Problem 2: Quantum Operators

In this problem you will work with the ladder operators for angular momentum:

$$L_+ = L_x + iL_y, \quad L_- = L_x - iL_y \tag{1}$$

where

$$\begin{aligned} L^2 &= L_x^2 + L_y^2 + L_z^2 \\ L^2|\ell, m\rangle &= \ell(\ell+1)\hbar^2|\ell, m\rangle \\ L_z|\ell m\rangle &= m\hbar|\ell, m\rangle \end{aligned} \tag{2}$$

(a) [1 pt] Show that the eigenvalues of any Hermitian operator are real.

(b) [2 pt] Is the the operator L_+L_-, the product of the angular momentum ladder operators, Hermitian? Show your work to justify your answer.

(c) [4 pt] Determine the results of the operations: $\hat{L}_+|\ell, m\rangle$ and $\hat{L}_-|\ell, m\rangle$. Show all of your work and make sure you determine all constants correctly.

Hint: The commutation relation $[L_z, L_\pm]$ and the matrix elements $\langle \ell, m|L_\pm L_\mp|\ell, m\rangle$ might be useful.

(d) [3 pt] Using the results from part (c), prove that $-\ell \leq m \leq +\ell$. Explain the physics of this result in terms of the operators L^2 and L_z.

Quantum #2

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a) The condition for Hermiticity is $A = A^{\dagger}$ for an operator A

Using: $A|\lambda\rangle = a|\lambda\rangle$, it must be true that $\langle\lambda|A^{\dagger} = a^{*}\langle\lambda|$

$= \langle\lambda|A$

$$\Rightarrow \langle\lambda|A|\lambda\rangle = a^{*}\langle\lambda|\lambda\rangle$$

$$a\langle\lambda|\lambda\rangle = a^{*}\langle\lambda|\lambda\rangle$$

$a = a^{*}$, which is only true if $a \in \mathbb{R}$

b) $L_+ = L_x + iL_y$

$L_- = L_x - iL_y$

$\Big\}$ We want $L_+L_- = (L_+L_-)^{\dagger} = L_-^{\dagger}L_+^{\dagger}$

$$\begin{aligned}
L_+L_- &= (L_x + iL_y)(L_x - iL_y) \\
&= (L_x^2 + iL_yL_x - iL_xL_y + L_y^2) \\
&= (L_x^2 + L_y^2 - i[L_x, L_y]) \\
&= (L_x^2 + L_y^2 - i(i\hbar L_z)) \\
&= (L_x^2 + L_y^2 + \hbar L_z) \\
&= (L^2 - L_z^2 + \hbar L_z)
\end{aligned}$$

$$\begin{aligned}
L_-^{\dagger}L_+^{\dagger} &= (L_x - iL_y)^{\dagger}(L_x + iL_y)^{\dagger} \\
&= (L_x^{\dagger} + iL_y^{\dagger})(L_x^{\dagger} - iL_y^{\dagger})
\end{aligned}$$

$*$ but $L_x^{\dagger} = L_x$, $L_y^{\dagger} = L_y$ by their status as observables

$$\begin{aligned}
&= (L_x + iL_y)(L_x - iL_y) \\
&= L^2 - L_z^2 + \hbar L_z
\end{aligned}$$

c) We must first determine $L_{\pm}|l,m\rangle$

$$\begin{aligned}
L_z(L_{\pm}|l,m\rangle) &= (L_{\pm}L_z \pm \hbar L_{\pm})|l,m\rangle \\
&= L_{\pm}(L_z \pm \hbar)|l,m\rangle \\
&= (m \pm \hbar)(L_{\pm}|l,m\rangle)
\end{aligned}$$

$\hookrightarrow$ $L_{\pm}$ increments the z-states of angular momentum

#2 (cont.)

c) Given the above, we know: $J_\pm |l,m\rangle = c_\pm |l, m\pm\hbar\rangle$

$\Rightarrow \langle l,m| L_+^\dagger L_+ |l,m\rangle = |c_+|^2 \langle l, m+\hbar \cancel{| l, m+\hbar\rangle}$ 1

$\langle l,m| L^2 - L_z^2 - \hbar L_z |l m\rangle = |c_+|^2$ (see part b for work)

$\hbar^2 l(l+1) - \hbar^2 m^2 - \hbar^2 m \cancel{\langle l, m | l, m\rangle} = |c_+|^2$ 1

$\hookrightarrow |c_+|^2 = \hbar^2[l(l+1) - m^2 - m]$

$c_+ = \hbar\sqrt{(l-m)(l+m+1)}$

Similarly for J_-

$\Rightarrow \langle l,m| J_-^\dagger J_- |l,m\rangle = |c_-|^2 \langle l, m-\hbar \cancel{| l, m-\hbar\rangle}$ 1

$\langle l,m| L^2 - L_z^2 + \hbar L_z |l,m\rangle = |c_-|^2$

$\hbar^2 l(l+1) - \hbar^2 m^2 + \hbar^2 m \langle l,m|l,m\rangle = |c_-|^2$

$\hookrightarrow |c_-|^2 = \hbar^2[l(l+1) - m^2 + m]$

$= \hbar\sqrt{(l+m)(l-m+1)}$

$\Rightarrow L_\pm |l,m\rangle = \hbar\sqrt{(l\mp m)(l\pm m+1)}\,|l, m\pm 1\rangle$

d) This part of the problem is effectively asking us to find the extremum values there fore we act upon the max/min states

$\Rightarrow L_+ |l, m_{max}\rangle = 0$

$L_- L_+ |l, m_{max}\rangle = 0$

$L^2 - L_z^2 - \hbar L_z |l, m_{max}\rangle = 0$

$\hbar^2 l(l+1) - \hbar^2 m^2 - \hbar^2 m |l, m_{max}\rangle = 0$

* assuming a non-zero ket

$l(l+1) = m(m+1) \Rightarrow m_{max} = l$

#2 (cont.)

d) $L_- |l, m_{min}\rangle = 0$

$L_+ L_- |l, m_{min}\rangle = 0$

$L^2 - L_z^2 + \hbar L_z |l, m_{min}\rangle = 0$

$\hbar^2(l+1)l - \hbar^2 m_{min}^2 + \hbar^2 m_{min} |l, m_{min}\rangle = 0$

* assuming a non-zero ket

$l(l+1) = m_{min}(m_{min} - 1)$

$m_{max}(m_{max} + 1) = m_{min}(m_{min} - 1)$

$\hookrightarrow m_{max} = -m_{min}$

$\hookrightarrow m_{min} = -l$

$\Rightarrow m \in [-l, l]$

Physically, the z-state of angular momentum can only contain as much angular momentum as the overall angular momentum of the whole system