Problem 4: Properties of the Hydrogen Atom

The wavefunctions for the ground state and first excited states of the hydrogen atom are given on the first page of this test.

(a) [2 pt] For the ground state of the hydrogen atom, determine the expectation value for the radial position of the electron, $\langle 1,0,0|r|1,0,0\rangle$.

(b) [3 pt] Define the radial probably density for the electron in a hydrogenic eigenstate: $P_{n,\ell,m}(r)dr$ as the probability of the electron being measured in the spherical shell between r and $r+dr$.

Write down expressions for $P_{1,0,0}(r)$ and $P_{2,1,1}(r)$, and sketch these as functions of r.

Hint: Remember that the integral of the probability density over r must be equal to one,

$$\int_0^\infty P_{n,\ell,m}(r)dr = 1 \tag{1}$$

(c) [3 pt] For the ground state of the hydrogen atom, determine the most probable radius for the electron. Compare your result to part (a) and explain the similarities and differences.

(d) [1 pt] What is the functional form for $P_{1,0,0}(r)$ in the limit as $r \to 0$? Explain your result considering that the ground state wavefunction is non-zero at $r = 0$.

(e) [1 pt] What are the functional forms of $P_{1,0,0}(r)$, $P_{2,1,1}(r)$, and $P_{200}(r)$ as $r \to 0$? Explain the similarities and differences.

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Quantum #4

a) $\psi_{n\ell m} = R_{n,r}(r)\, Y_\ell^m(\theta,\varphi)$

$$\psi_{100} = \frac{2}{\sqrt{a_0^3}} e^{-r/a_0} \frac{1}{\sqrt{4\pi}}$$

$$= \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

$$\langle r \rangle = \int dr^3 \frac{r}{\pi a_0^3} e^{-2r/a_0}$$

$$= \frac{4}{a_0^3} \int r^3 e^{-2r/a_0} dr$$

$$= \frac{4}{a_0^3} \frac{\Gamma(4)}{(2/a_0)^4}$$

$$= \frac{4 \cdot 3!\, a_0^4}{2^4 a_0^3}$$

$$= \frac{3a_0}{2}$$

b) $P_{n\ell m}(r)\, dr = \int_r^{r+dr} \psi_{n\ell m}^* \psi_{n\ell m} \cdot 4\pi r^2 dr$

$$P_{100}(r)\, dr = \int_r^{r+dr} \frac{4}{a_0^3} e^{-2r/a_0} r^2 dr$$

$$\psi_{211} = \frac{1}{\sqrt{8a_0^3}} \frac{r}{\sqrt{3}a_0} e^{-r/2a_0} \cdot -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\varphi}$$

$$= -\frac{r}{8\sqrt{\pi a_0^5}} e^{-r/2a_0} e^{i\varphi} \sin\theta$$

$$P_{211} = \int_r^{r+dr} \int_0^{\pi} \frac{r^4 \cdot 2\pi}{64\pi a_0^5} e^{-r/a_0} \sin^4\theta\, dr\, d\theta$$