Problem 4: Square Well Expansion

Consider a 1D quantum particle of mass m in a square well of width a:

$$\begin{aligned} V(x) &= 0, \quad |x| \leq \frac{a}{2} \\ V(x) &= \infty, \quad |x| > \frac{a}{2} \end{aligned} \tag{1}$$

(a) [1 pt] Write down the energy eigenvalues, E_n, and energy eigenstates, $\psi_n(x)$ for this well. You do not need to derive the states in all detail.

You might want to write the solutions for even and odd values of n separately.

(b) [2 pts] The well expands very suddenly to a new width $L > a$. The expansion is uniform about $x = 0$ so that for the new well, $V(x) = 0$ for $x \leq \frac{L}{2}$.

Assuming the particle is in the state n initially, for the well of width a, write an expression for the probability for the particle to be in the state n' after the expansion, for the well of width L. You don't have to solve for this probability yet, but write this expression in as much detail as you can. Explain why, for half of the possible values of n' this probability is zero.

(c) [2 pts] Consider the case where the particle is initially in the ground state of the well of width a. Show that the probability that the particle will end up in the ground state of the expanded well, of width L is

$$P_{11}\left(\frac{a}{L}\right) = \frac{16}{\pi^2}\frac{a}{L}\frac{\cos^2\left(\frac{\pi}{2}\frac{a}{L}\right)}{\left(1-\left(\frac{a}{L}\right)^2\right)^2} \tag{2}$$

(d) [3 pts] Calculate the limiting functional form for $P_{11}(a/L)$ from part (c) for $L >> a$, $\frac{a}{L} \to 0$. (Calculate the lowest order non-constant term in $\frac{a}{L}$.)

Calculate the limiting functional form for $P_{11}(a/L)$ from part (c) for $\frac{a}{L} \to 1$. It might be helpful to define $\frac{a}{L} = 1 - \delta$. (Calculate the lowest order non-constant term in δ.)

Explain physically why you would predict the two limiting values of the probability.

(e) [2 pts] Consider the case where the particle is initially in the ground state of the well and the potential well is completely removed suddenly ($V(x) = 0$ for all x).

Write down an expression that can be solved for the probability density of the particle having a momentum p after the well disappears. Just as in part (b), provide as much detail as you can, without actually solving for the probability.

Show that this will be very similar to the result in (b) so that calculating this probability would be a simple modification of the results in part (c).

Hint: The fact that $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ and $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$ might be useful.

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Quantum #4

a) For an infinite square well w/ $V = \begin{cases} 0 & -a/2 < x < a/2 \\ \infty & \text{elsewhere} \end{cases}$

$$\hookrightarrow \quad E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

$$\psi_n = \begin{cases} \sqrt{2/a}\, \sin\left(\frac{n\pi x}{a}\right) & n = \text{even} \\ \sqrt{2/a}\, \cos\left(\frac{n\pi x}{a}\right) & n = \text{odd} \end{cases}$$

b) For our expanded well with $L > a$, our solutions above are valid with $a \to L$. Therefore the probability of being in state n' after the expansion is:

$$|\langle n' | n \rangle|^2 = \left| \int_{-\infty}^{\infty} \psi_{n'}^* \psi_n \, dx \right|^2$$

For all values of n' even, we get an odd function, which integrates to 0 over symmetric bounds

c) $$P_{n',n} = |\langle n' | n \rangle|^2$$

$$= \left| \int_{-\infty}^{\infty} \psi_{n'}^* \psi_n \, dx \right|^2$$

$$= \left| \int_{-\infty}^{\infty} \sqrt{\tfrac{2}{L}} \cos\left(\tfrac{\pi x}{L}\right) \sqrt{\tfrac{2}{a}} \cos\left(\tfrac{\pi x}{a}\right) dx \right|^2$$

$$= \left| \int_{-a/2}^{a/2} \frac{2}{\sqrt{La}} \cos\left(\tfrac{\pi x}{L}\right) \cos\left(\tfrac{\pi x}{a}\right) \right|^2$$

$$= \left| \frac{2}{\sqrt{La}} \left[\frac{\sin\left(\left[\frac{\pi}{L} - \frac{\pi}{a}\right] x\right)}{2\left(\frac{\pi}{L} - \frac{\pi}{a}\right)} + \frac{\sin\left[\left(\frac{\pi}{L} + \frac{\pi}{a}\right) x\right]}{2\left(\frac{\pi}{L} + \frac{\pi}{a}\right)} \right] \right|_{-a/2}^{a/2}$$

