Physics 5153 Classical Mechanics
The Hamiltonian and Phase Space

1 Introduction

One of the reason to take different approaches to a given subject is see what new insights can be had. In this lecture, we will look at the concept of phase space. This concept allows us to get a global prospective on any system and allows a simple path to statistical mechanics, where one has to treat the properties of the system statistically.

When we used the Lagrangian, we defined the system in configuration space. The Lagrangian was defined in terms of the spatial coordinates and their derivatives, and the equations of motion are second order differential equations in the coordinates. This defines the configuration of the system. In the Hamiltonian approach, the Hamiltonian is defined in terms of the spatial coordinates and momenta, and the equations of motion are first order differential equations in the coordinates and momenta. This defines phase space with a total of $2n$ coordinates.

To see the advantages of phase space, we will start with an example. We will then look at the general properties of phase space.

1.1 Example

Let's consider a bead that is free to move inside a hollow ring that is rotating about a vertical axis at an angular rate ω as in Fig fig:ring. Rotation of the hollow ring is driven by an external source that keeps the angular velocity (ω) constant. The Lagrangian for this system in Cartesian coordinates is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \tag{1}$$

where the origin of the coordinate system is at the center of the ring. Since the system has only one degree of freedom, the Lagrangian can be written in terms of one variable, we select this the be the angle θ shown in the figure. The transformations to this generalized coordinate are

$$\left.\begin{aligned} x &= \ell \sin\theta \cos\omega t \\ x &= \ell \sin\theta \sin\omega t \\ z &= -\ell\cos\theta \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} \dot{x} &= \ell\left(\dot{\theta}\cos\theta\cos\omega t - \omega\sin\theta\sin\omega t\right) \\ \dot{y} &= \ell\left(\dot{\theta}\cos\theta\sin\omega t + \omega\sin\theta\cos\omega t\right) \\ \dot{z} &= \ell\dot{\theta}\sin\theta \end{aligned}\right. \tag{2}$$

The primary reason for starting with Cartesian coordinates is to show explicitly that the constraints are scleronomic (time dependent) and therefore the Hamiltonian will not be the total energy. In this case it is given by $H = T_2 - T_0 + V$, the total energy would be $H = T + V$. Carrying through the transformation, the Lagrangian is

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2\omega^2\sin^2\theta + mg\ell\cos\theta \tag{3}$$

We will proceed to calculate the Hamiltonian using the standard procedure. First we need the canonical momentum, which is

$$p_\theta = m\ell^2\dot{\theta} \tag{4}$$

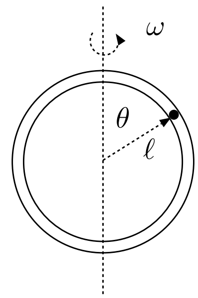

Figure 1: Rotating hollow ring with particle of mass m free to slide within hollow region. The contact between the ring and particle is massless.

Next, the Hamiltonian is given by

$$H = p_\theta \dot{\theta} - L = \frac{p_\theta^2}{2m\ell^2} - \frac{1}{2} m\ell^2\omega^2 \sin^2\theta - mg\ell\cos\theta \tag{5}$$

Therefore as we stated above, the Hamiltonian is not the total energy, but it is a constant of the motion as long as ω is constant.

Before we proceed to determine the motion of the bead, we will interpret the Hamiltonian. The work done by the external source to keep the ring rotating is

$$\delta W = N d\phi = \frac{dJ}{dt} d\phi = \omega dJ \tag{6}$$

where N is the applied torque, and J is the angular momentum about the vertical axis. The work done is equal to the change in energy of the system

$$dE = \omega dJ \quad \Rightarrow \quad d(E - \omega J) = J d\omega = 0 \tag{7}$$

where the last equality is due to ω being constant in this problem. The total energy for the bead is

$$E = \frac{p_\theta^2}{2m\ell^2} + \frac{1}{2} m\ell^2\omega^2 \sin^2\theta + mg\ell\cos\theta \tag{8}$$

The angular momentum about the vertical axis is

$$J = m\ell^2\omega^2 \sin^2\theta \tag{9}$$

which leads to

$$E - \omega J = \frac{p_\theta^2}{2m\ell^2} - \frac{1}{2} m\ell^2\omega^2 \sin^2\theta - mg\ell\cos\theta \tag{10}$$

which is the Hamiltonian. The Hamiltonian in this case is the difference in total energy and energy supplied by an external source.

