Problem 6: Hydrogenic Systems (10 pts):

(Note this problem is 2 pages and has 5 parts)

Consider the quantum system consisting of two charged particles interacting due to the Coulomb Potential:

$$H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} - \frac{qe^2}{|\vec{r}_1 - \vec{r}_2|}$$

$\vec{p}_1$ and $\vec{r}_1$ are the position and momentum of particle 1 with mass m_1. $\vec{p}_2$ and $\vec{r}_2$ are the position and momentum of particle 2 with mass m_2.

The charge of particle 1 is $-e$ and the charge of particle 2 is $+qe$, where q is an integer greater than or equal to 1.

a) (2 pts.) To solve this problem, you first want to convert to the center-of-mass and relative coordinates:

$$\vec{R} = \frac{m_1}{m_1 + m_2}\vec{r}_1 + \frac{m_2}{m_1 + m_2}\vec{r}_2 \ , \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

Derive the conjugate momenta to these spatial coordinates, $\vec{P}$ and $\vec{p}$, defined by:

$$[r_i, p_j] = i\hbar\delta_{i,j}, \quad [R_i, P_j] = i\hbar\delta_{i,j}, \quad [r_i, P_j] = [R_i, p_j] = 0$$

In these expressions, the subscripts indicate the vector components x, y, z, p_x, p_y, p_z, etc. Show your work.

Using these coordinates, the 2-particle Hamiltonian can be written:

$$H = \frac{\vec{P}^2}{2(m_1 + m_2)} + \frac{\vec{p}^2}{2\mu} - \frac{qe^2}{|\vec{r}|} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{1}$$

For the rest of the problem, assume $\vec{P} = 0$ (the center-of-mass reference frame).

b) (2 pts.) Define the wavefunction for the system as:

$$\Psi_{n,\ell,m}(\vec{r}) = \frac{u_{n,\ell}(r)}{r} Y_\ell^m(\theta, \phi)$$

where r, θ, ϕ are the usual spherical coordinates, and Y_ℓ^m the spherical harmonics.

Problem 6 continued

Show, in detail, that the Radial (Schrodinger) wave equation for the bound eigen-states, $u_{n,\ell}(r)$ can be written as:

$$\frac{\partial^2}{\partial r^2}u(r) - \frac{\ell(\ell+1)}{r^2}u(r) + \frac{2}{a_0}\frac{u(r)}{r} = \kappa_n^2 u(r)$$

What are a_0 and κ_n in terms of properties of the bound state system (μ, e, q, etc.)?

c) (3 pts.) Using the radial wave equation, determine the form of the function $u_{n,\ell}(r)$ in the limit as $r \to \infty$. How does $u_{n,\ell}(r)$ depend on the quantum number n for large values of r?

d) (2 pts.) In the limit that $r \to 0$, show that there are two possible solutions for $u_{n,\ell}(r)$, with the physical solution being $u_{n,\ell}(r) \propto r^{\ell+1}$. Do this for $\ell > 0$. (The $\ell = 0$ solution is a bit more complicated.)

e) (1 pt.) What are the ground-state energy and radius (Bohr radius) of the hydrogen-like system of a muon bound to an alpha particle?

Some potentially useful information:

- Fine structure constant - $\alpha = \frac{e^2}{\hbar c}$
- Bohr radius for a hydrogen atom - $a_B = \frac{\hbar}{\alpha m_e c}$
- Rydberg - $\frac{1}{2}\alpha^2 m_e c^2$
- Electron mass - $m_e c^2 = 0.51 MeV$
- Proton and Neutron mass - $m_N c^2 = 940 MeV$
- muon mass - $m_\mu c^2 = 106 MeV$.

$$a_0 = \frac{\hbar}{\frac{e^2}{\hbar c} m_e c}$$

$$= \frac{\hbar^2}{e^2 m_e}$$

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Quantum #6

a) $\vec{R} = \frac{m_1}{m_1+m_2}\vec{r}_1 + \frac{m_2}{m_1+m_2}\vec{r}_2$

$\vec{r} = \vec{r}_1 - \vec{r}_2$

$[r_i, p_j] = i\hbar\delta_{ij}$ $\quad [R_i, P_j] = i\hbar\delta_{ij}$ $\quad [r_i, P_j] = [R_i, p_j] = 0$

* Note: All commutators in expansion that are commutators like $[r_{1i}, p_{2j}] = 0$ b/c they act on different particles and thus automatically commute

* Assume $\underline{P} = a\vec{p}_1 + b\vec{p}_2$

$\vec{p} = c\vec{p}_1 + d\vec{p}_2$

$[r_i, p_j] = i\hbar\delta_{ij}$

$= c[r_{1i}, p_{1j}] - d[r_{2i}, p_{2j}]$

$= c\,i\hbar\delta_{ij} - d\,i\hbar\delta_{ij}$

$\hookrightarrow \boxed{c - d = 1}$

$[R_i, P_j] = \frac{am_1}{m_1+m_2}[r_{1i}, p_{1j}] + \frac{bm_2}{m_1+m_2}[r_{2i}, p_{2j}]$

