$\frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 0 \qquad \frac{1}{10}\begin{bmatrix}1 & 3\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix} = \frac{-8}{10} \cdot 1$

Problem 3: Two-State Problem (10 Points):

$\begin{bmatrix}-3 & 1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = 0$

Consider a two-state quantum system. In the orthonormal and complete set of basis vectors $|1\rangle$ and $|2\rangle$, the Hamiltonian operator for the system is represented by ($\omega > 0$)

$$\hat{H} = 10\hbar\omega|1\rangle\langle 1| - 3\hbar\omega|1\rangle\langle 2| - 3\hbar\omega|2\rangle\langle 1| + 2\hbar\omega|2\rangle\langle 2|$$

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Consider another complete and orthonormal basis $|\alpha\rangle$, $|\beta\rangle$, such that $\hat{H}|\alpha\rangle = E_1|\alpha\rangle$, and $\hat{H}|\beta\rangle = E_2|\beta\rangle$ (with $E_1 < E_2$). Let the action of operator $\hat{A}$ on the $|\alpha\rangle$, $|\beta\rangle$ basis vectors be given as

$$\hat{A}|\alpha\rangle = 2ia_0|\beta\rangle$$
$$\hat{A}|\beta\rangle = -2ia_0|\alpha\rangle - 3a_0|\beta\rangle$$

where $a_0 > 0$ is real.

✓a) Find the eigenvalues and eigenvectors of H in the $|1\rangle, |2\rangle$ basis (1 pt).

✓b) Find the eigenvalues and eigenvectors of $\hat{A}$ in the $|\alpha\rangle, |\beta\rangle$ basis (1 pt).

Suppose a measurement of $\hat{A}$ is carried out at t=0 on an arbitrary state and the largest possible value is obtained.

✓c) Calculate the probability P(t) that another measurement made at time t will yield the value as the one measured at t=0. (2 pts)

✓d) Calculate the time dependence of the expectation value $\langle\hat{A}\rangle$. What is the minimum value of $\langle\hat{A}\rangle$? At what time is the minimum value first achieved? (3 pts)

Now suppose that the average value obtained from a large number of measurements of $\hat{A}$ on identical quantum systems at a given time is -$a_0/4$.

e) (3 pts) Construct the most general normalized state vector (just before the measurement of $\hat{A}$) for your system consistent with this information in Dirac notation using the $|\alpha\rangle$, $|\beta\rangle$ basis. Express your answer as

$$|\Psi\rangle = C|\alpha\rangle + D|\beta\rangle$$

$\begin{bmatrix}1 & \frac{-i}{2}\end{bmatrix}\begin{bmatrix}1\\-\frac{2}{i}\end{bmatrix}$

$1 \cdot 1 + \frac{-i}{2} \cdot \frac{-2}{i}$

$\begin{bmatrix}1 & \frac{1}{2i}\end{bmatrix}\begin{bmatrix}1\\-\frac{2}{i}\end{bmatrix}$

$1 \cdot 1 + \frac{1}{2i} \cdot \frac{-2}{i} = \frac{-1}{i^2}$

$1 \cdot 1 + (2i)(-2i)$

$1 + 4(-i^2) =$

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Quantum #3

a) $H = 10\hbar\omega|1\rangle\langle 1| - 3\hbar\omega|1\rangle\langle 2| - 3\hbar\omega|2\rangle\langle 1| + 2\hbar\omega|2\rangle\langle 2|$

$$\doteq \begin{array}{c c} & \begin{array}{cc} |1\rangle & |2\rangle \end{array} \\ \begin{array}{c} \langle 1| \\ \langle 2| \end{array} & \begin{bmatrix} 10\hbar\omega & -3\hbar\omega \\ -3\hbar\omega & 2\hbar\omega \end{bmatrix} \end{array}$$

Using the eigenvalue equation $\det(H - \lambda \mathbb{I}) = 0$

$$\begin{vmatrix} 10\hbar\omega - \lambda & -3\hbar\omega \\ -3\hbar\omega & 2\hbar\omega - \lambda \end{vmatrix} = 0 = (10\hbar\omega - \lambda)(2\hbar\omega - \lambda) - (-3\hbar\omega)^2$$

$$= 20\hbar^2\omega^2 - 12\hbar\omega\lambda + \lambda^2 - 9\hbar^2\omega^2$$

$$= \lambda^2 - 12\hbar\omega\lambda + 11\hbar^2\omega^2$$

$$= (\lambda - \hbar\omega)(\lambda - 11\hbar\omega)$$

$\hookrightarrow \lambda_1 = \hbar\omega$

$\lambda_2 = 11\hbar\omega$

Using the eigenvector equation $H\vec{a} = \lambda\vec{a}$

$$\begin{bmatrix} 10\hbar\omega & -3\hbar\omega \\ -3\hbar\omega & 2\hbar\omega \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \begin{array}{l} 10\hbar\omega a_1 - 3\hbar\omega a_2 = \lambda a_1 \\ -3\hbar\omega a_1 + 2\hbar\omega a_2 = \lambda a_2 \end{array}$$

