8-10

Problem 1: Matrix Mechanics (10 points):

Consider a particle with mass m and one continuous degree of freedom (spatial coordinate z with associated momentum operator $\hat{p}_z = -\imath\hbar\frac{d}{dz}$) and two discrete internal (pseudo-) spin states described by the Hamiltonian operator $\hat{H}$:

$$\hat{H} = \frac{\hat{p}_z^2}{2m}\hat{I} + \frac{\hbar k_{so}}{m}\hat{\sigma}_z\hat{p}_z + \frac{\Omega}{2}\hat{\sigma}_x. \tag{1}$$

Here, $\hat{I}$ is the identity operator in spin space, k_{so} and Ω are constants, and $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are the usual Pauli spin operators for a spin-1/2 particle. Different from what you might be used to, the Hamiltonian $\hat{H}$, Eq. (1), couples the spin and spatial degrees of freedom.

✓ a) (1 pt) What are the units of k_{so} and Ω? Explain your answer.

✓ b) (1 pt) Choose a convenient basis that spans the spin space and express the Hamiltonian operator $\hat{H}$ in this spin basis (you should obtain a 2×2 matrix). Explain your reasoning.

? c) (1 pt) Show that the operator $\hat{p}_z$ commutes with every element of the 2×2 matrix obtained in b).

✓ d) (3 pts) (Use your results from parts b) and c) to determine the eigen energies $E(p_z)$ of $\hat{H}$. Here, p_z is not an operator but a number.

✓ e) (1 pt) What happens to the eigen energies in the large p_z limit?

? f) (3 pts) Plot the eigen energies obtained in d) as a function of p_z for:
2? i) vanishing Ω
ii) large Ω
iii) small Ω

Explain what the terms "large" and "small" mean in this context, i.e., identify the quantity that Ω needs to be compared with in both cases.

Jan 2018

Quantum #1

a) All terms in the Hamiltonian must have units of energy $\frac{\text{mass} \cdot \text{distance}^2}{\text{time}^2}$

$$\Rightarrow [\Omega] = \frac{\text{mass} \cdot \text{distance}^2}{\text{time}^2}$$

$$[k_{so}] = \frac{1}{\text{distance}}$$

b) We want to use the $|\uparrow\rangle$, $|\downarrow\rangle$ basis

$$\Rightarrow H = \begin{bmatrix} \frac{p_z^2}{2m} & 0 \\ 0 & \frac{p_z^2}{2m} \end{bmatrix} + \begin{bmatrix} \frac{\hbar k_{so}}{m} p_z & 0 \\ 0 & -\frac{\hbar k_{so} p_z}{m} \end{bmatrix} + \begin{bmatrix} 0 & \frac{\Omega}{2} \\ \frac{\Omega}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{p_z^2}{2m} + \frac{\hbar k_{so} p_z}{m} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \frac{p_z^2}{2m} - \frac{\hbar k_{so} p_z}{2m} \end{bmatrix}$$

c) $[p_z, p_z] = 0$ identically. Therefore $[p_z, p_z^2] = 0$ by application of $[A, BC]$ where all 3 terms are p_z. Also $[p_z, \sigma_i] = 0$ b/c linear momentum and spin exist in different Hilbert spaces

d) We can determine the eigenenergies by solving the eigenvalue equation $\det(H - \lambda \mathbb{I}) = 0$

$$\begin{vmatrix} \frac{p_z^2}{2m} + \frac{\hbar k_{so} p_z}{m} - \lambda & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \frac{p_z^2}{2m} - \frac{\hbar k_{so} p_z}{m} - \lambda \end{vmatrix} = 0$$

$$\hookrightarrow 0 = \left(\frac{p_z^2}{2m} + \frac{\hbar k_{so} p_z}{m} - \lambda\right)\left(\frac{p_z^2}{2m} - \frac{\hbar k_{so} p_z}{m} - \lambda\right) - \frac{\Omega^2}{4}$$

$$= \frac{p_z^4}{4m^2} + \cancel{\frac{\hbar k_{so} p_z^3}{2m^2}} - \lambda \frac{p_z^2}{2m} - \cancel{\frac{\hbar k_{so} p_z^3}{2m^2}} - \frac{\hbar^2 k_{so}^2 p_z^2}{m^2} - \cancel{\frac{\hbar k_{so} p_z \lambda}{m}} - \lambda \frac{p_z^2}{2m}$$
$$+ \cancel{\frac{\hbar k_{so} p_z \lambda}{m}} + \lambda^2 - \frac{\Omega^2}{4}$$

#1 (cont.)

d) $$0 = \lambda^2 - \frac{p_z^2}{2m}\lambda + \left(\frac{p_z^4}{4m^2} - \frac{\Omega^2}{4}\right)$$

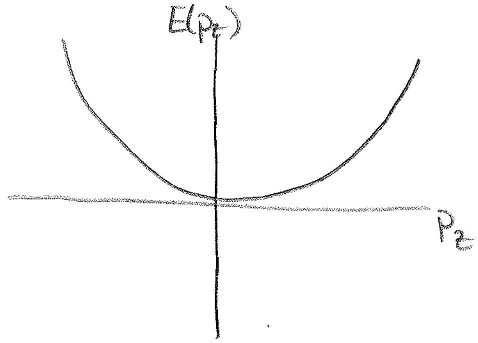
$$\hookrightarrow \lambda = \frac{\frac{p_z^2}{2m} \pm \sqrt{\frac{p_z^4}{m^2} - 4\left(\frac{p_z^4}{4m^2} - \frac{\Omega^2}{4}\right)}}{2}$$

$$= \frac{p_z^2}{4m} \pm \frac{\Omega}{2}$$

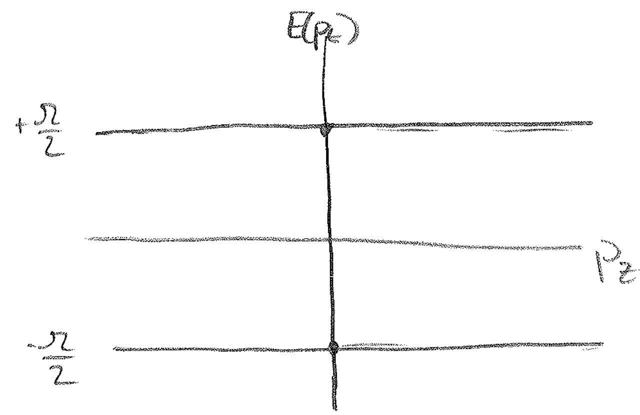
e) In the limit where p_z is large

$\lambda_\pm \approx \frac{p_z^2}{4m}$, eigenenergies will approach infinity as p increases

f) * For vanishing Ω



* For large Ω ($\Omega \gg p_z^2$)



* For small Ω ($\Omega \approx p_z^2$)