Problem 4: Operator Solutions to the Harmonic Oscillator

Consider the Harmonic Oscillator Hamiltonian in one dimension:

$$H_{ho} = \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2 \tag{1}$$

To simplify this problem, define the new observables:

$$p = \sqrt{\frac{1}{m\hbar\omega}} P \,, \quad q = \sqrt{\frac{m\omega}{\hbar}} X \tag{2}$$

This gives the dimensionless Hamiltonian,

$$H = \frac{1}{\hbar\omega} H_{ho} = \frac{1}{2}\left(p^2 + q^2\right) \tag{3}$$

(a) [1 pt] Calculate the commutation relation for these new variables, $[q, p]$. Be sure to show your work.

(b) [1 pt] Define the non-Hermitian operators $a = \frac{1}{\sqrt{2}}(q + ip)$, $a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$ and the Hermitian operator $n = a^\dagger a$. Compute $[a, a^\dagger]$, $[n, a^\dagger]$, and $[n, a]$

(c) [1 pt] Write the dimensionless Hamiltonian H in terms of a and $a^\dagger$. Write the dimensionless Hamiltonian H in terms of n.

(d) [3 pts] Define the eigenvalues and eigenvectors of n as:

$$n|\lambda\rangle = \lambda|\lambda\rangle. \tag{4}$$

and assume that these eigenvectors form a complete set.

Show that

$$\begin{aligned} a^\dagger|\lambda\rangle &= A|\lambda + 1\rangle \\ a|\lambda\rangle &= B|\lambda - 1\rangle \end{aligned} \tag{5}$$

Determine the normalization constants A and B.

(e) [2 pts.] Show that $n = a^\dagger a$ must have non-negative eigenvalues, $\lambda \geq 0$. Explain why this implies that there must be a state where $a|0\rangle = 0$ and that the eigenvalues of n must be non-negative integers.

(f) [2 pts.] Write the definition for the state $|0\rangle$

$$a|0\rangle = 0 \tag{6}$$

as a differential equation, in q, for the ground state wavefunction of H. Solve this expression for the normalized ground state wavefunction.

Jan 2013

Quantum #4

a) * Remember $[x_i, p_j] = i\hbar\delta_{ij}$

$$\Rightarrow [q,p] = qp - pq$$
$$= \sqrt{\frac{m\omega}{\hbar}}x\sqrt{\frac{1}{m\hbar\omega}}p - \sqrt{\frac{1}{m\hbar\omega}}p\sqrt{\frac{m\omega}{\hbar}}x$$
$$= \frac{1}{\hbar}(xp - px)$$
$$= \frac{1}{\hbar}[x,p]$$
$$= \frac{1}{\hbar}i\hbar$$
$$= i$$

b) $[a,a^\dagger] = aa^\dagger - a^\dagger a$

$$= \frac{1}{\sqrt{2}}(q+ip)\frac{1}{\sqrt{2}}(q-ip) - \frac{1}{\sqrt{2}}(q-ip)\frac{1}{\sqrt{2}}(q+ip)$$
$$= \frac{1}{2}(q^2 + ipq - iqp + p^2) - \frac{1}{2}(q^2 - ipq + iqp + p^2)$$
$$= \frac{i}{2}(pq - qp) - \frac{i}{2}(qp - pq)$$
$$= -i(pq + qp)$$
$$= -i([q,p])$$
$$= -i(i)$$
$$= 1$$

$$[n,a^\dagger] = [a^\dagger a, a^\dagger]$$
$$= a^\dagger a a^\dagger - a^\dagger a^\dagger a$$
$$= a^\dagger[aa^\dagger - a^\dagger a]$$
$$= a^\dagger[a,a^\dagger]$$
$$= a^\dagger$$

$$[n,a] = [a^\dagger a, a]$$
$$= a^\dagger a a - a a^\dagger a$$
$$= [a^\dagger a - a a^\dagger]a$$
$$= [a^\dagger, a]a$$
$$= -[a,a^\dagger]a$$
$$= -a$$

#4 (cont.)