To determine the motion, we will examine what happens for small displacements about $\theta = 0$. The Hamiltonian can be treated as the sum of the kinetic energy plus an effective potential. The effective potential can be approximated as

$$V = -\frac{1}{2} m\ell^2\omega^2 \sin^2\theta - mg\ell\cos\theta \approx \frac{1}{2} mg\ell \left[1 - (\omega^2\ell/g)\right] \theta^2 \tag{11}$$

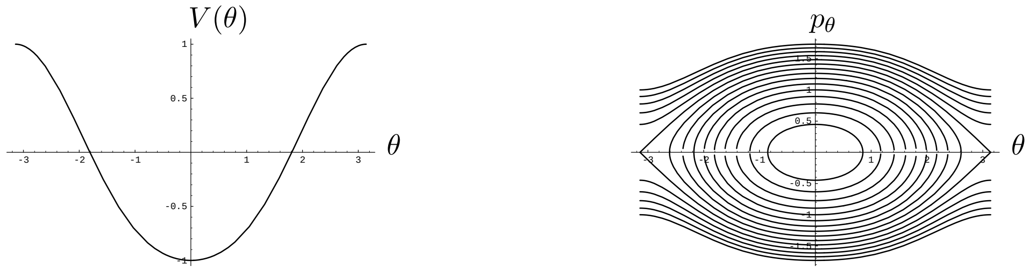

Figure 2: The figure on the left shows the effective potential for $\omega^2 < g/\ell$. The figure on the right is the phase space plot of the motion for the effective potential.

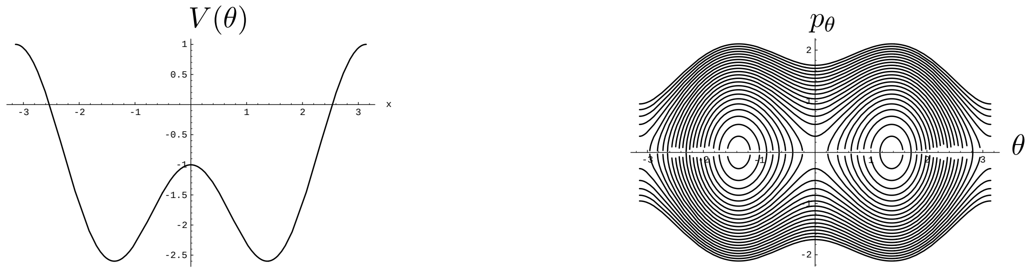

Figure 3: The figure on the left shows the effective potential for $\omega^2 > g/\ell$. The figure on the right is the phase space plot of the motion for the effective potential.

which is in the form of a harmonic oscillator potential. Notice that the stiffness for this potential can be either positive or negative depending on the values of ω. For $\omega^2 < g/\ell$ the stiffness is negative and there is a minimum at $\theta = 0$. If $\omega^2 > g/\ell$ then the stiffness is positive and a local maximum exists at $\theta = 0$. If we view the Hamiltonian in phase space (θ, p_θ), we find that extrema occur when

$$\dot{p}_q = -\frac{\partial H}{\partial \theta} = \frac{p_\theta}{m\ell^2} = 0 \quad \Rightarrow \quad p_\theta = 0 \tag{12}$$
$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = mg\ell \sin\theta \left[1 - (\omega^2 \ell/g)\cos\theta\right] 0 \quad \Rightarrow \quad \sin\theta = 0 \quad \text{and} \quad \cos\theta = \frac{g}{\ell\omega^2}$$

The second solution for θ only gives a real solution if $\omega^2 > g/\ell$. Therefore, we come to the same conclusion as before. We can now solve for p_θ and plot it verses θ for different values of h. Figure 4 shows the minimum and maximum of the effective potential, and where the minima bifurcate giving two minima. Figures 2 and 3 are the phase space plots showing the motion of the particle.

1.2 Phase Space

The concept of phase space and phase space plots provide a significant amount of information that can not be directly found using configuration space. For any point in configuration space, multiple trajectories are allowed since one point can correspond to multiple velocities can pass through it. On the other hand, each point in phase space is unique. Each corresponds to a single trajectory.

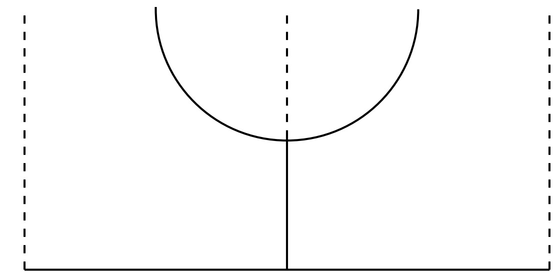

Figure 4: A plot of the extrema of the effective potential. The solid lines are minima, while the dashed lines are maxima or inflection points.

