Problem 4: Clebsh-Gordon Coefficients

Consider a system with two distinguishable spinless particles with angular momentum $j_1 = 1$ and $j_2 = 1$. Suppose the system is prepared in a state with total angular momentum $j = 2$ and total angular momentum projection $m = m_1 + m_2 = 0$. The state in the total j basis $|j_1, j_2; j, m\rangle$ is

$$|\psi\rangle \equiv |1, 1; j = 2, m = 0\rangle.$$

a) Express $|\psi\rangle$ in terms of products of single particle states, namely in the direct product basis $|j_1 = 1, m_1\rangle|j_2 = 1, m_2\rangle$. (4 Points).

b) If the angular momentum projection of particle 1 is measured along the z direction, what is the probability of finding a non-zero result? (2 Points)

c) If $\mathbf{J}_i$ is the angular momentum operator of each particle ($i = 1, 2$), compute the expectation value of $\mathbf{J}_1 \cdot \mathbf{J}_2$ in the $|\psi\rangle$ state. (2 Points)

d) If the $|\psi\rangle$ state is rotated by an infinitesimal angle $\delta\theta$ around the x direction, compute the probability of measuring the $|1, 1; j = 2, m = 1\rangle$ state in leading order in $\delta\theta$. (2 Points)

Raising and lowering angular momentum operators:

$$J_{\pm}|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

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Quantum #4

* Two distinguishable spinless particles,

a) with both particles having $j_i = 1$, $m_i = \{-1, 0, 1\}$

$$\hookrightarrow |1,1, j=2, m=0\rangle = A|1, m_1=0\rangle \otimes |1, m_2=0\rangle + B|1, m_1=1\rangle \otimes |1, m_1=-1\rangle$$
$$+ C|1, m_1=-1\rangle \otimes |1, m_2=1\rangle$$

* To determine the values of A, B, C (the Clebsh - Gordon) coefficients, we must start in either the highest or lowest state for two distinguishable spinless particles and use raising/lowering operators to reach our known state

$$J_- = J_{1,-} + J_{2,-} = J_{1,-} \otimes \mathbb{I}_2 + \mathbb{I}_1 \otimes J_{2,-}$$
$$J_-|a,b\rangle = \hbar\sqrt{(j+m)(j-m+1)}\,|a, b-\hbar\rangle$$

* Our maximum state is: $|1,1,2,2\rangle = |1,1\rangle \otimes |1,1\rangle$, $\hbar = 1$

$$J_-|1,1;2,2\rangle = |1,1;2,1\rangle \cdot \sqrt{(2+2)(2-2+1)}$$
$$= 2|1,1;2,1\rangle$$

$$J_-|1,1\rangle \otimes |1,1\rangle = \sqrt{2}\,\hbar|1,0\rangle \otimes |1,1\rangle + \sqrt{2}\,|1,1\rangle \otimes |1,0\rangle$$

$$\Rightarrow |1,1;2,1\rangle = \frac{1}{\sqrt{2}}\left[|1,0\rangle \otimes |1,1\rangle\right] + \frac{1}{\sqrt{2}}\left[|1,1\rangle \otimes |1,0\rangle\right]$$

* Repeating

$$J_-|1,1;2,1\rangle = \sqrt{(2+1)(2-1+1)}\,|1,1;2,0\rangle$$
$$= \sqrt{6}\,|1,1;2,0\rangle$$

$$J_-\left[\frac{1}{\sqrt{2}}(|1,0\rangle \otimes |1,1\rangle) + \frac{1}{\sqrt{2}}(|1,1\rangle \otimes |1,0\rangle)\right] = \frac{1}{\sqrt{2}}\Big(\sqrt{(1+0)(1-0+1)}\,|1,-1\rangle \otimes |1,1\rangle$$
$$+ \sqrt{(1+1)(1-1+1)}\,|1,0\rangle \otimes |1,0\rangle$$
$$+ \sqrt{(1+1)(1-1+1)}\,|1,0\rangle \otimes |1,0\rangle$$
$$+ \sqrt{(1+0)(1-0+1)}\,|1,1\rangle \otimes |1,-1\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(\sqrt{2}\,|1,-1\rangle \otimes |1,1\rangle + 2\sqrt{2}\,|1,0\rangle \otimes |1,0\rangle + \sqrt{2}\,|1,1\rangle \otimes |1,-1\rangle\Big)$$

#4 (cont.)

a) $\Rightarrow \quad |1,1;2,0\rangle = \frac{1}{\sqrt{6}}|1,-1\rangle \otimes |1,1\rangle + \sqrt{\frac{2}{3}}\,|1,0\rangle \otimes |1,0\rangle + \frac{1}{\sqrt{6}}|1,1\rangle \otimes |1,1\rangle$

b) $J_{z,1}|1,1;2,0\rangle = J_{z,1}\left[\frac{1}{\sqrt{6}}|1,-1\rangle \otimes |1,1\rangle + \sqrt{\frac{2}{3}}|1,0\rangle \otimes |1,0\rangle + \frac{1}{\sqrt{6}}|1,1\rangle \otimes |1,1\rangle\right]$

$$\underline{P}(J_{z,1} \neq 0) = \langle 1,1;2,0|J_{z,1}|1,1;2,0\rangle$$

$$= \sum |c_n|^2$$

$$= \frac{1}{6}(-1) + \frac{1}{6}(1) + \frac{2}{3}(0)$$

$\hookrightarrow \frac{1}{3}$ overall, $\frac{1}{6}$ each for $J_{z,1} = \pm 1$

c) $J_1 \cdot J_2 = \left(J^2 - J_1^2 + J_2^2\right)\cdot\frac{1}{2}$

$\hookrightarrow$ From $J^2 = (J_1 + J_2)\cdot(J_1 + J_2)$

$$J^2 = J_1^2 + 2J_1 \cdot J_2 + J_2^2$$

$$\langle J_1 \cdot J_2 \rangle = \langle 1,1;2,0|J_1 \cdot J_2|1,1;2,0\rangle$$

$$= \langle 1,1;2,0|\tfrac{1}{2}(J^2 - J_1^2 - J_2^2)|1,1;2,0\rangle$$

$$= \frac{1}{2}(2^2 - 1^2 - 1^2)$$

$$= \frac{1}{2}(4-1-1)$$

$$= 1$$

d)