# A Companion Paper to the Momentum Space RG Article by *Knops et al.*

Alex Kerr

June 24, 2015

## 1 Length Rescaling

To begin momentum space renormalization one must rescale the Hamiltonian via a length rescale in real space (and equivalently rescale the system in wave number space):

$$\begin{aligned} r' &= \frac{r}{1+\varepsilon} \\ &= k(1-\varepsilon+\varepsilon^2-...) \\ &\approx r(1-\varepsilon) \end{aligned} \tag{1}$$

$$k' = k(1+\varepsilon) \tag{2}$$

Accordingly, our Gaussian field also rescales:

$$\begin{aligned} X(\boldsymbol{k'}) &= \frac{1}{(1+\varepsilon)^2}X(\boldsymbol{k}) \\ &= X(\boldsymbol{k})(1-2\varepsilon+3\varepsilon^2-4\varepsilon^3+...) \\ &\approx X(\boldsymbol{k})(1-2\varepsilon) \end{aligned} \tag{3}$$

## 2 Hamiltonian for Gaussian field X(k)

We are given a rescaled Hamiltonian:

$$H'(J,y') = H_0(X(\boldsymbol{k})) + H_1(Y(\boldsymbol{k})) + \frac{2y'}{a^2}\int d^2r\cos(X(\boldsymbol{r})+Y(\boldsymbol{r})) \tag{4}$$

We want to use this model in a partition function, which in general is:

$$Z(J,y) = \int DX(\boldsymbol{r})e^{H(J,y)} \tag{5}$$

where

$$\int DX(\boldsymbol{r}) = \text{"sum over all states"} \tag{6}$$

Our partition function for our model corresponding to (4) is simply:

$$\begin{aligned} Z(J,y') &= \int DX(\boldsymbol{r}) \int DY(\boldsymbol{r}) e^{H'(J,y')} \\ &= \int DX(\boldsymbol{r}) \exp\left[H_0(X(\boldsymbol{k}))\right] \int DY(\boldsymbol{r}) \exp\left[H_1(Y(\boldsymbol{k})) + \frac{2y'}{a^2} \int d^2r \cos(X(\boldsymbol{r}) + Y(\boldsymbol{r}))\right] \\ &= \int DX(\boldsymbol{r}) e^{H_0(X(\boldsymbol{k}))} \int DY(\boldsymbol{r}) e^{H_1(Y(\boldsymbol{k}))} \exp\left[\frac{2y'}{a^2} I\right] \end{aligned} \tag{7}$$

where $I$ is the integral given in (4). We understand $\exp\{H_1(Y(\boldsymbol{k}))\}$ to be the probability density of $Y(\boldsymbol{k})$ states. Therefore we conclude that the following relationship holds:

$$\int DY(\boldsymbol{r}) e^{H_1(Y(\boldsymbol{k}))} \exp\left[\frac{2y'}{a^2} I\right] = \left\langle \exp\left[\frac{2y'}{a^2} I\right] \right\rangle_1 \tag{8}$$

where $\langle\rangle_1$ denotes the average with respect to the states modeled by $H_1(Y(\boldsymbol{k}))$.

The average of the exponential function is known as the moment-generating function due to its use in deriving the moments of a distribution. It can also be used to find the cumulants of a distribution:

$$\langle e^{kx} \rangle = \sum_{n=0}^{\infty} k^n \frac{\langle x^n \rangle}{n!} = \exp\left[\sum_{n=1}^{\infty} k^n \frac{\langle\langle x^n \rangle\rangle}{n!}\right] \tag{9}$$

Note that $\langle x^n \rangle$ and $\langle\langle x^n \rangle\rangle$ correspond to the $n$th moment and $n$th cumulant of the distribution respectively. With this knowledge we will expand (8) in terms of the cumulants of $Y(\boldsymbol{r})$:

$$\begin{aligned} \left\langle \exp\left[\frac{2y'}{a^2} I\right] \right\rangle_1 &= \exp\left[\sum_{n=1}^{\infty} \left(\frac{2y'}{a^2}\right)^n \frac{\langle\langle I^n \rangle\rangle_1}{n!}\right] \\ &\approx \exp\left[\left(\frac{2y'}{a^2}\right) \langle\langle I \rangle\rangle_1 + \frac{1}{2} \left(\frac{2y'}{a^2}\right)^2 \langle\langle I^2 \rangle\rangle_1\right] \end{aligned} \tag{10}$$

