PROBLEM 3: Dirac formulation of quantum mechanics

Let $\mathcal{E}_3$ be a three-dimensional Hilbert space that is spanned by the orthonormal basis $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$. The operator Ω acts in $\mathcal{E}_3$ as follows:

$$\begin{aligned} \Omega|u_1\rangle &= 3|u_1\rangle && (1) \\ \Omega|u_2\rangle &= 2|u_2\rangle - |u_3\rangle && (2) \\ \Omega|u_3\rangle &= -|u_2\rangle + 2|u_3\rangle && (3) \end{aligned}$$

(a) [5 pt] Prove that Ω is Hermitian. Find its eigenvalues, ω_1, ω_2, and ω_3, and write down each of the corresponding eigenvectors in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis.

(b) [1 pt] Does $\{\Omega\}$ constitute a complete set of commuting operators for $\mathcal{E}_3$? Why or why not?

(c) [2 pt] According to Eq. (1), $\mathcal{E}_3$ can be partitioned into eigensubspaces by letting $\mathcal{E}_a$ be the subspace spanned by $\{|u_1\rangle\}$ and $\mathcal{E}_b$ be its orthogonal supplement. Construct an orthonormal basis $\{|t_2\rangle, |t_3\rangle\}$ of $\mathcal{E}_b$, and write each basis vector in $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis. (Choose $|t_3\rangle$ to correspond to the *smallest* eigenvalue of Ω.)

(d) [2 pt] With $|t_1\rangle = |u_1\rangle$, the set $\{|t_1\rangle, |t_2\rangle, |t_3\rangle\}$ constitutes an alternate basis of $\mathcal{E}_3$. Find the matrix S, with elements $S_{i,k} = \langle u_i \mid t_k\rangle$, that transforms between $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ and $\{|t_1\rangle, |t_2\rangle, |t_3\rangle\}$.

Aug 2010.

Quantum #3

a) * Based on the given info, we can determine Ω has the form

$$\Omega = \begin{array}{c c} & \begin{array}{ccc} |u_1\rangle & |u_2\rangle & |u_3\rangle \end{array} \\ \begin{array}{c} \langle u_1| \\ \langle u_2| \\ \langle u_3| \end{array} & \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \end{array}$$

The condition for Hermitivity is that $A^+ = A$

$$\Rightarrow \Omega^+ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \Omega \checkmark$$

* To solve for eigenvalues

$$|\Omega - \lambda \mathbb{I}| = 0$$

$$\hookrightarrow 0 = (3-\lambda)[(2-\lambda)^2 - 1]$$

$$= (3-\lambda)[2-\lambda+1][2-\lambda-1]$$

$$\hookrightarrow \lambda = 3, 3, 1$$

*To solve for eigenvectors

$$\Omega\vec{v} = \lambda\vec{v} \Rightarrow \begin{aligned} 3x_1 &= \lambda x_1 \\ 2x_2 - x_3 &= \lambda x_2 \\ -x_2 + 2x_3 &= \lambda x_3 \end{aligned}$$

$\Rightarrow$ if $\lambda = 3$

$$3x_1 = 3x_1$$
$$2x_2 - x_3 = 3x_2$$
$$-x_2 + 2x_3 = 3x_3$$

$\Rightarrow |u_1\rangle = \langle 1, 0, 0\rangle$

$|u_2\rangle = \langle 0, 1, -1\rangle \cdot \frac{1}{\sqrt{2}}$

$\Rightarrow$ if $\lambda = 1$

$$3x_1 = x_1$$
$$2x_2 - x_3 = x_2$$
$$-x_2 + 2x_3 = x_3$$

$|u_3\rangle = \langle 0, 1, 1\rangle \cdot \frac{1}{\sqrt{2}}$

#3 (cont.)

b) Because Ω has degenerate eigenvalues, it cannot by definition be a complete set of commuting operators

c)