1 Length Rescaling

To begin momentum space renormalization one must rescale the Hamiltonian via a length rescale in real space (and equivalently rescale the system in wave number space):

\[ r' = r \frac{1}{1 + \varepsilon} \]
\[ = k(1 - \varepsilon + \varepsilon^2 - ...) \]
\[ \approx r(1 - \varepsilon) \] (1)

\[ k' = k(1 + \varepsilon) \] (2)

Accordingly, our Gaussian field also rescales:

\[ X(k') = \frac{1}{(1 + \varepsilon)^2} X(k) \]
\[ = X(k)(1 - 2\varepsilon + 3\varepsilon^2 - 4\varepsilon^3 + ...) \]
\[ \approx X(k)(1 - 2\varepsilon) \] (3)

2 Hamiltonian for Gaussian field X(k)

We are given a rescaled Hamiltonian:

\[ H'(J, y') = H_0(X(k)) + H_1(Y(k)) + \frac{2y'}{a^2} \int d^2r \cos(X(r) + Y(r)) \] (4)

We want to use this model in a partition function, which in general is:

\[ Z(J, y) = \int DX(r)e^{H(J, y)} \] (5)
where
\[
\int DX(r) = \text{"sum over all states"} \quad (6)
\]
Our partition function for our model corresponding to (4) is simply:
\[
Z(J, y') = \int DX(r) \int DY(r) e^{H'(J, y')}
\]
\[
= \int DX(r) \exp [H_0(X(k))] \int DY(r) \exp \left[ H_1(Y(k)) + \frac{2y'}{a^2} \int d^2r \cos(X(r) + Y(r)) \right]
\]
\[
= \int DX(r) e^{H_0(X(k))} \int DY(r) e^{H_1(Y(k))} \exp \left[ \frac{2y'}{a^2} I \right] \quad (7)
\]
where \( I \) is the integral given in (4). We understand \( \exp \{H_1(Y(k))\} \) to be the probability density of \( Y(k) \) states. Therefore we conclude that the following relationship holds:
\[
\int DY(r) e^{H_1(Y(k))} \exp \left[ \frac{2y'}{a^2} I \right] = \left\langle \exp \left[ \frac{2y'}{a^2} I \right] \right\rangle_1 \quad (8)
\]
where \( \left\langle \right\rangle_1 \) denotes the average with respect to the states modeled by \( H_1(Y(k)) \).

The average of the exponential function is known as the moment-generating function due to its use in deriving the moments of a distribution. It can also be used to find the cumulants of a distribution:
\[
\left\langle e^{kx} \right\rangle = \sum_{n=0}^{\infty} k^n \frac{\langle x^n \rangle}{n!} = \exp \left[ \sum_{n=1}^{\infty} k^n \frac{\langle x^n \rangle}{n!} \right] \quad (9)
\]
Note that \( \langle x^n \rangle \) and \( \langle \langle x^n \rangle \rangle \) correspond to the \( n \)th moment and \( n \)th cumulant of the distribution respectively. With this knowledge we will expand (8) in terms of the cumulants of \( Y(r) \):
\[
\left\langle \exp \left[ \frac{2y'}{a^2} I \right] \right\rangle_1 = \exp \left[ \sum_{n=1}^{\infty} \left( \frac{2y'}{a^2} \right)^n \frac{\langle I^n \rangle_1}{n!} \right]
\]
\[
\approx \exp \left[ \left( \frac{2y'}{a^2} \right) \langle I \rangle_1 + \frac{1}{2} \left( \frac{2y'}{a^2} \right)^2 \langle I^2 \rangle_1 \right] \quad (10)
\]
Recall:
\[
I = \int d^2r \cos(X(r) + Y(r)) \quad (11)
\]
We can redefine our original (total) Gaussian field to make our next few equations neater:

\[ Z(r) = X(r) + Y(r) \]  \hspace{1cm} (12)

such that:

\[ Z(r) \sim \mathcal{N}(0, \sigma_X^2 + \sigma_Y^2) \]  \hspace{1cm} (13)

Expanding the argument in (10):

\[ \left( \frac{2y'}{a^2} \right) \left( \langle \langle I \rangle \rangle_1 + \frac{1}{2} \left( \frac{2y'}{a^2} \right)^2 \langle \langle I^2 \rangle \rangle_1 \right) + \frac{1}{2} \left( \frac{2y'}{a^2} \right)^2 \int d^2r d^2r' \left[ \langle \cos Z(r) \cos Z(r') \rangle_1 - \langle \cos Z(r) \rangle_1 \langle \cos Z(r') \rangle_1 \right] \]  \hspace{1cm} (14)

