot \mathbf{J} + \hbar\delta\alpha_1] = \frac{1}{i\hbar}[\delta_1\boldsymbol{\omega} \cdot \mathbf{J}, \delta_2\boldsymbol{\omega} \cdot \mathbf{J}] = (\delta_1\boldsymbol{\omega} \times \delta_2\omega) \cdot \mathbf{J} + c_{[12]}1, \tag{17.24}$$

where $c_{[12]}$ is some number. Now the extra terms inside the commutator do not contribute, because they are multiples of the unit operator. What can $c_{[12]}$ be? It must depend on $\delta_1\boldsymbol{\omega}$ and $\delta_2\boldsymbol{\omega}$, and must be a scalar, so

$$c_{[12]} = C\delta_1\boldsymbol{\omega}\cdot\delta_2\boldsymbol{\omega}. \tag{17.25}$$

But the left-hand side of Eq. (17.24) must be antisymmetric in 1, 2. Therefore, $C = 0$. So the angular momentum commutations relations are preserved.

Similarly, although we might think that

$$\frac{1}{i\hbar}[\delta_1\boldsymbol{\epsilon}\cdot\mathbf{P}, \delta_2\boldsymbol{\epsilon}\cdot\mathbf{P}] = b_{[12]}1, \tag{17.26}$$

the number $b_{[12]}$ must be both antisymmetrical in 1 and 2, and of the form

$$b_{[12]} = B\delta_1\boldsymbol{\epsilon}\cdot\delta_2\boldsymbol{\epsilon}, \tag{17.27}$$

which can only be true if $B = 0$. But this new possibility does occur for transformations to a coordinate system that is moving relative to the original one.

17.1 Galilean Relativity

Galileo realized that one could not detect uniform motion—the physics is the same in two coordinate systems moving with constant velocity with respect to each other. Suppose the two coordinate systems coincide at $t = 0$. Then, a particle, described in the two coordinate systems at a given time, has position vectors

$$\bar{\mathbf{r}} = \mathbf{r} - \delta\mathbf{v}t. \tag{17.28}$$

This is a time-dependent displacement. The corresponding unitary transformation has the generator

$$G_{\delta\mathbf{v}} = \delta\mathbf{v}\cdot\mathbf{N}, \tag{17.29}$$

where $\mathbf{N}$ is sometimes called a *boost*. $\mathbf{N}$ is a vector, so under a rotation

$$\delta_{\delta\boldsymbol{\omega}}\mathbf{N} = \delta\boldsymbol{\omega}\times\mathbf{N} = \frac{1}{i\hbar}[\mathbf{N}, \delta\boldsymbol{\omega}\cdot\mathbf{J}]. \tag{17.30}$$

For example, this says that

$$\frac{1}{i\hbar}[N_x, J_y] = N_z. \tag{17.31}$$

Turn this around, to learn how $\mathbf{J}$ changes under a Galilean (boost) transformation:

$$\frac{1}{i\hbar}[\mathbf{J}, \delta\mathbf{v}\cdot\mathbf{N}] = \delta_{\delta\mathbf{v}}\mathbf{J} = \delta\mathbf{v}\times\mathbf{N}, \tag{17.32}$$

which should be compared to (16.6),

$$\frac{1}{i\hbar}[\mathbf{J}, \delta\boldsymbol{\epsilon}\cdot\mathbf{P}] = \delta_{\delta\boldsymbol{\epsilon}}\mathbf{J} = \delta\boldsymbol{\epsilon}\times\mathbf{P}. \tag{17.33}$$

Just as the order of translations does not matter,

$$\bar{\mathbf{r}} = \mathbf{r} - \delta\boldsymbol{\epsilon}, \tag{17.34}$$

so the order of Galilean transformations should not matter,

$$\bar{\mathbf{r}} = \mathbf{r} - \delta\mathbf{v}t, \tag{17.35}$$

since a boost is just a translation at a particular time. Thus, we can conclude

$$\frac{1}{i\hbar}[\delta_1\mathbf{v}\cdot\mathbf{N}, \delta_2\mathbf{v}\cdot\mathbf{N}] = d_{[12]}1, \tag{17.36}$$

