* Modified version of Sakurai 5.4

Problem 6: Perturbation Theory

An isotropic Harmonic oscillator in two dimensions has the Hamiltonian

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2),$$

where x and y are position operators in Cartesian coordinates x and y.

a) What is the energy of the *three* lowest energy levels and their respective degeneracies? (2 Points)

b) Consider a perturbative potential of the form:

$$V(x, y) = Am\omega^2 xy.$$

Compute the energy correction of the lowest level in the lowest order in perturbation theory where the result is non-zero. (3 Points)

c) Compute the energy splitting of the first excited energy level (which is degenerate), due to the perturbation. Compute the split ket states in terms of the original unperturbed kets. (3 Points)

d) Suppose that there are three indistinguishable spin 1/2 particles in the system. Compute the total energy of the ground state in first order in perturbation theory. (2 Points)

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Quantum #6

a) The energies of the isotropic oscillator are the sum of 2 1-D harmonic oscillators as the differential equation will be seperable by $\psi = X(x)\,Y(y)$

↳ $E_n = (n+\frac{1}{2})\hbar\omega$ in 1-D SHO

$\Rightarrow E_n = (n_x + n_y + \frac{1}{2})\hbar\omega$ in 2-D isotropic HO

Our 3 lowest levels are: $\frac{1}{2}\hbar\omega$ - $n_x = n_y = 0$

$\frac{3}{2}\hbar\omega$ - $(n_x=1,\ n_y=0),\ (n_x=0,\ n_y=1)$

$\frac{5}{2}\hbar\omega$ - $(n_x=2,\ n_y=0),\ (n_x=1,\ n_y=1),\ (n_x=0,\ n_y=2)$

b) $V'(x) = Am\omega^2 xy$

* We want lowest level $(n_x = n_y = 0)$, non-zero energy perturbation

↳ $\Delta E^{(1)}_{0,0} = \langle \psi_{00} | V' | \psi_{00} \rangle$

$= \langle 0,0 | Am\omega^2 xy | 0,0 \rangle$

$= Am\omega^2 \langle 0,0 | xy | 0,0 \rangle$

* Note that x, y can be written in terms of raising/lowering operators where:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$= Am\omega^2 \cdot \frac{\hbar}{2m\omega} \langle 0,0 | (a_x + a_x^+)(a_y + a_y^+) | 0,0 \rangle$

$= \frac{A\hbar\omega}{2} \langle 0,0 | a_x a_y + a_x^+ a_y + a_x a_y^+ + a_x^+ a_y^+ | 0,0 \rangle$

$= \frac{A\hbar\omega}{2}\left[0 \cancel{\langle 0,0|-1,-1\rangle}^{0} + 0\cancel{\langle 0,0|1,-1\rangle}^{0} + 0\cancel{\langle 0,0|-1,1\rangle}^{0} + 1\cancel{\langle 0,0|1,1\rangle}^{0}\right]$

* Note: First 3 terms physically impossible

$= 0$

#6 (cont.)

b) $$\Delta E^{(2)}_{0,0} = \sum_{k\neq n}' \frac{|\langle k|V'|n\rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$= \sum_{k\neq n}' \frac{|\langle k_x k_y | \frac{A\hbar\omega}{2} a_x^\dagger a_y^\dagger | 0,0\rangle|^2}{\frac{1}{2}\hbar\omega - E_k^{(0)}}$$

$$= \sum_{k\neq n}' \frac{|\langle k_x k_y | \frac{A\hbar\omega}{2} | 1,1\rangle|^2}{\frac{1}{2}\hbar\omega - E_k^{(0)}}$$

* Numerator $\neq 0$ only if $k_x = k_y = 1$ by orthogonality

$$= \frac{A^2\hbar^2\omega^2/4}{\frac{1}{2}\hbar\omega - \frac{3}{2}\hbar\omega}$$

$$= -\frac{A^2\hbar\omega}{8}$$

c) This problem can be achieved by diagonalizing the perturbation matrix for the first excited state

* Remember $E_1 = \frac{3}{2}\hbar\omega$, $(n_x = 1, n_y = 0)$ or $(n_x = 0, n_y = 1)$

$$V' = \begin{array}{c c} & \begin{array}{cc} |1,0\rangle & |0,1\rangle \end{array} \\ \begin{array}{c} |1,0\rangle \\ |0,1\rangle \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \cdot \frac{A\hbar\omega}{2}$$

$$xy = a_x a_y + a_x^\dagger a_y + a_x a_y^\dagger + a_x^\dagger a_y^\dagger$$

* To find the energy corrections, we find the eigenvalues

$$\begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0 = \lambda^2 - 1$$
$$= (\lambda+1)(\lambda-1)$$

$$\hookrightarrow \lambda = \pm 1$$

* To find the states that correspond to these energy corrections, we find the eigenvectors by $V\vec{a} = \lambda\vec{a}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \begin{array}{l} a_2 = \lambda a_1 \\ a_1 = \lambda a_2 \end{array}$$

#6 (cont.)

c) $*$ for $\lambda = 1$

$$\left.\begin{matrix} a_2 = a_1 \\ a_1 = a_2 \end{matrix}\right\} \vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

$*$ for $\lambda = -1$

$$\left.\begin{matrix} a_2 = -a_1 \\ a_1 = -a_2 \end{matrix}\right\} \vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

Orthogonality check: $\vec{a}_1 \circ \vec{a}_{-1} = 1 + -1 = 0$ ✓

$\hookrightarrow$ $\Delta E_1^{(1)} = \frac{A\hbar\omega}{2}$, $\quad |\psi\rangle = \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{\sqrt{2}}|0,1\rangle$

$\Delta E_2^{(1)} = -\frac{A\hbar\omega}{2}$, $\quad |\psi\rangle = \frac{1}{\sqrt{2}}|1,0\rangle - \frac{1}{\sqrt{2}}|0,1\rangle$

d)