c) We want to rewrite $H = \frac{1}{2}(p^2 + q^2)$ in terms of a and $a^\dagger$

$$\Rightarrow \sqrt{2}\, a = q + ip$$
$$\sqrt{2}\, a^\dagger = q - ip$$

$$\sqrt{2}(a + a^\dagger) = 2q \qquad \sqrt{2}(a - a^\dagger) = 2ip$$

$$\frac{1}{\sqrt{2}}(a + a^\dagger) = q \qquad \frac{-i}{\sqrt{2}}(a - a^\dagger) = p$$

$$\Rightarrow H = \frac{1}{2}\left[\left(\frac{-i}{\sqrt{2}}(a - a^\dagger)\right)^2 + \left(\frac{1}{\sqrt{2}}(a + a^\dagger)\right)^2\right]$$
$$= \frac{1}{2}\left[\frac{-1}{2}(aa - a^\dagger a - aa^\dagger + a^\dagger a^\dagger) + \frac{1}{2}(aa + a^\dagger a + aa^\dagger + a^\dagger a^\dagger)\right]$$
$$= \frac{1}{2}(a^\dagger a + aa^\dagger)$$
$$= \frac{1}{2}(n + aa^\dagger)$$
$$= \frac{1}{2}(n + 1 + a^\dagger a) \quad (\text{from } [a, a^\dagger] = 1)$$
$$= \frac{1}{2}(2n + 1)$$
$$= n + 1/2$$

d) * We must use the n-operator and its commutation relations to solve this problem

$$\Rightarrow a^\dagger |\lambda\rangle = A|\lambda + 1\rangle$$

$$\hookrightarrow n a^\dagger |\lambda\rangle = a^\dagger n + a^\dagger |\lambda\rangle$$
$$= (a^\dagger \lambda + a^\dagger)|\lambda\rangle$$
$$= a^\dagger(\lambda + 1)|\lambda\rangle$$
$$= (\lambda + 1)|\lambda\rangle$$

$$\Rightarrow \langle\lambda| a a^\dagger |\lambda\rangle = A^2 \langle\lambda + 1|\lambda + 1\rangle$$
$$\langle\lambda| a^\dagger a + 1 |\lambda\rangle = A^2$$
$$\langle\lambda| n + 1 |\lambda\rangle = A^2$$
$$\lambda + 1 = A^2 \Rightarrow \boxed{A = \sqrt{\lambda + 1}}$$

#4 (cont.)

d) Similiarly,

$$n(a|\lambda\rangle) = an - a\,|\lambda\rangle$$
$$= a(n-1)\,|\lambda\rangle$$
$$= a(\lambda-1)|\lambda\rangle$$
$$= (\lambda-1)(a|\lambda\rangle)$$

$$\Rightarrow \langle\lambda|a^{\dagger}a|\lambda\rangle = B^2\langle\lambda-1|\lambda-1\rangle$$
$$\langle\lambda|n|\lambda\rangle = B^2$$
$$\lambda = B^2 \rightarrow \boxed{\sqrt{\lambda} = B}$$

e)

#4 (cont.)

f) Given $a|0\rangle = 0$, where $a = \frac{1}{\sqrt{2}}(q + ip)$

$$\frac{1}{\sqrt{2}}(q+ip)|0\rangle = 0$$

$$\frac{1}{\sqrt{2}}\left(q + i\left(-i\hbar \frac{\partial}{\partial q}\right)\right)|0\rangle = 0$$

$$\frac{1}{\sqrt{2}}\left(q - \hbar\frac{\partial}{\partial q}\right)\psi_0 = 0$$

$$q\psi_0 - \hbar\frac{\partial \psi_0}{\partial q} = 0$$

$$\Rightarrow \frac{\partial \psi_0}{\partial q} = \frac{q}{\hbar}\psi_0$$

$$\int \frac{\partial \psi_0}{\psi_0} = \int \frac{q}{\hbar}\partial q$$

$$\ln(\psi_0) = -\frac{1}{2\hbar}q^2 + C$$

$$\psi_0 = \exp\left[\frac{-1}{2\hbar}q^2 + C\right]$$

$$= C\exp\left[\frac{-q^2}{2\hbar}\right]$$

*Checking our normalization

$$1 = c^2\int_{-\infty}^{\infty}\left|\exp\left[\frac{-q^2}{2\hbar}\right]\right|^2 dq$$

$$1 = C^2\int_{-\infty}^{\infty}\exp\left[-\frac{q^2}{\hbar}\right]dq$$

$$1 = C^2\sqrt{\pi\hbar}$$

$$\frac{1}{\sqrt{\pi\hbar}} = c^2$$

$$\hookrightarrow C = \left(\frac{1}{\pi\hbar}\right)^{1/4}$$

$$\Rightarrow \psi_0 = \left(\frac{1}{\pi\hbar}\right)^{1/4}\exp\left[\frac{-q^2}{2\hbar}\right]$$