PROBLEM 3: Matrix Mechanics

Let A, B and C be three ensembles that are represented in the orthonormal basis $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$,

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of A are doubly degenerated, $a = 1,\ 1,\ -1$, with eigenvectors

$$|a=1,1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad |a=1,2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |a=-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The eigenvalues of C are also doubly degenerate, $c = 2,\ 1,\ 1$, with eigenvectors:

$$|c=2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |c=1,1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |c=1,2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Assume that all particles in the ensemble are in the state $|\psi\rangle$,

$$|\psi\rangle = \frac{1}{2}|e_1\rangle - \frac{1}{2}|e_2\rangle + \frac{1}{\sqrt{2}}|e_3\rangle.$$

Answer the following questions:

a) Find the probability of measuring C and obtaining a value $c = 2$; then immediately measuring A and getting $a = 1$, *i.e.* find $P_{|\psi\rangle}(c = 2, a = 1)$. Identify the intermediate state $|\psi'\rangle$ after C is measured. (2 Points)

b) Now find the probability if those measurements are performed in the reverse order, *i.e.*, find $P_{|\psi\rangle}(a = 1, c = 2)$. Identify the intermediate state $|\psi''\rangle$ after A is measured. (2 Points)

c) Compare the results of parts a) and b) and explain why this happened. (1 Point)

d) If you are told that the eigenvalues of B are $b = -2, -2, 4$, justify whether or not the following 2 probabilities $P_{|\psi\rangle}(a = -1, b = 4)$ and $P_{|\psi\rangle}(b = 4, a = -1)$ will be equal (do NOT explicitly calculate the probabilities). Will the final states be the same or different? Explain. (2 Points)

e) Does $\{A, B\}$ constitute a complete set of commuting observables? Demonstrate explicitly. (3 Points)

Jan 2014

Quantum #3

a) * With the initial state $|\psi\rangle = \frac{1}{2}|e_1\rangle - \frac{1}{2}|e_2\rangle + \frac{1}{\sqrt{2}}|e_3\rangle$

$\Rightarrow$ Probability of measuring $C=2$

$\hookrightarrow$ $|\langle c=2|C|\psi\rangle|^2$

$\Rightarrow$ Immediately after, probability of measuring

$\hookrightarrow$ $|\langle a=1,1|A|c=2\rangle|^2 + |\langle a=1,2|A|c=2\rangle|$

* State is $|c=2\rangle$ immediately after first measurement, need both terms above b/c $a=1$ is doubly degenerate eigenvalue

$$\Rightarrow P_{|\psi\rangle}(c=2, a=1) = \left(|\langle a=1,1|A|c=2\rangle|^2 + |\langle a=1,2|A|c=2\rangle|\right)^2 |\langle c=2|C|\psi\rangle|^2$$

$$= \left[\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)^2 + \left(\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)^2\right]\left(\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/2\\-1/2\\1/\sqrt{2}\end{bmatrix}\right)$$

$$= \left[\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)^2 + \left(\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)^2\right]\left(\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}1\\-1/2\\1/\sqrt{2}\end{bmatrix}\right)^2$$

$$= \left(\frac{1}{2} + 0\right)(1)$$

$$= \frac{1}{2}$$

b) * Proceeding in a similar manner as above:

$$P_{|\psi\rangle}(a=1, c=2) = \left(|\langle c=2|C|a=1,1\rangle|^2 + \langle a=1,1|A|\psi\rangle|^2\right) + \left(|\langle c=2|C|a=1,2\rangle|^2 |\langle a=1,2|A|\psi\rangle|^2,\right.$$

$$= \left[\left(\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\frac{1}{\sqrt{2}}\right)^2\left(\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}1/\sqrt{2}\right)^2\right] + \left[\left(\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right)^2\left(\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\right.\right.$$

$$= \left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + (0)(0)\right]$$

$$= \frac{1}{4}$$

c) $$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&1&0\\2&0&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&2&0\\1&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$$

$\hookrightarrow$ non-comuting observables, thus order of observation matters

#1 (cont.)

d) Solving for the eigenvectors of B we see:

$B\vec{x} = \lambda\vec{x}$

Case $\lambda = 4$

$x_1 - 3x_2 = 4x_1 \rightarrow -x_2 = x_1$

$-3x_1 + x_2 = 4x_2 \rightarrow -x_1 = x_2$

$-2x_3 = 4x_3$

$\Rightarrow x_3 = 0$

$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$

Case $\lambda = -2$

$x_1 - 3x_2 = -2x_1$

$-3x_1 + x_2 = -2x_2$

$-2x_3 = -2x_3$

$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$

$\Rightarrow$ Same set of eigenvectors indicates commuting observables, and since both states under consideration have the same corresponding eigenvector, the eigenvalues are simultaneous. This will result in no difference in probability based on the order of observation and in both cases the particle will be in the same final state.

e) The criteria for $\{A, B\}$ to be a complete set of commuting observables is:

① All the observables commute in pairs

② If we specify the eigenvalues of all operators in the set, we identify a unique eigenvector in the Hilbert space

$[A,B] = AB - BA$

$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$= 0$ ✓ (Condition 1 statisfied)

$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = |a=1, b=-2\rangle$ $\quad \vec{x} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = |a=-1, b=4\rangle$ $\quad \vec{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = |a=-1, b=-2\rangle$

↳ Condition 2 statisfied