## Problem 1: Solving the Harmonic Oscillator

Solving the differential equation form of the time-independent Schrödinger equation for the eigenstates of the harmonic oscillator Hamiltonian in 1D requires solving a second order differential equation. By using operator algebra, it is possible to simplify the solution to this problem.

The 1D harmonic oscillator is described by the Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m}{2}\omega^2 X^2. \tag{1}$$

Define the unitless variables

$$x = \frac{X}{\lambda}, \quad p = \frac{\lambda}{\hbar}P, \quad \lambda = \sqrt{\frac{\hbar}{m\omega}}. \tag{2}$$

such that the Hamiltonian has the form

$$H = \frac{\hbar\omega}{2}\left(p^2 + x^2\right). \tag{3}$$

Note that $x$ and $p$ are conjugate observables, $[x, p] = i$

(a) [2 pt] Using the harmonic oscillator operators

$$\hat{a} = \frac{1}{\sqrt{2}}(x + ip), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(x - ip), \quad \hat{n} = \hat{a}^\dagger\hat{a}, \tag{4}$$

and their commutation relations, show that the Hamiltonian can be written as

$$H = \hbar\omega(\hat{n} + \frac{1}{2}). \tag{5}$$

(b) [2 pts] Define the eigenstates of the operator $\hat{n}$:

$$\hat{n}|n\rangle = n|n\rangle, \tag{6}$$

with $n$ some (unitless) numbers. Use the operator commutation relations to show that

$$\begin{aligned} \hat{a}|n\rangle &= c(n)|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= d(n)|n+1\rangle. \end{aligned} \tag{7}$$

Derive expressions for $c(n)$ and $d(n)$. Show your work.

(c) [3 pts] The potential, $V(x) = \frac{\hbar\omega}{2}x^2 \geq 0$ for all $x$. Explain why this implies that:

1. The eigenenergies of the Harmonic Oscillator must be positive
2. The eigenvalues of $\hat{n}$ must be non-negative integers
3. There is a lowest eigenstate of $\hat{n}$, $|0\rangle$ defined by $\hat{a}|0\rangle = 0$.

(d) [2 pts] Show that results above define a first order differential equation in $X$ that can be solved for the ground state harmonic oscillator wavefunction $\psi_0(X)$. Determine this equation and solve for this wavefunction.

(e) [1 pt] Use the result from (e) and the operators to determine the first excited state wavefunction for the harmonic oscillator, $\psi_1(X)$.

# Quantum #1



a) Given: $H = \frac{\hbar\omega}{2}(p^2 + x^2)$ , $[x, p] = i$

$a = \frac{1}{\sqrt{2}}(x + ip)$ $\quad a^+ = \frac{1}{\sqrt{2}}(x - ip)$ $\quad n = a^+ a$

* Rewrite $H$ in terms $a, a^+$ then $n$

$$\Rightarrow \sqrt{2}\,a = (x + ip) \qquad \qquad x = \frac{1}{\sqrt{2}}(a + a^+)$$
$$\sqrt{2}\,a^+ = (x - ip) \quad \Longleftrightarrow \quad p = \frac{-i}{\sqrt{2}}(a - a^+)$$

$$\Rightarrow H = \frac{\hbar\omega}{2}\left(\left[\frac{-i}{\sqrt{2}}(a - a^+)\right]^2 + \left[\frac{1}{\sqrt{2}}(a + a^+)\right]^2\right)$$
$$= \frac{\hbar\omega}{2}\left(\frac{-1}{2}(\cancel{aa} - a^+a - aa^+ + \cancel{a^+a^+}) + \frac{1}{2}(\cancel{aa} + aa^+ + a^+a + \cancel{a^+a^+})\right)$$
$$= \frac{\hbar\omega}{2}(aa^+ + a^+a)$$
$$= \frac{\hbar\omega}{2}(a^+a + 1 + a^+a) \quad (\text{from } [a, a^+] = 1)$$
$$= \hbar\omega(n + 1/2) \checkmark$$

b) * Remember that $[n, a] = -a$ , $[n, a^+] = a^+$

$\Rightarrow$ Solve by using the above commutator relations

$$n(a|n\rangle) = (an - a)|n\rangle$$
$$= a(n-1)|n\rangle$$
$$= (n-1)(a|n\rangle) \qquad \Rightarrow a|n\rangle = c_n|n-1\rangle$$

$$n(a^+|n\rangle) = (a^+n + a^+)|n\rangle$$
$$= a^+(n+1)|n\rangle$$
$$= (n+1)(a^+|n\rangle) \quad \Rightarrow \quad a^+|n\rangle = d_n|n+1\rangle$$

$\Rightarrow$ Determine $c_n$ and $d_n$ through normalization

$$\langle n|a^+a|n\rangle = |c_n|^2\langle n-1|n-1\rangle$$
$$n\langle n|n\rangle = |c_n|^2$$
$$n = |c_n|^2 \quad \Rightarrow \quad \boxed{c_n = \sqrt{n}}$$

