Problem 4: Matrix Mechanics (10 pts)

Consider a system governed by a Hamiltonian H, with an observable C. The Hamiltonian is represented in the $|e_i\rangle$ basis as:

$$H = \hbar\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{Where } |e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvalues and eigenvectors of H are

$$|E_1 = -\hbar\omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, |E_2 = \hbar\omega, 1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, |E_2 = \hbar\omega, 2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let C be represented in the $|e_i\rangle$ basis as

$$C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

At t=0, the system is in the state: $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$

a) At time t=0, the observable C is measured. What results are possible and with what probabilities? (2 pts)

b) Determine the representation of the time evolution operator $U(t, t_0 = 0)$ in the $|e_i\rangle$ representation. (2 pts)

c) Determine $|\Psi(t)\rangle$ in the $|e_i\rangle$ basis. (2 pts)

d) If C is measured at some later time t, what results are possible and with what probabilities? (2 pts)

e) Are your probabilities time dependent or time independent? Explain (2 pts)

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Quantum #4

a) * Read question as: Starting in $|\Psi(t=0)\rangle$, what is the probability of obtaining each eigenvalue of C

* Determine eigenvalues

$$|C - \lambda \mathbb{I}| = 0$$

$$\Rightarrow \quad 0 = \begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix}$$

$$0 = -\lambda\left[(1-\lambda)(-\lambda) - 0\right] - 0\left[0(-\lambda) - 0(2)\right] + 2\left[0(0) - (1-\lambda)(2)\right]$$

$$= (-\lambda)^2(1-\lambda) - 4(1-\lambda)$$

$$= (1-\lambda)\left[\lambda^2 - 4\right]$$

$$= (1-\lambda)(\lambda+2)(\lambda-2)$$

$$\hookrightarrow \lambda = 1, 2, -2$$

* Determine eigenvectors

$$C\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\hookrightarrow \quad 2x_3 = \lambda x_1$$

$$x_2 = \lambda x_2$$

$$2x_1 = \lambda x_3$$

* for $\lambda = 1$

$$2x_3 = x_1$$

$$x_2 = x_2$$

$$2x_1 = x_3$$

$$\Rightarrow \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

* for $\lambda = 2$

$$2x_3 = 2x_1$$

$$x_2 = 2x_2$$

$$2x_1 = 2x_3$$

$$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

* for $\lambda = -2$

$$2x_3 = -2x_1$$

$$x_2 = -2x_2$$

$$2x_1 = -2x_3$$

$$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

#4 (cont.)

a) * Rewriting $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle)$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(|\lambda=1\rangle + \frac{1}{2}[\,|\lambda=2\rangle + |\lambda=-2\rangle]\right)$$

* Probabilities of form

$$|\langle i|C|\Psi(t=0)\rangle|^2$$

$$C|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|\lambda=1\rangle + \frac{1}{2}(2|\lambda=2\rangle - 2|\lambda=-2\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}|\lambda=1\rangle + \frac{1}{\sqrt{2}}|\lambda=-2\rangle - \frac{1}{\sqrt{2}}|\lambda=-2\rangle$$

$$\Rightarrow |\langle\lambda=1|C|\Psi(t=0)\rangle|^2 = \frac{1}{2}$$

$$|\langle\lambda=+2|C|\Psi(t=0)\rangle|^2 = \frac{1}{4}$$

$$|\langle\lambda=-2|C|\Psi(t=0)\rangle|^2 = \frac{1}{4}$$

b) $U(t, t_0=0) = e^{-iHt/\hbar}$

$$= e^{-i\omega t}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_3| + |e_3\rangle\langle e_2|)$$

c) $|\Psi(t)\rangle = U(t,t_0=0)|\Psi(t=0)\rangle$

$$= \frac{1}{\sqrt{2}}\left[e^{-i\omega t}|e_1\rangle + e^{-i\omega t}|e_3\rangle\right]$$

d) * Rewriting $|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega t}(|e_1\rangle + e_3\rangle)$

$$= \frac{1}{\sqrt{2}}e^{-i\omega t}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$= e^{-i\omega t}|\lambda=2\rangle$$