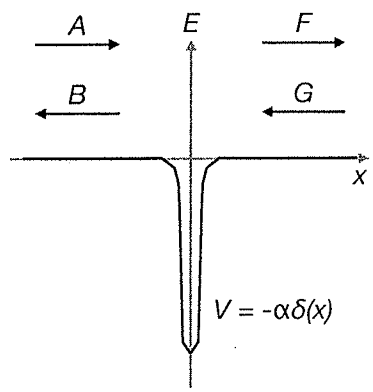
2

## Problem 5: Transmission across delta functions(10 points):

a) (1 pt) Consider the potential $V(x) = -\alpha\delta(x)$. Show that the derivative of the wave function is discontinuous across the potential.
i.e $lim_{\epsilon\to 0}\left(\left(\frac{\partial\psi(x)}{\partial x}\right)_{x=\epsilon} - \left(\frac{\partial\psi(x)}{\partial x}\right)_{x=-\epsilon}\right) = -\frac{2m\alpha}{\hbar^2}\psi(0)$

b) (2 pts) A particle with $E > 0$ is incident on the delta function potential from $x < 0$. Determine the probability that the particle will be transmitted across the potential. Can the probability of transmission = 1?




c) (3 pts) One can define a transfer matrix $M$, which gives the amplitudes to the right of the potential in terms of those on the left.

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

Construct the $M$-matrix for scattering from a single delta-function potential at point $a$.

$$V(x) = -\alpha\delta(x-a)$$

d) (1 pt) Show that if you have a potential consisting of 2 isolated pieces, the M-matrix for the combination is the product of the two M-matrices for each section separately.

$$M = M_2 M_1$$

e) (3 pts) Now consider a double delta function potential

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)]$$

Determine the probability of transmission across the double delta function potential ($T = \frac{1}{|M_{22}|^2}$). Can the probability of transmission = 1?

Jan 2018

# Quantum #5

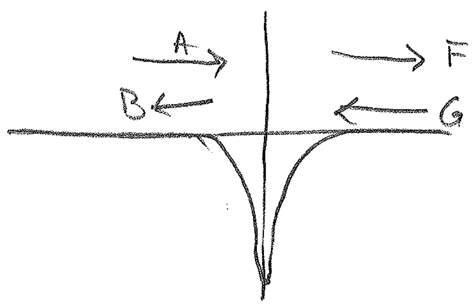
a) We can show this discontinuity by integrating the Schrödinger equation from $-\epsilon$ to $\epsilon$

$$\Rightarrow \int_{-\epsilon}^{\epsilon} -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = \int_{-\epsilon}^{\epsilon} E\psi$$

$$\lim_{\epsilon\to 0} \left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} = \left[+E\cancel{\int_{-\epsilon}^{\epsilon}\psi}^{\,0} - \int_{-\epsilon}^{\epsilon} -\alpha\psi\delta(x)\,dx\right]\frac{-2m}{\hbar^2}$$

$$\left.\frac{d\psi}{dx}\right|_{\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon} = \frac{-2m\alpha}{\hbar^2}\psi(0)$$

b)




Our boundary conditions are:

① $Ae^{ikx} + Be^{-ikx} = Fe^{ikx}$

② $-A(ik)e^{ikx} + B(ik)e^{-ikx} + F(ik)e^{ikx} = \frac{-2m\alpha}{\hbar^2}F$

Solving ① at $\psi(0)$ ($\psi$ must be continuous)

$$A + B = F \Leftrightarrow B = F - A$$

Solving ② ($d\psi/dx$ discontinuous by $\frac{-2m\alpha}{\hbar^2}\psi(0)$)

$$(F - A + B)\,ik = \frac{-2m\alpha}{\hbar^2}F$$

$$F - A + (F - A) = \frac{i2m\alpha}{\hbar^2 k}F$$

$$F - A = \frac{im\alpha}{\hbar^2 k}F$$

$$F\left(1 - \frac{im\alpha}{\hbar^2 k}\right) = A$$

$$\frac{F}{A} = \frac{1}{1 - \frac{im\alpha}{\hbar^2 k}}$$

$$T = \left|\frac{F}{A}\right|^2 = \frac{1}{1 + \frac{m^2\alpha^2}{\hbar^4 k^2}} \Rightarrow T \neq 1 \text{ at any point}$$

#5 (cont.)

c) $$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

*We must use our boundary conditions to determine this matrix

① $Ae^{ikx} + Be^{-ikx} = Fe^{ikx} + Ge^{-ikx}$ (continuous $\psi$)

② $ik[Fe^{ikx} - Ge^{-ikx}] - ik[Ae^{ikx} - Be^{-ikx}] = \frac{-2m\alpha}{\hbar^2}(Ae^{ikx} + Be^{ikx})$ (discontinuous $\frac{d\psi}{dx}$)

*We can rewrite these as:

① $Ae^{2ikx} + B = Fe^{2ikx} + G$

② $Fe^{2ikx} - G = +Ae^{2ikx} - B + \frac{2im\alpha}{\hbar^2 k}(Ae^{2ikx} + B)$

*Adding ① and ② yields

$$2Fe^{2ikx} = 2Ae^{2ikx} + \frac{2im\alpha}{\hbar^2 k}(Ae^{2ikx} + B)$$

*let $\frac{m\alpha}{\hbar^2 k} = \beta$

$\hookrightarrow F = (1 + i\beta)A + i\beta e^{-2ikx} B$

$\Rightarrow M_{11} = 1 + i\beta$

$M_{12} = i\beta e^{-2ikx}$

*Subtracting 2 from ①

$$2G = 2B - 2i\beta(Ae^{2ikx} + B)$$

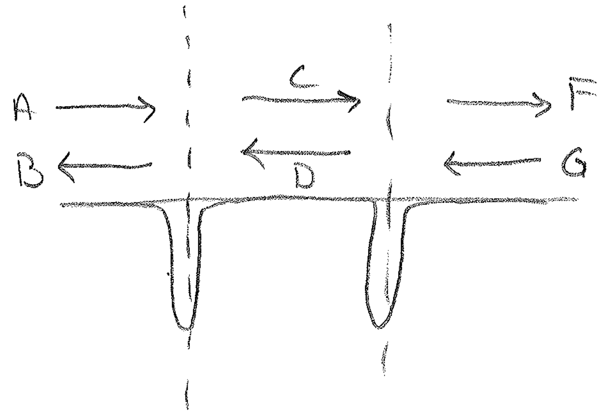
$$G = (1 - i\beta)B - i\beta e^{2ikx} A$$

$\Rightarrow M_{22} = 1 - i\beta$

$M_{21} = -i\beta e^{2ikx}$

#5 (cont.)

d) * For a double - $\delta$ potential




$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} F \\ G \end{bmatrix} = m'm \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} F \\ G \end{bmatrix} = \underline{M} \begin{bmatrix} A \\ B \end{bmatrix} \quad \text{where } \underline{M} = m'm$$

e) $T = \left| \frac{1}{\underline{M}_{22}} \right|^2$

$$\begin{aligned} \underline{M}_{22} &= m'_{21} m_{12} + m'_{22} m_{22} \\ &= (-i\beta e^{2ikx})(i\beta e^{-2ikx}) + (1-i\beta)(1-i\beta) \\ &= \beta^2 + 1 - 2i\beta - \beta^2 \\ &= 1 - 2i\beta \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{|1-2i\beta|^2} \\ &= \frac{1}{1+4\beta^2} \end{aligned}$$