#1 (cont.)

b) $\langle n | a a^\dagger | n \rangle = |d_n|^2 \langle n+1 | n+1 \rangle$

$\langle n | a^\dagger a + 1 | n \rangle = |d_n|^2$

$(n+1) \langle n | n \rangle = |d_n|^2$

$n+1 = |d_n|^2 \Rightarrow d_n = \sqrt{n+1}$

c) ① Given a potential $V(x) = \frac{1}{2}\hbar\omega x^2 \geq 0$ for all $x$ and combined with the fact that $T \geq 0$ (since kinetic energy can never be negative), then $(H = T + V) \geq 0$ at all times, thus its eigenenergies must also be positive

② We can rewrite our potential as: $V = \frac{1}{2}\hbar\omega\left(\frac{a+a^\dagger}{\sqrt{2}}\right)^2$

$\Rightarrow V = \frac{\hbar\omega}{4}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)$

$V|n\rangle = \frac{\hbar\omega}{4}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)|n\rangle$

$= \frac{\hbar\omega}{4}\left[\sqrt{n(n-1)}\,|n-2\rangle + \sqrt{n+1}\sqrt{n+1}\,|n\rangle + \sqrt{n}\sqrt{n}\,|n\rangle + \sqrt{(n+1)(n+2)}\,|n+2\rangle\right]$

$= \frac{\hbar\omega}{4}\left[\sqrt{n(n-1)}\,|n-2\rangle + (2n+1)|n\rangle + \sqrt{(n+1)(n+2)}\,|n+2\rangle\right]$

* For the above equation to be greater than or equal to 0, n must also be greater than or equal to 0

③ In order for the above to be true $a|0\rangle = 0$ because $n \geq 0$. Following our work from part b

$n(a|0\rangle) = (an - a)|0\rangle$

$= a(0-1)|0\rangle$

$= -a|0\rangle = c_n|-1\rangle$

$\langle 0| -a^\dagger \, -a|0\rangle = |c_n|^2 \langle -1|-1\rangle$

$0\,\cancel{\langle 0|0\rangle}^{1} = |c_n|^2 \langle -1|-1\rangle$

$0 = |c_n|^2 \langle -1|-1\rangle$

$\hookrightarrow \langle -1|-1\rangle = 0$

#1 (cont.)

d) If we let $|0\rangle = \psi_0$

$$a\psi_0 = 0$$

$$\frac{1}{\sqrt{2}}(x + ip)\psi_0 = 0$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\lambda}x + i\frac{\lambda}{\hbar}p\right)\psi_0 = 0$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\lambda}x + i\frac{\lambda}{\hbar}\left(\frac{-i\hbar\partial}{\partial x}\right)\right)\psi_0 = 0$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\lambda}x + \lambda\frac{\partial}{\partial x}\right)\psi_0 = 0$$

$$\frac{1}{\lambda}x\psi_0 = -\lambda\frac{\partial\psi_0}{\partial x}$$

$$\int x\,\partial x = \int -\lambda^2\frac{\partial\psi_0}{\psi_0}$$

$$\frac{-1}{2\lambda^2}x^2 + C = \ln(\psi_0)$$

$$\exp\left[\frac{-x^2}{2\lambda^2} + C\right] = \psi_0$$

$$\hookrightarrow \psi_0 = C\exp\left[\frac{-m\omega x^2}{2\hbar}\right]$$

* Normalizing our wavefunction

$$1 = \int |\psi_0|^2 dx$$

$$1 = C^2\int \left|\exp\left[\frac{-m\omega}{2\hbar}x^2\right]\right|^2 dx$$

$$1 = C^2\int \exp\left[\frac{-m\omega}{\hbar}x^2\right]dx$$

$$1 = C^2\sqrt{\frac{\hbar\pi}{m\omega}}$$

$$\hookrightarrow C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\Rightarrow \psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[\frac{-m\omega}{2\hbar}x^2\right]$$

#1 (cont.)

e) * Following a similar progression to part d

$$|1\rangle = \psi_1 = a^\dagger |0\rangle$$

$$\psi_1 = \frac{1}{\sqrt{2}}(x - ip)\psi_0$$

$$= \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - i\frac{\lambda}{\hbar}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\psi_0$$

$$= \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - \lambda\frac{\partial}{\partial x}\right)\psi_0$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,x - \sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial x}\right)\left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[\frac{-m\omega}{2\hbar}x^2\right]\right]$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,x\psi_0 - \sqrt{\frac{\hbar}{m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-x\frac{m\omega}{\hbar}\right)\exp\left[\frac{-m\omega}{2\hbar}x^2\right]\right)$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,x\psi_0 + \sqrt{\frac{m\omega}{\hbar}}\,x\psi_0\right)$$

$$= \sqrt{\frac{2m\omega}{\hbar}}\,x\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[\frac{-m\omega}{2\hbar}x^2\right]$$