4?

Problem 2: Variational Method (10 Points)

The Hamiltonian of a one-dimensional harmonic oscillator is

$$H = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2} \,.$$

The ground state energy is $E_0 = \hbar\omega/2$.

Let us employ the variational method with the following trial function as the ground-state wave function

$$\langle x|\psi\rangle = \psi(x) = N e^{-\beta|x|} \,.$$

a. Determine the constant N by applying the normalization condition. (2 points)

b. Find the value of β that minimizes $\langle\psi|H|\psi\rangle$. (2 points)

c. What is the ground-state energy calculated with the variational method? (5 points)
N.B. *The derivative of the trial function has a discontinuity.*

d. How close do you get to the true ground-state energy? (1 points)

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Quantum #2

a) The normalization condition is: $1 = \int_{-\infty}^{\infty} |\psi|^2 dx$

$$\Rightarrow 1 = N^2 \int_{-\infty}^{\infty} e^{-2\beta|x|}$$

$$1 = N^2\left[\int_{-\infty}^{0} e^{2\beta x}dx + \int_{0}^{\infty} e^{-2\beta x}dx\right]$$

$$1 = N^2\left[\frac{1}{2\beta} e^{2\beta x}\Big|_{-\infty}^{0} + \frac{-1}{2\beta} e^{-2\beta x}\Big|_{0}^{+\infty}\right]$$

$$1 = N^2\left[\frac{1}{2\beta}\left(\cancel{e^{2\beta(0)}}^{1} - \cancel{e^{2\beta(-\infty)}}^{0} - \cancel{e^{-2\beta(\infty)}}^{0} + \cancel{e^{-2\beta(0)}}^{1}\right)\right]$$

$$1 = N^2 \frac{1}{2\beta}(2)$$

$$1 = \frac{N^2}{\beta} \Rightarrow N = \sqrt{\beta}$$

b) $\langle\psi|H|\psi\rangle = \langle\psi|x'\rangle\langle x'|H|x\rangle\langle x|\psi\rangle$

$$= \int dx' \int dx\ \psi^*(x') H\ \psi(x)$$

$$= \int dx' \int dx\ \psi^*(x')\left[\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right]\psi(x)$$

$$= \int dx' \int dx\ \psi^*(x')\left[\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\delta(x-x')\right)^2 + \frac{m\omega^2}{2}x^2\delta(x-x')\right]\psi(x)$$

$$= \int_{-\infty}^{\infty} dx\ \psi^*(x)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2\right]\psi(x)$$

* minimum occurs when $\frac{d}{d\beta}\langle\psi|H|\psi\rangle = 0$

$$0 = \frac{d}{d\beta}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\beta}}e^{-\beta|x|}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2\right]\frac{1}{\sqrt{\beta}}e^{-\beta|x|}dx$$

$$= \frac{d}{d\beta}\int_{-\infty}^{\infty}\frac{1}{\beta}e^{\beta x}\left[\frac{-\hbar^2\beta^2}{2m}e^{\beta x} + \frac{m\omega^2}{2}x^2\right]e^{\beta x}dx + \int_0^{\infty}\frac{1}{\beta}e^{-\beta x}\left[\frac{-\hbar^2\beta^2}{2m}e^{-\beta x} + \frac{m\omega^2}{2}x^2 e^{-\beta x}\right]dx$$

$$= \frac{d}{d\beta}\int_{-\infty}^{0}\frac{-\hbar^2\beta}{2m}e^{2\beta x} + \frac{m\omega^2 x^2}{2\beta}e^{2\beta x}dx + \int_0^{\infty}\frac{-\hbar^2\beta}{2m}e^{-2\beta x} + \frac{m\omega^2 x^2}{2\beta}e^{-2\beta x}dx$$

$$= \frac{d}{d\beta}\left[\frac{-\hbar^2}{4m}e^{2\beta x}\Big|_{-\infty}^{0} + \frac{m\omega^2}{2}\frac{1}{4\beta^4} + \frac{\hbar^2}{2m}\cdot\frac{1}{2}e^{-2\beta x}\Big|_0^{\infty} + \frac{m\omega^2}{2}\frac{1}{4\beta^4}\right.$$

$$= \frac{d}{d\beta}\left[\frac{-\hbar^2}{4m} + \frac{m\omega^2}{8\beta^4} - \frac{\hbar^2}{4m} + \frac{m\omega^2}{8\beta}\right]$$

$$= \frac{d}{d\beta}\left[\frac{-\hbar^2}{2m} + \frac{m\omega^2}{4\beta^4}\right]$$

#2 (cont.)

b) $0 = \frac{-m\omega^2}{\beta^5}$