PROBLEM 2: Dirac Notation in Quantum Mechanics

Consider the kets $|a_n\rangle$ as the eigenstates of an observable operator $\mathbf{A}$

$$\mathbf{A}|a_n\rangle = a_n|a_n\rangle\,.$$

Assume that $|a_n\rangle$ form a discrete orthonormal basis in the vector space. Define an operator $U(m,n)$ as

$$U(m,n) = |a_m\rangle\langle a_n|.$$

(a) Show that $U(m,n)$ is an Hermitian operator. Calculate the commutator $[A, U(m,n)]$. [2 Points]

(b) For a generic operator with matrix elements $B_{mn} = \langle a_m|B|a_n\rangle$, show that

$$B = \sum_{mn} B_{mn} U(m,n).$$

[2 Points]

(c) Assume the Hamiltonian of a three-level system

$$\mathbf{H} = H_{12}U(1,2) + H_{21}U(2,1) + H_{23}U(2,3) + H_{32}U(3,2)$$

where $H_{12} = H_{23}$, and $H_{21} = H_{32}$ are complex numbers with dimension of energy. Find the eigenvectors and the eigenvalues of the Hamiltonian in the $|a_n\rangle$ basis. [4 Points]

(d) Assuming the Hamiltonian above, and $n = 1,2,3$, find the condition where the observable operator A is time independent. [2 Points]

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Quantum #2

a) * The condition for Hermiticity is $A^\dagger = A$

$$U(m,n)^\dagger = \left[|a_m\rangle\langle a_n| \right]^\dagger$$

$$= |a_m\rangle^\dagger \langle a_n|^\dagger$$

$$= |a_n\rangle\langle a_m|$$

$$= U(n,m) \quad \text{as expected}$$

* To evaluate the commutator, we act it upon the state $|a_n\rangle$

$$\Rightarrow [A, U(m,n)]|a_n\rangle = AU(m,n) - U(m,n)A\,|a_n\rangle$$

$$= AU(m,n)|a_n\rangle - U(m,n)A|a_n\rangle$$

$$= A\,|a_m\rangle\langle a_n|a_n\rangle - |a_m\rangle\langle a_n|A|a_n\rangle$$

$$= A\,|a_m\rangle - a_n|a_m\rangle\langle a_n|a_n\rangle$$

$$= a_m|a_m\rangle - a_n|a_m\rangle$$

$$= (a_n - a_m)|a_n\rangle \quad \text{(indexes arbitrary, so we switch order)}$$

$$\hookrightarrow [A, U(m,n)] = a_n - a_m$$

b) We want to show: $B = \sum_{mn} B_{mn} U(m,n)$ where $B_{mn} = \langle a_m|B|a_n\rangle$

$$\hookrightarrow \sum_{mn} B_{mn} U(m,n) = \sum_{mn} \langle a_m|B|a_n\rangle |a_m\rangle\langle a_n|$$

* To determine matrix elements of B, we use completeness relation

$$B = \sum_{m,n} |a_m\rangle\langle a_m|B|a_n\rangle\langle a_n|$$

$$= \sum_{m,n} \langle a_m|B|a_n\rangle |a_m\rangle\langle a_n|$$

$$\Rightarrow B = \sum_{mn} B_{mn} U(m,n)$$

#2 (cont.)

c) Given $H = H_{12}\,U(1,2) + H_{21}\,U(2,1) + H_{23}\,U(2,3) + H_{32}\,U(3,2)$, we can rewrite H as:

$$
H \doteq
\begin{array}{c}
\begin{array}{ccc} |a_1\rangle & |a_2\rangle & |a_3\rangle \end{array} \\
\begin{bmatrix} 0 & H_{12} & 0 \\ H_{21} & 0 & H_{23} \\ 0 & H_{32} & 0 \end{bmatrix}
\end{array}
$$

$$
0 = \begin{vmatrix} -\lambda & H_{12} & 0 \\ H_{21} & -\lambda & H_{23} \\ 0 & H_{32} & -\lambda \end{vmatrix}.
$$

$$0 = -\lambda(\lambda^2 - H_{23}H_{32}) - H_{12}(-H_{21}\lambda)$$

$$= -\lambda^3 + H_{23}H_{32}\lambda + H_{12}H_{21}\lambda$$

$$= -\lambda^3 + 2H_{12}H_{21}\lambda$$

$$= -\lambda(\lambda^2 - 2H_{12}H_{21})$$

$$\hookrightarrow \lambda = 0,\ +\sqrt{2H_{12}H_{21}},\ -\sqrt{2H_{12}H_{21}}$$

* To find eigenvectors:

$$H\vec{v} = \lambda\vec{v}$$

$$
\begin{bmatrix} 0 & H_{12} & 0 \\ H_{21} & 0 & H_{12} \\ 0 & H_{21} & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\Rightarrow
\begin{aligned}
H_{12}x_2 &= \lambda x_1 \\
H_{21}x_1 + H_{12}x_3 &= \lambda x_2 \\
H_{21}x_2 &= \lambda x_3
\end{aligned}
$$

* for $\lambda = 0$

$$H_{12}x_2 = 0$$

$$H_{21}x_1 + H_{12}x_3 = 0$$

$$H_{21}x_2 = 0$$

* for $\lambda = \sqrt{2H_{12}H_{21}}$

$$H_{12}x_2 = \sqrt{2H_{12}H_{21}}\,x_1$$

$$H_{21}x_1 + H_{12}x_3 = \sqrt{2H_{12}H_{21}}\,x_2$$

$$H_{21}x_2 = \sqrt{2H_{12}H_{21}}\,x_3$$

* for $\lambda = -\sqrt{2H_{12}H_{21}}$

$$H_{12}x_2 = -\sqrt{2H_{12}H_{21}}\,x_1$$

$$H_{21}x_1 + H_{12}x_3 = -\sqrt{2H_{12}H_{21}}\,x_2$$

$$H_{21}x_2 = -\sqrt{2H_{12}H_{21}}\,x_3$$

$$\Rightarrow \vec{v} = \begin{bmatrix} H_{12} \\ 0 \\ H_{21} \end{bmatrix} \frac{1}{\sqrt{H_{12}H_{21}}}$$

$$= \frac{1}{\sqrt{H_{12}H_{21}}}\left(H_{12}|a_1\rangle + H_{21}|a_3\rangle\right)$$