Recall:

$$I = \int d^2r \cos(X(\boldsymbol{r}) + Y(\boldsymbol{r})) \tag{11}$$

We can redefine our original (total) Gaussian field to make our next few equations neater:

$$Z(\boldsymbol{r}) = X(\boldsymbol{r}) + Y(\boldsymbol{r}) \tag{12}$$

such that:

$$Z(\boldsymbol{r}) \sim \mathcal{N}(0, \sigma_X^2 + \sigma_Y^2) \tag{13}$$

Expanding the argument in (10):

$$\left(\frac{2y'}{a^2}\right)\langle\langle I\rangle\rangle_1 + \frac{1}{2}\left(\frac{2y'}{a^2}\right)^2\langle\langle I^2\rangle\rangle_1 = \left(\frac{2y'}{a^2}\right)\int d^2r\langle\cos Z(\boldsymbol{r})\rangle_1 + \frac{1}{2}\left(\frac{2y'}{a^2}\right)^2\int d^2rd^2r'\left[\langle\cos Z(\boldsymbol{r})\cos Z(\boldsymbol{r'})\rangle_1 - \langle\cos Z(\boldsymbol{r})\rangle_1\langle\cos Z(\boldsymbol{r'})\rangle_1\right] \tag{14}$$

We return to the partition function with (10) in mind:

$$\begin{aligned} Z &= \int DX(\boldsymbol{r})e^{H_0(X(\boldsymbol{k}))}\left\langle\exp\left[\frac{2y'}{a^2}I\right]\right\rangle_1 \\ &= \int DX(\boldsymbol{r})e^{H_0(X(\boldsymbol{k}))}\exp\left[\left(\frac{2y'}{a^2}\right)\langle\langle I\rangle\rangle_1 + \frac{1}{2}\left(\frac{2y'}{a^2}\right)^2\langle\langle I^2\rangle\rangle_1\right] \\ &= \int DX(\boldsymbol{r})\exp\left[H_0(X(\boldsymbol{k})) + \left(\frac{2y'}{a^2}\right)\langle\langle I\rangle\rangle_1 + \frac{1}{2}\left(\frac{2y'}{a^2}\right)^2\langle\langle I^2\rangle\rangle_1\right] \end{aligned} \tag{15}$$

We retrieve our Hamiltonian from the argument in the partition function:

$$H = H_0(X(\boldsymbol{k})) + \left(\frac{2y'}{a^2}\right)\int d^2r\langle\cos Z(\boldsymbol{r})\rangle_1 + \frac{1}{2}\left(\frac{2y'}{a^2}\right)^2\int d^2rd^2r'\left[\langle\cos Z(\boldsymbol{r})\cos Z(\boldsymbol{r'})\rangle_1 - \langle\cos Z(\boldsymbol{r})\rangle_1\langle\cos Z(\boldsymbol{r'})\rangle_1\right] \tag{16}$$

We will attempt to evaluate the first-order term in (16):

$$\begin{aligned} \langle\cos Z(\boldsymbol{r})\rangle_1 &= \langle\cos(X(\boldsymbol{r}) + Y(\boldsymbol{r}))\rangle_1 \\ &= \langle\cos X(\boldsymbol{r})\cos Y(\boldsymbol{r}) + \sin X(\boldsymbol{r})\sin Y(\boldsymbol{r})\rangle_1 \\ &= \langle\cos X(\boldsymbol{r})\cos Y(\boldsymbol{r})\rangle_1 + \langle\sin X(\boldsymbol{r})\sin Y(\boldsymbol{r})\rangle_1 \\ &= \cos X(\boldsymbol{r})\langle\cos Y(\boldsymbol{r})\rangle_1 + \sin X(\boldsymbol{r})\langle\sin Y(\boldsymbol{r})\rangle_1 \end{aligned} \tag{17}$$

