PROBLEM 2: Particle in a Box

A particle of mass m is in the ground state of a one dimension box of length L. At $t = 0$, the box suddenly expands *symmetrically* to *three* times its size, leaving the wavefunction of the particle undisturbed. Assume the particle was in the ground state before the expansion.

a) Solve the Schrodinger equation and calculate the eigenenergies and eigenfunctions in the box before and *after* the expansion (show all your work). (3 Points)

b) What is the probability of finding the particle in the ground state immediately after the expansion? (4 Points)

c) Compute the wave function of the particle $\psi(x, t)$ for $t \geq 0$. Hint: express your answer as a superposition of eigenstates. (3 Points)

Hint: $\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \, \cos\theta \cos(q\theta) = \frac{2}{1-q^2} \cos\left(q\frac{\pi}{2}\right)$,

$\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \, \cos\theta \sin(q\theta) = 0.$

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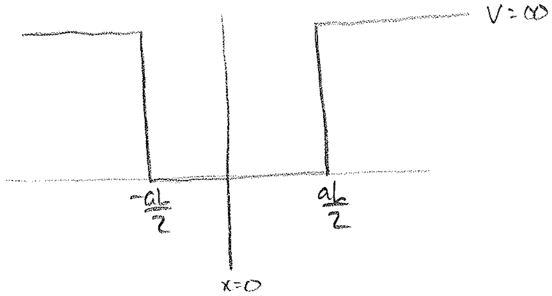
Quantum #2

* Due to symmetric expansion, we choose our edges to by symmetric about $x=0$

a)



$H\psi = E\psi$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

$$= Ae^{ikx} + Be^{-ikx}$$

$$= A\sin(kx) + B\cos(kx)$$

* Solving for our boundary conditions, we know $\psi(-\frac{aL}{2}) = 0 = \psi(\frac{aL}{2})$

$$\pm\frac{kaL}{2} = \frac{n\pi}{2} \quad \Rightarrow \quad k = \frac{n\pi}{aL}$$

↳ if $n =$ even, $\sin(kL) = 0 \rightarrow B = 0$

$n =$ odd, $\cos(kL) = 0 \rightarrow A = 0$

$$\Rightarrow \psi(x) = \begin{cases} A\sin\left(\frac{n\pi x}{aL}\right) & n \text{ even} \\ B\cos\left(\frac{n\pi x}{aL}\right) & n \text{ odd} \end{cases}$$

* Checking normalization

$$1 = A^2\int_{-\frac{aL}{2}}^{aL/2} \sin^2\left(\frac{n\pi x}{aL}\right)dx$$

$$= \frac{A^2}{2}\int_{-aL/2}^{aL/2}\left(1 - \cos\left(\frac{2n\pi x}{aL}\right)\right)dx$$

$$= \frac{A^2}{2}\left[x - \frac{aL}{2n\pi}\sin\left(\frac{2n\pi x}{aL}\right)\right]\Bigg|_{-aL/2}^{aL/2}$$

#2 (cont.)

a) $1 = \frac{A^2}{2}(aL - 0)$ b/c n = even, sin → 0

$A^2 = \frac{2}{aL} \rightarrow A = \sqrt{\frac{2}{aL}}$

$1 = B^2 \int_{-aL/2}^{aL/2} \cos^2\left(\frac{n\pi x}{aL}\right) dx$

$1 = \frac{B^2}{2} \int_{-aL/2}^{aL/2} \left(1 + \cos\left(\frac{2n\pi x}{aL}\right)\right) dx$

$1 = \frac{B^2}{2} \left[x + \sin\left(\frac{2n\pi x}{aL}\right) \cdot \frac{La}{2n\pi} \right] \Big|_{-aL/2}^{aL/2}$

$1 = \frac{B^2}{2}[aL + 0]$ b/c n = odd, cos → 0

$B^2 = \frac{2}{aL} \rightarrow B = \sqrt{\frac{2}{aL}}$

$$\Rightarrow \psi(x) = \begin{cases} \sqrt{\frac{2}{aL}} \sin\left(\frac{n\pi x}{aL}\right) & n = \text{even} \\ \sqrt{\frac{2}{aL}} \cos\left(\frac{n\pi x}{aL}\right) & n = \text{odd} \end{cases}$$