* if $\lambda = \hbar\omega$

$10a_1 - 3a_2 = a_1$

$-3a_1 + 2a_2 = a_2$

$\hookrightarrow a_1 = \frac{1}{3}a_2$

$|\lambda = \hbar\omega\rangle = \langle 1, 3\rangle \frac{1}{\sqrt{10}}$

* if $\lambda = 11\hbar\omega$

$10a_1 - 3a_2 = 11a_1$

$-3a_1 + 2a_2 = 11a_2$

$\hookrightarrow -3a_2 = a_1$

$|\lambda = 11\hbar\omega\rangle = \langle -3, 1\rangle \cdot \frac{1}{\sqrt{10}}$

* Dot product verifies orthogonality $\frac{1}{10}(1 \cdot -3 + 3 \cdot 1) = 0$

b) $A|\alpha\rangle = 2ia_0|\beta\rangle$

$A|\beta\rangle = -2ia_0|\alpha\rangle - 3a_0|\beta\rangle$

$$\Rightarrow A \doteq \begin{bmatrix} 0 & 2ia_0 \\ -2ia_0 & -3a_0 \end{bmatrix}$$

#3 (cont.)

b) Similarly to part a:

$$\begin{vmatrix} 0-\lambda & 2ia_0 \\ -2ia_0 & -3a_0-\lambda \end{vmatrix} = 0 = -\lambda(-3a_0-\lambda) - (2ia_0)(-2ia_0)$$

$$= \lambda^2 + 3a_0\lambda - 4a_0^2$$

$$= (\lambda+4a_0)(\lambda-a_0)$$

$$\Rightarrow \lambda = -4a_0, +a_0$$

Using the eigenvector equation:

$$\begin{bmatrix} 0 & 2ia_0 \\ -2ia_0 & -3a_0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \begin{aligned} 2ia_0 a_2 &= \lambda a_1 \\ -2ia_0 a_1 - 3a_0 a_2 &= \lambda a_2 \end{aligned}$$

* if $\lambda = 4a_0$

$2ia_0 a_2 = -4a_0 a_1$

$-2ia_0 a_1 - 3a_0 a_2 = -4a_0 a_2$

$\hookrightarrow \quad ia_2 = -2a_1$

$-2ia_1 = -a_2$

$\Rightarrow |\lambda = -4a_0\rangle = \langle -i, 2\rangle$

* if $\lambda = a_0$

$2ia_0 a_2 = a_0 a_1$

$-2ia_0 a_1 - 3a_0 a_2 = a_0 a_2$

$\hookrightarrow \quad 2ia_2 = a_1$

$-ia_1 = 2a_2$

$\Rightarrow |\lambda = a_0\rangle = \langle 2i, 1\rangle$

c) To obtain the largest possible value of A at $t=0$, $|\psi\rangle = |\lambda = a_0\rangle$

But since $U(t,t_0) = \exp[-iHt/\hbar]$, we must convert $|\lambda = a_0\rangle$ to the basis of the Hamiltonian.

$|\lambda_A = a_0\rangle = \frac{1}{\sqrt{5}}\langle 2i, 1\rangle$

Hamiltonian basis vectors: $|\lambda_H = \hbar\omega\rangle = \frac{1}{\sqrt{10}}\langle 1, 3\rangle$

$|\lambda_H = 11\hbar\omega\rangle = \frac{1}{\sqrt{10}}\langle -3, 1\rangle$

$$\frac{2i}{\sqrt{5}} = a\frac{1}{\sqrt{10}} + b\frac{-3}{\sqrt{10}}$$

$$\frac{1}{\sqrt{5}} = a\frac{3}{\sqrt{10}} + b\frac{1}{\sqrt{10}}$$

$2\sqrt{2}\, i = a - 3b$

$\sqrt{2} = 3a + b \quad \Rightarrow b = \sqrt{2} - 3a$

$2\sqrt{2}\, i = a - 3(\sqrt{2} - 3a)$

$2\sqrt{2}\, i = 10a - 3\sqrt{2} \quad \Rightarrow \quad a = \frac{3\sqrt{2} - 2\sqrt{2}\, i}{10}$

$$b = \frac{-\sqrt{2} + 6\sqrt{2}\, i}{10}$$

#3 (cont.)

c) $\Rightarrow |\lambda_A = a_0\rangle = \frac{3\sqrt{2} - 2\sqrt{2}i}{10} |\lambda_H = \hbar\omega\rangle + \frac{\sqrt{2} + 6\sqrt{2}i}{10} |\lambda_H = 11\hbar\omega\rangle$

$$|\lambda_A = a_0 (t)\rangle = U(t, t_0) |\lambda_A = a_0\rangle$$

$$= \exp[-iHt/\hbar]\left[\frac{3\sqrt{2} - 2\sqrt{2}i}{10} |\lambda_H = \hbar\omega\rangle + \frac{\sqrt{2} + 6\sqrt{2}i}{10} |\lambda_H = 11\hbar\omega\rangle\right]$$

$$= \exp[-i\omega t] \left(\frac{3\sqrt{2} - 2\sqrt{2}i}{10}\right) |\lambda_H = \hbar\omega\rangle + \exp[-11i\omega t] \left(\frac{\sqrt{2} + 6\sqrt{2}i}{10}\right) |\lambda_H = 11\hbar\omega\rangle$$

$$P(t) = \langle \lambda_a = a_0 | A | \lambda_a = a_0 (t)\rangle$$

$$= \left[\frac{3\sqrt{2} + 2\sqrt{2}i}{10} \langle \lambda_H = \hbar\omega |\right.$$