$$= \frac{4}{La} \left| \left(\frac{\sin\left(\left[\frac{\pi}{L} - \frac{\pi}{a}\right] \frac{a}{2}\right)}{2\left(\frac{\pi}{L} - \frac{\pi}{a}\right)} + \frac{\sin\left(\left(\frac{\pi}{L} + \frac{\pi}{a}\right) \frac{a}{2}\right)}{2\left(\frac{\pi}{L} + \frac{\pi}{a}\right)} \right) - \left(\frac{\sin\left(\left(\frac{\pi}{L} - \frac{\pi}{a}\right) \frac{-a}{2}\right)}{2\left(\frac{\pi}{L} - \frac{\pi}{a}\right)} + \frac{\sin\left(\left[\frac{\pi}{L} - \frac{\pi}{a}\right] ?\right.}{2\left(\frac{\pi}{L} + \frac{\pi}{a}\right)} \right.$$

$$= \frac{4}{La} \left| \frac{\sin\left(\frac{\pi a}{2L} - \frac{\pi}{2}\right)}{\frac{\pi}{L} - \frac{\pi}{a}} + \frac{\sin\left(\frac{\pi a}{2L} + \frac{\pi}{2}\right)}{\left(\frac{\pi}{L} + \frac{\pi}{a}\right)} \right|^2$$

#4 (cont.)

c) $P_{1,1} = \frac{4}{La}\left|\frac{1}{\left(\frac{\pi^2}{L^2}-\frac{\pi^2}{a^2}\right)}\left[\left(\frac{\pi}{L}+\frac{\pi}{a}\right)\sin\left(\frac{\pi a}{2L}-\frac{\pi}{2}\right)+\left(\frac{\pi}{L}-\frac{\pi}{a}\right)\sin\left(\frac{\pi a}{2L}+\frac{\pi}{2}\right)\right]\right|^2$

$= \frac{4}{La}\left|\frac{1}{\frac{\pi^2}{L^2}-\frac{\pi^2}{a^2}}\left[\left(\frac{\pi}{L}+\frac{\pi}{a}\right)\sin\left(\frac{\pi a}{2L}-\frac{\pi}{2}\right)-\left(\frac{\pi}{L}-\frac{\pi}{a}\right)\sin\left(\frac{\pi a}{2L}-\frac{\pi}{2}\right)\right]\right|^2$

$= \frac{4}{La}\left|\frac{1}{\frac{\pi^2}{L^2}-\frac{\pi}{a^2}}\cdot\frac{2\pi}{a}\sin\left(\frac{\pi a}{2L}-\frac{\pi}{2}\right)\right|^2$

$= \frac{16\pi^2}{La^2}\sin^2\left(\frac{\pi a}{2L}-\frac{\pi}{2}\right)\cdot\frac{1}{\left(\frac{\pi^2}{L^2}-\frac{\pi^2}{a^2}\right)^2}$

$= \frac{16\pi^2}{La^2}\cos^2\left(\frac{\pi a}{2L}\right)\cdot\frac{1}{\left(\frac{\pi^2}{L^2}-\frac{\pi^2}{a^2}\right)^2}$

$= \frac{16\cos^2\left(\frac{\pi a}{2L}\right)}{\pi^2 La^2\left(\frac{1}{L^2}-\frac{1}{a^2}\right)^2}$

$= \frac{16\cos^2\left(\frac{\pi a}{2L}\right)}{\pi^2 L\left(1-\left(\frac{a}{L}\right)^2\right)^2 a^6}$ off by factor a^7??

d) Using $P_{1,1} = \frac{16a\cos^2\left(\frac{\pi}{2}\frac{a}{L}\right)}{\pi^2 L\left(1-(a/L)^2\right)^2}$, if $L \gg a$, $\frac{a}{L}\to 0$

$P_{11} = \frac{16\cos^2\left(\frac{\pi}{2}\frac{a}{L}\right)}{\pi^2}\cdot\left(\frac{a}{L}\right)$ $\quad\left(1-\left(\frac{a}{L}\right)^2\right)^2 \to 1$