$= i\hbar\delta_{ij}$

$= \frac{am_1}{m_1+m_2}(i\hbar\delta_{ij}) + \frac{bm_2}{m_1+m_2}(i\hbar\delta_{ij})$

$\hookrightarrow \boxed{\frac{am_1}{m_1+m_2} + \frac{bm_2}{m_1+m_2} = 1}$

$[r_i, P_j] = 0$

$= a[r_{1i}, p_{1j}] - b[r_{2i}, p_{2j}]$

$= a(i\hbar\delta_{ij}) - b(i\hbar\delta_{ij})$

$\hookrightarrow \boxed{a - b = 0}$

#6 (cont.)

a) $[R_i, p_j] = 0$

$$= \frac{m_1 c}{m_1+m_2}[r_{1i}, P_{1j}] + \frac{m_2 d}{m_1+m_2}[r_{2i}, P_{2j}]$$

$$= \left(\frac{m_1 c}{m_1+m_2} + \frac{m_2 d}{m_1+m_2}\right) i\hbar\,\delta_{ij}$$

$$\hookrightarrow \boxed{\frac{m_1 c}{m_1+m_2} + \frac{m_2 d}{m_1+m_2} = 0}$$

* Solving the 4 boxed equations yields:

$$a = b = 1$$

$$c = \frac{m_2}{m_1+m_2}$$

$$d = \frac{-m_1}{m_1+m_2}$$

$$\Rightarrow \quad \vec{P} = \vec{P}_1 + \vec{P}_2$$

$$\vec{p} = \frac{m_2}{m_1+m_2}\vec{P}_1 - \frac{m_1}{m_1+m_2}\vec{P}_2$$

b) Given $\psi_{n\ell m}(\vec{r}) = \frac{U_{n\ell}(r)}{r} Y_\ell^m(\theta,\varphi)$, the Schrödinger Eqn says:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi - \frac{L^2}{\hbar^2 r^2}\psi\right) - \frac{qe^2}{r}\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial^2 U(r)}{\partial r^2}Y_\ell^m - \frac{L^2}{\hbar^2 r^3}U(r)Y_\ell^m\right) - \frac{qe^2}{r^2}UY_\ell^m = \frac{E}{r}UY_\ell^m$$

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial^2 U(r)}{\partial r^2} - \frac{\ell(\ell+1)\hbar^2}{\hbar^2 r^3}U(r)\right)Y_\ell^m - \frac{qe^2}{r^2}UY_\ell^m = \frac{E}{r}UY_\ell^m$$

$$-\frac{\hbar^2}{2mr}\frac{\partial^2 U}{\partial r^2} + \frac{\hbar^2\ell(\ell+1)}{2mr^3}U - \frac{qe^2}{r^2}U - \frac{E}{r}U = 0$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 U}{\partial r^2} + \left(\frac{\hbar^2\ell(\ell+1)}{2mr^2} - \frac{qe^2}{r} - E\right)U = 0$$

#6 (cont.)

b) $$\frac{\partial^2 u}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}u - \frac{2mqe^2}{\hbar^2 r}u + \frac{2mE}{\hbar^2}u = 0$$

$$* \text{let } a_0 = \frac{\hbar^2}{mqe^2}, \quad \kappa_n^2 = \frac{-2mE}{\hbar^2}$$

$$\frac{\partial^2 u}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}u - \frac{2}{a_0}u = \kappa_n^2 u$$

c) In the limit $r \to \infty$

$$\frac{\partial^2 u}{\partial r^2} \approx \kappa_n^2 u$$

$$\hookrightarrow u(r) \propto e^{-\kappa_n r} + \cancel{e^{+\kappa_n r}}^{\,0}$$ b/c unphysical ($\to \infty$ as $r \to \infty$)

* But since $\kappa_n^2 = \frac{1}{a_0 n}$

$$u(r) \propto e^{-r/a_0 n}$$

Therefore as $n \to \infty$, the function will approch 0 faster

d) In the limit $r \to 0$, our equation becomes since $\ell > 0$

$$\frac{\partial^2 u}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}u \approx \kappa_n^2 u$$

$$\hookrightarrow u(r) \propto r^{\ell+1} + \cancel{r^{-\ell}}^{\,0}$$ b/c ψ will not be normalizable ($\psi \propto r^{-(\ell+1)}$)

* To check our solutions, first with $u \propto r^{\ell+1}$

$$\frac{\partial^2 u}{\partial r^2} = \ell(\ell+1)r^{\ell-1}, \quad \frac{\ell(\ell+1)}{r^2}r^{\ell+1} = \ell(\ell+1)r^{\ell-1} \checkmark$$

* Now for $u \propto r^{-\ell}$

$$\frac{\partial^2 u}{\partial r^2} = \ell(\ell+1)r^{-\ell-2}, \quad \frac{\ell(\ell+1)}{r^2}r^{-\ell} = \ell(\ell+1)r^{-\ell-2} \checkmark$$