Aug 2008

Problem 3: The Harmonic Oscillator(10 Points)

A one dimensional harmonic oscillator has a potential given by

$$V(x) = m\omega^2 x^2/2.$$

where ω is the oscillator frequency and m is its mass. Derive all results.

a. Write the Schrodinger equation for a single particle in a one dimensional harmonic oscillator potential. **(1 Point)**

b. Consider the raising and lowering operators

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\hbar\omega}}$$

and

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\hbar\omega}},$$

respectively, where p is the momentum operator. If Ψ_E is an eigenvector of the Hamiltonian with energy eigenvalue E, find the energy eigenvalues of $a^\dagger\Psi_E$ and $a\Psi_E$. (You may need to use the fact that $[x,p] = i\hbar$). **(2 Points)**

c. Using the raising and lowering operators find the energy eigenvalues for a single particle in a one dimensional harmonic oscillator potential. **(2 Points)**

d. Find the normalized ground state wave function. **(2 Points)**

e. The harmonic oscillator models a particle attached to an ideal spring. If the spring can only be stretched, and not compressed, so that $V = \infty$ for $x < 0$, what will be the energy levels of this system? **(3 Points)**

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Quantum #3

a) The general form of the Schrödinger equation is: $i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$

For a 1-D harmonic oscillator, the equation becomes:

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \Psi \quad \text{(Time Dependent)}$$

$$E\psi = \frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \psi \quad \text{(Time Independent)}$$

b) Given: $a^+ = \sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{ip}{\sqrt{2m\hbar\omega}} \Rightarrow \sqrt{2m\hbar\omega}\, a^+ = m\omega x - ip$

$a = \sqrt{\frac{m\omega}{2\hbar}}\, x + \frac{ip}{\sqrt{2m\hbar\omega}} \Rightarrow \sqrt{2m\hbar\omega}\, a = m\omega x + ip$

* Rewriting the TISE in terms of momentum will allow us later to define the Hamiltonian in terms of raising/lowering operators

TISE: $\frac{p^2}{2m}\psi + \frac{1}{2}m\omega^2 x^2 \psi = E\psi \Rightarrow H|\psi\rangle = E|\psi\rangle$

* Remember, we want to solve: $H(a^+|\psi\rangle) = A(a^+|\psi\rangle)$

$H(a|\psi\rangle) = B(a|\psi\rangle)$

⇒ Rewriting our Hamiltonian:

$m\omega x = \sqrt{2m\hbar\omega}\, a^+ + ip$

$\sqrt{2m\hbar\omega}\, a - ip = \sqrt{2m\hbar\omega}\, a^+ + ip$

$\sqrt{2m\hbar\omega}\,(a - a^+) = 2ip$

$-i\sqrt{\frac{m\hbar\omega}{2}}\,(a - a^+) = p$

$\sqrt{2m\hbar\omega}\, a - m\omega x = ip$

$\sqrt{2m\hbar\omega}\, a - m\omega x = m\omega x - \sqrt{2m\hbar\omega}\, a^+$

$\sqrt{2m\hbar\omega}\,(a + a^+) = 2m\omega x$

$\sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+) = x$

* Substituting into our Hamiltonian we see:

$$H = \frac{\left(-i\sqrt{\frac{m\hbar\omega}{2}}[a - a^+]\right)^2}{2m}\psi + \frac{1}{2}m\omega^2\left(\sqrt{\frac{\hbar}{2m\omega}}[a + a^+]\right)^2$$

#3 (cont.)

b) $$H = \frac{-\frac{m\omega\hbar}{2}[a-a^{\dagger}]^2}{2m} + \frac{m\omega^2\hbar}{4m\omega}[a+a^{\dagger}]^2$$

$$= -\frac{\omega\hbar}{4}\left[aa - a^{\dagger}a - aa^{\dagger} + a^{\dagger}a^{\dagger}\right] + \frac{\hbar\omega}{4}\left[aa + a^{\dagger}a + aa^{\dagger} + a^{\dagger}a^{\dagger}\right]$$

$$= \frac{\hbar\omega}{2}\left[a^{\dagger}a + aa^{\dagger}\right]$$

* substituting $aa^{\dagger} = a^{\dagger}a + 1$ (from $[a, a^{\dagger}] = 1$)

$$= \hbar\omega\left(a^{\dagger}a + \tfrac{1}{2}\right)$$

* Using this, we can determine $[H, a^{\dagger}]$ and $[H, a]$ which will allow us to act H on $|\psi\rangle$ while maintaining $a^{\dagger}|\psi\rangle$ and $a|\psi\rangle$ kets

