PROBLEM 3: Perturbation Theory

Consider a particle of mass m trapped inside a 1D parabolic potential

$$V(x) = \frac{1}{2} m\omega^2 x^2,$$

where ω sets the frequency of oscillation inside the potential.

a) If the particle is perturbed by a *static* potential

$$V_I = \alpha x,$$

with α small, compute energy correction of the energy levels in the lowest order where the result is non-zero. (3 Points)

b) What is the pertubed ket in the ground state? Compute the expectation value $\langle x \rangle$ in this state. Interpret the sign of $\langle x \rangle$. (3 Points)

c) Assume from now on that $\alpha = 0$. Imagine that the particle is charged and sits in the ground state at $t = -\infty$. Suppose an electric field is gradually tuned on, increases to a maximum at $t = 0$ and then slowly dies away,

$$V_I'(t) = -e|\mathbf{E}| x \, \mathrm{e}^{-t^2/\tau^2},$$

where e is the electric charge, and $\mathbf{E}$ is the electric field . Write down the general expression for the amplitude of transition from a generic level i to level f. (Do not solve the integral yet) (2 Points).

d) Evaluate the probability of having the particle in the first excited state at $t = +\infty$. (2 Points).

Hint: $\int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{-t^2/\tau^2} \mathrm{e}^{i\omega t} = \sqrt{\pi}\tau \, \mathrm{e}^{-\omega^2\tau^2/4}$

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Quantum #3

a) In general, our first order energy correction is $\Delta E^{(1)} = \langle n^{(0)} | V' | n^{(0)} \rangle$

$\hookrightarrow V = \frac{1}{2} m\omega^2 x^2 \rightarrow$ SHO

$V' = \alpha x = \alpha \left(\sqrt{\frac{\hbar}{2m\omega}} [a^\dagger + a] \right)$

$$\Rightarrow \Delta E^{(1)} = \langle n | \alpha \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) | n \rangle$$

$$= \alpha \sqrt{\frac{\hbar}{2m\omega}} \langle n | a^\dagger + a | n \rangle$$

$$= \alpha \sqrt{\frac{\hbar}{2m\omega}} \left[\xcancel{\langle n | a^\dagger | n \rangle}^{0} + \xcancel{\langle n | a | n \rangle}^{0} \right]$$

*results will go to 0 by orthogonality $\langle n | m \rangle = \delta_{nm}$

Our second order correction is generally: $\Delta E^{(2)} = \sum'_{k \neq n} \frac{|V_{kn}|^2}{E_n^{(0)} - E_k^{(0)}}$

$$\Rightarrow \Delta E^{(2)} = \sum'_{k \neq n} \frac{|\langle k | V' | n \rangle|^2}{\hbar\omega [(n + 1/2) - (k + 1/2)]}$$

$$= \sum'_{k \neq n} \frac{|\langle k | \alpha \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) | n \rangle|^2}{\hbar\omega (n - k)}$$

$$= \frac{\alpha^2 \hbar}{2m\omega} \cdot \frac{1}{\hbar\omega} \sum'_{k \neq n} \frac{|\langle k | a^\dagger a | n \rangle|^2}{(n - k)}$$

$$= \frac{\alpha^2}{2m\omega^2} \sum'_{k \neq n} \frac{|\langle k | n+1 \rangle \sqrt{n+1} + \langle k | n-1 \rangle \sqrt{n}|^2}{n - k}$$

$$= \frac{\alpha^2}{2m\omega^2} \sum'_{k \neq n} \frac{|\sqrt{n+1}\, \delta_{k,n+1} + \sqrt{n}\, \delta_{k,n-1}|^2}{n - k}$$

$$= \frac{\alpha^2}{2m\omega} \left[\frac{n+1}{n - (n+1)} + \frac{n}{n - (n-1)} \right]$$

$$= \frac{\alpha^2}{2m\omega} \left[-(n+1) + n \right]$$

$$= \frac{-\alpha^2}{2m\omega}$$

#3 (cont.)

b) The formula for the first order correction to the wave function is:

$$|n^{(1)}\rangle = \sum_{k\neq n}' \frac{\langle k|V'|n\rangle}{E_n - E_k} |k^{(0)}\rangle$$

$$= \sum_{k\neq n}' \frac{\alpha\sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1}\,\delta_{k,n+1} + \sqrt{n}\,\delta_{k,n-1}}{\hbar\omega(n-k)} |k^{(0)}\rangle$$

$$= \left(\frac{\alpha^2}{2m\hbar\omega^3}\right)^{1/2} \left(\frac{\sqrt{n+1}}{n-(n+1)}|n+1\rangle + \frac{\sqrt{n}}{n-(n-1)}|n-1\rangle\right)$$

$$= \left(\frac{\alpha^2}{2m\hbar\omega^3}\right)^{1/2}\left(\sqrt{n+1}\,|n+1\rangle - \sqrt{n}\,|n-1\rangle\right)$$

