Problem 6: Measurements and Probability (10 points)

A three-level quantum system has a non-degenerate ground state and a two-fold degenerate excited state, defined by:

$$H|0\rangle = 0 \ , \qquad H|a\rangle = \epsilon|a\rangle \ , \qquad H|b\rangle = \epsilon|b\rangle$$

where ϵ is a positive constant energy.

(a) (1 pt) Write down the matrix representation of H in the basis $|0\rangle$, $|a\rangle$, $|b\rangle$.

(b) (2 pts.) Define the observable C by its operation on the eigenstates of H.

$$C|0\rangle = \gamma|a\rangle \ , \quad C|a\rangle = \gamma|0\rangle \ , \quad C|b\rangle = -\gamma|b\rangle \qquad (3)$$

$\gamma > 0$. What are all the possible outcomes of a measurement of C?

(c) (2 pts.) For each of the eigenstates of H, calculate the probability of measuring the different possible values for C if the system is in that eigenstate.

(d) (1 pts.) Do H and C have common eigenstates? Are H and C compatible observables? Explain.

(e) (2 pts.) At time $t = 0$, the system is in the eigenstate of C with the largest eigenvalue. Calculate the probabilities, as functions of time, of obtaining the different possible results of a measurement of C.

(f) (2 pt.) At time $t = 0$, the system is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle + |b\rangle\right)$. Calculate the probabilities, as functions of time, of obtaining the different possible results of a measurement of C. Explain the differences in this result and what was found in part (e).

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a) $$H \doteq \begin{array}{c|ccc} & |0\rangle & |a\rangle & |b\rangle \\ \hline \langle 0| & 0 & 0 & 0 \\ \langle a| & 0 & \epsilon & 0 \\ \langle b| & 0 & 0 & \epsilon \end{array}$$

b) $$C \doteq \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & -\gamma \end{bmatrix} \qquad C - \lambda \mathbb{I} = \begin{bmatrix} -\lambda & \gamma & 0 \\ \gamma & -\lambda & 0 \\ 0 & 0 & -\gamma-\lambda \end{bmatrix}$$

* Reading the question as asking us to determine the eigenvalues of C

$$|C - \lambda \mathbb{I}| = 0 = -\lambda(-\lambda(-\lambda-\gamma)-0) - \gamma(\gamma(-\gamma-\lambda)-0)$$

$$= -\lambda^2(\lambda+\gamma) + \gamma^2(\lambda+\gamma)$$

$$\doteq (\lambda+\gamma)(\gamma^2-\lambda^2)$$

$$\hookrightarrow \boxed{\lambda = -\gamma, -\gamma, +\gamma}$$

c) * To do this, we must rewrite eigenstates of H in the eigenbasis of C

$$C\vec{x} = \lambda\vec{x}$$

$$\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & -\gamma \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{aligned} \gamma x_2 &= \lambda x_1 \\ \gamma x_1 &= \lambda x_2 \\ -\gamma x_3 &= \lambda x_3 \end{aligned}$$

* for $\lambda = \gamma$

$$\gamma x_2 = \gamma x_1$$
$$\gamma x_1 = \gamma x_2$$
$$-\gamma x_3 = \gamma x_3$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$$

$$= |\lambda_c = \gamma\rangle$$

* for $\lambda = -\gamma$

$$\gamma x_2 = -\gamma x_1$$
$$\gamma x_1 = -\gamma x_2$$
$$-\gamma x_3 = -\gamma x_3$$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\frac{1}{\sqrt{2}}$$

$$= |\lambda_c = -\gamma, 1\rangle \text{ or } |\lambda_c = -\gamma, 2\rangle$$

#6 (cont.)

c) $\Rightarrow |0\rangle = \frac{1}{\sqrt{2}}\left[|\lambda_c=\gamma\rangle + |\lambda_c=-\gamma,2\rangle\right]$ $\quad P(\gamma)=\frac{1}{2} \quad P(-\gamma)=\frac{1}{2}$

