Aug 2008

Problem 4: The Infinite Square Well: (10 Points)

A single particle is in a one dimensional infinite well whose potential $V(x)$ is given by:

$$V(x) = \begin{cases} 0, & \text{if -L} \leq \text{x} \leq \text{L} \\ \infty, & \text{otherwise} \end{cases}$$

a. Find the allowed energies (E_n) and the normalized eigenfunctions ($\Phi_n(x)$) to Schrodinger's Equation for this potential. Show all your work. **(2 Points)**

Assume the particle is in the ground state and a position measurement of the particle is made. Since any measuring apparatus has a finite resolution, the exact location of the particle cannot be detemined. We therefore only know the location of the particle within some resolution ϵ. After making the position measurement the wave function $\Psi(x)$ is:

$$\Psi(x) = \frac{1}{\sqrt{\epsilon}} \qquad |x| < \frac{\epsilon}{2}$$

$$\Psi(x) = 0 \qquad |x| > \frac{\epsilon}{2}$$

b. What is the probability that the particle has energy E_n? **(2 Points)**

c. If $\epsilon = 2L$, we know that the particle is somewhere in the box. What is the probability that the particle is in the ground state? **(1 Point)**

d. Before the position measurement we knew the particle was in the box and in the ground state. If after the measurement and $\epsilon = 2L$ we know that the particle is in the box, why is probability that the particle is in the ground state not 1? **(1 Point)**

For parts e), f) and g) now assume that the particle is in the potential $V(x)$

$$V(x) = \begin{cases} 0, & \text{if -L} \leq \text{x} \leq \text{L} \\ \infty, & \text{otherwise} \end{cases}$$

and in the ground state. The position of the walls are quickly increased to

$$V(x) = \begin{cases} 0, & \text{if -L'} \leq \text{x} \leq \text{L'} \\ \infty, & \text{otherwise} \end{cases}$$

where $|L'| > |L|$

e. After the expansion, what is the probability that the particle has energy E_n? You do not need to solve the integral. **(2 Points)**

f. Before the walls of the potential are increased, does $|\Psi(x,t)|^2$ (where $\Psi(x,t)$ is a solution to Schrodinger's equation before the expansion) have any time dependance? Explain **(1 Point)**

g. After the position of the walls are increased to L', does $|\Psi(x,t)|^2$ (where $\Psi(x,t)$ is a solution to Schrodinger's equation after the expansion) have any time dependance? Explain. **(1 Point)**

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Quantum #4

a) $H\psi = E\psi$

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi$$

$$\frac{\partial^2}{\partial x^2}\psi = \frac{-2mE}{\hbar^2}\psi$$

* let $k = \sqrt{\frac{2mE}{\hbar^2}}$

$$\frac{\partial^2}{\partial x^2}\psi = -k^2\psi$$

$\hookrightarrow \psi = A\sin(kx) + B\cos(kx)$

* Our boundary conditions are $\psi(-L) = \psi(L) = 0$

$$0 = A\sin(kL) + B\cos(kL)$$

$$0 = A\sin(-kL) + B\cos(-kL)$$

$$= -A\sin(kL) + B\cos(kL)$$

* The above equations are true when $kL = \frac{n\pi}{2} \Rightarrow k = \frac{n\pi}{2L}$

$\hookrightarrow$ if n is even:

$$0 = A\,\cancel{\sin(kL)}^{\,0} + B\cos(kL)$$

$\hookrightarrow B = 0$

$\hookrightarrow$ if n is odd:

$$0 = A\sin(kL) + B\,\cancel{\cos(kL)}^{\,0}$$

$\hookrightarrow A = 0$

$$\Rightarrow \psi(x) = \begin{cases} A\sin\left(\frac{n\pi}{2}\right) & n \text{ even} \\ B\cos\left(\frac{n\pi}{2}\right) & n \text{ odd} \end{cases}$$

* Normalizing the above wave function yields

$$1 = \int_{-L}^{L} A^2\sin^2(kx)\,dx$$

$$= A^2\int_{-L}^{L}\frac{1}{2}(1-\cos(2kx))\,dx$$

$$= \frac{A^2}{2}\left[x - \frac{1}{2k}\sin(2kx)\right]\Big|_{-L}^{L}$$

$$= A^2L \quad \Rightarrow \quad A = \frac{1}{\sqrt{L}} \quad \text{(same for B)}$$

#4 (cont.)

a) Therefore: $\psi(x) = \begin{cases} \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi}{2}\right) & n \text{ even} \\ \sqrt{\frac{1}{L}} \cos\left(\frac{n\pi}{2}\right) & n \text{ odd} \end{cases}$

* Returning to the energy

$$k = \sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{2L}$$

$$\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{4L^2}$$

$$E_n = \frac{n^2\pi^2\hbar^2}{8mL^2}$$

b) $P = \left| \int_{-\epsilon/2}^{\epsilon/2} \sqrt{\frac{1}{L}} \cos\left(\frac{n\pi x}{2L}\right) \frac{1}{\sqrt{\epsilon}} dx \right|^2$

$$= \frac{1}{\epsilon L} \left| \int_{-\epsilon/2}^{\epsilon/2} \cos\left(\frac{n\pi x}{2L}\right) dx \right|^2$$

$$= \frac{1}{\epsilon L} \left| \frac{2L}{n\pi} \sin\left(\frac{n\pi x}{2L}\right) \Big|_{-\epsilon/2}^{\epsilon/2} \right|^2$$

$$= \frac{4L}{\epsilon n^2\pi^2} \left(2 \sin\left(\frac{n\pi\epsilon}{4L}\right) \right)^2$$

$$= \frac{16L}{\epsilon n^2\pi^2} \sin^2\left(\frac{n\pi\epsilon}{4L}\right)$$

c) If $\epsilon = 2L$, $n = 1$:

$$P = \frac{8}{\pi^2}$$

d) The act of measuring the particle has perturbed the system, thus altering the state of the system

e) After expansion, our wavefunction becomes

$$\psi_n(x) = \begin{cases} \sqrt{1/L'} \sin\left(\frac{n\pi x}{2L'}\right) & n' \text{ even} \\ \sqrt{1/L'} \cos\left(\frac{n\pi x}{2L'}\right) & n' \text{ odd} \end{cases}$$

$$\Rightarrow P = \left| \int_{-L'}^{L'} \sqrt{\frac{1}{L'}} \cos\left(\frac{n\pi x}{2L'}\right) \sqrt{\frac{1}{L}} \cos\left(\frac{n\pi x}{2L}\right) dx \right|^2$$

#4 (cont.)

f) The eigenstates of the infinite square well are stationary states, thus $|\Psi(x,t)|^2$ has no time dependence

g) See part f