where, once again, we conclude that $d_{[12]} = 0$ because it must be both proportional to $\delta_1\mathbf{v}\cdot\delta_2\mathbf{v}$ and antisymmetrical in 1, 2. So the boost generators commute,

$$[N_k, N_l] = 0. \tag{17.37}$$

What if we follow a boost by a translation, and vice versa. Geometrically, the order makes no difference, but we have once again the phase ambiguity,

$$\frac{1}{i\hbar}[\delta\boldsymbol{\epsilon}\cdot\mathbf{P}, \delta\mathbf{v}\cdot\mathbf{N}] = M\delta\boldsymbol{\epsilon}\cdot\delta\mathbf{v}, \tag{17.38}$$

where M is some scalar number. M does not have to be zero, since there is no symmetry here. The previous cases all involved two different examples of the same transformation, which is not so here. In other words, the action of a boost on the momentum is

$$\frac{1}{i\hbar}[\mathbf{P}, \delta\mathbf{v}\cdot\mathbf{N}] = \delta_{\delta\mathbf{v}}\mathbf{P} = M\delta\mathbf{v}. \tag{17.39}$$

This suggests we call M the "mass" of the system. Of course, if you change the state of motion by going to a relatively moving coordinate system, the momentum changes.

Another way of writing this is

$$\frac{1}{i\hbar}[P_k, N_l] = M\delta_{kl}. \tag{17.40}$$

Now how does $\mathbf{R}$ change? Recall under a translation,

$$\delta_{\delta\boldsymbol{\epsilon}}\mathbf{R} = \delta\boldsymbol{\epsilon} = \frac{1}{i\hbar}[\mathbf{R}, \delta\boldsymbol{\epsilon}\cdot\mathbf{P}], \tag{17.41}$$

so with $\delta\boldsymbol{\epsilon} = \delta\mathbf{v}t$ this suggests

$$\delta_{\delta\mathbf{v}}\mathbf{R} = \delta\mathbf{v}t = \frac{1}{i\hbar}[\mathbf{R}, \delta\mathbf{v}\cdot\mathbf{N}]. \tag{17.42}$$

So, to some extent, $\mathbf{N}$ acts like $\mathbf{P}t$, that is, a boost is a translation growing in time. On the other hand, we also must have Eq. (17.40), which from

$$\frac{1}{i\hbar}[P_k, R_l] = -\delta_{kl}, \tag{17.43}$$

would occur if $\mathbf{N}$ contained the term $-M\mathbf{R}$. So this suggests the following construction of the boost generator:

$$\mathbf{N} = \mathbf{P}t - M\mathbf{R}. \tag{17.44}$$

We see here a hint of the mixing of space and time, characteristic of relativity.

Let's check that this works: First, we must have Eq. (17.37), or

$$0 = [N_k, N_l] = [P_k t - MR_k, P_l t - MR_l] = -Mt\delta_{kl} + Mt\delta_{kl} = 0. \tag{17.45}$$

Let's also see if Eq. (17.32), or

$$\delta_{\delta\mathbf{v}}\mathbf{J} = \delta\mathbf{v} \times \mathbf{N} \tag{17.46}$$

is true when we use the construction

$$\mathbf{J} = \mathbf{R} \times \mathbf{P} + \mathbf{S}. \tag{17.47}$$

So

$$\delta_{\delta\mathbf{v}}\mathbf{J} = \delta\mathbf{v}t \times \mathbf{P} + \mathbf{R} \times M\delta\mathbf{v} + \delta_{\delta\mathbf{v}}\mathbf{S} = \delta\mathbf{v} \times (\mathbf{P}t - M\mathbf{R}) + \delta_{\delta\mathbf{v}}\mathbf{S} \tag{17.48}$$

indeed equals the required $\delta\mathbf{v} \times \mathbf{N}$ provided

$$0 = \delta_{\delta\mathbf{v}}\mathbf{S} = \frac{1}{i\hbar}[\mathbf{S}, \delta\mathbf{v} \cdot (\mathbf{P}t - M\mathbf{R})] = 0, \tag{17.49}$$

which is true, because $\mathbf{S}$ commutes with both $\mathbf{P}$ and $\mathbf{R}$,

$$[S_k, P_l] = 0, \quad [S_k, R_l] = 0. \tag{17.50}$$

Now that we have introduced time, let's recognize that we also have the freedom to translate the origin of time, or make a time displacement:

$$\bar{t} = t - \delta t, \tag{17.51}$$

where δt is a constant. Quantum mechanically, this must be represented by a unitary transformation, or by a generator,

$$G_{\delta t} = -\delta t H, \tag{17.52}$$

where H is the Hamiltonian or the energy operator. The minus sign appears here because otherwise physical energies would turn out to be negative. [Another hint of (Einsteinian) relativity.]

