PROBLEM 1: Eigenvalue Equation and Time Evolution

The Hamiltonian for a certain three-level system is represented by the matrix

$$H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix},$$

where a, b, and c are real numbers and $a - c \neq \pm b$. $\Rightarrow$ $c \neq a \mp b$

(a) Find the eigenvalues E_n and normalized eigenvectors $|E_n\rangle, n = 1, 2, 3$ of H. [4 points]

(b) If the system starts out in the state

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

what is $|\psi(t)\rangle$? [3 points]

(c) If the system starts out in the state

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

what is $|\psi(t)\rangle$? [3 points]

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Quantum #1

a) $H = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{bmatrix}$

* To find eigenvalues, we solve characteristic equation

$$\det(H - \lambda \mathbb{I}) = 0$$

$$\begin{vmatrix} a-\lambda & 0 & b \\ 0 & c-\lambda & 0 \\ b & 0 & a-\lambda \end{vmatrix} = a-\lambda\left[(c-\lambda)(a-\lambda) - 0\right] - 0\left[0(a-\lambda) - 0(b)\right] + b\left[0 - b(c-\lambda)\right]$$

$$= (a-\lambda)^2(c-\lambda) - b^2(c-\lambda)$$

$$= (c-\lambda)\left[(a-\lambda)^2 - b^2\right]$$

$$= (c-\lambda)(a-\lambda+b)(a-\lambda-b)$$

$$\hookrightarrow \boxed{\lambda = c,\ a-b,\ a+b}$$

* To find eigenvectors, we solve $H\vec{v} = \lambda\vec{v}$

$$\Rightarrow \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_3 \\ cx_2 \\ bx_1 + ax_3 \end{bmatrix}$$

* Case $\lambda = c$

$$ax_1 + bx_3 = cx_1$$
$$cx_2 = cx_2$$
$$bx_1 + ax_3 = cx_3$$

$\hookrightarrow$ $b = 0$

$c = 1 \neq a$

$$\Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Case $\lambda = a-b$

$$ax_1 + bx_3 = (a-b)x_1$$
$$cx_2 = (a-b)x_2$$
$$bx_1 + ax_3 = (a-b)x_3$$

$b = 0,\ c \neq a$

$\hookrightarrow x_2 = 0$

$bx_3 = -bx_1$

$bx_1 = -bx_3$

$$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case $\lambda = a+b$

$$ax_1 + bx_3 = (a+b)x_1$$
$$cx_2 = a+b\,x_2$$
$$bx_1 + ax_3 = (a+b)x_3$$

$b = 0,\ c \neq a$

$\hookrightarrow x_2 = 0$

$bx_3 = bx_1$

$bx_1 = bx_3$

$$v_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ +1 \end{bmatrix}$$

#1 (cont)

b) $U(t,0) = \exp\left[-\frac{i}{\hbar} H t\right]$

$$\Rightarrow |\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle$$

$$= e^{-iHt/\hbar}\,|\psi(0)\rangle$$

$\ast$ substituting $|\psi(0)\rangle = |c\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$= e^{-iHt/\hbar}\,|c\rangle$$

$$= e^{-ict/\hbar}\,|c\rangle$$ (after taylor expansion to act out operator)

c) $|\psi(0)\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \sqrt{2}\left(|a+b\rangle - |a-b\rangle\right)$ where $|a+b\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $|a-b\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$= \sqrt{2}$$

$$\hookrightarrow |\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle$$

$$= e^{-iHt/\hbar}\left(\sqrt{2}\,|a+b\rangle - \sqrt{2}\,|a-b\rangle\right)$$

$$= \sqrt{2}\left[e^{-i(a+b)t/\hbar}\,|a+b\rangle - e^{-i(a-b)t/\hbar}\,|a-b\rangle\right]$$ (taylor expand exponential to act operator as above)