Problem 5: Magnetic Moments and Spin (10 pts)

Consider a spin 1/2 particle with a magnetic moment. We can write the interaction between the spin and an external magnetic field using the Hamiltonian:

$$H = -\gamma \vec{B} \cdot \vec{S} \tag{1}$$

where $\vec{B}$ is the external field, $\vec{S}$ is the spin operator for the particle, and γ is a real positive constant. In this problem, use the usual basis states that are eigenstates of S_z

$$S_z \chi_\pm = \pm\frac{\hbar}{2}\chi_\pm, \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

For this problem, assume the magnetic field lies in the x-z plane:

$$\vec{B} = B_x \hat{e}_x + B_z \hat{e}_z \tag{3}$$

(a) [1 pt] Solve for the eigenenergies for the Hamiltonian, showing your work. Explain the physics of your results.

(b) [2 pts] Any state of the spin can be written in the $\chi_\pm$ basis as:

$$\Psi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \tag{4}$$

Using the Hamiltonian, derive the first-order coupled differential equations that give the time dependence for $\alpha(t)$ and $\beta(t)$. In other words, derive the equations for $\dot{\alpha}(t)$ and $\dot{\beta}(t)$.

(c) [2 pts] Show that you can re-write your results from part (b) as two uncoupled second-order differential equations:

$$\begin{aligned} \ddot{\alpha}(t) &= -\frac{\gamma^2 B_T^2}{4}\alpha(t) \\ \ddot{\beta}(t) &= -\frac{\gamma^2 B_T^2}{4}\beta(t) \end{aligned} \tag{5}$$

where $B_T = \sqrt{B_x^2 + B_z^2}$ is the magnitude of the total magnetic field. How is this result related to what you found in part (a)?

Of course, the solutions to these equations are:

$$\begin{aligned} \alpha(t) &= C_1 \cos(\omega t) + C_2 \sin(\omega t) \\ \beta(t) &= C_3 \cos(\omega t) + C_4 \sin(\omega t) \end{aligned} \tag{6}$$

with $\omega = \frac{\gamma B_T}{2}$.

(d) [3 pts] Consider the situation where the spin is in the spin-up S_z state χ_+ at time $t = 0$. Using the boundary conditions at time $t = 0$, determine the values for the constants C_1, C_2, C_3, C_4 that will solve for the time-dependence of the state. Remember that the equations in part (c) are second-order, so you need two boundary conditions at $t = 0$ for each.

(e) [2 pt] Write down the time-dependent probabilities, $P_\pm$ of the spin being in the spin-up and spin-down S_z states. Show that your results are correct in the two cases where $B_x = 0$ and $B_z = 0$.

Jan 2016

Quantum #5

a) Given $H = -\gamma \vec{B} \cdot \vec{S}$

$$= -\gamma (B_x S_x + B_z S_z)$$

$$= -\gamma \left(B_x \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + B_z \hbar/2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

$$= -\gamma \frac{\hbar}{2} \begin{bmatrix} B_z & B_x \\ B_x & -B_z \end{bmatrix}$$

We can determine the energy eigenvalues by $\det(H - \lambda \mathbb{I}) = 0$

$$\begin{vmatrix} -\frac{\gamma\hbar B_z}{2} - \lambda & -\frac{\gamma\hbar B_x}{2} \\ -\frac{\gamma\hbar B_x}{2} & \frac{\gamma\hbar B_z}{2} - \lambda \end{vmatrix} = \left(-\frac{\gamma\hbar B_z}{2} - \lambda\right)\left(\frac{\gamma\hbar B_z}{2} - \lambda\right) - \frac{\gamma^2\hbar^2 B_x^2}{4}$$

$$= -\frac{\gamma^2\hbar^2 B_z^2}{4} \cancel{- \frac{\lambda\gamma\hbar B_z}{2} + \frac{\lambda\gamma\hbar B_z}{2}} + \lambda^2 - \frac{\gamma^2\hbar^2 B_x^2}{4}$$

$$0 = \lambda^2 - \frac{\gamma^2\hbar^2 (B_z^2 + B_x^2)}{4}$$

$$\lambda^2 = \frac{\gamma^2\hbar^2}{4} (B_z^2 + B_x^2)$$

$$\lambda = \pm \frac{\gamma\hbar}{2} (B_z^2 + B_x^2)^{1/2}$$

b) The time-dependent Schrödinger eqn states:

$$i\hbar \frac{\partial}{\partial t} \Psi = H\Psi$$

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} = -\frac{\gamma\hbar}{2} \begin{bmatrix} B_z & B_x \\ B_x & -B_z \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix}$$

$$i \frac{\partial \alpha}{\partial t} = \frac{-\gamma}{2} (B_z \alpha(t) + B_x \beta(t))$$

$$i \frac{\partial \beta}{\partial t} = \frac{-\gamma}{2} (B_x \alpha(t) - B_z \beta(t))$$

or

$$\dot{\alpha}(t) = \frac{i\gamma}{2} (B_z \alpha(t) + B_x \beta(t))$$

$$\dot{\beta}(t) = \frac{i\gamma}{2} (B_x \alpha(t) - B_z \beta(t))$$

#5 (cont.)