Keep in mind the following for $X \sim \mathcal{N}(0, \sigma^2)$:

$$\langle x^n \rangle = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \sigma^n (n-1)!! & \text{if } n \text{ is even} \end{cases} \tag{18}$$

We investigate the terms in (17):

$$\begin{aligned} \langle \cos Y \rangle_1 &= \left\langle 1 - \frac{Y^2}{2!} + \frac{Y^4}{4!} - ... \right\rangle_1 \\ &= \langle 1 \rangle_1 - \frac{1}{2}\langle Y^2 \rangle_1 + \frac{1}{24}\langle Y^4 \rangle_1 - ... \\ &= 1 - \frac{1}{2}\langle Y^2 \rangle_1 + \frac{3}{24}\langle Y^2 \rangle_1^2 - ... \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[ -\frac{1}{2}\langle Y^2 \rangle_1 \right]^n \\ &= \exp\left( -\frac{1}{2}\langle Y^2 \rangle_1 \right) \end{aligned} \tag{19}$$

The 3rd equality holds because $Y(\boldsymbol{r})$ is a Gaussian with zero mean. Similarly:

$$\begin{aligned} \langle \sin Y \rangle_1 &= \left\langle Y - \frac{Y^3}{3!} + \frac{Y^5}{5!} - ... \right\rangle_1 \\ &= \langle Y \rangle_1 - \frac{1}{6}\langle Y^3 \rangle_1 + \frac{1}{120}\langle Y^5 \rangle_1 - ... \\ &= 0 \end{aligned} \tag{20}$$

Remember that odd-numbered moments vanish when the mean is zero. Now we seek the find the average of $Y^2$ as in (19). We know this to be equivalent to the variance of $Y$ if $\langle Y \rangle_1$ is 0. Note Eq. 2.4 from the paper:

$$H_1(Y(\boldsymbol{k})) = -\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \left[ \frac{Jk^2}{\psi(k)} Y(\boldsymbol{k}) Y(-\boldsymbol{k}) + E_1(k) \right] \tag{21}$$

We can retrieve the variance from this expression because the distribution of a Gaussian variable is of the form:

$$\rho_Y(y) \sim \exp\left[ -\frac{(y-\mu)^2}{2\sigma^2} \right] \tag{22}$$

Therefore:

$$\begin{aligned}
\langle Y^2\rangle_1 &= \int DY(\boldsymbol{r})Y^2 \exp\left[H_1\right] \\
&= \int \frac{d^2k}{(2\pi)^2}\frac{\psi(k)}{Jk^2} \\
&= \frac{1}{(2\pi)^2 J}\int_0^{2\pi} d\phi \int_0^{\infty} dk \frac{\psi(k)}{k} \\
&= -\frac{\varepsilon}{2\pi J}\int_0^{\infty} \frac{d\phi(k)}{dk} \\
&= -\frac{\varepsilon}{2\pi J}\left[\phi(\infty)-\phi(0)\right] \\
&= \frac{\varepsilon}{2\pi J}
\end{aligned} \tag{23}$$

where we have used Eq. 2.5 from the article to relate the cut-off functions and we know the cut-off functions must go to zero at infinity and to unity at zero.

Our result for the first order integrand is then:

$$\begin{aligned}
\langle \cos(X(\boldsymbol{r})+Y(\boldsymbol{r}))\rangle_1 &= \cos X(\boldsymbol{r}) \exp\left(-\frac{1}{2}\langle Y^2\rangle_1\right) \\
&= \cos X(\boldsymbol{r}) \exp\left(-\frac{\varepsilon}{4\pi J}\right) \\
&\approx \cos X(\boldsymbol{r}) \left[1-\frac{\varepsilon}{4\pi J}\right]
\end{aligned} \tag{24}$$