Jan 2008

Problem 4: Measurement of Hermitian Observables: (10 Points)

Consider a system with three Hermetian observables that are represented in a three-dimensional Hilbert space using the orthonormal basis $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$

with

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The system at time t=0 is in the state:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{6}}|e_1\rangle - \frac{1}{\sqrt{6}}|e_2\rangle + \sqrt{\frac{2}{3}}|e_3\rangle$$

1. Find the eigenvalues and normalized eigenvectors of B and C. **(1 Point)**
2. Find the probability of measuring B at time $t = 0$ with the eigenvalue $b = 1$, and then immediately measuring C and finding the eigenvalue $c = 1$, i.e. find $P_{|\Psi(0)\rangle}(b = 1, c = 1)$. **(2 Points)**
3. Now find the probability if these measurements are performed in reverse order at $t = 0$, i.e. find $P_{|\Psi(0)\rangle}(c = 1, b = 1)$. **(2 Points)**
4. Are the probabilities obtained in part 1. and part 2. the same or different? Explain in detail. **(2 Points)**
5. Use the Generalized Uncertainty Principle to determine a lower bound on $\Delta B \Delta C$ for the system in the initial state $|\Psi(0)\rangle$. Discuss your results. **(2 Points)**
6. Discuss in detail, the conditions that would result in obtaining a lower bound of zero when using the Generalized Uncertainty Principle. Would you expect to get zero for a particular pair of the observables, A, B, and C in this problem? Or for other conditions? **(1 Point)**

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Quantum #4

a) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{bmatrix}$

$\det(B - \lambda \mathbb{I}) = 0$

$\Rightarrow (1-\lambda)\left[(1-\lambda)^2 - (2i)(-2i)\right] = 0$

$0 = (1-\lambda)^3 - 4(1-\lambda)$

$= \left[(1 - 2\lambda + \lambda^2) - 4\right](1-\lambda)$

$= (\lambda^2 - 2\lambda - 3)(1-\lambda)$

$= (1-\lambda)(\lambda-3)(\lambda+1)$

$\Rightarrow \lambda = 1, 3, -1$

$C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(C - \lambda \mathbb{I}) = 0$

$0 = -\lambda(-\lambda(1-\lambda) - 0) - 1 \cdot ((1-\lambda) - 0)$

$= \lambda^2(1-\lambda) - (1-\lambda)$

$= [\lambda^2 - (1 \quad)](1-\lambda)$

$\Rightarrow \lambda = 1, 1, -1$

$Bx = \lambda x$

* for $\lambda = 1$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$x_1 = x_1$

$x_2 + 2ix_3 = x_2$

$-2ix_2 + x_3 = x_3$

$\Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

* for $\lambda = 3$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$x_1 = 3x_1$

$x_2 + 2ix_3 = 3x_2$

$-2ix_2 + x_3 = 3x_3$

$2ix_3 = 2x_2$

$ix_3 = x_2$

$\cancel{-2i(ix_3) + x_3 = 3x_3}$

$\cancel{-2x_3 + x_3 =}$

$\Rightarrow \vec{x} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$

* for $\lambda = -1$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$x_1 = -x_1$

$x_2 + 2ix_3 = -x_2$

$-2ix_2 + x_3 = -x_3$

$2ix_3 = -2x_2$

$ix_3 = -x_2$

$\Rightarrow \vec{x} = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$

a) $Cx = \lambda x$

* for $\lambda = -1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_2 = -x_1$$
$$x_1 = -x_2$$
$$x_3 = -x_3$$
$$\Rightarrow \vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

* for $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_2 = x_1$$
$$x_1 = x_2$$
$$x_3 = x_3$$
$$\Rightarrow \vec{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{|\lambda_c = 1, 2\rangle} \text{ or } \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{|\lambda_c = 1, 1\rangle}$$

b) * Convert $|\psi(0)\rangle$ into B eigenbasis

$$|\psi(0)\rangle = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{6}}\left(|\lambda=1\rangle + \frac{i}{2}\left[|\lambda=3\rangle - |\lambda=-1\rangle\right] + \left[|\lambda=3\rangle + |\lambda=-1\rangle\right]\right)$$

$$= \frac{1}{\sqrt{6}}\left(|\lambda=1\rangle + \left(1+\frac{i}{2}\right)|\lambda=3\rangle + \left(1-\frac{i}{2}\right)|\lambda=-1\rangle\right)$$

