PROBLEM 2: Harmonic Oscillator with Two Particles

Consider a Hamiltonian for two non-interacting particles:

$$\begin{aligned} H(1,2) &= \frac{P_1^2}{2m} + \frac{1}{2}m\omega_1^2 X_1^2 + \frac{P_2^2}{2m} + \frac{1}{2}m\omega_2^2 X_2^2 \\ &= H_1 + H_2 \end{aligned}$$

where $\omega_2 = 2\omega_1 = 2\omega$.

Defining the raising and lowering operators:

$$\begin{aligned} a_n &= \frac{1}{\sqrt{2}}(\bar{X}_n + i\bar{P}_n) \\ a_n^\dagger &= \frac{1}{\sqrt{2}}(\bar{X}_n - i\bar{P}_n) \end{aligned}$$

where $n = 1, 2$ and

$$\begin{aligned} \bar{X}_n &= \left(\frac{m\omega_n}{\hbar}\right)^{1/2} X_n \\ \bar{P}_n &= \left(\frac{1}{\hbar m\omega_n}\right)^{1/2} P_n \end{aligned}$$

such that $[a_m, a_n^\dagger] = \delta_{mn}, m, n = 1, 2$.

Answer the following questions:

(a) [2 points] Write the Hamiltonian in terms of raising and lowering operators.

(b) [2 points] Write the eigenvector $|\psi_{n_1,n_2}\rangle$ in terms of the ground state $\psi_{0,0}\rangle = |\phi_{n_1=0}\rangle|\phi_{n_2=0}\rangle$ where $|\phi_{n_1}\rangle$ is the eigenvector for particle 1, i.e.,

$$H_1|\phi_{n_1}\rangle = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1|\phi_{n_1}\rangle$$

and similarly for particle 2.

(c) [1 points] Write a formula for the energy levels of this oscillator, E_n with n defined in terms of n_1 and n_2.

(d) [1 points] Determine a formula for the degeneracy, g_n , of an energy level E_n.

(e) [2 points] Using your results from part (d) determine the degeneracy g_n for the energy, $E = 15/2\hbar\omega$ and list all the eigenfunctions $|\psi_{n_1,n_2}\rangle$ that have this energy.

(f) [2 points] Determine ΔX_1, the uncertainty in X_1 for the state $|\psi_{n_1=1,n_2=2}\rangle$ using raising and lowering operators. Discuss the dependence of ΔX_1, on the frequency ω_1 and explain why it makes sense physically.

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Quantum #2

a) Given $H(1,2) = \frac{P_1^2}{2m} + \frac{1}{2}m\omega_1^2 X_1^2 + \frac{P_2^2}{2m} + \frac{1}{2}m\omega_2^2 X_2^2$

$$\bar{X}_n = \sqrt{\frac{m\omega_n}{\hbar}}\, X_n \qquad \bar{P}_n = \sqrt{\frac{1}{\hbar m\omega_n}}\, P_n$$

$$a_n = \frac{1}{\sqrt{2}}(\bar{X}_n + i\bar{P}_n) \qquad a_n^+ = \frac{1}{\sqrt{2}}(\bar{X}_n - i\bar{P}_n)$$

$$\Rightarrow H(1,2) = \frac{(\sqrt{\hbar m\omega_1}\,\bar{P}_1)^2}{2m} + \frac{1}{2}m\omega_1^2\left(\sqrt{\frac{\hbar}{m\omega_1}}\,\bar{X}_1\right)^2 + \frac{(\sqrt{\hbar m\omega_2}\,\bar{P}_2)^2}{2m} + \frac{1}{2}m\omega_2^2\left(\sqrt{\frac{\hbar}{m\omega_2}}\,\bar{X}_2\right)^2$$

$$= \frac{\hbar\omega_1\bar{P}_1^2}{2} + \frac{1}{2}\hbar\omega_1\bar{X}_1^2 + \frac{\hbar\omega_2\bar{P}_2}{2} + \frac{1}{2}\hbar\omega_2\bar{X}_2^2$$

$$= \frac{1}{2}\hbar\omega_1(\bar{X}_1^2 + \bar{P}_1^2) + \frac{1}{2}\hbar\omega_2(\bar{X}_2^2 + \bar{P}_2^2)$$

*Notice: $(a_n + a_n^+)\frac{1}{\sqrt{2}} = \bar{X}_n$

$\frac{-i}{\sqrt{2}}(a_n - a_n^+) = \bar{P}_n$

$$= \frac{1}{2}\hbar\omega_1\left[\frac{1}{2}(a_1+a_1^+)^2 - \frac{1}{2}(a_1-a_1^+)^2\right] + \frac{1}{2}\hbar\omega_2\left[\frac{1}{2}(a_2+a_2^+)^2 - \frac{1}{2}(a_2-a_2^+)^2\right]$$

$$= \frac{1}{4}\hbar\omega_1[a_1a_1 + a_1a_1^+ + a_1^+a_1 + a_1^+a_1^+ - a_1a_1 + a_1a_1^+ + a_1^+a_1 - a_1^+a_1^+]$$

$$+ \frac{1}{4}\hbar\omega_2[a_2a_2 + a_2a_2^+ + a_2^+a_2 + a_2^+a_2^+ - a_2a_2 + a_2a_2^+ + a_2^+a_2 - a_2^+a_2^+]$$

$$= \frac{1}{2}\hbar\omega_1(a_1a_1^+ + a_1^+a_1) + \frac{1}{2}\hbar\omega_2(a_2a_2^+ + a_2^+a_2)$$

$$= \frac{1}{2}\hbar\omega_1(2a_1^+a_1 + 1) + \frac{1}{2}\hbar\omega_2(2a_2^+a_2 + 1)$$

$$= \hbar\omega_1(N_1 + 1) + \hbar\omega_2(N_2 + 1), \quad N_n = a_n^+a$$

b) We know: $|\psi_n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|\psi_0\rangle$

$$\hookrightarrow |\psi_{n_1,n_2}\rangle = \frac{(a_1^+)^{n_1}(a_2^+)^{n_2}}{\sqrt{n_1!\,n_2!}}|\psi_{n_1=0}\rangle|\psi_{n_2=0}\rangle$$

$$= \frac{(a_1^+)^{n_1}(a_2^+)^{n_2}}{\sqrt{n_1!\,n_2!}}|\psi_{00}\rangle$$

#2 (cont.)

c) $E_{sys} = E_1 + E_2$

$= \hbar\omega_1(n_1 + \frac{1}{2}) + \hbar\omega_2(n_2 + \frac{1}{2})$

$= \hbar\omega_1(n_1 + 2n_2 + \frac{3}{2})$

$= \hbar\omega_1(N + \frac{3}{2})$, $N = n_1 + 2n_2$

d) $E_{11} = \frac{9\hbar\omega_1}{2}$ $g =$

$E_{21} = \frac{11\hbar\omega}{2}$ $g =$

$E_{12} = \frac{13\hbar\omega}{2}$ $g =$

$E_{22} = \frac{15\hbar\omega}{2}$

$a = n_1 + 2n_2 + \frac{3}{2}$

$\boxed{g = \frac{a - 3/2}{2} + 1}$

e) $E_{n_1 n_2} = \frac{15\hbar\omega}{2}$

$\frac{15}{2} = n_1 + 2n_2 + \frac{3}{2}$

$6 = n_1 + 2n_2 \longrightarrow (n_1, n_2) = \{(0,3); (6,0); (2,2), (4,1)\}$

$g = 4$

f)