Problem 1: Clebsh-Gordon coefficients (10 pts)

A system of two particles with spins $s_1 = \frac{3}{2}$ and $s_2 = \frac{1}{2}$ is described by the Hamiltonian

$$\mathcal{H} = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2$$

with α a constant and $\mathbf{S}_i$ $(i = 1, 2)$ is the spin operator of the i-th particle.

a) What are the allowed values for the quantum numbers of the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$? (2 Points)

b) Calculate the energy levels of the Hamiltonian. (2 Points)

c) Let us define the basis of eigenstates of the $\mathbf{S}_1^2$, $\mathbf{S}_2^2$, S_{1z}, S_{2z} operators, $|s_1 s_2; m_1 m_2\rangle$, where m_1 and m_2 are the quantum numbers of the projection operators S_{1z} and S_{2z} respectively. The system at time $t = 0$ is initially in the state

$$\left| s_1 s_2; \frac{1}{2}, \frac{1}{2} \right\rangle .$$

Find the state of the system at times $t > 0$. (4 Points)

d) Assuming the initial state above, what is the probability of finding the system in the state

$$\left| s_1 s_2; \frac{3}{2}, -\frac{1}{2} \right\rangle$$

at $t > 0$? (2 Points)

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Quantum #1

a) The allowed S values are:

$$|S_1 - S_2| < S < S_1 + S_2$$
$$\left|\tfrac{3}{2} - \tfrac{1}{2}\right| < S < \tfrac{3}{2} + \tfrac{1}{2}$$
$$\hookrightarrow S = 1, 2$$

b) $H = \alpha S_1 \cdot S_2$

* Remember $S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2$

$$\hookrightarrow S_1 \cdot S_2 = \tfrac{1}{2}(S^2 - S_1^2 - S_2^2)$$

$$H|s_1, s_2; S, S_z\rangle = \tfrac{\alpha}{2}(S^2 - S_1^2 - S_2^2)\,|3/2, 1/2; 2, S_z\rangle = \tfrac{\alpha\hbar^2}{2}\left(2(2+1) - \tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right)|\,\rangle$$

or

$$\tfrac{\alpha}{2}(S^2 - S_1^2 - S_2^2)\,|3/2, 1/2; 1, S_z\rangle = \tfrac{\alpha\hbar^2}{2}\left(1(1+1) - \tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right)|\,\rangle$$

$\Rightarrow$ Our energy states are $E = \frac{\alpha\hbar^2}{2}\left(6 - \frac{15}{4} - \frac{3}{4}\right) = \frac{3\alpha\hbar^2}{2}$ $\quad S = 2$

$E = \frac{\alpha\hbar^2}{2}\left(2 - \frac{15}{4} - \frac{3}{4}\right) = -\frac{\alpha\hbar^2}{2}$ $\quad S = 1$

c) $S_1 \cdot S_2 = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}$

$$= \tfrac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+}) + S_{1z}S_{2z}$$

To determine $|s_1, s_2; \frac{1}{2}, \frac{1}{2}\rangle$ in basis of H, we must start in max S state $|s_1, s_2; S, S_z\rangle$ and lower to appropriate state

$$|2, 2\rangle = |3/2, 1/2\rangle$$

$$S_-|2, 2\rangle = \sqrt{(2+2)(2-2+1)}\,|2, 1\rangle$$
$$= 2|2, 1\rangle$$

$$S_-|3/2, 1/2\rangle = S_{1-}|3/2, 1/2\rangle + S_{2-}|3/2, 1/2\rangle$$
$$= \sqrt{(\tfrac{3}{2}+\tfrac{3}{2})(\tfrac{3}{2}-\tfrac{3}{2}+1)}\,|1/2, 1/2\rangle + \sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\,|3/2, -1/2\rangle$$
$$= \sqrt{3}\,|1/2, 1/2\rangle + |3/2, -1/2\rangle$$

#1(cont.)

c) $\Rightarrow |2,1\rangle = \sqrt{3/4}\,|1/2, 1/2\rangle + \frac{1}{\sqrt{4}}|3/2, -1/2\rangle$

*Note: The $|1,1\rangle$ state is also a linear combination of $|1/2, 1/2\rangle$ and $|3/2, -1/2\rangle$

$$|1,1\rangle = \frac{-1}{\sqrt{4}}|1/2, 1/2\rangle + \frac{\sqrt{3}}{\sqrt{4}}|3/2, -1/2\rangle$$

$$-\sqrt{3}\,|2,1\rangle + |1,1\rangle = -|1/2, 1/2\rangle$$

$$\Downarrow$$

$$|1/2, 1/2\rangle = \sqrt{\tfrac{3}{4}}|2,1\rangle - \sqrt{\tfrac{1}{4}}|1,1\rangle$$

$|\psi(t)\rangle = U(t,t_0)|\psi(0)\rangle$, $U(t,t_0) = \exp\left[-\frac{iHt}{\hbar}\right]$

$\hookrightarrow |1/2, 1/2\,(t)\rangle = \sqrt{\tfrac{3}{4}}\exp\left[-\frac{i3\alpha\hbar t}{2}\right]|2,1\rangle - \sqrt{\tfrac{1}{4}}\exp\left[\frac{i\alpha\hbar t}{2}\right]|1,1\rangle$

d) $|\langle 3/2, -1/2 \,|\, 1/2, 1/2\,(t)\rangle|^2$

$$|3/2, -1/2\rangle = \sqrt{\tfrac{1}{4}}|2,1\rangle + \sqrt{\tfrac{3}{4}}|1,1\rangle$$

$$\left|\left[\sqrt{\tfrac{3}{4}}\langle 1,1| + \sqrt{\tfrac{1}{4}}\langle 2,1|\right]\left[\sqrt{\tfrac{3}{4}}\exp\left[-\frac{i3\alpha\hbar t}{2}\right]|2,1\rangle - \sqrt{\tfrac{1}{4}}\exp\left[\frac{i\alpha\hbar t}{2}\right]|1,1\rangle\right]\right|^2$$

$$\left|\frac{\sqrt{3}}{4}\exp\left[\frac{i\alpha\hbar t}{2}\right] + \frac{\sqrt{3}}{4}\exp\left[-\frac{i3\alpha\hbar t}{2}\right]\right|^2$$

$$\frac{3}{16}\left(\exp\left[-\frac{i\alpha\hbar t}{2}\right] - \exp\left[\frac{i3\alpha\hbar t}{2}\right]\right)\left(\exp\left[\frac{i\alpha\hbar t}{2}\right) - \exp\left[-\frac{i3\alpha\hbar t}{2}\right]\right)$$

$$\frac{3}{16}\left(2 - \exp[2i\alpha\hbar t] - \exp[-2i\alpha\hbar t]\right)$$