Problem 5: Interaction Picture of Quantum Mechanics

The "Interaction Picture" of quantum mechanics is in some ways in-between the Schrödinger formulation and the Heisenberg formulation.

Consider a system with the Hamiltonian $H = H_0 + V(t)$ where H_0 is independent of time and $V(t)$ may or may not be time dependent. The Interaction Picture is defined by the transformation of the Schrödinger states:

$$\begin{aligned} |\psi\rangle_I &= {U_0}^{-1}|\psi\rangle_S \\ U_0 &= e^{-\frac{i}{\hbar}(t-t_0)H_0}. \end{aligned} \tag{1}$$

The subscripts I and S refer to the Interaction Picture and Schrödinger Picture respectively. t_0 is a time when the pictures coincide, and we will set $t_0 = 0$ for this problem.

(a) [1 pt] Show that U_0 is a unitary operator. Why is it important for the transformation between pictures be unitary?

(b) [3 pts] The transformation between $|\psi\rangle_S$ and $|\psi\rangle_I$ implies that there is also a transformation of the observables between the pictures. If A_S and A_I are operators for an observable in the Schrödinger and Interaction pictures respectively, derive the relation between A_S and A_I. Show that this implies that H_0 is the same in the two pictures.

(c) [3 pts] Derive the differential equation that determines the time dependence of the Interaction Picture states, $|\psi(t)\rangle_I$. Be sure to show and explain your work. Explain why the Interaction Picture may be particularly useful when $V(t)$ is "small".

(d) [1 pt] Define the eigenstates of H_0 to be

$$H_0|\lambda\rangle_S = E_\lambda|\lambda\rangle_S \tag{2}$$

Show that if $V(t) = 0$, the Interaction Picture energy eigenstates $|\lambda\rangle_I$ are equal to $|\lambda(t=0)\rangle_S$ and independent of time.

(e) [2 pts] Consider a potential of the form

$$V(t) = 0, \quad t \leq 0 \quad V(t) \neq 0, \quad t > 0 \tag{3}$$

The system is in a state $|\psi_0\rangle_I$ for $t < 0$. For $t > 0$ the Interaction Picture state will depend on time. It can be expanded as:

$$|\psi(t)\rangle_I = \sum_\lambda c_\lambda(t)|\lambda(0)\rangle_I \tag{4}$$

In this expression, $c_\lambda(t)$ are time-dependent expansion coefficients for the state and $|\lambda(0)\rangle_I$ is the complete set of time-independent eigenstates of H_0 in the interaction picture.

Use the time dependence found in part (c) to derive a set of coupled equations relating $c_\lambda(t)$ and $\partial_t c_\lambda(t)$.

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Quantum #5

a) An operator is unitary if: $U^{+}U = \mathbb{I}$

$$\hookrightarrow U = \exp\left[\frac{-i(t-t_0)H_0}{\hbar}\right]$$

$$U^{+} = \exp\left[\frac{i(t-t_0)H_0}{\hbar}\right]$$

$$U^{+}U = \exp\left[\frac{i(t-t_0)H_0}{\hbar}\right]\exp\left[\frac{-i(t-t_0)H_0}{\hbar}\right]$$

$$= 1 \checkmark$$

Our transformation must be a unitary operator b/c it preserves lengths of vectors, as well as the angles b/w them

b) This transformation is best seen in terms of an expectation value. In the Schrödinger picture:

$$\langle A \rangle = {}_S\langle\psi| A_S |\psi\rangle_S$$

Since we can only multiply by 1 and we want $|\psi\rangle_I = U_0^{-1}|\psi\rangle_S$

$$= {}_S\langle\psi| U_0 U_0^{-1} A_S U_0 U_0^{-1} |\psi\rangle_S$$

$$= {}_I\langle\psi| U_0^{-1} A_S U_0 |\psi\rangle_I$$

$$\hookrightarrow U_0^{-1} A_S U_0 = A_I$$

*Not sure I've fully answered this part

c) $|\psi\rangle_I = \exp\left[\frac{i(t-t_0)H_0}{\hbar}\right]|\psi\rangle_S$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle_I = i\hbar\frac{\partial}{\partial t}\left(\exp\left[\frac{i(t-t_0)H_0}{\hbar}\right]|\psi\rangle_S\right)$$

$$= i\hbar\left(\frac{H_0}{-i\hbar}\exp\left[\frac{i(t-t_0)H_0}{\hbar}\right]|\psi\rangle_S + \left[\frac{i(t-t_0)H_0}{\hbar}\right]\frac{\partial}{\partial t}|\psi\rangle_S\right)$$

$$= -H_0\exp\left[\frac{i(t-t_0)H_0}{\hbar}\right]|\psi\rangle_S + (H_0+V)|\psi_S\rangle\exp\left[\frac{i(t-t_0)H_0}{\hbar}\right]$$

$$= V\exp\left[\frac{i(t-t_0)H_0}{\hbar}\right]|\psi\rangle_S$$

$$= V|\psi\rangle_I$$

#5 (cont.)

c) $i\hbar \frac{\partial}{\partial t} \langle n | \psi \rangle_I = \langle n | V | \psi \rangle_I$

$$i\hbar \frac{\partial}{\partial t} c_n(t) = \sum_m \langle n | V | m \rangle \langle m | \psi \rangle_I$$

$$= \sum_m c_m V_{nm}$$

expanding $c_n(t)$ as $c_n^{(0)}(t) + \lambda c_n^{(1)}(t) + \ldots$

$$i\hbar \frac{\partial}{\partial t} \left(c_n^{(0)}(t) + \lambda c_n^{(1)}(t) + \ldots \right) = \sum_m \left[c_m^{(0)}(t) + \lambda c_n^{(1)}(t) \right.$$