PROBLEM 4: Two Particles in a 1D Box

Consider two noninteracting particles of mass m inside a 1D box,

$$V(x) = \begin{cases} 0 & , 0 < |x| < a \\ \infty & , \text{otherwise} \end{cases} .$$

Make sure to consider the spin part of the wavefunction in this problem.

a) Let n_1 and n_2 be the quantum numbers of particle 1 and 2 respectively. What are the wavefunctions of the single particle states for the each particle in the box? What are the single particle energies? (2 Points)

b) If the particles are distinguishable what is the two-particle wavefunction that describes the state? What is the energy? Write out explicitly the state (or states) and energies for the ground state and first excited states of the system. (2 Points)

c) If the two particles are identical spin 0 bosons what are the ground state and first excited state wavefunctions and energies? (2 Points)

d) If the two particles are identical spin 1/2 fermions what are the ground state and first excited state wavefunctions and energies? (2 Points)

e) Write down the Hamiltonian for the two particles in the box and show that when the particles are identical H commutes with the exchange operator. (2 Points)

#4 (cont.)

a) * Similarly, if $n = \text{odd}$: $A = 0, \quad B = \sqrt{\frac{1}{a}}$

$\Rightarrow$ For any single particle:

$$\boxed{\psi(x) = \begin{cases} \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi x}{2a}\right) & n = \text{even} \\ \sqrt{\frac{1}{a}} \cos\left(\frac{n\pi x}{2a}\right) & n = \text{odd} \end{cases}}$$

$$k = \frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{2a}$$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$

b) Assuming distinguishable particles, our two particle wave functions will be the product of the two single particle states, plus a spin function

$$\psi_{sys} = \psi_{n_1}\psi_{n_2}\psi_{spin}, \qquad E_{sys} = \frac{(n_1^2 + n_2^2)\pi^2\hbar^2}{2ma^2}$$

The ground state occurs when $n_1 = n_2 = 1$

$$\psi_{sys} = \frac{1}{a}\cos\left(\frac{\pi x_1}{2a}\right)\cos\left(\frac{\pi x_2}{2a}\right)$$

$$E = \frac{\pi^2\hbar^2}{ma^2}$$

The first excited state occurs when $(n_1 = 2, n_2 = 1)$ or $(n_1 = 1, n_2 = 2)$

$$\psi_{sys} = \frac{1}{a}\sin\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{2a}\right) \qquad E = \frac{5\pi^2\hbar^2}{2ma^2}$$

or

$$\psi_{sys} = \frac{1}{a}\cos\left(\frac{\pi x_1}{2a}\right)\sin\left(\frac{\pi x_2}{a}\right) \qquad E = \frac{5\pi^2\hbar^2}{2ma^2}$$

c) Our spin function now becomes important. Bosons must have symmetric spin functions which we will denote ψ_{spin}^{sym}. Since our bosons are identical, and thus indistinguishable, it will be a superposition of the two possible single particle states

$\Rightarrow$ In the ground state, $E_{sys} = \frac{\pi^2\hbar^2}{ma^2}$

$$\psi_{sys} = A\left[\frac{1}{a}\cos\left(\frac{\pi x_1}{2a}\right)\cos\left(\frac{\pi x_2}{2a}\right) + \frac{1}{a}\cos\left(\frac{\pi x_1}{2a}\right)\cos\left(\frac{\pi x_2}{2a}\right)\right]$$

$$= \frac{2A}{a}\cos\left(\frac{\pi x_1}{2a}\right)\cos\left(\frac{\pi x_2}{2a}\right),$$ A is normalization constant

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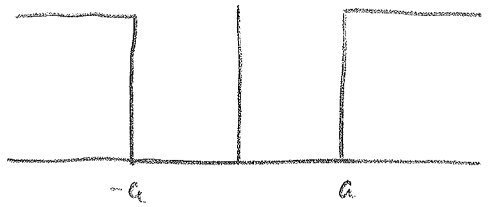
Quantum #4

a) For two non-interacting particles in a box, where

$$V(x) = \begin{cases} 0 & 0 < |x| < a \\ \infty & \text{otherwise} \end{cases}$$



