PROBLEM 4: Spin Physics

Spin-1/2 objects generally have magnetic moments that affect their energy levels and dynamics in magnetic fields. The interaction between the magnetic moment and a magnetic field, $\vec{B}$ can be written as:

$$H = -\mu \vec{S} \cdot \vec{B} \tag{1}$$

where $\vec{S}$ is the spin of the particle

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \tag{2}$$

where the σ_i's are Pauli matrices.

In this problem we'll be using as our basis the eigenstates of S_z,

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

with eigenvalues $\pm\frac{\hbar}{2}$.

(a) [1 point] If a particle is in the spin state $|+\rangle$, compute the expectation values of S_x, S_y, and S_z.

(b) [1 point] If a particle is in the spin state $|+\rangle$, what are the uncertainties of S_x, S_y, and S_z? ($\Delta S_i^2 = \langle S_i^2\rangle - \langle S_i\rangle^2$.) Explain the physics of your results in terms of the eigenvalues and measurement probabilities of the spin in the x, y, and z directions.

(c) [3 points] A large ensemble of particles are all prepared to be in the spin state $|+\rangle$ at time $t = 0$ when a magnetic field in the x-direction is switched on, $\vec{B} = B_0\hat{e}_x$. Solve for the time-dependent probabilities, $P_\pm(t)$, of measuring S_z to be $\pm\hbar/2$.

(d) [2 points] For the experiment described in part (c), what are the probabilities for measuring S_x to be $\pm\hbar/2$? Explain the differences between the results for S_z and S_x.

(e) [3 points] Consider the case where the magnetic field is $\vec{B} = \frac{B_0}{\sqrt{2}}(\hat{e}_x + \hat{e}_z)$. In this case what is the time-dependent probability of measuring S_z to be $+\hbar/2$?

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Quantum #4

a) $S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\quad S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\langle S_x \rangle = {}_z\langle + | S_x | + \rangle_z$$

$$= [1\,0]\begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= 0$$

$$\langle S_y \rangle = {}_z\langle + | S_y | + \rangle_z$$

$$= [1\,0]\begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= 0$$

$$\langle S_z \rangle = {}_z\langle + | S_z | + \rangle_z$$

$$= [1\,0]\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \hbar/2$$

b) $\Delta S_i^2 = \langle S_i^2 \rangle - \langle S_i \rangle^2$

$$\langle S_x^2 \rangle = [1\,0]\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= [1\,0]\begin{bmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \hbar^2/4$$

$$\langle S_y^2 \rangle = [1\,0]\begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= [1\,0]\begin{bmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \hbar^2/4$$

$$\langle S_z^2 \rangle = [1\,0]\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= [1\,0]\begin{bmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \hbar^2/4$$

$$\Rightarrow \Delta S_x^2 = \frac{\hbar^2}{4} - (0)^2 = \hbar^2/4$$

$$\Delta S_y^2 = \frac{\hbar^2}{4} - (0)^2 = \frac{\hbar^2}{4}$$

$$\Delta S_z^2 = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2 = 0$$

* Because we are working in the S_z basis, there is no uncertainty b/c we know the state of the particle. However, the S_x and S_z eigenstates are linear combinatio of the S_z states, thus there is some uncertainty as to which eigenvalue is preferred

#4 (cont.)

c) $H = -\mu \vec{S} \cdot \vec{B}$ $\qquad |\psi(t=0)\rangle = |+\rangle_z$

$= -\mu(S_x B_0)$

* To act time evolution operator, convert to S_x basis

$|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z)$

$|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_z - |-\rangle_z)$ $\quad > \quad |+\rangle_z = \frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x)$

$|\psi(t)\rangle = U(t, t_0=0)|\psi(t=0)\rangle$

$= e^{-iHt/\hbar}\left(\frac{1}{\sqrt{2}}[|+\rangle_x + |-\rangle_x]\right)$ $\qquad E_\pm = \pm\frac{\mu_0 B\hbar}{2}$

$= \frac{1}{\sqrt{2}}\left(e^{-iE_+t/\hbar}|+\rangle_x + e^{-iE_-t/\hbar}|-\rangle_x\right)$

$P(S_z = +\hbar/2) = |{}_z\langle+|\psi(t)\rangle|^2$

$= \left| \frac{1}{\sqrt{2}}[\langle+|_x + \langle-|_x] \frac{1}{\sqrt{2}}\, e^{-iE_+t/\hbar}|+\rangle_x + e^{-iE_-t/\hbar}|-\rangle_x\right.$

