Jan 2006

Problem 2: The Harmonic Oscillator (10 Points):

The normalized wave functions for the one-dimensional quantum harmonic oscillator can be written as,

$$\Psi_n(x) = \left(\frac{\sqrt{\alpha}}{2^n n! \sqrt{\pi}}\right)^{1/2} e^{-\alpha x^2/2} H_n(\sqrt{\alpha}x),$$

where n is the principle quantum number of the oscillator, H_n is the n^{th} order Hermite polynomial, $\alpha = \omega m/\hbar$, ω is the oscillator frequency, and m is its mass. The following equations may be useful,

$$H_{n+1}(q) + 2nH_{n-1}(q) - 2qH_n(q) = 0$$

$$\frac{dH_n(q)}{dq} = 2nH_{n-1}(q)$$

and

$$\begin{aligned} \langle H_n | qH_{n+1} \rangle &= 2^n(n+1)!\sqrt{\pi} \\ \langle H_n | qH_n \rangle &= 0 \\ \langle H_n | qH_{n-1} \rangle &= 2^{n-1} n! \sqrt{\pi} \end{aligned}$$

1. Calculate the expectation value of x and x^2 for the n^{th} state of the harmonic oscillator, where x is the position. **(2 Points)**

2. Calculate the expectation value of p and p^2 for the n^{th} state of the harmonic oscillator, where p is the momentum. **(2 Points)**

3. Calculate Δx and Δp for the n^{th} state. What is the uncertainty product $(\Delta x \Delta p)$ for the oscillator?**(2 Points)**

4. Calculate the expectation value of the kinetic energy and the potential energy of the n^{th} state of the oscillator. Show that the sum of the expectation value of the kinetic and potential energies are equal to the total energy of the n^{th} state. **(2 Points)**

5. How does the uncertainty principle relate to the fact that the energy is not zero in the ground state? Explain and interpret your answer to receive credit.**(2 Points)**

Jan 2008

Quantom #2

a) Given: $\Psi_n(x) = \left(\frac{\sqrt{\alpha}}{2^n n! \sqrt{\pi}}\right)^{1/2} \exp\left[-\frac{1}{2}\alpha x^2\right] H_n(\sqrt{\alpha}x)$

Find: $\langle x \rangle_n$, $\langle x^2 \rangle_n$

* Using raising/lowering operators, we know:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+) \qquad a|\Psi_n\rangle = \sqrt{n}\,|\Psi_{n-1}\rangle \qquad \langle \Psi_n | \Psi_m \rangle = \delta_{mn}$$

$$p = -i\sqrt{\frac{\hbar m\omega}{2}}\,(a - a^+) \qquad a^+|\Psi_n\rangle = \sqrt{n+1}\,|\Psi_{n+1}\rangle$$

$$\Rightarrow \langle x \rangle_n = \langle \Psi_n | x | \Psi_n \rangle$$
$$= \langle \Psi_n | \sqrt{\frac{\hbar}{2m\omega}}(a + a^+) | \Psi_n \rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle \Psi_n | a | \Psi_n \rangle + \langle \Psi_n | a^+ | \Psi_n \rangle\right]$$
$$= \sqrt{\hbar/2m\omega}\left[\langle \Psi_n | \sqrt{n} | \Psi_{n-1} \rangle + \langle \Psi_n | \sqrt{n+1} | \Psi_{n+1} \rangle\right]$$
$$= 0$$

$$\langle x^2 \rangle_n = \langle \Psi_n | x^2 | \Psi_n \rangle$$
$$= \langle \Psi_n | \left(\frac{\hbar}{2m\omega}\right)(aa + aa^+ + a^+a + a^+a^+) | \Psi_n \rangle$$
$$= \frac{\hbar}{2m\omega}\left[\langle \Psi_n | aa | \Psi_n \rangle + \langle \Psi_n | aa^+ | \Psi_n \rangle + \langle \Psi_n | a^+a | \Psi_n \rangle + \langle \Psi_n | a^+a^+ | \Psi_n \rangle\right]$$
$$= \frac{\hbar}{2m\omega}\left[\langle \Psi_n | \sqrt{n}\sqrt{n-1} | \Psi_{n-2} \rangle + \langle \Psi_n | \sqrt{n+1}\sqrt{n+1} | \Psi_n \rangle + \langle \Psi_n | \sqrt{n}\sqrt{n} | \Psi_n \rangle + \langle \Psi_n | \sqrt{n+1}\sqrt{n+2} | \Psi_{n+2} \rangle\right]$$
$$= \frac{\hbar}{2m\omega}\left[2n + 1\right]$$

