Problem 6: Harmonic Oscillators in 1D

A quantum harmonic oscillator is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

where p is momentum, x is position, m is mass, and ω is the oscillation frequency.

The Hamiltonian has the usual eigenstates and energies:

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle, \quad n = 0, 1, 2, ... \tag{2}$$

Let the system be perturbed by a potential in the form $V = Ax^2$ where A is a real constant.

(a) [2 pt] What is the change in the energy of the unperturbed eigenstates $|n\rangle$ to first order in A? Show your work.

(b) [2 pt] If the perturbation is time-dependent, $V(t) = A(t)x^2$, it can cause transitions between the harmonic oscillator states. To study these transitions, it is helpful to use the time-dependent expansion:

$$|\psi(t)\rangle = \sum_{n'} c_{n'}(t)e^{-\frac{i}{\hbar}E_{n'}t}|n'\rangle \tag{3}$$

The $c_{n'}(t)$ are time-dependent probability amplitudes for the states $|n'\rangle$ and the energies $E_{n'}$ are the unperturbed eigenenergies. Use the Schroedinger equation to show that the expansion amplitudes satisfy a set of coupled equations:

$$i\hbar\frac{\partial}{\partial t}c_n(t) = \sum_{n'} c_{n'}(t)e^{-\frac{i}{\hbar}(E_{n'}-E_n)t}\langle n|V(t)|n'\rangle \tag{4}$$

(c) [3 pt] Consider the case where the oscillator starts at time $t = 0$ in the ground state, $c_n(t=0) = \delta_{n,0}$. Use the result from (b) to write down the time dependence of the excited state probability amplitudes to first order in V, $c_n^{(1)}(t)$, $n > 0$. This will be an integral equation, as we have not yet defined $A(t)$.

Show that, to first order, there is a transition only to the $n = 2$ excited state.

(d) [3 pt] Finally, consider a time dependent perturbation with $A(t)$ of the form

$$A(t) = Ae^{-i\Omega t}e^{-\Gamma t} \tag{5}$$

Ω and Γ being real and positive.

Compute the probability that the $n = 2$ state is populated for $t \to \infty$, and explain the dependence of your result on Ω and Γ.

Note: In this problem, it is useful to use

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - i\frac{\lambda}{\hbar}p\right), \quad a = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} + i\frac{\lambda}{\hbar}p\right) \tag{6}$$

where $\lambda = \sqrt{\frac{\hbar}{m\omega}}$ is the length scale in the problem.

You do not need to derive the properties of these two operators, but you should state the results you are using.

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Quantum #6

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \qquad V' = Ax^2, \; A \in \mathbb{R}$$

$$H|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

a) $V' = Ax^2$

* but we know the raising/lowering operators

$$a^+ = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - \frac{i\lambda}{\hbar} p\right)$$

$$a = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} + \frac{i\lambda}{\hbar} p\right)$$

$$a + a^+ = \frac{2x}{\sqrt{2}\lambda} \rightarrow x^2 = \frac{\lambda^2}{2}(a + a^+)^2$$

$$\Delta E_n^{(1)} = \langle n^{(0)} | V' | n^{(0)} \rangle$$

$$= A\langle n| \frac{\lambda^2}{2}(aa + aa^+ + a^+a + a^+a^+)|n\rangle$$

$$= \frac{A\lambda^2}{2}\left[\xrightarrow{0}\cancel{\langle n|\sqrt{n(n-1)}|n-2\rangle} + \langle n|(n+1)|n\rangle + \langle n|n|n\rangle + \xrightarrow{0}\cancel{\langle n|\sqrt{(n+1)(n+2)}|n+2\rangle}\right]$$

$$= \frac{A\lambda^2}{2}(2n+1)$$

$$= \frac{A\hbar}{2m\omega}(2n+1)$$

b)