Problem 3: Vector Spaces and Dirac Notation

Consider a quantum system that can be described by three basis states, $|n\rangle$, $n = 1, 2, 3$, and an operator defined by its action on these three states:

$$\begin{aligned} A|1\rangle &= -i\alpha|3\rangle \\ A|2\rangle &= \alpha|2\rangle \\ A|3\rangle &= i\alpha|1\rangle \end{aligned} \tag{1}$$

where α is real.

(a) [2 pts] Write the operator A as a matrix using these basis states:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

(b) [1 pt] Show that A is Hermitian.

(c) [3 pts] Compute the eigenvalues and corresponding eigenvectors of A.

(d) [2 pts] In your result for part (c), you found one non-degenerate eigenstate, call it $|\gamma\rangle$, with eigenvalue γ. The other eigenstates are degenerate.

Define the projection operator $\mathcal{P}_\gamma = |\gamma\rangle\langle\gamma|$. Write the operator $\mathcal{P}_\gamma$ as a matrix using the basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$.

Check your results to show that this matrix form for the projection operator is correct.

(e) [2 pts] Consider the system in the state:

$$|\phi\rangle = \frac{2}{3}|1\rangle + \frac{2}{3}|2\rangle - \frac{i}{3}|3\rangle \tag{3}$$

Write down an expression for the probability that a measurement of A would result in the value γ in terms of the projection operator $\mathcal{P}_\gamma$. Solve for this probability.

Aug 2015

Quantum #3

a) Using the given basis vectors, A can be written as:

$$A = \begin{bmatrix} 0 & 0 & -i\alpha \\ 0 & \alpha & 0 \\ i\alpha & 0 & 0 \end{bmatrix}$$

b) The condition for Hermiticity is that $A^\dagger A = AA^\dagger = \mathbb{I}$

$$\Rightarrow A^\dagger = \begin{bmatrix} 0 & 0 & -i\alpha \\ 0 & \alpha & 0 \\ i\alpha & 0 & 0 \end{bmatrix}$$

$$\hookrightarrow A^\dagger A = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix} = \alpha^2 \mathbb{I}$$

c) Solve eigenvalue equation $\det(A - \mathbb{I}\lambda) = 0$

$$\begin{vmatrix} -\lambda & 0 & -i\alpha \\ 0 & \alpha-\lambda & 0 \\ i\alpha & 0 & -\lambda \end{vmatrix} = -\lambda[(-\lambda)(\alpha-\lambda)-0] - 0 + -i\alpha[(0)-(i\alpha)(\alpha-\lambda)]$$

$$\begin{aligned} 0 &= \lambda^2(\alpha-\lambda) - \alpha^2(\alpha-\lambda) \\ &= (\alpha-\lambda)(\lambda^2-\alpha^2) \\ &= (\alpha-\lambda)(\lambda+\alpha)(\lambda-\alpha) \end{aligned}$$

$$\hookrightarrow \lambda = \alpha, \alpha, -\alpha$$

Solve eigenvector equation $A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 0 & 0 & -i\alpha \\ 0 & \alpha & 0 \\ i\alpha & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} -i\alpha x_3 &= \lambda x_1 \\ \alpha x_2 &= \lambda x_2 \\ i\alpha x_1 &= \lambda x_3 \end{aligned}$$

Case: $\lambda = -\alpha$

$$\begin{aligned} -i\alpha \overset{-ix_1}{x_3} &= -\alpha x_1 \\ \alpha x_2 &= -\alpha x_2 \\ i\alpha x_1 &= -\alpha x_3 \end{aligned} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$$

Case: $\lambda = \alpha$

$$\begin{aligned} -i\alpha x_3 &= \alpha x_1 \\ \alpha x_2 &= \alpha x_2 \\ i\alpha x_1 &= \alpha x_3 \end{aligned} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$$

d) $P_\gamma = |\gamma\rangle\langle\gamma|$, where $|\gamma\rangle = |2\rangle$

$$\Rightarrow P_\gamma = |2\rangle\langle 2|$$
$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

e) $|\varphi\rangle = \frac{2}{3}|1\rangle + \frac{2}{3}|2\rangle - \frac{5}{3}|3\rangle$

$$P(\lambda = \gamma) = |\langle\gamma|A|\varphi\rangle|^2$$
$$= \langle\varphi|A^\dagger|\gamma\rangle\langle\gamma|A|\varphi\rangle$$
$$= \Big[\left(\frac{2}{3}\langle 1| + \frac{2}{3}\langle 2| - \frac{5}{3}\langle 3|\right)$$