Jan 2009

Problem 5: Measurement and Probability (10 points) 5

Consider the following two observables, H and C, whose representation in the unit basis $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$ is:

$$H = \hbar\omega \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

where:

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Assume that at time t=0 the ensemble of particles is in the state:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$$

The eigenvalues of H are given by $\lambda = 2, 1, -1$ with normalized eigenvectors given by $(1,1,1)/\sqrt{3}$, $(1,0,-1)/\sqrt{2}$ and $(1,-2,1)/\sqrt{6}$ respectively.

The eigenvalues of C are given by $\lambda = 1, 1, -1$ with normalized eigenvectors given by $(1,0,-1)/\sqrt{2}$, $(0,1,0)$ and $(1,0,1)/\sqrt{2}$ respectively.

a) What is the probability of measuring H and obtaining $E = \hbar\omega$? What state is the particle in after the measurement? (2 pts)

b) If one immediately measures C after the measurement of H in part b), what is the probability of obtaining $c = 1$? (1 pt)

c) What is the probability of measuring H first and getting $E = \hbar\omega$, then measuring C and getting $c = 1$, i.e. what is $P_{|\Psi(0)\rangle}(E = \hbar\omega, c = 1)$? (1 pt)

d) If the system is allowed to evolve in time after the measurement of H and before C is measured, will your answer to part c) change? Explain your reasoning. (1 pt)

e) With the ensemble of particles all in the original state: $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$, reverse the order of the above measurements and answer the same questions:

i) What is the probability of obtaining $c = 1$ if C is measured first? What state is the particle in after C is measured? (1 pt)

ii) If one immediately measures H after C is measured in part i), what is the probability of obtaining $E = \hbar\omega$? (1 pt) **(question continues on next page...)**

iii) What is the composite probability $P_{|\Psi(0)\rangle}(c = 1, E = \hbar\omega)$? (1 pt)

iv) If the system had been allowed to evolve in time after the measurement of C and before H is measured, would your answer to part ii) be different? Explain. (1 pt)

f) Are H and C compatible observables? Why?

Jan 2009

Quantum #5

$A^2 |\langle \psi | \varphi \rangle|^2$

Given! $H = \hbar\omega \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow$ $|\lambda_H = 2\rangle = \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle$

$|\lambda_H = 1\rangle = \frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle$

$|\lambda_H = -1\rangle = \frac{1}{\sqrt{6}}\langle 1, -2, 1\rangle$

$C = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \Rightarrow$ $|\lambda_C = 1, 1\rangle = \frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle$

$|\lambda_C = 1, 2\rangle = \langle 0, 1, 0\rangle$

$|\lambda_C = -1\rangle = \frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle$

$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle$

a) $P(H=1) = |\langle \lambda_H = 1 | H | \psi(t=0)\rangle|^2$

$= \left| \frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \right|^2$

$= \frac{1}{4} \left| \langle 1, 0, -1\rangle \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right|^2$

$= \frac{1}{4} |1|^2 = \frac{1}{4}$

$\Rightarrow$ The state after measurement is $|\lambda_H = 1\rangle = \frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle$ automatically b/c eigenvalues of H are non-degenerate

b) * From above, we know our starting state is: $|\lambda_H = 1\rangle = \frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle$

$P(C=1) = |\langle \lambda_C = 1, 1 | \lambda_H = 1\rangle|^2 + |\langle \lambda_C = 1, 2 | \lambda_H = 1\rangle|^2$

$= \left| \frac{1}{\sqrt{2}} [1 \ 0 \ -1] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \right|^2 + \left| [0 \ 1 \ 0] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \right|^2$

$= \left| \frac{1}{2} \cdot 2 \right|^2 + \left| \frac{1}{\sqrt{2}} \cdot 0 \right|^2$

$= 1$

c) $P(H=1, C=1) = P(H=1)P(C=1)$

$= \frac{1}{4} \cdot 1$

$= \frac{1}{4}$

#5 (cont.)

d) Evolving the system in time after measuring H will have no impact on the measurement of C b/c the time evolution operator is a function of H and the eigenstates of H are thus stationary states

e) $P(c=1) = |\langle \lambda_c = 1,1 | \psi(0) \rangle|^2 + |\langle \lambda_c = 1,2 | \psi(0) \rangle|^2$

$$= \left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \right|^2 + \left| \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \right|^2$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

* Our beginning state is either $|\lambda_c = 1,1\rangle$ or $|\lambda_c = 1,2\rangle$

$\Rightarrow P(H=1) = |\langle \lambda_H = 1 | \lambda_c = 1,1 \rangle|^2 + |\langle \lambda_H = 1 | \lambda_c = 1,2 \rangle|^2$

$$= \left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right|^2$$

$$= \left| \frac{1}{2} \cdot 2 \right|^2 + 0^2$$

$$= 1$$

$P(c=1, H=1) = P(H=1)\,P(c=1)$

$$= 1 \cdot 3/4$$

$$= 3/4$$

* Allowing the system to evolve in time b/w measuring C and H (in that order) will result in a change in the probability of finding $E = \hbar\omega$ as the two possible eigenstates of $\lambda_c = 1$ not <u>both</u> eigenstates of H, thus they are non-stationary + will be changed after being acted upon by the time evolution operator

f) * Observables are compatible if $[A, B] = 0$, and also if they have a common, complete set of eigenvectors. Since H and C <u>do not</u> share the same eigenbasis, they are not compatible