Problem 6: Electron in a Finite Square Well (10 pts)

Consider an electron of energy E incident from x=$-\infty$ on a symmetric one-dimensional square well of depth V_0 and width L.

$$V(x) = \begin{cases} 0, & \text{x < -L/2} \\ -V_0, & \text{-L/2 < x < L/2} \\ 0, & \text{x > L/2} \end{cases}$$

a) Write down the solutions to the time-independent Schrodinger Equation for this situation. There should be five integration constants (2 points)

b) Apply boundary conditions to find the probability that the electron is transmitted past the finite well (4 points)

c) For what values of E is there a 100% probability for transmission past the well? (2 points)

d) Consider a potential well with V_0 large enough for there to be two bound states. For this well, what is the smallest electron energy (E > 0) for which there is a 100% probability for transmission? Your answer will depend on V_0 and other parameters in the problem. (2 points)

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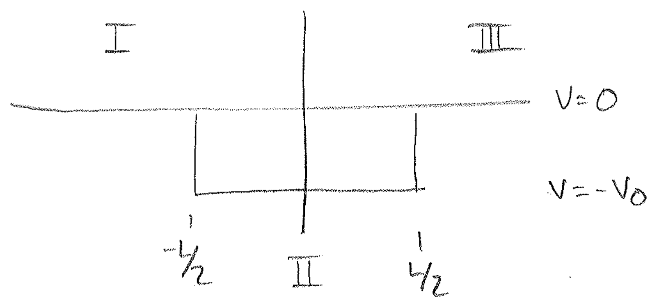
Quantum #6

a)



The time-independent Schrödinger equation states:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V\psi = E\psi$$

Regions I and III:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{-2mE}{\hbar^2}\psi$$

$$\text{let } \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

$$\frac{d^2\psi}{dx^2} = \kappa^2\psi$$

$$\hookrightarrow \psi_I = Ae^{\kappa x} + Be^{-\kappa x}$$

$$\psi_{III} = Fe^{\kappa x} + Ge^{-\kappa x}$$

Region II:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{-2m(E+V_0)}{\hbar^2}\psi$$

$$\text{let } k = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi \Rightarrow \psi_{II} = Ce^{ikx} + De^{-ikx}$$

#6 (cont.)

a) * Note: The above derivation assumes we have a bound state $(-V_0 < E < 0)$

If $E > 0$, then our wave functions become

$$\psi_I = Ae^{ikx} + Be^{-ikx} \qquad k_I = k_{III} = \frac{\sqrt{2mE}}{\hbar} = k$$

$$\psi_{II} = Ce^{ikx} + De^{ikx} \qquad k_{II} = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$$\psi_{III} = Fe^{ikx}$$ (Assume no incoming wave from left)

b) Our boundary conditions are that ψ and $\frac{d\psi}{dx}$ are continuous

↳ at $-L/2$:

$$Ae^{-ikL/2} + Be^{ikL/2} = C\sin\left(-k_{II}\tfrac{L}{2}\right) + D\cos\left(-\tfrac{k_{II}L}{2}\right)$$

$$ik_I\left(Ae^{-ik\cdot L/2} - Be^{ik\cdot L/2}\right) = k_{II}\left[C\cos\left(-k_{II}\tfrac{L}{2}\right) - D\sin\left(-\tfrac{k_{II}L}{2}\right)\right]$$

↳ at $L/2$:

$$C\sin\left(k_{II}\tfrac{L}{2}\right) + D\cos\left(k_{II}\tfrac{L}{2}\right) = Fe^{ikL/2}$$

$$k_{II}\left[C\cos\left(k_{II}\tfrac{L}{2}\right) - D\sin\left(k_{II}\tfrac{L}{2}\right)\right] = ikFe^{ikL/2}$$

* In the end, the transmission probability $T = \frac{|F|^2}{|A|^2}$

⇒ After using $\frac{L}{2}$ B.C's to eliminate C and D and substituting them into our $-\frac{L}{2}$ B.C, with waste of time algebra we find:

$$F = \frac{\exp[-ikL]}{\cos(k_{II}L) - i\frac{k^2+k_{II}^2}{2kk_{II}}\sin(k_{II}L)}A$$

$(-i)(i) = 1$

$$\Rightarrow T = \frac{1}{\left|\cos(k_{II}L) - i\frac{k^2+k_{II}^2}{2kk_{II}}\sin(k_{II}L)\right|^2}$$

$$= \frac{1}{\cos^2(k_{II}L) + \left(\frac{k^2+k_{II}^2}{2kk_{II}}\right)^2\sin^2(k_{II}L)}$$

#6 (cont.)

c) Perfect transmission will occur when $F = A$

$$\hookrightarrow E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} - V_0$$ (from Griffiths)

d) $E = \frac{2\pi^2\hbar^2}{mL^2} - V_0$ for 2 bound states