$$[H, a^{\dagger}] = [\hbar\omega(a^{\dagger}a + 1/2), a^{\dagger}]$$
$$= \hbar\omega(a^{\dagger}a + 1/2)a^{\dagger} - a^{\dagger}(\hbar\omega[a^{\dagger}a + \tfrac{1}{2}])$$
$$= \hbar\omega a^{\dagger}aa^{\dagger} + \cancel{\tfrac{1}{2}\hbar\omega a^{\dagger}} - \hbar\omega a^{\dagger}a^{\dagger}a - \cancel{\tfrac{1}{2}\hbar\omega a^{\dagger}}$$
$$= \hbar\omega(a^{\dagger}aa^{\dagger} - a^{\dagger}a^{\dagger}a)$$
$$= \hbar\omega[a^{\dagger}(a^{\dagger}a + 1) - a^{\dagger}a^{\dagger}a]$$
$$= \hbar\omega a^{\dagger}$$

$$[H, a] = [\hbar\omega(a^{\dagger}a + 1/2), a]$$
$$= \hbar\omega(a^{\dagger}a + \tfrac{1}{2})a - a(\hbar\omega[a^{\dagger}a + 1/2])$$
$$= \hbar\omega a^{\dagger}aa + \cancel{\tfrac{1}{2}\hbar\omega a} - \hbar\omega aa^{\dagger}a - \cancel{\tfrac{1}{2}\hbar\omega a}$$
$$= \hbar\omega(a^{\dagger}aa - aa^{\dagger}a)$$
$$= \hbar\omega(a^{\dagger}aa - (a^{\dagger}a + 1)a)$$
$$= \hbar\omega a$$

#3 (cont.)

b) * Applying these operators to the kets, we see:

$$H(a^\dagger|\psi\rangle) = (a^\dagger H + \hbar\omega a^\dagger)|\psi\rangle$$
$$= a^\dagger(E + \hbar\omega)|\psi\rangle$$
$$\hookrightarrow \boxed{A = E + \hbar\omega}$$

$$H(a|\psi\rangle) = (aH - \hbar\omega a)|\psi\rangle$$
$$= a(E - \hbar\omega)|\psi\rangle$$
$$\hookrightarrow \boxed{B = E - \hbar\omega}$$

c) Using the number operator N, where $N = a^\dagger a$ and $N|\psi_n\rangle = n|\psi_n\rangle$

$$\Rightarrow H|\psi_n\rangle = E_n|\psi_n\rangle$$
$$= \hbar\omega(a^\dagger a + 1/2)|\psi_n\rangle$$
$$= \hbar\omega(N + 1/2)|\psi_n\rangle$$
$$= \hbar\omega(n + 1/2)|\psi_n\rangle$$
$$\hookrightarrow \boxed{E_n = \hbar\omega(n + 1/2)}$$

d) To find the ground state wavefunction, we use the fact that: $a|\psi_0\rangle = 0$

$$\Rightarrow \left(\sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{iP}{\sqrt{2m\hbar\omega}}\right)\psi_0 = 0$$

$$\left(\sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i(-i\hbar\,\partial/\partial x)}{\sqrt{2m\hbar\omega}}\right)\psi_0 = 0$$

$$\sqrt{\frac{m\omega}{2\hbar}}\,x\,\psi_0 + \sqrt{\frac{\hbar}{2m\omega}}\,\frac{\partial\psi_0}{\partial x} = 0$$

$$\sqrt{\frac{\hbar}{2m\omega}}\,\frac{\partial}{\partial x}\psi_0 = -\sqrt{\frac{m\omega}{2\hbar}}\,x\,\psi_0$$

$$\frac{\partial}{\partial x}\psi_0 = \frac{-m\omega}{\hbar}\,x\,\psi_0$$

$$\frac{\partial\psi_0}{\psi_0} = \frac{-m\omega}{\hbar}\,x\,\partial x$$

#3 (cont.)

d) $\ln(\psi_0) = -\frac{m\omega}{2\hbar}x^2 + C$

$$\psi_0 = \exp\left[-\frac{m\omega}{2\hbar}x^2 + C\right]$$

$$\psi_0 = C\exp\left[-\frac{m\omega}{2\hbar}x^2\right]$$

* Checking the normalization

$$1 = \int_{-\infty}^{\infty} |\psi_0|^2 dx$$

$$= C^2 \int_{-\infty}^{\infty} \exp\left[-\frac{m\omega}{\hbar}x^2\right] dx$$

$$= C^2\left(\sqrt{\frac{\hbar\pi}{m\omega}}\right)$$

$$\sqrt{\frac{m\omega}{\hbar\pi}} = C^2$$

$$\Rightarrow C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$\Rightarrow$ our normalized wavefunction is: $\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{\hbar}x^2\right]$

e) * Our potential now becomes: $V(x) = \begin{cases} \infty, & x < 0 \\ \frac{1}{2}m\omega^2x^2, & x > 0 \end{cases}$