Problem 1: Harmonic Oscillator (10 Points)

Consider the quantum mechanical simple harmonic oscillator.

a. Using the raising and lower operators, $\hat{a}$ and $\hat{a}^\dagger$ find the average value of X and P for the state $|n>$. **(1 Points)**

b. Using the raising and lower operators, $\hat{a}$ and $\hat{a}^\dagger$, find the average value of X^2 and P^2 for the state $|n>$. **(2 Points)**

c. Using the raising and lower operators, $\hat{a}$ and $\hat{a}^\dagger$ find the root mean square deviations of X and P for the state $|n>$. **(2 Points)**

d. Find the uncertainty product for the state $|n>$ **(2 Points)**

e. Fine the average potential energy and average kinetic energy for the oscillator when it is in state $|n>$ **(3 Points)**

Jan 2017

Quantum #1

a) * Remember that: $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right)$

$a^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right)$

$$\Rightarrow \quad x = \frac{1}{\sqrt{2}}\sqrt{\frac{2\hbar}{m\omega}}(a+a^{+}) = \sqrt{\frac{\hbar}{2m\omega}}(a+a^{+}) \qquad p = \frac{m\omega}{2i}\sqrt{\frac{2\hbar}{m\omega}}(a-a^{+}) = -i\sqrt{\frac{m\omega\hbar}{2}}(a-a^{+})$$

$$\Rightarrow \langle x\rangle = \langle n|\sqrt{\frac{\hbar}{2m\omega}}(a+a^{+})|n\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|a|n\rangle + \langle n|a^{+}|n\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\langle n|n-1\rangle + \sqrt{n+1}\langle n|n+1\rangle\right]$$

$$= 0$$

$$\langle p\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\left[\langle n|a-a^{+}|n\rangle\right]$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\left[\langle n|a|n\rangle - \langle n|a^{+}|n\rangle\right]$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\left[\sqrt{n}\langle n|n-1\rangle - \sqrt{n+1}\langle n|n+1\rangle\right]$$

$$= 0$$

b) $$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left[\langle n|aa + a^{+}a + aa^{+} + a^{+}a^{+}|n\rangle\right]$$

$$= \frac{\hbar}{2m\omega}\left[\sqrt{n(n-1)}\langle n|n-2\rangle + \sqrt{(n+1)^2}\langle n|n\rangle + \sqrt{n^2}\langle n|n\rangle + \sqrt{(n+1)(n+2)}\langle n|n+2\rangle\right]$$

$$= \frac{\hbar}{2m\omega}(2n+1)$$

$$\langle p^2\rangle = \frac{-m\omega\hbar}{2}\left[\langle n|aa - aa^{+} - a^{+}a + a^{+}a^{+}|n\rangle\right]$$

$$= \frac{-m\omega\hbar}{2}\left[\sqrt{n(n-1)}\langle n|n-2\rangle - (n+1)\langle n|n\rangle - \sqrt{n^2}\langle n|n\rangle + \sqrt{(n+1)(n+2)}\langle n|n+2\rangle\right]$$

$$= \frac{m\omega\hbar}{2}(2n+1)$$

c) * In general, the RMS value of an operator is defined by $\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$

$$\Rightarrow \langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}(2n+1) \qquad \langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \frac{m\omega\hbar}{2}(2n+1)$$

#1 (cont.)

d) $\sqrt{\langle(\Delta x^2)\rangle\langle(\Delta p)^2\rangle} = \sqrt{\frac{\hbar}{2m\omega}(2n+1)\frac{m\omega\hbar}{2}(2n+1)}$

$= \frac{\hbar}{2}(2n+1)$

e) $T = \frac{p^2}{2m}$

$\langle T\rangle = \langle \frac{p^2}{2m}\rangle$

$= \frac{1}{2m}\left(\frac{m\omega\hbar}{2}(2n+1)\right)$

$= \frac{\hbar\omega}{4}(2n+1)$

$V = \frac{1}{2}m\omega^2x^2$

$\langle V\rangle = \langle \frac{1}{2}m\omega^2x^2\rangle$

$= \frac{m\omega^2}{2}\left(\frac{\hbar}{2m\omega}\right)(2n+1)$

$= \frac{\hbar\omega}{4}(2n+1)$