$|a\rangle = \frac{1}{\sqrt{2}}\left[|\lambda_c=\gamma\rangle - |\lambda_c=-\gamma,2\rangle\right]$ $\quad P(\gamma)=\frac{1}{2} \quad P(-\gamma)=\frac{1}{2}$

$|b\rangle = |\lambda_c=-\gamma,1\rangle$ $\quad P(\gamma)=0 \quad P(-\gamma)=1$

d) No, H and C are not compatible b/c they do not share an eigenbasis

e) $|\psi(t=0)\rangle = |\lambda_c=\gamma\rangle$

$|\psi(t)\rangle = U(t,t_0=0)|\psi(t=0)\rangle, \quad U(t,t_0)\ e^{-itH/\hbar}$

$\hookrightarrow |\psi(t)\rangle = \exp[-\frac{i}{\hbar}Ht]\,|\lambda_c=\gamma\rangle$

$$= \exp[-\tfrac{i}{\hbar}Ht]\left(\tfrac{1}{\sqrt{2}}\left[|0\rangle + |a\rangle\right]\right)$$

$$= \tfrac{1}{\sqrt{2}}\left[|0\rangle + e^{-i\epsilon t/\hbar}|a\rangle\right]$$

$$= \tfrac{1}{2}\left[|\lambda_c=\gamma\rangle + |\lambda_c=-\gamma,2\rangle + e^{-i\epsilon t/\hbar}\left(|\lambda_c=\gamma\rangle - |\lambda_c=-\gamma,2\rangle\right)\right]$$

$$= \tfrac{1}{2}\left[\left(1+e^{-i\epsilon t/\hbar}\right)|\lambda_c=\gamma\rangle + \left(1-e^{-i\epsilon t/\hbar}\right)|\lambda_c=-\gamma,2\rangle\right]$$

$$P(\gamma) = \tfrac{1}{4}\left|1+e^{-i\epsilon t/\hbar}\right|^2$$
$$= \tfrac{1}{4}\left(1 + e^{-i\epsilon t/\hbar} + e^{i\epsilon t/\hbar} + 1\right)$$
$$= \tfrac{1}{4}\left(2 + 2\cos\left(\tfrac{\epsilon t}{\hbar}\right)\right)$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\cos\left(\tfrac{\epsilon t}{\hbar}\right)$$

$$P(-\gamma) = \tfrac{1}{4}\left|1-e^{-i\epsilon t/\hbar}\right|^2$$
$$= \tfrac{1}{4}\left(1 - e^{-i\epsilon t/\hbar} - e^{i\epsilon t/\hbar} + 1\right)$$
$$= \tfrac{1}{4}\left(2 - 2\cos\left(\tfrac{\epsilon t}{\hbar}\right)\right)$$
$$= \tfrac{1}{2} - \tfrac{1}{2}\cos\left(\tfrac{\epsilon t}{\hbar}\right)$$

f) $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle + |b\rangle\right)$

$$|\psi(t)\rangle = \tfrac{1}{\sqrt{2}}e^{-iHt/\hbar}\left(|a\rangle + |b\rangle\right)$$
$$= \tfrac{1}{\sqrt{2}}e^{-i\epsilon t/\hbar}\left(|a\rangle + |b\rangle\right)$$
$$= \tfrac{1}{\sqrt{2}}e^{-i\epsilon t/\hbar}\left(\tfrac{1}{\sqrt{2}}\left[|\lambda_c=\gamma\rangle - |\lambda_c=-\gamma,2\rangle\right]\right) + \tfrac{1}{\sqrt{2}}e^{-i\epsilon t/\hbar}|\lambda_c=-\gamma,1\rangle$$

#6 (cont)

f) $P(\gamma) = \left|\frac{1}{2}e^{-i\epsilon t/\hbar}\right|^2$

$= \frac{1}{4}$

$P(-\gamma) = \left|\frac{1}{2}e^{i\epsilon t/\hbar}\right|^2 + \left|\frac{1}{\sqrt{2}}e^{-i\epsilon t/\hbar}\right|^2$

$= \frac{1}{4} + \frac{1}{2}$

$= 3/4$

$\Rightarrow$ In this case, we have probabilities that are not time dependent.