PROBLEM 2: Oscillator Model of Angular Momentum

Arbitrary angular momentum can be constructed from spin-1/2. The latter can be described in terms of the Pauli matrices

$$\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}.$$

The construction of a general angular momentum can be done by introducing two sets of independent harmonic oscillators, in terms of creation ($a_\zeta^\dagger$) and annihilation (a_ζ) operators,

$$[a_+, a_-] = 0, \quad [a_+^\dagger, a_-^\dagger] = 0, \quad [a_\zeta, a_{\zeta'}^\dagger] = \delta_{\zeta,\zeta'},$$

with $\zeta, \zeta' = \pm$ indexing oscillators of type $\pm$. Now define

$$\mathbf{J} = \frac{\hbar}{2} a^\dagger \boldsymbol{\sigma} a,$$

where a is a two component operator,

$$a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}.$$

a) Given the form of the Pauli matrices, give the explicit form for J_x, J_y, J_z in terms of $a_\zeta^\dagger$ and a_ζ operators (2 Points).

b) Show that $J_\pm = J_x \pm iJ_y$ have particularly simple forms in terms of a_ζ and $a_\zeta^\dagger$ operators (1 Point).

c) Compute the commutator $[J_x, J_y]$. How is this generalized for the other components? (2 Points)

d) Show that

$$J^2 = J_z^2 + J_+J_- + i[J_x, J_y],$$

and then write this in terms of the number operators for the two harmonic oscillators,

$$n_+ = a_+^\dagger a_+, \quad n_- = a_-^\dagger a_-.$$

Show that this implies that the eigenvalues of J^2 are $j(j+1)\hbar^2$, where j is an integer or an integer plus $\frac{1}{2}$ (Hint: apply the J^2 operator in the two harmonic oscillator state $|n_+, n_-\rangle$) (3 Points).

e) Using the properties of the harmonic oscillators, show that the state in which J^2 has the eigenvalue $j(j+1)\hbar$ and $J_z = m\hbar$ can be constructed from the state in which both n_+ and n_- have the value zero, $|0\rangle$, by

$$|jm\rangle = \frac{(a_+^\dagger)^{j+m}}{\sqrt{(j+m)!}} \frac{(a_-^\dagger)^{j-m}}{\sqrt{(j-m)!}} |0\rangle.$$

(2 Points)

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Quantum #2

a) Given $\vec{J} = \frac{\hbar}{2} a^\dagger \vec{\sigma} a$ where $a = \langle a_+, a_- \rangle$

$$J_x = \frac{\hbar}{2} a^\dagger \sigma_x a$$
$$= \frac{\hbar}{2} [a_+^\dagger \; a_-^\dagger] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_+ \\ a_- \end{bmatrix}$$
$$= \frac{\hbar}{2} [a_+^\dagger \; a_-^\dagger] \begin{bmatrix} a_- \\ a_+ \end{bmatrix}$$
$$= \frac{\hbar}{2} (a_+^\dagger a_- + a_-^\dagger a_+)$$

$$J_y = \frac{\hbar}{2} a^\dagger \sigma_y a$$
$$= \frac{\hbar}{2} [a_+^\dagger \; a_-^\dagger] \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a_+ \\ a_- \end{bmatrix}$$
$$= \frac{\hbar}{2} [a_+^\dagger \; a_-^\dagger] \begin{bmatrix} -i a_- \\ i a_+ \end{bmatrix}$$
$$= \frac{\hbar}{2} (-i a_+^\dagger a_- + i a_-^\dagger a_+)$$

$$J_z = \frac{\hbar}{2} [a_+^\dagger \; a_-^\dagger] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_+ \\ a_- \end{bmatrix}$$
$$= \frac{\hbar}{2} [a_+^\dagger \; a_-^\dagger] \begin{bmatrix} a_+ \\ -a_- \end{bmatrix}$$
$$= \frac{\hbar}{2} [a_+^\dagger a_+ - a_-^\dagger a_-]$$

