Aug 2008

Problem 5: Time Evolution (10 Points)

Consider the Hamiltonian and a second observable, B, for a system that can be represented in a 3-dimensional Hilbert space using the orthonormal basis: $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$

with

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

as:

$$H = \hbar\omega \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

The system at time t=0 is in the state:

$$|\Psi(0)\rangle = |e_2\rangle$$

a) Calculate the eigenvalues and normalized eigenvectors of H and B. **(2 Point)**

b) Determine $|\Psi(t)\rangle$, the wavefunction at a later time. **(1 Point)**

c) Determine $P_{|\Psi(t)\rangle}(b = 2)$, the probability of obtaining $b = 2$ if b is measured at an arbitrary time. **(1 Points)**

d) Is your probability in part c) time-dependent or time-independent? Discuss in detail. **(1 Point)**

e) Derive an expression for $\frac{\partial}{\partial t}\langle B\rangle$ where $\langle B\rangle = \langle\Psi(t)|B|\Psi(t)\rangle$ by explicit differentiation using the Time-Dependent Schrodinger Equation. **(2 Points)**

f) Use your expression in part b) to find $\frac{\partial}{\partial t}\langle B\rangle$ for this system using the $|\Psi(t)\rangle$ you found in part a). **(2 Points)**

g) Without doing further calculations describe what result you would expect for $\frac{\partial}{\partial t}\langle B\rangle$ if the initial wavefunction $|\Psi(0)\rangle = |e_2\rangle$ changes to:

$$|\Psi(0)\rangle = |e_1\rangle$$

Explain your answer in detail. **(1 Point)**

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Quantum #5

a) Starting w/ $H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\Rightarrow$ find the eigenvalues from: $\det(H - \lambda \mathbb{I}) = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (2-\lambda)[(\lambda^2 - 1)]$$

$$0 = (2-\lambda)(\lambda+1)(\lambda-1)$$

$\hookrightarrow \lambda = 2, -1, 1$

$\Rightarrow$ find eigenvectors from $H\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 &= \lambda x_1 \\ x_3 &= \lambda x_2 \\ x_2 &= \lambda x_3 \end{aligned}$$

* for $\lambda = 2$

$$\begin{aligned} 2x_1 &= 2x_1 \\ x_3 &= 2x_2 \\ x_2 &= 2x_3 \end{aligned} \Rightarrow \vec{V} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

* for $\lambda = -1$

$$\begin{aligned} 2x_1 &= -x_1 \\ x_3 &= -x_2 \\ x_2 &= -x_3 \end{aligned} \Rightarrow \vec{V} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$$

* for $\lambda = 1$

$$\begin{aligned} 2x_1 &= x_1 \\ x_3 &= x_2 \\ x_2 &= x_3 \end{aligned} \Rightarrow V = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

* Similarly for $B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)[(2-\lambda)(1-\lambda) - 0] - 1(0 + (2-\lambda)) = 0$$

$$\begin{aligned} \Rightarrow 0 &= (2-\lambda)(1-\lambda)^2 - (2-\lambda) \\ &= (2-\lambda)[(1-\lambda)^2 - 1] \\ &= (2-\lambda)[(1-\lambda)+1][(1-\lambda)-1] \end{aligned}$$

$\hookrightarrow \lambda = 2, 2, 0$

#5 (cont.)

a) $B\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{aligned} x_1 - x_3 &= \lambda x_1 \\ 2x_2 &= \lambda x_2 \\ -x_1 + x_3 &= \lambda x_3 \end{aligned}$$

* for $\lambda = 2$

$$\begin{aligned} x_1 - x_3 &= 2x_1 \\ 2x_2 &= 2x_2 \\ -x_1 + x_3 &= 2x_3 \end{aligned} \Rightarrow \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\cdot\frac{1}{\sqrt{2}}$$

$|\lambda_B = 2, 1\rangle$ $\quad |\lambda_B = 2, 2\rangle$

* for $\lambda = 0$

$$\begin{aligned} x_1 - x_3 &= 0 \\ 2x_2 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\cdot\frac{1}{\sqrt{2}}$$

b) Given $|\psi(0)\rangle = \langle 0, 1, 0\rangle$, we most first convert this to H basis before acting time-evolution operator

$$\Rightarrow |\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\lambda_H = -1\rangle + |\lambda_H = 1\rangle\right)$$

$$|\psi(t)\rangle = U(t, t_0 = 0)\,|\psi(0)\rangle, \text{ where } U(t, t_0 = 0) = e^{-iHt/\hbar}$$

$$= e^{-iHt/\hbar}\left(\frac{1}{\sqrt{2}}\left[|\lambda_H = -1\rangle + |\lambda_H = 1\rangle\right)\right.$$

$$= \frac{1}{\sqrt{2}}\left(e^{-itw}\,|\lambda_H = 1\rangle + e^{itw}\,|\lambda_H = -1\rangle\right)$$

c) $$P(b=2) = |\langle\lambda_b = 2, 1|\psi(t)\rangle|^2 + |\langle\lambda_b = 2, 2|B|\psi(t)\rangle|^2$$

* convert kets from B basis to H basis

$$|\lambda_b = 2, 1\rangle = \frac{1}{\sqrt{2}}\left(|\lambda_H = -1\rangle + |\lambda_H = 1\rangle\right)$$

$$|\lambda_b = 2, 2\rangle = \frac{1}{\sqrt{3}}\left(|\lambda_H = 2\rangle + \sqrt{2}\,|\lambda_H = -1\rangle - \sqrt{2}\,|\lambda_H = 1\rangle\right) = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \end{bmatrix}$$

$$P(b=2) = \left|\frac{1}{\sqrt{2}}\left(\langle\lambda_H = -1| + \langle\lambda_H = 1|\right)\left(\frac{1}{\sqrt{3}}e^{-itw}|\lambda_H = 1\rangle + e^{itw}|\lambda_H = -1\rangle\right)\right|^2 + \ldots$$

$$= \left|\frac{1}{2}\left(e^{itw} + e^{-itw}\right)\right|^2 + \left|\frac{2}{3}\left(e^{itw} - e^{-itw}\right)\right|^2$$

$$= \left(\frac{1}{2}\left(e^{-itw} + e^{itw}\right)\left(e^{itw} + e^{-itw}\right) + \frac{2}{3}\left(e^{-itw} - e^{itw}\right)\left(e^{itw} + e^{-itw}\right)\right)$$

$$= \left(\frac{1}{2}\left(1 + e^{-2itw} + e^{2iwt} + 1\right) + \frac{2}{3}\left(1 - e^{2iwt} - e^{-2iwt} + 1\right)\right)$$

$$= 1 + \frac{4}{3}$$