c) To get 2nd order differential equations, we take another set of time derivatives

$$\ddot{\alpha}(t) = \frac{i\gamma}{2}\left(B_z\,\dot{\alpha}(t) + B_x\,\dot{\beta}(t)\right)$$

$$\ddot{\beta}(t) = \frac{i\gamma}{2}\left(B_x\,\dot{\alpha}(t) - B_z\,\dot{\beta}(t)\right)$$

Substituting our first order differential equations into the above equations yield:

$$\ddot{\alpha}(t) = \frac{-\gamma^2}{4}\left(B_z\left[B_z\alpha(t) + B_x\beta(t)\right] + B_x\left[B_x\alpha(t) - B_z\beta(t)\right]\right)$$

$$\ddot{\beta}(t) = \frac{-\gamma^2}{4}\left(B_x\left[B_z\alpha(t) + B_x\beta(t)\right] - B_z\left[B_x\alpha(t) - B_z\beta(t)\right]\right)$$

Simplifying and letting $B_T^2 = B_x^2 + B_z^2$

$$\ddot{\alpha}(t) = \frac{-\gamma^2 B_T^2}{4}\,\alpha(t)$$

$$\ddot{\beta}(t) = \frac{-\gamma^2 B_T^2}{4}\,\beta(t)$$

d) If we are in the spin-up state at $t=0$

$$1 = C_1\cos(\omega t) + C_2\sin(\omega t) = \alpha(t)$$

$$0 = C_3\cos(\omega t) + C_4\sin(\omega t) = \beta(t)$$

$\Rightarrow$ From this, we immediately determine $C_1 = 1$, $C_3 = 0$ b/c $\cos(\omega t) = 1$ at $t=0$

Our other condition comes from the first order differential equations

$\hookrightarrow$ $\dot{\alpha}(0) = \frac{i\gamma}{2}B_z$ $\qquad$ $\dot{\beta}(0) = \frac{i\gamma}{2}B_x$

$$\dot{\alpha}(t) = -\omega C_1\sin(\omega t) + \omega C_2\cos(\omega t) \rightarrow \dot{\alpha}(0) = \omega C_2$$

$$\dot{\beta}(t) = -\omega C_3\sin(\omega t) + \omega C_4\cos(\omega t) \rightarrow \dot{\beta}(0) = \omega C_4$$

$$\Rightarrow C_2 = \frac{i\gamma}{2\omega}B_z = \frac{iB_z}{B_T}$$

$$C_4 = \frac{i\gamma}{2\omega}B_x = \frac{iB_x}{B_T}$$

#5 (cont.)

e) We now know our time-dependent initial state

$$|\psi\rangle = \begin{bmatrix} \cos(\omega t) + \frac{iB_z}{B_T}\sin(\omega t) \\ \frac{iB_x}{B_T}\sin(\omega t) \end{bmatrix}$$

$$P_{\pm} = |\langle \chi_{\pm} | \psi(t)\rangle|^2$$

$$P_{+} = |\langle \chi_{+} | \psi(t)\rangle|^2$$

$$= \left| [1 \ \ 0] \begin{bmatrix} \cos(\omega t) + \frac{iB_z}{B_T}\sin(\omega t) \\ \frac{iB_x}{B_T}\sin(\omega t) \end{bmatrix} \right|^2$$

$$= \left| \cos(\omega t) + \frac{iB_z}{B_z}\sin(\omega t) \right|$$

$$= \cos^2(\omega t) + \cancel{\frac{iB_z}{B_T}\sin(\omega t)\cos(\omega t)} - \cancel{\frac{iB_z}{B_T}\sin(\omega t)\cos(\omega t)} + \frac{B_z^2}{B_T^2}\sin^2(\omega t)$$

$$= \cos^2(\omega t) + \frac{B_z^2}{B_T^2}\sin^2(\omega t)$$

$$P_{-} = |\langle \chi_{-} | \psi(t)\rangle|^2$$

$$= \left| [0 \ \ 1] \begin{bmatrix} \cos(\omega t) + \frac{iB_z}{B_T}\sin(\omega t) \\ \frac{iB_x}{B_T}\sin(\omega t) \end{bmatrix} \right|^2$$

$$= \frac{B_x^2}{B_T^2}\sin^2(\omega t)$$

$$P_{+} + P_{-} = \cos(\omega t) + \frac{B_z^2}{B_T^2}\sin^2(\omega t) + \frac{B_x^2}{B_T^2}\sin^2(\omega t)$$

$$= \cos^2(\omega t) + \cancelto{1}{\frac{B_z^2 + B_x^2}{B_x^2 + B_z^2}}\sin^2(\omega t)$$

$$= \cos^2(\omega t) + \sin^2(\omega t)$$

$$= 1 \quad \Rightarrow \text{ Valid at all times}$$

* if $B_x = 0$

$P_+ = 1, \quad P_- = 0$

* if $B_z = 0$

$P_+ = \cos^2(\omega t)$

$P_- = \sin^2(\omega t)$