Problem 3: Identical particles (10 pts)

Two non-interacting particles of mass m are trapped in a 1-dimensional infinite box of length L situated between $x = 0$ and $x = L$. (In the cases you are considering fermions, assume them to all be spin up.)

(a) [1 points] Write down the single particle energy eigenvalues and wavefunctions.

(b) [1 points] Write down the energy eigenvalues and wavefunctions for two distinguishable particles. Label the states by n_1 for particle 1 and n_2 for particle 2.

(c) [2 points] An energy measurement of the *two identical particle* system yields $E = \hbar^2\pi^2/mL^2$. Write down the state vector/wave function of the system.

(d) [2 points] Suppose instead the energy of the two identical particle system is measured to be $E = 5\hbar^2\pi^2/mL^2$. What is the wave function? *Hint: there are two possibilities.*

(e) [2 points] Show that the fermion state you found in part (d) is an eigenfunction of the Hamiltonian, with the appropriate eigenvalue.

(f) [1 points] Write down the wavefunction for two identical spin-up fermions in the $n_1 = 2$ and $n_2 = 2$ state.

(g) [1 points] If instead you had three particles in the orthonormal states Ψ_1, Ψ_2, and Ψ_3, construct the three particle state for identical fermions.

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Quantum #3

a) For an infinite well b/w 0 and L, our solution is:

$$\psi_n(x) = \sqrt{\tfrac{2}{L}} \sin\left(\tfrac{n\pi x}{L}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

for any single particle

b) Assuming the particles are distinguishable, we simply use the above equations

$$\psi_{n_1} = \sqrt{\tfrac{2}{L}} \sin\left(\tfrac{n_1\pi x}{L}\right) \qquad E_{n_1} = \frac{n_1^2\pi^2\hbar^2}{2mL^2}$$

$$\psi_{n_2} = \sqrt{\tfrac{2}{L}} \sin\left(\tfrac{n_2\pi x}{L}\right) \qquad E_{n_2} = \frac{n_2^2\pi^2\hbar^2}{2mL^2}$$

c) Considering two identical spin-up fermions, the exclusion principle prevents both particles from being in the same state. Since the only combination that yields $E_{n_1 n_2} = \frac{(n_1^2+n_2^2)\pi^2\hbar^2}{2mL^2} = \frac{\pi^2\hbar^2}{mL^2}$ is $n_1 = n_2 = 1$, this state is disallowed by the exclusion principle, therefore

$$\psi_{1,1} = 0$$

d) If $E_{n_1,n_2} = \frac{5\pi^2\hbar^2}{mL^2}$, our possible configurations are $n_1 = 1, n_2 = 3$; $n_1 = 3, n_2 = 1$

Our general wavefunction for identical fermions is:

$$\psi_{n_1,n_2} = \frac{1}{\sqrt{2}}\left(\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)\right)$$

This yields the following potential wave functions:

$$\psi_{13} = \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1)\right)$$

$$= \frac{\sqrt{2}}{L}\left(\sin\left(\tfrac{\pi x_1}{L}\right)\sin\left(\tfrac{3\pi x_2}{L}\right) - \sin\left(\tfrac{\pi x_2}{L}\right)\sin\left(\tfrac{3\pi x_1}{L}\right)\right)$$

$$\psi_{31} = \frac{\sqrt{2}}{L}\left(\sin\left(\tfrac{3\pi x_1}{L}\right)\sin\left(\tfrac{\pi x_2}{L}\right) - \sin\left(\tfrac{3\pi x_2}{L}\right)\sin\left(\tfrac{\pi x_1}{L}\right)\right)$$

#3 (cont.)

e) We know that $H\psi_n = E_n \psi_n$ where $H = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$

$\Rightarrow$ For the ψ_{13} state;

$$\frac{\partial^2}{\partial x_1^2}\psi_{13} = \frac{\sqrt{2}}{L}\left(\frac{-\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \left(\frac{3\pi}{L}\right)^2\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right)$$

$$\frac{\partial^2}{\partial x_2^2}\psi_{13} = \frac{\sqrt{2}}{L}\left(-\left(\frac{3\pi}{L}\right)^2\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right)$$

$$\hookrightarrow H\psi_{13} = \frac{-\hbar^2}{2m}\left(\frac{\sqrt{2}}{L}\left[\frac{-\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \frac{9\pi^2}{L^2}\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right.\right.$$

$$+ \frac{\sqrt{2}}{L}\left[-\frac{9\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \frac{\pi^2}{L^2}\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right)$$

$$= \frac{\hbar^2 10\pi^2}{2mL^2}\left(\frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right]\right)$$

$$= \frac{10\hbar^2\pi^2}{2mL^2}\psi_{13} \checkmark$$

* A similar process will reach the same conclusion for the ψ_{31} state

f) The $n_1 = n_2 = 2$ state is disallowed by the exclusion principle

$\hookrightarrow \psi_{22} = 0$

g) My guess is this follows something like a cyclic permutation

$$\Rightarrow \psi_{n_1 n_2 n_3} = \frac{1}{\sqrt{6}}\left[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) - \psi_{n_1}(x_2)\psi_{n_2}(x_3)\psi_{n_3}(x_1)\right.$$

$$- \psi_{n_1}(x_3)\psi_{n_2}(x_1)\psi_{n_3}(x_2) + \psi_{n_1}(x_3)\psi_{n_2}(x_2)\psi_{n_3}(x_1)$$

$$\left.+ \psi_{n_1}(x_2)\psi_{n_2}(x_1)\psi_{n_3}(x_3) + \psi_{n_1}(x_1)\psi_{n_2}(x_3)\psi_{n_3}(x_2)\right]$$