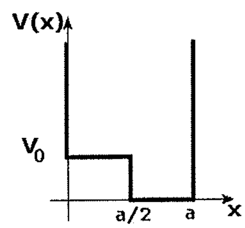
Problem 3: Double Step Potential

Consider a single particle of mass m in a one dimensional well of width a and a potential, $V(x)$, given by:

$$V(x) = \begin{cases} \infty, & x < 0 \\ V_0, & 0 < x < \frac{a}{2} \\ 0, & \frac{a}{2} < x < a \\ \infty, & x > a \end{cases} \tag{1}$$



In this question, you will consider the special cases where this potential well has a bound state at the energy $E = V_0$. There are only certain values of V_0 and a where this will happen.

In this problem, use the constant

$$k = \sqrt{\frac{2mV_0}{\hbar^2}} \tag{2}$$

(a) [2 pts] For the energy $E = V_0$ in this potential, determine the general eigenfunction solutions to the time-independent Schrödinger equation in all regions of x. Show your work.

(b) [3 pts] Apply boundary conditions to determine relationships between the constants you introduced in writing the wave functions in part (a).

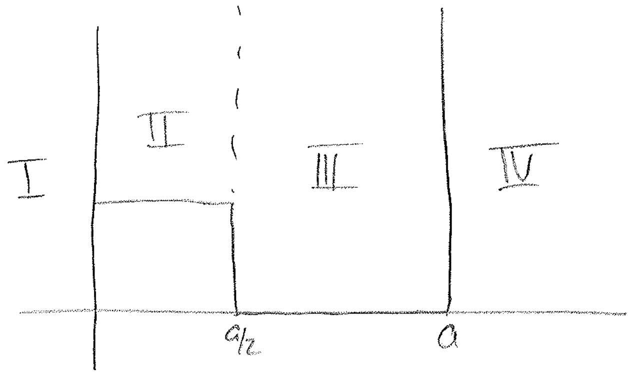
(c) [2 pts] From your results above, derive a transcendental equation that gives the values of V_0 where there is an energy eigenstate with $E = V_0$, for a fixed well width a. This equation will have the form $z = f(z)$ with $z = k\frac{a}{2}$. Plot this function and determine a relationship between the first energy V_0 that satisfies this equation and the bound state energies of a square well of width a.

(d) [2 pts] Qualitatively sketch the wave function that corresponds to the smallest value of V_0 that satisfies the transcendental equation from part (c), for a fixed value of a.

(e) [1 pt] Finally, consider the case where the width of the well is fixed but the potential step, V_0, can be changed. There are an infinite number of possible values of V_0 where the well contains an energy eigenstate with $E = V_0$. Describe, qualitatively, the changes in the wavefunctions of these eigenstates as V_0 gets larger.

Jan 2015

Quantum #3



a) $H\psi_i = E\psi_i$

* In regions I + IV: $\psi_I = 0 = \psi_{IV}$ (due to infinite potential)

* In region III: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{-2mE}{\hbar^2}\psi$$

* if $k = \sqrt{\frac{2mV_0}{\hbar}}$ ($E = V_0$ as stated in problem)

$$\frac{\partial^2\psi}{\partial x^2} = -k^2\psi$$

$\hookrightarrow \psi_{III}(x) = Ae^{-ikx} + Be^{ikx}$

* In region II: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = (E-V)\psi$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{-2m(E-V)}{\hbar^2}\psi$$

$\hookrightarrow$ let $k = \sqrt{\frac{2m(E-V)}{\hbar^2}}$, $E = V_0$ as stated in problem

$$\frac{\partial^2\psi}{\partial x^2} = 0$$

$\hookrightarrow \psi_{II} = Cx + D$

#3 (cont.)

b) * Remember, our two boundary conditions are that: ψ is continuous

$\frac{d\psi}{dx}$ is continuous where $V \neq \infty$

$0 = C(0) + D$

$\hookrightarrow \boxed{D = 0}$

$0 = Ae^{-ika} + Be^{ika}$

$= A + Be^{2ika}$

$\hookrightarrow \boxed{A = -Be^{2ika}}$

$C\left(\frac{a}{2}\right) = -Be^{3ika/2} + Be^{ika/2}$

$C = \frac{2B}{a}\left(e^{ika/2} - e^{3ika/2}\right)$

$C = ikA\left(e^{ika/2} + e^{3ika/2}\right)$