Now time and spatial translations are unrelated, so we would expect

$$\frac{1}{i\hbar}[\delta\boldsymbol{\epsilon} \cdot \mathbf{P}, -\delta t H] = 0. \tag{17.53}$$

A possible multiple of the unit operator on the right-hand side of this equation is precluded, since it would have to be a scalar constructed bilinearly from $\delta\boldsymbol{\epsilon}$ and δt, and there is none such. Thus

$$[P_k, H] = 0, \tag{17.54}$$

which says that H does not change under a displacement of the coordinate system,

$$\delta_{\delta\boldsymbol{\epsilon}} H = 0. \tag{17.55}$$

Likewise, time displacements and spatial rotations are independent,

$$\frac{1}{i\hbar}[\delta\boldsymbol{\omega}\cdot\mathbf{J}, -\delta t H] = 0, \tag{17.56}$$

or

$$[J_k, H] = 0, \tag{17.57}$$

which is to say H is a scalar,

$$\delta_{\delta\boldsymbol{\omega}} H = 0. \tag{17.58}$$

But Galilean transformations, boosts, involve both time and space. Under a time translation,

$$\bar{t} = t - \delta t, \quad \bar{\mathbf{r}} = \mathbf{r}, \tag{17.59}$$

while under a boost,

$$\bar{t} = t, \quad \bar{\mathbf{r}} = \mathbf{r} - \delta\mathbf{v}t. \tag{17.60}$$

Consider the following sequence of these transformations, 1, 2, 1^{-1}, 2^{-1}:

$$\begin{aligned}
1: &\quad t_1 = t - \delta t, \quad \mathbf{r}_1 = \mathbf{r}; && (17.61a)\\
2: &\quad t_2 = t_1 = t - \delta t, \quad \mathbf{r}_2 = \mathbf{r}_1 - \delta\mathbf{v}t_1 = \mathbf{r} - \delta\mathbf{v}(t - \delta t); && (17.61b)\\
1^{-1}: &\quad t_{1^{-1}} = t_2 + \delta t = t - \delta t + \delta t = t, &&\\
&\quad \mathbf{r}_{1^{-1}} = \mathbf{r}_2 = \mathbf{r} - \delta\mathbf{v}(t - \delta t); && (17.61c)\\
2^{-1}: &\quad t_{2^{-1}} = t_{1^{-1}} = t, \quad \mathbf{r}_{2^{-1}} = \mathbf{r}_{1^{-1}} + \delta\mathbf{v}t_{1^{-1}} = \mathbf{r} + \delta\mathbf{v}\delta t. && (17.61d)
\end{aligned}$$

So we see a net spatial displacement of $-\delta\mathbf{v}\delta t$, or, in terms of the generators [see Eq. (17.21)],

$$\frac{1}{i\hbar}[-\delta t H, \delta\mathbf{v}\cdot\mathbf{N}] = -\delta t\delta\mathbf{v}\cdot\mathbf{P}, \tag{17.62}$$

or

$$\frac{1}{i\hbar}[H, \delta\mathbf{v}\cdot\mathbf{N}] = \delta\mathbf{v}\cdot\mathbf{P}. \tag{17.63}$$

A Galilean transformation changes the state of motion, and thus changes the energy.

We summarize the statements we have learned about how the energy operator changes under Galilean relativity:

$$\begin{aligned}
\frac{1}{i\hbar}[\mathbf{P}, H] &= 0, && (17.64a)\\
\frac{1}{i\hbar}[\mathbf{J}, H] &= 0, && (17.64b)\\
\frac{1}{i\hbar}[\mathbf{N}, H] + \mathbf{P} &= 0. && (17.64c)
\end{aligned}$$

In the next chapter, we will review the dynamics resulting from the Hamiltonian.