Jan 2009

Problem 1: Spin $\frac{1}{2}$ particles (10 points)

Consider a system made up of spin 1/2 particles. If one measures the spin of the particles, one can only measure spin up or spin down. The general spin state of a spin 1/2 particle can be expressed as a two-element column matrix.

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

The spin matrices are:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) Can one simultaneously measure S_x, S_y and S_z? Explain your answer. (1 pt)

b) Can one simultaneously measure S^2 and S_z? Explain your answer. (1 pt)

c) Show S_z is Hermetian. (1 pt)

d) Calculate the normalized eigenvectors and eigenvalues of S_z. (2 pts)

Suppose a spin 1/2 particle is in the state

$$\chi = A\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

e) Normalize the state in order to determine A (1 pt)

f) If one measures S_z, what is the probability of getting -$\hbar/2$? (1 pt)

g) If one measures S_x, what is the probability of getting $+\hbar/2$? (2 pts)

h) What is the expectation value of S_y (1 pt)

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Quantum #1

a) Simultaneous measurements can only occur if two or more operators have the same eigenbasis. Said another way, if the commutator b/w two operators is 0, then they can be simultaneously measured. It is common knowledge that for the spin operators,

$$[S_i, S_j] = i\hbar \epsilon_{ijk} S_k$$

Thus S_x, S_y, and S_z <u>cannot</u> be measured simultaneously.

b) Similarly to part a, S^2 and S_z can only be measured simultaneously if $[S^2, S_z] = 0$. Again, it is well known that $[S^2, S_i] = 0$ where $i = \{x, y, z\}$. Thus S^2 and S_z <u>can</u> be measured simultaneously.

c) The condition of Hermitivity is $A = A^+$

$$\Rightarrow S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S_z^+ = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$>$ S_z is Hermitian

d) Begin by finding eigenvalues

$\hookrightarrow$ b/c S_z is diagonalized eigenvalues are $\pm \hbar/2$

By similar logic, the corresponding eigenvectors are

$$\frac{\hbar}{2} : \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \frac{-\hbar}{2} : \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as they would be for any diagonalized 2x2 matrix

e) Normalization Condition: $1 = \langle \chi | \chi \rangle$

$$\hookrightarrow 1 = A^2 \begin{bmatrix} 1+i & 2 \end{bmatrix} \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

$$= A^2(1+1+4)$$

$$= 6A^2$$

$$\hookrightarrow A = \sqrt{\frac{1}{6}}$$

#1 (cont.)

f) $P(S_z = \frac{\hbar}{2}) = |\langle S_z = \frac{-\hbar}{2} | \chi \rangle|^2$

$= \left| [0 \quad 1] \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{6}} \right|^2$

$= \frac{1}{6} \cdot |2|^2$

$= \frac{2}{3}$

g) * We must first find eigenvectors of S_x using $S_x \vec{v} = \lambda \vec{v}$

$\begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \hbar/2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow \quad \frac{\hbar}{2} x_2 = \frac{\hbar}{2} x_1 \quad \rightarrow \quad \vec{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\frac{\hbar}{2} x_1 = \frac{\hbar}{2} x_1$

$P(S_x = \frac{\hbar}{2}) = |\langle S_x = \frac{\hbar}{2} | \chi \rangle|^2$

$= \left| \frac{1}{\sqrt{2}} [1 \quad 1] \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \frac{1}{\sqrt{6}} \right|^2$

$= \frac{1}{12} |3+i|^2$

$= \frac{1}{12} (9+1)$

$= \frac{10}{12} = \frac{5}{6}$

h) $\langle S_y \rangle = \langle \chi | S_y | \chi \rangle$

$= \frac{1}{6} [1-i \quad 2] \begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix} \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

$= \frac{1}{6} [1-i \quad 2] \begin{bmatrix} -i\hbar \\ -\frac{\hbar}{2} + \frac{i\hbar}{2} \end{bmatrix}$

$= \frac{1}{6} (-i\hbar - \hbar \quad -\hbar + i\hbar)$

$= -\hbar/3$