Aug 2009

Problem 5: Quantum statistics (10 points)

1. Write down the energy eigenvalues and wave functions for a particle of mass m in an infinite square well, with $V = 0$ for $-L/2 < x < L/2$ and $V = \infty$ for $|x| > L/2$. (2 pts)

2. What is the ground state energy and wave-function if 2 identical non-interacting bosons are in the well? (4 pts)

3. What is the ground state energy and wave-function if 2 identical non-interacting spin-up fermions are in the well? (4 pts)

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Quantum #5

a) $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$$\psi_n = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & n \text{ even} \\ \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right. & n \text{ odd} \end{cases}$$

b) For bosons, there is no exclusion principle rule to follow, thus multiple bosons can occupy the same state in a system simultaneously, but the wavefunction must be symmetric

$$\Rightarrow E_{sys} = E_{1,1} + E_{2,1}$$
$$= \frac{\pi^2\hbar^2}{2m_1L^2} + \frac{\pi^2\hbar^2}{2m_2L^2}$$
$$= \frac{\pi^2\hbar^2}{mL^2}$$

$$\Rightarrow \psi_{sys} = \frac{1}{\sqrt{2}}\left(\psi_{1,1}(x_1)\psi_{2,1}(x_2) + \psi_{1,1}(x_2)\psi_{2,1}(x_1)\right)$$
$$= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{L}}\cos\left(\frac{\pi x_1}{L}\right)\sqrt{\frac{2}{L}}\cos\left(\frac{\pi x_2}{L}\right) + \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x_2}{L}\right)\sqrt{\frac{2}{L}}\cos\left(\frac{\pi x_1}{L}\right)\right]$$
$$= \frac{4}{L\sqrt{2}}\left[\cos\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{\pi x_2}{L}\right)\right]$$

* Normalizing the above wavefunction

$$1 = \frac{8}{L^2}A^2\int_{-L/2}^{L/2} dx_1 \cos^2\left(\frac{\pi x_1}{L}\right)\int_{-L/2}^{L/2} dx_2 \cos^2\left(\frac{\pi x_2}{L}\right)$$
$$= \frac{8A^2}{L^2}\cdot\frac{L^2}{4}$$
$$= 2A^2 \quad \rightarrow \quad A = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \psi_{sys} = \frac{2}{L}\cos\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{\pi x_2}{L}\right)$$

c) For fermions, we must follow the exclusion principle, thus our two particles cannot occupy the same state simultaneously and the wavefunction must be antisymmetric

$$E_{sys} = E_{1,1} + E_{2,2}$$
$$= \frac{\pi^2\hbar^2}{2mL^2} + \frac{4\pi^2\hbar^2}{2mL^2}$$
$$= \frac{5\pi^2\hbar^2}{2mL^2}$$

#5 (cont.)

c)

$$\psi_{sys} = \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\,\psi_2(x_2) - \psi_1(x_2)\,\psi_2(x_1)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{L}}^{\,2}\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x_2}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x_1}{L}\right)\right]$$

$$= \frac{\sqrt{2}}{L}\left[\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \cos\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{2\pi x_1}{L}\right)\right]$$