## Problem 2: WKB approximation (10 Points):

The one-dimensional Schrodinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

can be rewritten as

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\cancel{2m}\,\hbar^2}\psi\,,$$

where

$$p(x) \equiv \sqrt{2m[E - V(x)]}\,.$$

The wave function $\psi(x)$ is often expressed as $\psi(x) = A(x)e^{i\phi(x)}$ where $A(x)$ is the amplitude and $\phi(x)$ is the phase. Both $A(x)$ and $\phi(x)$ can be real.

(a) Show that the amplitude is $A = \frac{C}{\sqrt{\phi'}}$ where $C$ is a constant and prime is the derivative with respect to $x$. (2 points)

(b) (3 points) Let us assume that $A''/A \ll (\phi')^2$ and $A''/A \ll p^2/\hbar^2$. Show that the wave function in the WKB approximation is

$$\psi(x) \simeq \frac{C}{\sqrt{p(x)}}e^{\pm\frac{i}{\hbar}\int p(x)dx}\,.$$

In parts (c)–(e), the potential energy of the one-dimensional harmonic oscillator is

$$V(x) = \frac{1}{2}m\omega^2x^2\,.$$

(c) Find the classical turning points $x_1 < x_2$ for an energy $E$. (1 points)

(d) Evaluate the phase $\phi$ in terms of $E$ and $\omega$ with the WKB method. (3 points)

(e) Apply the eigenvalue condition $\phi = \left(n + \frac{1}{2}\right)\pi\hbar$ and find energy eigenvalues $E_n$. (1 points)

Aug 2017

# Quantum #2

a) To show $A = \frac{c}{\sqrt{\varphi'}}$, assuming $\psi(x) = A(x)e^{i\varphi(x)}$, we substitute $\psi$ into the rewritten Schrödinger eqn: $\frac{d^2\psi}{dx^2} = \frac{-p^2}{\hbar^2}\psi$, $p = \sqrt{2m(E-V(x))}$

$$\hookrightarrow \frac{d\psi}{dx} = A'e^{i\varphi(x)} + Ai\varphi' e^{i\varphi(x)}$$

$$\frac{d^2\psi}{dx^2} = A''e^{i\varphi(x)} + iA'\varphi' e^{i\varphi(x)} + iA'\varphi' e^{i\varphi(x)} + iA\varphi'' e^{i\varphi(x)} - A(\varphi')^2 e^{i\varphi(x)}$$

$$e^{i\varphi(x)} \cdot \left[A'' - A(\varphi')^2 + i\left[2A'\varphi' + A\varphi''\right]\right] = \frac{-p^2}{\hbar^2} A(x) e^{i\varphi(x)}$$

Real: $\frac{-p^2}{\hbar^2} A = A'' - A(\varphi')^2$

Imaginary: $0 = 2A'\varphi' + A\varphi''$

$$0 = \frac{d}{dx}(A^2\varphi')$$

$$\boxed{\begin{aligned} &\tfrac{d}{dx}(A^2\varphi') = 2AA'\varphi' + A^2\varphi'' = 0 \\ &\text{*divide by } A \\ &2A'\varphi' + A\varphi'' = 0 \end{aligned}}$$

$$\hookrightarrow c^2 = A^2\varphi'$$

$$\hookrightarrow A = \frac{c}{\sqrt{\varphi'}} \checkmark$$

b) Assuming $A''/A << (\varphi')^2$ and $A''/A << p^2/\hbar^2$, we can now use the real equation from part A to show $\psi(x) \approx \frac{c}{\sqrt{p(x)}} e^{\pm\frac{i}{\hbar}\int p(x)dx}$

$$\frac{-p^2}{\hbar^2} A = A'' - A(\varphi')^2$$

$$\frac{p^2}{\hbar^2}\cancel{\frac{c}{\sqrt{\varphi'}}} = \cancel{\frac{c}{\sqrt{\varphi'}}}(\varphi')^2$$

$$\varphi' = \pm\frac{p}{\hbar} \Rightarrow \varphi = \pm\frac{1}{\hbar}\int_0^x p(x)dx$$

$$\hookrightarrow \psi(x) = Ae^{i\varphi(x)}$$

$$= \frac{c}{\sqrt{\varphi'}} \exp\left[\frac{\pm i}{\hbar}\int p(x)dx\right]$$

#2 (cont.)

c) The classical turning points occur when $E = V(x)$

$\hookrightarrow E = \frac{1}{2} m w^2 x^2$

$\hookrightarrow x = \pm \sqrt{\frac{2E}{m}}\, w$

d) Note: In the region where $E < V$, $p(x)$ is imaginary
$E > V$, $p(x)$ is real
$E = V$ $p(x)$ is $0$

* If $p(x) = 0$, $\psi(x) = 0$

* If $p(x)$ is real $(E < V)$

$$\psi(x) = \int_{x_1}^{x_2} p(x)\,dx$$

$$= \int_{x_1}^{x_2} \left[2m\left(E - \tfrac{1}{2} m w^2 x^2\right)\right]^{1/2} dx$$

$$= \int_{x_1}^{x_2} \left(2mE - m^2 w^2 x^2\right)^{1/2} dx$$

*let $a = \sqrt{2mE}$, $u = mwx$

$$= \int_{x_1}^{0} \sqrt{a^2 - u^2}\,\frac{du}{mw} + \int_{0}^{x_2} \sqrt{a^2 - u^2}\,du/mw$$

$$= \int_{0}^{x_2} \frac{1}{mw}\sqrt{a^2 - u^2}\,du - \int_{0}^{x_1} \frac{1}{mw}\sqrt{a^2 - u^2}\,du$$

$$= \frac{1}{mw}\left[\frac{\pi x_2^2}{4} - \frac{\pi x_1^2}{4}\right]$$

$$= \frac{\pi}{4mw}\left[x_2^2 - x_1^2\right]$$

$$= \frac{\pi}{4mw^2}$$

See Griffiths QM ex. 8.4