We return to the partition function with (10) in mind:

\[ Z = \int DX(r)e^{H_0/X(k)} \left\langle \exp \left[ \left( \frac{2y'}{a^2} I \right) \right] \right\rangle_1 \]

\[ = \int DX(r)e^{H_0/X(k)} \exp \left[ \left( \frac{2y'}{a^2} \right) \langle \langle I \rangle \rangle_1 + \frac{1}{2} \left( \frac{2y'}{a^2} \right)^2 \langle \langle I^2 \rangle \rangle_1 \right] \]

\[ = \int DX(r) \exp \left[ H_0(X(k)) + \left( \frac{2y'}{a^2} \right) \langle \langle I \rangle \rangle_1 + \frac{1}{2} \left( \frac{2y'}{a^2} \right)^2 \langle \langle I^2 \rangle \rangle_1 \right] \]  \hspace{1cm} (15)

We retrieve our Hamiltonian from the argument in the partition function:

\[ H = H_0(X(k)) + \left( \frac{2y'}{a^2} \right) \int d^2r \langle \cos Z(r) \rangle_1 + \frac{1}{2} \left( \frac{2y'}{a^2} \right)^2 \int d^2r d^2r' \left[ \langle \cos Z(r) \cos Z(r') \rangle_1 - \langle \cos Z(r) \rangle_1 \langle \cos Z(r') \rangle_1 \right] \]  \hspace{1cm} (16)

We will attempt to evaluate the first-order term in (16):

\[ \langle \cos Z(r) \rangle_1 = \langle \cos (X(r) + Y(r)) \rangle_1 \]

\[ = \langle \cos X(r) \cos Y(r) + \sin X(r) \sin Y(r) \rangle_1 \]

\[ = \langle \cos X(r) \cos Y(r) \rangle_1 + \langle \sin X(r) \sin Y(r) \rangle_1 \]

\[ = \cos X(r) \langle \cos Y(r) \rangle_1 + \sin X(r) \langle \sin Y(r) \rangle_1 \]  \hspace{1cm} (17)
Keep in mind the following for $X \sim \mathcal{N}(0, \sigma^2)$:

\[
\langle x^n \rangle = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
\sigma^n(n-1)!! & \text{if } n \text{ is even}
\end{cases}
\] (18)

We investigate the terms in (17):

\[
\langle \cos Y \rangle_1 = \left\langle 1 - \frac{Y^2}{2!} + \frac{Y^4}{4!} - \ldots \right\rangle_1 \\
= \langle 1 \rangle_1 - \frac{1}{2} \langle Y^2 \rangle_1 + \frac{1}{24} \langle Y^4 \rangle_1 - \ldots \\
= 1 - \frac{1}{2} \langle Y^2 \rangle_1 + \frac{3}{24} \langle Y^2 \rangle_1^2 - \ldots \\
= \sum_{n=0}^{\infty} \frac{1}{n!} \left[ -\frac{1}{2} \langle Y^2 \rangle_1 \right]^n \\
= \exp \left( -\frac{1}{2} \langle Y^2 \rangle_1 \right)
\] (19)

The 3rd equality holds because $Y(r)$ is a Gaussian with zero mean. Similarly:

\[
\langle \sin Y \rangle_1 = \left\langle Y - \frac{Y^3}{3!} + \frac{Y^5}{5!} - \ldots \right\rangle_1 \\
= \langle Y \rangle_1 - \frac{1}{6} \langle Y^3 \rangle_1 + \frac{1}{120} \langle Y^5 \rangle_1 - \ldots \\
= 0
\] (20)

Remember that odd-numbered moments vanish when the mean is zero. Now we seek the find the average of $Y^2$ as in (19). We know this to be equivalent to the variance of $Y$ if $\langle Y \rangle_1$ is 0. Note Eq. 2.4 from the paper:

\[
H_1(Y(k)) = -\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left[ \frac{Jk^2}{\psi(k)} Y(k)Y(-k) + E_1(k) \right]
\] (21)

We can retrieve the variance from this expression because the distribution of a Gaussian variable is of the form:

\[
\rho_Y(y) \sim \exp \left[ -\frac{(y - \mu)^2}{2\sigma^2} \right]
\] (22)
Therefore:

\[ \langle Y^2 \rangle_1 = \int D Y(r) Y^2 \exp [H_1] \]
\[ = \int \frac{d^2 k}{(2\pi)^2} \frac{\psi(k)}{J^2} \]
\[ = \frac{1}{(2\pi)^2 J} \int_0^{2\pi} d\phi \int_0^\infty dk \frac{\psi(k)}{k} \]
\[ = -\frac{\varepsilon}{2\pi J} \int_0^\infty \frac{d\phi(k)}{dk} \]
\[ = -\frac{\varepsilon}{2\pi J} [\phi(\infty) - \phi(0)] \]
\[ = \frac{\varepsilon}{2\pi J} \]

where we have used Eq. 2.5 from the article to relate the cut-off functions and we know the cut-off functions must go to zero at infinity and to unity at zero.

Our result for the first order integrand is then:

\[ \langle \cos(X(r) + Y(r)) \rangle_1 = \cos X(r) \exp \left( \frac{1}{2} \langle Y^2 \rangle_1 \right) \]
\[ = \cos X(r) \exp \left( -\frac{\varepsilon}{4\pi J} \right) \]
\[ \approx \cos X(r) \left[ 1 - \frac{\varepsilon}{4\pi J} \right] \]

\[ (24) \]