* but since we are in the ground state $n=0$, and $|-1\rangle = 0$

$$= -\left(\frac{\alpha^2}{2m\hbar\omega^3}\right)^{1/2} |1\rangle$$

* Our full state is now $|\psi\rangle = |0\rangle - \left(\frac{\alpha^2}{2m\hbar\omega^3}\right)^{1/2}|1\rangle$

$$\Rightarrow \langle x\rangle = \langle\psi|x|\psi\rangle \,; \quad x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$

$$= \overset{0}{\cancel{\langle 0|x|0\rangle}} - \left(\frac{\alpha^2}{2m\hbar\omega^3}\right)^{1/2}\langle 0|x|1\rangle - \left(\frac{\alpha^2}{2m\hbar\omega}\right)^{1/2}\langle 1|x|0\rangle + \frac{\alpha^2}{2m\hbar\omega^3}\overset{0}{\cancel{\langle 1|x|1\rangle}}$$

$$= -\left(\frac{\alpha^2}{2m\hbar\omega^3}\right)^{1/2}\left[\langle 0|\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)|1\rangle + \langle 1|\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)|0\rangle\right]$$

$$= -\left(\frac{\alpha^2}{4m^2\omega^4}\right)^{1/2}\left[\overset{0}{\cancel{\langle 0|a^\dagger|1\rangle}} + \langle 0|a|1\rangle + \langle 1|a^\dagger|0\rangle + \overset{0}{\cancel{\langle 1|a|0\rangle}}\right]$$

$$= \frac{-\alpha}{2m\omega^2}\left[\langle 0|0\rangle + \langle 1|1\rangle\right]$$

$$= \frac{-\alpha}{m\omega^2}$$

* The expectation value being negative implies that the potential is deeper on the negative side than the unperturbed potential

#3 (cont.)

c) The transition probability is: (assumes a two state problem of i, f as states

$$c_n^{(1)} = \frac{-i}{\hbar}\int_{t_0}^{t} e^{-i\omega_{ni}t'} V_{ni}(t')\,dt', \quad \omega_{ni} = \omega_n - \omega_i, \quad E = \hbar\omega$$

$$V = -e|\vec{E}|x e^{-t^2/\tau^2}$$

*Simplifying the equation, we see:

$$V_{ni} = \langle f|V|i\rangle$$

$$= \langle f|-e|\vec{E}|x e^{-t^2/\tau^2}|i\rangle$$

$$= -e|\vec{E}|e^{-t^2/\tau^2}\langle f|\sqrt{\tfrac{\hbar}{2m\omega}}(a^\dagger + a)|i\rangle$$

$$= -eEe^{-t^2/\tau^2}\sqrt{\tfrac{\hbar}{2m\omega}}\left[\langle f|a^\dagger|i\rangle + \langle f|a|i\rangle\right]$$

d) $P = |c_n^{(1)}|^2$, now specifying $|f\rangle = |1\rangle$, $|i\rangle = |0\rangle$

$$V_{ni} = -eEe^{-t^2/\tau^2}\sqrt{\tfrac{\hbar}{2m\omega}}\left[\langle 1|a^\dagger|0\rangle + \cancel{\langle 1|a|0\rangle}^{\,0}\right]$$

$$= -eEe^{-t^2/\tau^2}\sqrt{\tfrac{\hbar}{2m\omega}}$$

$$\hookrightarrow c_{10}^{(1)} = \frac{-i}{\hbar}\int_{-\infty}^{\infty} e^{-i\omega_{10}t'}\left(-eEe^{-t^2/\tau^2}\right)dt' \cdot \sqrt{\tfrac{\hbar}{2m\omega}}$$

$$= \frac{ieE}{\hbar}\sqrt{\tfrac{\hbar}{2m\omega}}\int_{-\infty}^{\infty} e^{-i\omega_{10}t'} e^{-t'^2/\tau^2}\,dt'$$

$$= \frac{ieE}{\hbar}\sqrt{\tfrac{\hbar}{2m\omega}}\left(\sqrt{\pi}\,\tau\,\exp[-\omega_{10}^2\tau^2/4]\right)$$

$$\underline{P} = \frac{e^2E^2}{\hbar^2}\left(\frac{\hbar}{2m\omega}\right)\pi\tau^2\exp[-\omega_{10}^2\tau^2/2], \quad \omega_{10} = \frac{E_1^{(0)} - E_0^{(1)}}{\hbar}$$