Problem 5: Simple Harmonic Oscillator with External Perturbations

Consider a one-dimensional simple harmonic oscillator of mass m with a natural angular frequency ω. If there is no external perturbation, the Hamiltonian for this system is

$$H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2, \quad H_0|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle \tag{1}$$

(a) [2 pts] Consider the case where there is an external potential on the oscillator of the form $V_1(x) = \gamma_1 x$. Calculate the exact eigenenergies of $H_0 + V_1$.

Describe the difference between the new eigenstates of this total Hamiltonian and the eigenstates of H_0.

(Hint: The new Hamiltonian can be transformed back into a harmonic oscillator of frequency ω plus an extra term).

(b) [4 pts] Using perturbation theory to the first non-zero order, calculate the perturbed eigenenergies of $H_0 + V_1$. How do these compare with the exact solutions from (a)?

(c) [1 pts] Now consider the case where there is an external potential on the oscillator of the form $V_2(x) = \gamma_2 x^2$. Calculate the exact eigenenergies of $H_0 + V_2$.

Describe the new eigenstates of this total Hamiltonian, comparing them with the eigenstates of H_0.

(d) [3 pts] Using perturbation theory to the first non-zero order, calculate the perturbed eigenenergies of $H_0 + V_2$. How do these compare with the exact solutions from (c)?

Aug 2015

Quantum #5

a) Adding the potential $V_1(x) = \gamma_1 x$ to the SHO yields

$$H_0 + V_1 = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2 + \gamma_1 x$$

* To rewrite this as a version of SHO, we shift variables such that

$$x = y - \frac{\gamma_1}{m\omega^2}, \quad \frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy}, \quad \frac{dy}{dx} = 1 \rightarrow \frac{d^2}{dx^2} = \frac{d^2}{dy^2}$$

$$\hookrightarrow H_0 + V_1 = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{m\omega^2}{2}\left(y - \frac{\gamma_1}{m\omega^2}\right)^2 + \gamma_1\left(y - \frac{\gamma'}{m\omega^2}\right)$$

$$= \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{m\omega^2}{2}\left(y^2 - \frac{2\gamma_1 y}{m\omega^2} + \frac{\gamma_1^2}{m^2\omega^4}\right) + \gamma_1 y - \frac{\gamma_1^2}{m\omega^2}$$

$$= \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{m\omega^2 y^2}{2} - \cancel{\gamma_1 y} + \frac{\gamma_1^2}{2m\omega^2} + \cancel{\gamma_1 y} - \frac{\gamma_1^2}{m\omega}$$

$$= \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{m\omega^2 y^2}{2} - \frac{\gamma_1^2}{2m\omega}$$

* If we move our extra term to other side, and call $E + \frac{\gamma_1^2}{2m\omega} = E'$ we return our expected SHO

$$\hookrightarrow E_n' = \hbar\omega(n + 1/2) + \frac{\gamma_1^2}{2m\omega}$$

* Our eigenstates will be shifted along the x axis by $+\frac{\gamma'}{m\omega^2}$

b) Our first order energy corrections are determined by:

$$\Delta E^{(1)} = \langle n^{(0)} | V_1 | n^{(0)} \rangle$$

$$V_1 = \gamma_1 x = \gamma_1 \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a) \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$= \langle n | \gamma_1 \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a) | n \rangle$$

$= 0$ by the orthogonality of $|n\rangle$ states ($\langle m|n\rangle = \delta_{mn}$)

#5 (cont.)

b) Our second order energy corrections are determined by:

$$\Delta E^{(2)} = \sum_{k\neq n}' \frac{|\langle k|V_1|n\rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$= \sum_{k\neq n}' \frac{|\langle k|\gamma_1\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)|n\rangle|^2}{\hbar\omega(n-k)}$$

$$= \frac{\gamma_1^2 \hbar}{2m\omega} \cdot \frac{1}{\hbar\omega} \sum_{k\neq n}' |\langle k|a^\dagger|n\rangle + \langle k|a|n\rangle|^2/(n-k)$$

$$= \frac{\gamma_1^2}{2m\omega^2} \sum_{k\neq n}' |\sqrt{n+1}\langle k|n+1\rangle + \sqrt{n}\langle k|n-1\rangle|^2/n-k$$

$$= \frac{\gamma_1^2}{2m\omega^2}\left[\frac{n+1}{n-(n+1)} + \frac{n}{n-(n-1)}\right]$$

$$= \frac{\gamma_1^2}{2m\omega^2}\left[-(n+1) + n\right]$$

$$= \frac{-\gamma_1^2}{2m\omega^2}$$

$$\boxed{E_n = E_n' - \frac{\gamma_1^2}{2m\omega^2}} \Rightarrow \text{Matches our exact solution}$$

c) For $V_2 = \gamma_2 x^2$, our Hamiltonian becomes

$$H_0 + V_2 = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}x^2\underbrace{\left(\omega^2 + \frac{2\gamma_2}{m}\right)}_{\omega_1^2}$$

$$= \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_1^2 x^2$$

$$\hookrightarrow E_n'' = \hbar\omega_1(n + 1/2)$$

$$= \hbar(n+1/2)\left(\omega^2 + \frac{2\gamma_2}{m}\right)^{1/2}$$

$$= \hbar\omega(n+1/2)\left(1 + \frac{2\gamma_2}{m\omega^2}\right)^{1/2} \quad * \text{ if } \gamma_2/\omega^2 \ll 1$$

$$= \hbar\omega(n+1/2)\left(1 + \frac{\gamma_2}{m\omega^2}\right)$$

#5 (cont.)

d) Again, our first order energy corrections are:

$$\Delta E^{(1)} = \langle n^{(0)} | V_2 | n^{(0)} \rangle$$

$$= \langle n | \tfrac{\gamma}{2} x^2 | n^{(0)} \rangle$$

$$x^2 = \frac{\hbar}{2m\omega}(a^\dagger a^\dagger + a^\dagger a + a a^\dagger + aa)$$

$$= \frac{\gamma_2 \hbar}{2m\omega}\left[\cancelto{0}{\langle n | a^\dagger a^\dagger | n \rangle} + \langle n | a^\dagger a | n \rangle + \langle n | a a^\dagger | n \rangle + \cancelto{0}{\langle n | aa | n \rangle}\right]$$

$$= \frac{\gamma_2 \hbar}{2m\omega}\left[n + n+1\right]$$

$$= \frac{\gamma_2 \hbar}{m\omega}(n + 1/2)$$

* Matches exact solution