Problem 3: Angular Momentum Hamiltonian (10 points)

Consider the following Hamiltonian for a spinless particle with orbital angular momentum ℓ=2.

$$\hat{H} = \frac{3a}{2\hbar}\hat{L}_z - \frac{a}{\hbar^2}(\hat{L}_x^2 + \hat{L}_y^2)$$

where a is a constant greater than 0 and $\hat{L}_i$ denotes the i^{th} component of the angular momentum operator.

✓ a) Calculate the energy spectrum of this Hamiltonian (2 pts)

b) Suppose a particle with this Hamiltonian has the wavefunction

$$\Psi(\theta, \phi) = A(\sin\theta\cos\theta\cos\phi + \sin^2\theta\sin\phi\cos\phi)$$

where θ is the polar angle, ϕ is the azimuthal angle, and A is a normalization constant. What is the average energy obtained in energy measurements on an ensemble of particles described by the wavefunction above? (3 pts)

c) Assume the particle is in the lowest energy state (with ℓ=2) for $t < 0$. Starting at t=0, an external magnetic field is applied with

$$\hat{V}(t) = \frac{\lambda}{\hbar}\hat{L}_x e^{-t/\tau}$$

where τ is the decay constant and λ is a constant. Calculate the transition probabilities to possible excited states after a very long time ($\tau \ll t \to \infty$) using first order time-dependent perturbation theory. (5 pts)

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Quantum #3

a) We know that we can write simultaneous eigenkets of L^2, L_z as $|l, m\rangle$ in systems where angular momentum is under investigation

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L_\pm = L_x \pm i L_y$$

$$\Rightarrow H = \frac{3a}{2\hbar} L_z - \frac{a}{\hbar^2}(L_x^2 + L_y^2)$$

$$= \frac{3a}{2\hbar} L_z - \frac{a}{\hbar^2}(L^2 - L_z^2)$$

* We know that $H|l,m\rangle = E|l,m\rangle$

$$\hookrightarrow \quad \frac{3a}{2\hbar} L_z - \frac{a}{\hbar^2}(L^2 - L_z^2)|l,m\rangle = \frac{3a}{2\hbar} m - \frac{a}{\hbar^2}\left(l(l+1) - m^2\right)|l,m\rangle$$

$$\Rightarrow E = \frac{3am}{2\hbar} - \frac{a^2}{\hbar^2}\left(l(l+1) - m^2\right)$$

b) $\Psi(\theta,\varphi) = A(\sin\theta\cos\theta\cos\varphi + \sin^2\theta\sin\varphi\cos\varphi)$

$$\hookrightarrow Y_{2,\pm 1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\, e^{\pm i\varphi}\cos\theta\sin\theta = \mp\sqrt{\frac{15}{8\pi}}(\cos\varphi \pm i\sin\varphi)\cos\theta\sin\theta$$

$$Y_{2,\pm 2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\, e^{\pm 2i\varphi}\sin^2\theta = \sqrt{\frac{15}{32\pi}}(\cos 2\varphi \pm i\sin 2\varphi)\sin^2\theta$$

* From wolfram, $\int d\theta\, d\varphi\, |\Psi|^2 = \frac{7\pi^2 A^2}{32} = 1 \Rightarrow A = \sqrt{\frac{32}{7\pi^2}}$

* Notice that: $Y_{2,-1} - Y_{2,+1} = 2\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta\cos\varphi$

$$Y_{2,+2} - Y_{2,-2} = 2i\sqrt{\frac{15}{8\pi}}\sin^2\theta\sin\varphi\cos\varphi$$

$$\Rightarrow \Psi, \theta, \varphi) = \langle r,\theta,\varphi|\left(\frac{1}{2}\sqrt{\frac{8\pi}{15}}\left[|2,-1\rangle - |2,1\rangle\right] + \frac{1}{2i}\sqrt{\frac{8\pi}{15}}\left[|2,+2\rangle - |2,-2\rangle\right]\right)$$

$$\hookrightarrow \langle\Psi|H|\Psi\rangle = \left(\frac{1}{2}\sqrt{\frac{8\pi}{15}}\right)^2\left[\langle 2,-1| - \langle 2,1| + i\langle 2,2| - i\langle 2,-2|\right] H\left[|2,-1\rangle - |2,1\rangle - i|2,2\rangle + i|2,-2\rangle\right]$$

$$= \left(\frac{1}{2}\sqrt{\frac{8\pi}{15}}\right)^2\left[\langle 2,-1|H|2,-1\rangle - \langle 2,1|H|2,1\rangle + \langle 2,2|H|2,2\rangle + \langle 2,-2|H|2,-2\rangle\right.$$

* All other terms 0 by orthogonality of spherical harmonics as kets are unaltered by Hamiltonian

#3 (cont.)

b) $$\langle \Psi | H | \Psi \rangle = \left(\frac{1}{2}\sqrt{\frac{8\pi}{15}}\right)^2 \left[\left(\cancel{-\frac{3a}{2\hbar}} - \cancel{\frac{a}{\hbar^2}[6+1]}\right) - \left(\cancel{\frac{3a}{2\hbar}} - \cancel{\frac{a}{\hbar^2}[6+1]}\right) + \left(\cancel{\frac{6a}{2\hbar}} - \frac{a}{\hbar^2}[6+4]\right) + \frac{6a}{2\hbar} - \frac{a}{\hbar^2}[6+4]\right]$$

$$= \left(\frac{1}{2}\sqrt{\frac{8\pi}{15}}\right)^2 \left[\frac{6a}{2\hbar} - \frac{20a}{\hbar^2}\right]$$

$$= \frac{8\pi}{60}\left(\frac{6a}{2\hbar} - \frac{20a}{\hbar^2}\right)$$

c) First order time dependant perturbation theory (for two states) says: