Problem 4: Finite Quantum System

Consider a quantum system that can be described by three basis states, $|n\rangle$, $n = 1, 2, 3$, and the Hamiltonian in this basis:

$$H = \frac{\hbar\omega}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix} \tag{1}$$

(a) [3 pts] Solve for the energy eigenvalues and eigenstates of this system.

(b) [2 pts] If the system starts in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \tag{2}$$

determine the time-dependence of the state $|\psi(t)\rangle$. You may write your answer in terms of either the states $|n\rangle$ or the eigenstates you found in part (a).

(c) [3 pts] Calculate the time dependent probabilities for measuring the system to be in each of the states $|1\rangle$, $|2\rangle$, and $|3\rangle$, if the system starts in the state given in part (b). Explain why the different states can or cannot be measured and the frequency of the oscillations you found.

(d) [2 pts] Finally, assume that the states $|n\rangle$ are the eigenstates of some observable O where

$$O|n\rangle = (-1)^n n|n\rangle \tag{3}$$

If, again, the system starts in the state given in part (b), what is the time dependent expectation value of O, $\langle O\rangle(t)$?

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Quantum #4

a) $H = \frac{\hbar\omega}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix}$

* Solving eigenvalue equation $\det(H - \lambda \mathbb{I}) = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & i \\ 0 & -i & 1-\lambda \end{vmatrix} = 0 = (2-\lambda)\left[(1-\lambda)^2 - (-i)(i)\right]$$

$$\Rightarrow \quad 0 = (2-\lambda)\left[(1-\lambda^2) - 1\right]$$
$$= (2-\lambda)\left[(1-\lambda) - 1\right]\left[(1-\lambda) + 1\right]$$

$$\hookrightarrow \lambda = 2, 2, 0$$

* Solving eigenvector equation: $H\vec{x} = \lambda\vec{x}$

Case $\lambda = 0$:

$2x_1 = 0$

$x_2 + ix_3 = 0 \rightarrow x_2 = -ix_3$

$-ix_2 + x_3 = 0 \rightarrow x_3 = ix_2$

$\rightarrow \vec{x} = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$

Case $\lambda = 2$

$2x_1 = 2x_1$

$x_2 + ix_3 = 2x_2 \rightarrow ix_3 = x_2$

$-ix_2 + x_3 = 2x_3 \qquad -ix_2 = x_3$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$

b) * Proceed according to: $|1\rangle = \langle 1,0,0\rangle$, $\lambda_1 = 2$

$|2\rangle = \frac{1}{\sqrt{2}}\langle 0, i, 1\rangle$, $\lambda_2 = 2$

$|3\rangle = \frac{1}{\sqrt{2}}\langle 0, -i, 1\rangle$, $\lambda_3 = 0$

* for rest of problem

$$\Rightarrow |\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)$$

$|\Psi(t)\rangle = U(t,0)\,|\Psi(0)\rangle$, where $U(t,0) = e^{-iHt/\hbar}$

$$= e^{-iHt/\hbar}\left(\frac{1}{\sqrt{2}}\left[|1\rangle + |2\rangle\right]\right)$$
$$= \frac{1}{\sqrt{2}} e^{-i2t/\hbar}\left(|1\rangle + |2\rangle\right)$$

#4 (cont.)

c) We want to calculate $|\langle n|\psi(t)\rangle|^2$ where $n \in \{1,2,3\}$

$$|\langle 1|\tfrac{1}{\sqrt{2}}e^{-i2t/\hbar}(|1\rangle + |2\rangle)|^2$$

$$= \tfrac{1}{2}|\langle 1|1\rangle + \langle 1|2\rangle|^2$$

$$= \tfrac{1}{2}$$

$$|\langle 3|\tfrac{1}{\sqrt{2}}e^{-i2t/\hbar}(|1\rangle + |2\rangle)|^2$$

$$= \tfrac{1}{2}|\langle 3|1\rangle + \langle 3|2\rangle|^2$$

$$= 0$$

$$|\langle 2|\tfrac{1}{\sqrt{2}}e^{-i2t/\hbar}(|1\rangle + |2\rangle)|^2$$

$$= \tfrac{1}{2}|\langle 2|1\rangle + \langle 2|2\rangle|^2$$

$$= \tfrac{1}{2}$$

$\Rightarrow$ Oscillations will only occur if $|2\rangle$ has $\lambda = 0$ as this will prevent $e^{-i2t/\hbar}e^{i2t/\hbar} = 1$ term from forming. In an abstract notation, you would find oscillations with frequency $\frac{1}{\hbar}(\lambda_i - \lambda_j)$ where $i,j \in \{1,2,3\}$ and $i \neq j$

d) * Expectation values are by definition time-independent

$$\Rightarrow |\psi(0)\rangle = \tfrac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$

$$\langle O(t)\rangle = \langle \psi(0)|U^{\dagger}OU|\psi(0)\rangle$$

$$= \tfrac{1}{2}\left[(\langle 1| + \langle 2|)U^{\dagger}OU(|1\rangle + |2\rangle)\right]$$

$$= \tfrac{1}{2}\left[\langle 1|U^{\dagger}OU|1\rangle + \langle 1|U^{\dagger}OU|2\rangle + \langle 2|U^{\dagger}OU|1\rangle + \langle 2|U^{\dagger}OU|2\rangle\right)$$

$$= \tfrac{1}{2}\left[\langle 1|O|1\rangle + \langle 1|O|2\rangle + \langle 2|O|1\rangle + \langle 2|O|2\rangle\right]$$

$$= \tfrac{1}{2}\left[(-1)^1 1 + (-1)^2(2) + (-1)^1 1 + (-1)^2(1)\right]$$

$$= \tfrac{1}{2}\left[-1 + 2 - 1 + 2\right]$$

$$= 1$$

* See note from part c about oscillatory term if states $|2\rangle$ and $|3\rangle$ are mislabled. In the context of this problem, $e^{-i2t/\hbar}$ term introduced in cross terms

$$\hookrightarrow \langle O(t)\rangle = \tfrac{1}{2}\left[1 + e^{-i2t/\hbar}\right]$$