This can be seen as follows. For a conservative system we have equations of the form

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\} \quad \Rightarrow \quad \dot{\vec{\mathbf{x}}} = \vec{\mathbf{X}}(\vec{\mathbf{x}}) \tag{13}$$

where $\vec{\mathbf{x}}$ corresponds to the n position coordinates and the n momentum coordinates. This equation states that the tangent to a given phase point is unique, since the tangent depends only on the position of the point[1].

From Figs. 2 and 3, we see that the density of contours varies similar in form to contours of a potential or fluid flow. Let's consider a holonomic system. Suppose we follow a group of phase points as they describe trajectories in a $2n$ dimensional phase space. We can think of the points within a small volume element $dV = dq_1 \cdots dq_n dp_1 \cdots dp_n$ as constituting the moving particles of a fluid known as a phase fluid. The density of this fluid is given as the number of points in a given volume

$$\rho = \frac{dN}{dV} \tag{14}$$

where we define the density at a given point and region of phase space G_1. At some later time, the Hamiltonian transforms the fluid into a region G_2 (see Fig. **??**). According to Liouville's theorem, the density remains the same as long as the boundary obeys the canonical equations of motion.

To prove this, let's consider a an infinitesimal volume in phase space, and consider the number of points entering and leaving the volume per unit time (see Fig. 5). The number entering the volume per unit time through the face AD is

$$\rho \dot{q}_i dp_i dV_i \quad \text{where} \quad dV_i = \prod_{j \neq i}^{n} dp_j dq_j \tag{15}$$

The number of phase space point leaving the surface BC is given by

$$\left(\rho \dot{q}_i + \frac{\partial}{\partial q_i}(\rho \dot{q}_i) dq_i\right) dp_i dV_i \tag{16}$$

[1] By position we mean position in phase space, which is given by spatial position and momentum.

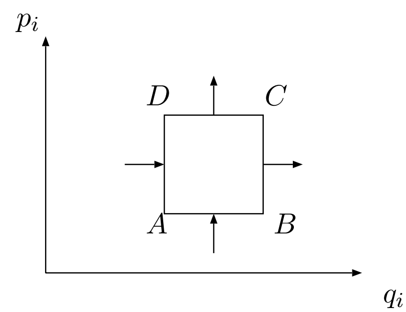

Figure 5: Flow through 4 sufraces in a $2n$ dimensional phase space volume.

The same can be done on the remaining surfaces. The number entering AB is

$$\rho\dot{p}_i dq_i dV \tag{17}$$

and leaving DC is

$$\left(\rho\dot{p}_i + \frac{\partial}{\partial p_i}(\rho\dot{p}_i)dp_i\right) dq_i dV \tag{18}$$

Next we add all the terms together, sum over all i and realize that the the sum is the change in density with respect to time

$$\frac{\partial\rho}{\partial t} = -\sum_{i=1}^{n}\left(\frac{\partial}{\partial q_i}(\rho\dot{q}_i) + \frac{\partial}{\partial p_i}(\rho\dot{p}_i)\right) \tag{19}$$

This has the form of a continuity equation

$$\boldsymbol{\nabla}\cdot(\rho\dot{\vec{\mathbf{x}}}) + \frac{\partial\rho)}{\partial t} = 0 \tag{20}$$

The phase velocity $\vec{\mathbf{v}}$ of a fluid particle is given in terms of its $2n$ components $(\dot{q}_i, \dot{p}_i)$. It can be expressed as functions of the q's, p's, and t by the canonical equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{21}$$

This expression can be expanded out to give

$$\sum_{i=1}^{n}\left(\frac{\partial\rho}{\partial q_i}\dot{q}_i + \rho\frac{\partial\dot{q}_i}{\partial q_i} + \frac{\partial\rho}{\partial p_i}\dot{p}_i + \rho\frac{\partial\dot{p}_i}{\partial p_i}\right) + \frac{\partial\rho}{\partial t} = 0 \tag{22}$$

Using the canonical equations of motion, we get

$$\begin{array}{l}\dfrac{\partial\dot{q}_i}{\partial q_i} = \dfrac{\partial^2 H}{\partial q_i\partial p_i}\\ \dfrac{\partial\dot{p}_i}{\partial p_i} = -\dfrac{\partial^2 H}{\partial q_i\partial p_i}\end{array} \quad\Rightarrow\quad \frac{\partial\dot{q}_i}{\partial q_i} + \frac{\partial\dot{p}_i}{\partial p_i} = 0 \tag{23}$$

leading to

$$\sum_{i=1}^{n}\left(\frac{\partial\rho}{\partial q_i}\dot{q}_i + \frac{\partial\rho}{\partial p_i}\dot{p}_i\right) + \frac{\partial\rho}{\partial t} = 0 \quad \Rightarrow \quad \frac{d\rho}{dt} = 0 \tag{24}$$

The space space density is conserved.