b) Similarly to above:

$$\langle p \rangle_n = \langle \Psi_n | p | \Psi_n \rangle$$
$$= \langle \Psi_n | -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^+) | \Psi_n \rangle$$
$$= -i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle \Psi_n | a | \Psi_n \rangle - \langle \Psi_n | a^+ | \Psi_n \rangle\right]$$
$$= -i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle \Psi_n | \sqrt{n} | \Psi_{n-1} \rangle - \langle \Psi_n | \sqrt{n+1} | \Psi_{n+1} \rangle\right]$$
$$= 0$$

#2 (cont.)

b) $\langle p_n^2 \rangle = \langle \psi_n | p^2 | \psi_n \rangle$

$= \langle \psi_n | -\frac{\hbar m \omega}{2}(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger) | \psi_n \rangle$

$= -\frac{\hbar m \omega}{2}\left[\langle \psi_n | aa | \psi_n \rangle - \langle \psi_n | aa^\dagger | \psi_n \rangle - \langle \psi_n | a^\dagger a | \psi_n \rangle + \langle \psi_n | a^\dagger a^\dagger | \psi_n \rangle\right]$

$= -\frac{\hbar m \omega}{2}\left[\langle \psi_n | \sqrt{n-1}\sqrt{n} | \psi_{n-2} \rangle - \langle \psi_n | \sqrt{n+1}\sqrt{n+1} | \psi_n \rangle - \langle \psi_n | \sqrt{n}\sqrt{n} | \psi_n \rangle + \langle \psi_n | \sqrt{n+2}\sqrt{n+1} | \psi_{n+2} \rangle\right]$

$= \frac{\hbar m \omega}{2}[2n+1]$

c) Generally $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$

$\Rightarrow \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

$= \sqrt{\frac{\hbar}{2m\omega}[2n+1] - 0^2}$

$= \sqrt{\frac{\hbar}{2m\omega}[2n+1]}$

$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

$= \sqrt{\frac{\hbar m \omega}{2}[2n+1] - 0^2}$

$= \sqrt{\frac{\hbar m \omega}{2}(2n+1)}$

$\Rightarrow \Delta x \Delta p = \sqrt{\frac{\hbar}{2m\omega}[2n+1]}\sqrt{\frac{\hbar m \omega}{2}[2n+1]}$

$= \frac{\hbar}{2}[2n+1]$

d) $\langle T \rangle = \langle \psi_n | T | \psi_n \rangle$

$= \langle \psi_n | \frac{p^2}{2m} | \psi_n \rangle$

$= \frac{\hbar\omega}{4}[2n+1]$

$\langle U \rangle = \langle \psi_n | U | \psi_n \rangle$

$= \langle \psi_n | \frac{1}{2}m\omega^2 x^2 | \psi_n \rangle$

$= \frac{\hbar\omega}{4}[2n+1]$

$\hookrightarrow \langle T \rangle + \langle U \rangle = \frac{\hbar\omega}{2}[2n+1]$ which matches what we know to be the energy of the n^{th} state; $E_n = \hbar\omega(n + 1/2)$

e) * From the above formula, we know $E_0 = \frac{\hbar\omega}{2}$ and that $\Delta x \Delta p = \hbar/2$

$\Rightarrow$ Rewriting the total energy in terms of the uncertainties, we see:

$\Delta E = \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2$

#2 (cont.)

e) $\quad * \text{but} \quad \Delta p = \frac{\hbar}{2\Delta x}$

$$\Rightarrow \Delta E = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2$$

$\frac{d(\Delta E)}{d(\Delta x} = 0$ will give minimum of energy

$$\Rightarrow 0 = \frac{-\hbar^2}{4m(\Delta x^3)} + m\omega^2(\Delta x)$$

$$\frac{\hbar^2}{4m(\Delta x)^3} = m\omega^2(\Delta x)$$

$$\frac{\hbar^2}{4m^2\omega^2} = \Delta x^4 \quad \Rightarrow \quad \Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Rightarrow \Delta E = \frac{\hbar^2}{8m}\left(\frac{2m\omega}{\hbar}\right) + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)$$

$$= \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4}$$

$$= \frac{\hbar\omega}{2}$$

$\Rightarrow$ The uncertainly principle directly implies a non-zero ground state energy