* To find probability

$$|\langle \lambda_b = 1| B |\psi(0)\rangle|^2 = \left|\langle \lambda_b = 1|\left(\frac{1}{\sqrt{6}}\left[1|\lambda=1\rangle + 3\left(1+\frac{i}{2}\right)|\lambda=3\rangle - \left(1-\frac{i}{2}\right)|\lambda=-1\rangle\right]\right)\right|^2$$

$$= \frac{1}{6}$$

$$|\langle \lambda_c = 1| C |\lambda_b = 1\rangle|^2 = \left|\langle \lambda_c = 1| C \left|\frac{1}{\sqrt{2}}\left(|\lambda_c = 1, 1\rangle - |\lambda_c = -1\rangle\right)\right|^2$$

$$= \left|\langle \lambda_c = 1, 1| \frac{1}{\sqrt{2}}\left(|\lambda_c = 1, 1\rangle + |\lambda_c = -1\rangle\right)\right|^2$$

$$= \frac{1}{2}$$

* Note: Only need $\langle \lambda_c = 1, 1|$ case b/c of orthogonality of eigenkets, i.e. probability 0 in $\langle \lambda_c = 1, 2|$ case

Overall probability: $1/12$

#4 (cont.)

c) * Reversing the order from part b, we see:

$$|\psi(0)\rangle = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{6}}\left(2|\lambda_c = 1,2\rangle - |\lambda_c = -1\rangle\right)$$

$$|\langle \lambda_c = 1 | C | \psi(0)\rangle|^2 = \left|\langle \lambda_c = 1| \frac{1}{\sqrt{6}}\left(2|\lambda_c = 1,2\rangle + |\lambda_c = -1\rangle\right)\right|^2$$

* only need $\langle \lambda_c = 1| = \langle \lambda_c = 1,2|$ case b/c of orthogonality of eigenvectors

$$= \left|\frac{2}{\sqrt{6}}\right|^2$$

$$= \frac{1}{4}$$

* Converting $|\lambda_c = 1,2\rangle$ to B eigenbasis: $|\lambda_c = 1,2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|\lambda_b = 3\rangle + |\lambda_b = -1\rangle\right)$

$$\left|\langle \lambda_b = 1 | B \left|\left(\frac{1}{\sqrt{2}}|\lambda_b = 3\rangle + \frac{1}{\sqrt{2}}|\lambda_b = -1\rangle\right)\right|^2 = 0$$

Overall probability: 0

d) The probabilities in parts b + c are different because the two observables are not commutatable. They have different eigenbasis and therefore the system is affected in different ways depending upon which operator is acted first

e) $\langle (\Delta B)^2\rangle \langle (\Delta C)^2\rangle \geq \frac{1}{4}|\langle [B,C]\rangle|^2$, where $\Delta A = \langle A^2\rangle - \langle A\rangle^2$

$$\cancel{B^2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -4i \\ 0 & 4i & 5 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\langle B^2\rangle = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -4i \\ 0 & 4i & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -5-8i \\ 10-4i \end{bmatrix} = \frac{1}{6}(1+5+20) \quad +8i-8i$$

$$= \frac{26}{6}$$

$$\langle C^2\rangle = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{6}(6) = 1$$

#4 (cont.)

e) Taking square root of above equation yields: $(\Delta B)(\Delta C) \geq \frac{1}{2}|\langle [B,C] \rangle|$

$$\Rightarrow BC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2i \\ -2i & 0 & 1 \end{bmatrix}$$

$$CB = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2i \\ 1 & 0 & 0 \\ 0 & -2i & 1 \end{bmatrix}$$