* To determine energies

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{aL} \Rightarrow E = \frac{n^2\pi^2\hbar^2}{2ma^2L^2}$$

* Pre-expansion, $a = 1$

$$\hookrightarrow \psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & n = \text{even} \\ \sqrt{2/L} \cos\left(\frac{n\pi x}{L}\right) & n = \text{odd} \end{cases} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

* Post-expansion, $a = 3$

$$\hookrightarrow \psi_m(x) = \begin{cases} \sqrt{2/3L} \sin\left(\frac{n\pi x}{3L}\right) & n = \text{even} \\ \sqrt{2/3L} \cos\left(\frac{n\pi x}{3L}\right) & n = \text{odd} \end{cases} \qquad E_m = \frac{m^2\pi^2\hbar^2}{18mL^2}$$

#2 (cont.)

b) * Both before and after expansion, the ground state corresponds to n=1

$$P = \left| \langle \psi_{m=1} | \psi_{n=1} \rangle \right|^2$$

$$= \left| \int \psi^*_{m=1} \psi_{n=1} \, dx \right|^2$$

$$= \left| \int_{-L/2}^{L/2} \cos\left(\frac{\pi x}{3L}\right) \cos\left(\frac{\pi x}{L}\right) dx \cdot \frac{2}{L\sqrt{3}} \right|$$

$$= \left| \frac{1}{L\sqrt{3}} \int_{-L/2}^{L/2} \cos\left(\frac{\pi x}{3L} - \frac{\pi x}{L}\right) + \cos\left(\frac{\pi x}{3L} + \frac{\pi x}{L}\right) dx \right|^2$$

$$= \left| \frac{1}{L\sqrt{3}} \int_{-L/2}^{L/2} \cos\left(-\frac{2\pi x}{3L}\right) + \cos\left(\frac{4\pi x}{3L}\right) dx \right|^2$$

$$= \left| \frac{1}{L\sqrt{3}} \left[\frac{3L}{-2\pi} \sin\left(\frac{-2\pi x}{3L}\right) + \frac{3L}{4\pi} \sin\left(\frac{4\pi x}{3L}\right) \right] \Big|_{-L/2}^{L/2} \right|^2$$

$$= \left| \frac{\sqrt{3}}{\pi} \left[-\frac{1}{2}\left(\sin\left(-\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right) + \frac{1}{4}\left(\sin\left(\frac{2\pi}{3}\right) - \sin\left(-\frac{2\pi}{3}\right)\right) \right] \right|^2$$

$$= \left| \frac{\sqrt{3}}{\pi} \left[\sin(\pi/3) + \frac{1}{2} \sin(2\pi/3) \right] \right|^2$$

$$= \left| \frac{\sqrt{3}}{\pi} \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) \right) \right|^2$$

$$= \left| \frac{\sqrt{3}}{\pi} \left(\frac{3\sqrt{3}}{4} \right) \right|^2$$

$$= \left| \frac{9}{4\pi} \right|^2$$

$$= \frac{81}{16\pi^2}$$

#2 (cont.)

c) $|\psi_n(t)\rangle = e^{-iHt/\hbar}|\psi_n\rangle$

* to write this as an expansion of eigenstates

$$\sum_m' |\psi_m\rangle\langle\psi_m|\psi_n\rangle = \sum_m' C_m |\psi_m\rangle$$

In integral form

$$C_m = \int \psi_m^* \psi_n \, dx$$

$$\hookrightarrow \psi_n = \sum_m' \left(\int \psi_m^* \psi_n \, dx \right) \psi_m(x)$$

$$\psi_n(t) = \sum_m' \int \psi_m^* \psi_n \, dx \, e^{-iHt/\hbar} \psi_m(x)$$

$$= \sum_m' \int \psi_m^* \psi_n \, dx \, e^{-iE_m t/\hbar} \psi_m(x)$$