PROBLEM 2: Generalized Uncertainty Principle

Consider the spin 1/2 operator

$$\mathbf{S} = \frac{\hbar}{2}\vec{\sigma},$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli matrices, which are defined in the basis of the S_z operator eigenvectors,

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) Compute the commutator $[S_i, S_j]$, with $i, j = x, y, z$. [2 Points]

(b) Compute the expectation values $\langle(\delta S_x)^2\rangle$ and $\langle(\delta S_y)^2\rangle$ for the state

$$|\alpha\rangle = \cos(\alpha)|+\rangle + \sin(\alpha)|-\rangle,$$

where $\delta\mathbf{S} = \mathbf{S} - \langle\mathbf{S}\rangle$. Show explicitly that the relation

$$\langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle \geq \frac{1}{4}\left|\langle[S_x, S_y]\rangle\right|^2$$

is satisfied. What does it physically mean? [4 Points]

(c) Find the states that maximize and minimize the product $\langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle$. Interpret the results. [2 Points]

(d) Suppose one performs an experiment which filters the $+\hbar/2$ eigenstate of the S_z operator from the initially prepared state $|\alpha\rangle$. Then the S_x component of the spin is measured. Compute the expectation value of this measurement in the state $|\alpha\rangle$. [2 Points]

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Quantum #2

a) It is well known that $[S_i, S_j] = i\hbar S_k$

$$\Rightarrow S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad S_y = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Proof! $[S_x, S_y] = S_x S_y - S_y S_x$

* Continues as such for other pairs and can be proved quickly on exam

$$= \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\frac{\hbar^2}{4}$$

$$= \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\right)\frac{\hbar^2}{4}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$$

$$= \frac{i\hbar^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= i\hbar S_z$$

b) Using $|\alpha\rangle = \cos(\alpha)|+\rangle + \sin(\alpha)|-\rangle$, find $\langle(\delta S_x)^2\rangle \langle(\delta S_y)^2\rangle$

* Note! $\langle(\delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$

$$\Rightarrow \langle(\delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$$

$$= [\cos(\alpha)\ \sin(\alpha)]\begin{bmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{bmatrix}\begin{bmatrix} \cos(\alpha) \\ \sin\alpha \end{bmatrix} - \left([\cos(\alpha)\ \sin(\alpha)]\begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}\right)^2$$

$$= [\cos\alpha\ \sin\alpha]\begin{bmatrix} \hbar^2/4\,\cos\alpha \\ \hbar^2/4\,\sin\alpha \end{bmatrix} - \left([\cos\alpha\ \sin\alpha]\begin{bmatrix} \hbar/2\,\sin\alpha \\ \hbar/2\,\cos\alpha \end{bmatrix}\right)^2$$

$$= \hbar^2/4 - (\cdot\,\hbar \sin\alpha\cos\alpha)^2$$

$$= \hbar^2\left(\frac{1}{4} - \sin^2\alpha\cos^2\alpha\right)$$

$$= \frac{\hbar^2}{4}\left(1 - \frac{1-\cos(4\theta)}{2}\right)$$

$$= \frac{\hbar^2}{4}\left(\frac{1}{2} - \cos(4\alpha)\right)$$

$$= \frac{\hbar^2}{8}(1 - \cos(4\alpha))$$

#2 (cont.)

b) $\langle(\delta S_y)^2\rangle = \langle S_y^2\rangle - \langle S_y\rangle^2$

$$= \hbar^2/4 - \left(\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}0 & -\frac{i\hbar}{2}\\ \frac{i\hbar}{2} & 0\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}\right)^2$$

$$= \frac{\hbar^2}{4} - \left(\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}-\frac{i\hbar}{2}\sin\alpha\\ \frac{i\hbar}{2}\cos\alpha\end{bmatrix}\right)^2$$

$$= \frac{\hbar^2}{4} - \left(-\frac{i\hbar}{2}\sin\alpha\cos\alpha + \frac{i\hbar}{2}\cos\alpha\sin\alpha\right)^2$$

$$= \frac{\hbar^2}{4}$$

$$\frac{1}{4}\left|\langle[S_x, S_y]\rangle\right|^2 = \frac{1}{4}\left|\langle S_z\rangle\right|^2$$

$$= \frac{1}{4}\left|\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}\hbar/2 & 0\\ 0 & -\hbar/2\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}\right|^2$$

$$= \frac{1}{4}\left|\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}\hbar/2\cos\alpha\\ -\hbar/2\sin\alpha\end{bmatrix}\right|^2$$

$$= \frac{1}{4}\left|\frac{\hbar}{2}(\cos^2\alpha - \sin^2\alpha)\right|^2$$

$$= \frac{\hbar^2}{16}(\cos^4\alpha - 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha)$$

$$\Rightarrow \langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle \overset{?}{\geq} \frac{1}{4}\left|\langle[S_x, S_y]\rangle\right|^2$$

$$\frac{\hbar^2}{4}\left(\frac{\hbar^2}{4} - \hbar^2\sin^2\alpha\cos^2\alpha\right) \overset{?}{\geq} \frac{\hbar^2}{16}(\cos^4\alpha - 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha)$$

$$\frac{\hbar^4}{16} - \frac{\hbar^4}{4}\sin^2\alpha\cos^2\alpha \overset{?}{\geq} \frac{\hbar^2}{16}(\cos^4\alpha + \sin^4\alpha) - \frac{\hbar^2}{8}\cos^2\alpha\sin^2\alpha$$

#2 (cont.)

c) $A = \langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle \Rightarrow$ max/min @ $\frac{dA}{d\alpha} = 0$

$$\Rightarrow 0 = \frac{dA}{d\alpha} = \frac{d}{d\alpha}\left(\frac{\hbar^4}{16} - \frac{\hbar^4}{4}\sin^2\alpha\cos^2\alpha\right)$$

$$0 = -\frac{\hbar^4}{4}\left(2\sin\alpha\cos^3\alpha - 2\cos\alpha\sin^3\alpha\right)$$ $\Rightarrow$ any multiple of $\pi/2$ also zero's function

$$0 = \frac{-\hbar^2}{2}\sin\alpha\cos^3\alpha + \frac{\hbar^2}{2}\cos\alpha\sin^3\alpha$$

$$\sin\alpha\cos^3\alpha = \cos^\alpha\sin^3\alpha$$

$$\cos^2\alpha = \sin^2\alpha$$

$$1-\sin^2\alpha = \sin^2\alpha$$

$$1 = 2\sin^2\alpha$$

$$\frac{1}{2} = \sin^2\alpha$$

$$\pm\frac{1}{\sqrt{2}} = \sin\alpha$$

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \alpha = n\frac{\pi}{4}, \quad n = 1, 3, 5 \ldots$$

$\Rightarrow$ minimums at multiples of $n\pi/2$, maximums at multiples of $n\pi/4$

↳ Minimal States: $|\alpha\rangle = |+\rangle$
$= |-\rangle$

↳ Maximal states: $|\alpha\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$
$= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$
$= \frac{1}{\sqrt{2}}(-|+\rangle + |-\rangle)$
$= \frac{1}{\sqrt{2}}(-|+\rangle - |-\rangle)$

d) $|\alpha\rangle = |+\rangle$ by virtue of the experiment

$$\Rightarrow \langle S_x\rangle = \langle +|S_x|+\rangle$$

$$= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$= 0$$