$H\psi = E\psi$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{-\sqrt{2mE}}{\hbar}\psi$$

$$= -k^2\psi$$

$\hookrightarrow \psi = A\sin(kx) + B\cos(kx)$

*Our boundary conditions are $\psi(-a) = 0 = \psi(a)$

$$0 = A\sin(-ka) + B\cos(-ka)$$

$$0 = A\sin(ka) + B\cos(ka)$$

$\hookrightarrow$ Our trig functions will be 0 when $ka = \frac{n\pi}{2} \iff k = \frac{n\pi}{2a}$

$\sin\left(\frac{n\pi}{2}\right) = 0$ if $n =$ even $(0, 2, 4, \text{etc})$

$\cos\left(\frac{n\pi}{2}\right) = 0$ if $n =$ odd $(1, 3, 5, \text{etc})$

*if $n =$ even,

$$\psi = A(0) + B\cos\left(\frac{n\pi}{2}\right) \Rightarrow B = 0$$

$$1 = A^2\int_{-a}^{a}\sin^2\left(\frac{n\pi x}{2}\right)dx$$

$$1 = \frac{A^2}{2}\int_{-a}^{a}\left[1 - \cos\left(\frac{n\pi x}{2a}\right)\right]dx$$

$$1 = \frac{A^2}{2}\left[x - \sin\left(\frac{n\pi x}{a}\right)\right]\Big|_{-a}^{a}$$

$$1 = \frac{A^2}{2}\left[\left(a - \frac{1}{n\pi}\cancel{\sin\left(\frac{n\pi a}{a}\right)}^{0}\right) - \left(-a - \frac{1}{n\pi}\cancel{\sin\left(\frac{-n\pi a}{a}\right)}^{0}\right)\right]$$

$$1 = \frac{A^2}{2}[2a] \Rightarrow A = \sqrt{\frac{1}{a}}$$

#4 (cont)

c) $\Rightarrow$ In the first excited state, $E_{sys} = \frac{5\pi^2\hbar^2}{2ma^2}$

$$\psi_{sys} = A\left[\frac{1}{a}\sin\left(\frac{\pi x_1}{2a}\right)\cos\left(\frac{\pi x_2}{a}\right) + \frac{1}{a}\cos\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{2a}\right)\right]$$

d) With two identical spin ½ fermions, we need an anti symmetric spin function, denoted ψ^{asym}_{spin}, and an anti symmetric wave function

$\Rightarrow$ In the ground state, $E_{sys} = \frac{\pi^2\hbar^2}{2ma^2}$ (1 particle spin up, 1 spin down will violate exclusion principle)

$$\psi_{sys} = A\left[\frac{1}{a}\sin\left(\frac{\pi x_1}{2a}\right)\sin\left(\frac{\pi x_2}{2a}\right) - \frac{1}{a}\sin\left(\frac{\pi x_1}{2a}\right)\sin\left(\frac{\pi x_2}{2a}\right)\right]$$

$$= 0$$

Therefore, our ground state becomes $E_{sys} = \frac{5\pi^2\hbar^2}{2ma^2}$

$$\psi_{sys} = A\left[\frac{1}{a}\sin\left(\frac{\pi x_1}{2a}\right)\cos\left(\frac{\pi x_2}{a}\right) - \frac{1}{a}\cos\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{2a}\right)\right]$$

$\Rightarrow$ Now the first excited state occurs when $(n_1 = 1, n_2 = 3)$ or $(n_1 = 3, n_2 = 1)$. with $E_{sys} = \frac{5\pi^2\hbar^2}{ma^2}$

$$\psi_{sys} = A\left[\frac{1}{a}\sin\left(\frac{\pi x_1}{2a}\right)\sin\left(\frac{3\pi x_2}{2a}\right) - \frac{1}{a}\sin\left(\frac{3\pi x_1}{2a}\right)\sin\left(\frac{\pi x_2}{2a}\right)\right]$$

e) The Hamiltonian for the system is:

$$H = \frac{-\hbar^2}{2m}\left[\frac{d^2}{dx_1^2} + \frac{d^2}{dx_2^2}\right]$$