Problem 5: Angular Momentum (10 Points):

10

Suppose an electron is in a state described by the wave function

$$\psi = \frac{1}{\sqrt{4\pi}}(e^{i\phi}\sin\theta + \cos\theta)g(r)$$

where $\int_0^\infty |g(r)|^2 r^2 dr = 1$
and ϕ, θ are the azimuth and polar angles respectively.

✓(a) Express ψ in terms of spherical harmonics functions. (2 pts.)

✓(b) What are the possible results of a measurement of the z-component L_z of the angular momentum of the electron in this state? (2 pts.)

✓(c) Determine if $\int |\psi|^2 \, d^3\vec{r} = 1$. (2 pts.)

✓(d) Use the result in (c) to find the probability of obtaining each of the possible results in part (b). (2 pts.)

✓(e) What is the expectation value of L_z? (2 pts.)

$\cos\varphi = \frac{1}{2} e^{i\varphi} + e^{-i\varphi}$

$\sin\varphi = \frac{1}{2i} e^{i\varphi} - e^{-i\varphi}$

$\cos\varphi - i\sin\varphi = e^{-i\varphi}$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

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Quantum #5

a) $\psi = \frac{1}{\sqrt{4\pi}} \left(e^{i\varphi}\sin\theta + \cos\theta\right) g(r)$

* But we know $Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta$

$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin\theta$

$\hookrightarrow \psi = \frac{1}{\sqrt{4\pi}} \cdot \left(-\sqrt{\frac{8\pi}{3}} Y_{1,1} + \sqrt{\frac{4\pi}{3}} Y_{1,0}\right) g(r)$

$= \left(\frac{1}{\sqrt{3}} Y_{1,0} - \sqrt{\frac{2}{3}} Y_{1,1}\right) g(r)$

* Check normalization

$1 = \frac{1}{4\pi} A^2 \int_0^\infty r^2 dr \int_0^{2\pi} d\varphi \int_0^\pi d\theta\, |g(r)|^2 \left(e^{-i\varphi}\sin\theta + \cos\theta\right)\left(e^{i\varphi}\sin\theta + \cos\theta\right)$

$1 = \frac{1}{4\pi} A^2 \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta\, d\theta \cdot \underbrace{\sin^2\theta + \cos^2\theta}_{1} + e^{-i\varphi}\sin\theta\cos\theta + e^{i\varphi}\sin\theta\cos\theta$

$1 = \frac{1}{4\pi} A^2 \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta\, d\theta \cdot \left(1 + \sin\theta\cos\theta\left(e^{-i\varphi} + e^{i\varphi}\right)\right)$

$1 = \frac{1}{4\pi} A^2 \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta + 2\sin^2\theta\cos\theta\cos\varphi\, d\theta$

$1 = \frac{1}{4\pi} A^2 \int_0^{2\pi} d\varphi \left[-\cos\theta + \frac{2}{3}\sin^3\theta\cos\varphi\right]\Big|_0^\pi$

$1 = \frac{1}{4\pi} A^2 \int_0^{2\pi} d\varphi \left[(1 + 0) - (-1 + 0\right.$

$1 = \frac{1}{2\pi} A^2 \int_0^{2\pi} d\varphi$

$1 = A^2 \Rightarrow A = 1$ ✓

b) Rewriting our function in bra-ket notation

$\psi = \frac{1}{\sqrt{3}} |1,0\rangle - \sqrt{\frac{2}{3}} |1,1\rangle$

$\hookrightarrow L_z|\psi\rangle = L_z \cdot \frac{1}{\sqrt{3}}|1,0\rangle - \sqrt{\frac{2}{3}} L_z |1,1\rangle$

* possible measurements are $L_z = 0, 1$

#5 (cont.)

c) See work from part a checking normalization

d) $\langle \psi | L_z | \psi \rangle = 0 \cdot \frac{1}{3} \cancelto{1}{\langle 1,0|1,0\rangle} + \frac{2}{3} \cancelto{1}{\langle 1,1|1,1\rangle} \cdot 1$ (Other terms ignored due to orthogonality)

↳ $L_z = 0$ $\frac{1}{3}$ of the time

$L_z = 1$ $\frac{2}{3}$ of the time

(Expectation value is weighted sum of possible measurements)

e) $\langle \psi | L_z | \psi \rangle = \frac{2}{3}$