$= \frac{1}{4}\left|e^{iE_+t/\hbar} + e^{-iE_-t/\hbar}\right|^2$

$= \frac{1}{4}\left(2 + e^{-i(E_+-E_-)t/\hbar} + e^{+i(E_+-E_-)t/\hbar}\right)$

$= \frac{1}{4}\left(2 + 2\cos\left(\frac{\Delta E t}{\hbar}\right)\right)$, $\quad \Delta E = E_+ - E_-$

$= \frac{1}{2}\left(1 + \cos\left(\frac{\Delta E t}{\hbar}\right)\right)$

$P(S_z = -\hbar/2) = 1 - P(S_z = +\hbar/2)$

$= 1 - \frac{1}{2}\left(1 + \cos\left(\frac{\Delta E t}{\hbar}\right)\right)$

$= \frac{1}{2}\left(1 - \cos\left(\frac{\Delta E t}{\hbar}\right)\right)$

<u>#4 (cont.)</u>

d) * utilizing work done in part c;

$$P(S_x = \hbar/2) = |{}_x\langle +|\psi(t)\rangle|^2$$

$$= |{}_x\langle +|(\tfrac{1}{\sqrt{2}}[e^{-iE_+t/\hbar}|+\rangle_x + e^{-iE_-t/\hbar}|-\rangle_x])|^2$$

$$= \frac{1}{2}|e^{-iE_+t/\hbar}|^2$$

$$= \frac{1}{2}$$

$$P(S_x = -\hbar/2) = 1 - P(S_x = \hbar/2)$$

$$= \frac{1}{2}$$

* In this case, our final states are in the same eigenbasis as our original state, which is a linear combination of the two states where the coefficients evolve in time in a coupled manner. But since we are operating in only 1 eigenbasis, we must have time independent probabilities, as the basis states are stationary states.

e) With $\vec{B} = \frac{B_0}{\sqrt{2}}(\hat{e}_x + \hat{e}_z)$, our Hamiltonian now becomes $-\frac{\mu B_0 \hbar}{2\sqrt{2}}(S_x + S_z)$

* if $A = \frac{-\mu B_0 \hbar}{2\sqrt{2}}$

$$\Rightarrow H = A\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

* Determining eigenvalues + eigenvectors

$$0 = (1-\lambda)(-1-\lambda) - 1$$

$$= -1 + \cancel{\lambda} - \cancel{\lambda} + \lambda^2 - 1$$

$$= \lambda^2 - 2$$

$$\hookrightarrow \lambda = \pm\sqrt{2}$$

$$H\vec{v} = \lambda\vec{v} \rightarrow \quad x_1 + x_2 = \lambda x_1$$
$$x_1 - x_2 = \lambda x_2$$

* if $\lambda = +\sqrt{2}$

$$x_1 + x_2 = \sqrt{2}\,x_1$$
$$x_1 - x_2 = \sqrt{2}\,x_2$$
$$\hookrightarrow x_1 = 1+\sqrt{2},\ x_2 = 1$$
$$\vec{v}_1 = \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}$$
$$= |+\rangle_{xz}$$

* if $\lambda = -\sqrt{2}$

$$x_1 + x_2 = -\sqrt{2}\,x_1$$
$$x_1 - x_2 = -\sqrt{2}\,x_2$$
$$\hookrightarrow x_1 = 1-\sqrt{2},\ x_2 = 1$$
$$\vec{v}_2 = \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}$$
$$= |-\rangle_{xz}$$

$$A^2[(1+\sqrt{2})^2 + 1] = 1$$

$$A^2(1 + 2\sqrt{2} + 2 + 1) = 1$$

$$A^2 = \frac{1}{4+2\sqrt{2}}\left(\frac{4-2\sqrt{2}}{4-2\sqrt{2}}\right)$$

$$A^2 = \frac{4-2\sqrt{2}}{16+4(2)} = \frac{4-2\sqrt{2}}{24}$$

$$= \left(\frac{2-\sqrt{2}}{12}\right)^{1/2}$$

$$A^2[(1-\sqrt{2})^2 + 1] = 1$$

$$A^2[1 - 2\sqrt{2} + 2 + 1] = 1$$

$$A^2 = \frac{1}{[4-2\sqrt{2}]}$$

$$A^2 = \frac{4+2\sqrt{2}}{16-4(2)}$$

$$A^2 = \frac{2+2\sqrt{2}}{4}$$

$$A = \left(\frac{2+2\sqrt{2}}{4}\right)^{1/2}$$

#4 (cont.)

e) * To act time evolution operator, convert $|\psi(t=0)\rangle$ to H-basis

$$|\psi(t=0)\rangle = |+\rangle_z = \left(\left(\frac{1}{4+2\sqrt{2}}\right)^{1/2} |+\rangle_{xz} + \left(\frac{1}{4-2\sqrt{2}}\right)^{1/2} |-\rangle_{xz} \right)$$

* Not worth time + effort