* Modified version of Sakurai 5.11

## Problem 5: Two Level Systems

Consider the Hamiltonian for a two-state system:

$$H = \begin{pmatrix} \epsilon & \lambda\Delta \\ \lambda\Delta & -\epsilon \end{pmatrix} \tag{1}$$

where $\lambda$ (a unitless parameter) determines the strength of the perturbation on the two-level system and $\epsilon$ and $\Delta$ are constants with the unit of energy.

The energy eigenvectors for the unperturbed Hamiltonian ($\lambda = 0$) are

$$\psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

(a) [2 pt] Solve for the energy eigenvalues $E_1$ and $E_2$ for the full Hamiltonian (for any $\lambda$). What is the functional form of the eigenenergies in the limits $\lambda \to 0$ and $\lambda \to \infty$?

(b) [2 pt] For the case that $\lambda|\Delta| \ll \epsilon$, solve for the energy eigenvalues to first order and second order in $\lambda$.

Compare these results with the exact results obtained in part (a) and show that they are in agreement.

(c) [1 pt] For the case that $\lambda|\Delta| \ll \epsilon$, what is the change in the unperturbed eigenstate $\psi_+$ to first order in $\lambda$?

(d) [2 pt] For the case that the unperturbed Hamiltonian is nearly degenerate, $\epsilon \ll \lambda|\Delta|$ show that the exact results obtained in part (a) agree with the results of applying first order degenerate perturbation theory with $\epsilon = 0$.

(e) [3 pts] For the case that $\epsilon \ll \lambda|\Delta|$, it would advantageous to use a different set of basis states to describe the system. Using basis states that are approximately eigenstates of the Hamiltonian for small $\epsilon$, determine the Hamiltonian matrix in this new basis. Show that the exact solutions for the eigenenergies are the same as in part (a) in this basis.

Aug 2013

# Quantum # 5

a) Given $H = \begin{bmatrix} \epsilon & \lambda\Delta \\ \lambda\Delta & -\epsilon \end{bmatrix}$ we want the energy eigenvalues

Using the eigenvalue equation $\det(H - a\mathbb{I}) = 0$

$$\begin{vmatrix} \epsilon - a & \lambda\Delta \\ \lambda\Delta & -\epsilon - a \end{vmatrix} = 0 = (\epsilon - a)(-\epsilon - a) - \lambda^2\Delta^2$$
$$= -\epsilon^2 + a^2 - \lambda^2\Delta^2$$

$$\hookrightarrow 0 = a^2 - [\lambda^2\Delta^2 + \epsilon^2]$$

$$0 = \left(a + \sqrt{\lambda^2\Delta^2 + \epsilon^2}\right)\left(a - \sqrt{\lambda^2\Delta^2 + \epsilon^2}\right)$$

$$\hookrightarrow E = \pm\sqrt{\lambda^2\Delta^2 + \epsilon^2}$$

* in the limit $\lambda \to 0$, $E = \pm\epsilon$

$\lambda \to \infty$ $E = \pm\lambda\Delta$ (assumes $\lambda^2\Delta^2 \gg \epsilon^2$)

b) Note $|\psi_+\rangle = \langle 1, 0\rangle$ $E_+ = \epsilon$

$|\psi_-\rangle = \langle 0, 1\rangle$ $E_- = -\epsilon$

$$\Delta E_\pm^{(1)} = \langle \psi_\pm | V | \psi_\pm \rangle \quad \text{where} \quad V = \begin{bmatrix} 0 & \lambda\Delta \\ \lambda\Delta & 0 \end{bmatrix}$$

$$\Delta E_+^{(1)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda\Delta \\ \lambda\Delta & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\hookrightarrow E_+ \approx \epsilon + \cancel{\lambda\Delta}$$

$$\Delta E_- = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \lambda\Delta \\ \lambda\Delta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$\hookrightarrow E_- \approx -\epsilon + \cancel{\lambda\Delta}$$

$$\Delta E_\pm^{(2)} = \frac{|V_{kn}|^2}{E_n - E_k} = \frac{\Delta^2\lambda^2}{\pm 2\epsilon}$$

#5 (cont.)

c) * Assuming $\lambda|\Delta| \ll \epsilon$

$$|\psi_+^{(1)}\rangle = \sum_{k\neq n}' \frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}} |\psi_k\rangle$$

$$= \frac{\Delta\lambda}{2\epsilon} |\psi_-^{(0)}\rangle$$

d)