b) $J_\pm = J_x \pm i J_y$

$$\hookrightarrow J_+ = \frac{\hbar}{2} [a_+^\dagger a_- + a_-^\dagger a_+] + i \left(\frac{-i\hbar}{2} [a_+^\dagger a_- + a_-^\dagger a_+] \right)$$
$$= \frac{\hbar}{2} [a_+^\dagger a_- + a_-^\dagger a_+] + \frac{\hbar}{2} [a_+^\dagger a_- - a_-^\dagger a_+]$$
$$= \hbar [a_+^\dagger a_-]$$

$$\hookrightarrow J_- = \frac{\hbar}{2} [a_+^\dagger a_- + a_-^\dagger a_+] - i \left(\frac{-i\hbar}{2} [a_+^\dagger a_- - a_-^\dagger a_+] \right)$$
$$= \frac{\hbar}{2} [a_+^\dagger a_- + a_-^\dagger a_+] - \frac{\hbar}{2} [a_+^\dagger a_- - a_-^\dagger a_+]$$
$$= \hbar [a_-^\dagger a_+]$$

c) $[J_x, J_y] = J_x J_y - J_y J_x$

$$= \frac{i\hbar^2}{4} \left([a_+^\dagger a_- + a_-^\dagger a_+][a_-^\dagger a_+ - a_+^\dagger a_-] - [a_-^\dagger a_+ - a_+^\dagger a_-][a_+^\dagger a_- + a_-^\dagger a_+] \right)$$
$$= \frac{i\hbar^2}{4} \Big(a_+^\dagger a_- a_-^\dagger a_+ + \cancel{a_-^\dagger a_+ a_-^\dagger a_+} - a_+^\dagger \cancel{a_- a_+^\dagger a_-} - a_-^\dagger a_+ a_+^\dagger a_- - a_-^\dagger a_+ a_+^\dagger a_-$$
$$- \cancel{a_-^\dagger a_+ a_-^\dagger a_+} + a_+^\dagger \cancel{a_- a_+^\dagger} a_- + a_+^\dagger a_- a_-^\dagger a_+ \Big)$$
$$= \frac{i\hbar^2}{4} \left(2 a_+^\dagger a_- a_-^\dagger a_+ - 2 a_-^\dagger a_+ a_+^\dagger a_- \right)$$

#2 (cont.)

c) $[J_x, J_y] = \frac{i\hbar^2}{2}\left(a_+^\dagger \cancel{a_- a_-^\dagger}^{1} a_+ - a_-^\dagger \cancel{a_+ a_+^\dagger}^{1} a_-\right)$

$= \frac{i\hbar^2}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$

$= i\hbar J_z$

$\Rightarrow$ Angular momentum operators will generalize as $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$

d) $J^2 = J_x^2 + J_y^2 + J_z^2$

$= J_z^2 + (J_x^2 + J_y^2)$

$= J_z^2 + (J_x + iJ_y)(J_x - iJ_y) \underbrace{- iJ_yJ_x + iJ_xJ_y}$ eliminate cross terms in expansion

$= J_z^2 + J_+ J_- + i[J_x, J_y]$

* Rewriting this in terms of the oscillators

$= \frac{\hbar^2}{4}[a_+^\dagger a_+ - a_-^\dagger a_-]^2 + \hbar^2 a_+^\dagger a_- a_-^\dagger a_+ + i(i\hbar J_z)$

$= \frac{\hbar^2}{4}[a_+^\dagger a_+ a_+^\dagger a_+ - a_+^\dagger a_+ a_-^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_+ + a_-^\dagger a_- a_-^\dagger a_-] + \hbar^2 a_+^\dagger a_- a_-^\dagger a_+$

$- \frac{\hbar^2}{2}[a_+^\dagger a_+ - a_-^\dagger a_-]$

$= \frac{\hbar^2}{4}[a_+^\dagger a_+ - a_+^\dagger a_+ a_-^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_+ + a_-^\dagger a_-] + \hbar^2 a_+^\dagger a_+ - \frac{\hbar^2}{2}[a_+^\dagger a_+ - a_-^\dagger a_-]$

$= \frac{\hbar^2}{4}[n_+ - n_+n_- - n_-n_+ + n_-] + \hbar^2 n_+ - \frac{\hbar^2}{2}[n_+ - n_-]$

$= \frac{\hbar^2}{4}[3n_+ - n_+n_- - n_-n_+ - n_-]$

$= \ldots$??