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Problem 6: Spin $\frac{1}{2}$ System (10 points)

Consider a spin $\frac{1}{2}$ particle in the state space E_s. This space can be spanned by the 2 eigenvectors of S_x, S_y, or S_z, the components of the spin operator $S = S_x\hat{i} + S_y\hat{j} + S_z\hat{k}$. The matrix representation of S_x, S_y and S_z in the eigenbasis $|+\rangle_z$, $|-\rangle_z$ of S_z are given below:

$$S_x = \hbar/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \hbar/2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \hbar/2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $S_z|+\rangle_z = \hbar/2|+\rangle_z$ and $S_z|-\rangle_z = -\hbar/2|-\rangle_z$.

Assume that the state of the system at time $t = 0$ is: $|\Psi(0)\rangle = |-\rangle_z$.

a) If the observable S_x is measured at time $t = 0$, what results can be found and with what probabilities? (1 pt)

Now assume that a magnetic field is applied in the x **direction**: $\vec{B} = B_0\hat{i}$. The original wave function $|\Psi(0)\rangle = |-\rangle_z$ is allowed to evolve in time. The Hamiltonian governing the evolution is:

$$H_{spin} = \vec{S} \cdot \vec{B}$$

b) Set up the time evolution operator for this system, $U(t, 0)$. (1 pt)

c) Find $|\Psi(t)\rangle$, the wave function at a later time t. (1 pt)

d) At time $t > 0$ after $|\Psi(0)\rangle$ has evolved, S_x is measured. What is the probability of obtaining $+\hbar/2$? Is your answer time dependent or time independent? Explain correctly for credit. (1 pt)

e) Now let $|\Psi(0)\rangle$ evolve again and measure S_z at time t. Determine the probability of measuring S_z at time t and obtaining $-\hbar/2$. Is your answer time dependent or time independent? Explain correctly for credit. (1 pt)

f) Without explicitly finding the probabilities, discuss whether you expect the following probabilities to be equal or not. Give a brief explanation of your reasoning for any credit. The symbol $P_{|\Psi(t)\rangle}(a, c)$ represents the probability of starting with an ensemble in the state $|\Psi(t)\rangle$, measuring A first and getting eigenvalue "a" and then measuring C and getting eigenvalue "c". Assume that the eigenvalues of H_{spin} are E_+ and E_-. (1 pt)

i) Is $P_{|\Psi(0)\rangle}(+\hbar/2 \text{ for } S_y, -\hbar/2 \text{ for } S_x) = P_{|\Psi(0)\rangle>}(-\hbar/2 \text{ for } S_x, +\hbar/2 \text{ for } S_y)$? All measurements are taken at $t = 0$, i.e. the second measurement is taken immediately after the first measurement in each case. (1 pt)

ii) Is $P_{|\Psi(0)\rangle}(E_+, -\hbar/2 \text{ for } S_x) = P_{|\Psi(0)\rangle}(-\hbar/2 \text{ for } S_x, E_+)$? The first measurement in each case is taken at $t = 0$; the second measurement is taken immediately after the first measurement in each case. (1 pt)

iii) Is the probability $P_{|\Psi(0)\rangle}(+\hbar/2 \text{ for } S_x \text{ at } t, -\hbar/2 \text{ for } S_y \text{ at } t')$ time dependent or time independent in regards to the time t of the first measurement? Same question for the time t' of the second measurement. Discuss your reasoning in each case. (2 pts)

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Quantum #6

a) * To operate S_x on $|-\rangle_z$, we must rewrite S_x as:

$$S_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left(|-\rangle_z\langle+|_z + |+\rangle_z\langle-|_z\right)\frac{\hbar}{2}$$

$$\Rightarrow S_x|-\rangle_z = \frac{\hbar}{2}\left(|-\rangle_z\langle+|_z + |+\rangle_z\langle-|_z\right)|-\rangle_z$$

$$= \frac{\hbar}{2}|+\rangle_z$$

$\hookrightarrow$ $|+\rangle_z$ w/ $P = 1$

b) $|\psi(t)\rangle = U(t, t_0=0)\,|\psi(0)\rangle$ where $U(t, t_0=0) = \exp\left[-\frac{i}{\hbar}Ht\right]$

$$H = \vec{S}\cdot\vec{B} = S_x B_0$$

$$= \frac{B_0\hbar}{2}\left(|-\rangle_z\langle+|_z + |+\rangle_z\langle-|_z\right)$$

c) $\hookrightarrow$ $|\psi(t)\rangle = e^{-iB_0t/2}\,|+\rangle_z$

d) $S_x|\psi(t)\rangle = S_x\left(e^{-iB_0t/2}|+\rangle_z\right)$

$$= e^{-iB_0t/2}\,|-\rangle_z \qquad \text{w/ probability } P = 1$$

The answer is time independent b/c there is only one allowed state. Alternatively, $|c_n|^2$ determines the probability of that state. $|e^{-iB_0t/2}|^2 = 1$

e) $S_z|\psi(t)\rangle = \left(|+\rangle_z\langle+|_z - |-\rangle_z\langle-|_z\right)e^{-iB_0t/2}\,|+\rangle_z$

$$= e^{-iB_0t/2}\,|+\rangle_z$$

$P(S_z = -\hbar/2) = 0$ By same logic as above

f) i – $S_y = \frac{\hbar}{2}\left(-i|-\rangle\langle+| + i|+\rangle\langle-|\right)$

$\hookrightarrow$ No, $[S_x, S_y] \neq 0$, therefore they are not compatible observables

ii – yes, b/c H is simply a multiple of S_x, therefore $[S_x, H] = 0$ and the observables are compatible.

#6 (cont.)

f) iii - Both probabilities are time dependent