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Problem 4: Coherent States and the Harmonic Oscillator (10 pts)

Consider a one dimensional harmonic oscillator with mass m and frequency ω

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \, x^2$$

The raising and lowering operators are useful for harmonic oscillator problems:

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - i \frac{p}{m\omega} \right) \qquad a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i \frac{p}{m\omega} \right) \quad .$$

√ (a) (1 pt) Verify that the Hamiltonian can be recast to the form $H = \hbar\omega(N + \frac{1}{2})$, where $N = a^\dagger a$. Be sure to show your work.

(b) (3 pts) Prove by induction that

$$[a, (a^\dagger)^n] \; = \; n(a^\dagger)^{n-1}$$

$a a^{+n} - a^{+n} a = n(a^+)^{n-1}$

$a a^{+n} - n(a^+)^{n-1}$

where $n \geq 1$ denotes a positive integer.

√ (c) (4 pts) Define a state

$$|f\rangle \; = \; e^{-|f|^2/2} \times e^{f a^\dagger} |0\rangle$$

where f is a complex number. This state is called a coherent state. Starting from your results in part (b) of this problem, show that

$$a|f\rangle \; = \; f|f\rangle \; .$$

√ (d) (2 pts) Check that

$$\langle f|f\rangle \; = \; 1$$

If needed, you can use the fact that $(a^\dagger)^n|0\rangle \; = \; \sqrt{n!}|n\rangle$ for level n of the harmonic oscillator.

$(a+ib)(a-ib) = a^2+b^2$

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Quantum #4

a) $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

* Note: We can define x and p in terms of a, a^+ as:

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a^+ + a)$$

$$p = i\sqrt{\frac{\hbar m\omega}{2}}\,(a^+ - a)$$

$$\Rightarrow x^2 = \frac{\hbar}{2m\omega}\left(a^+a^+ + aa^+ + a^+a + aa\right)$$

$$p^2 = \frac{-\hbar m\omega}{2}\left(a^+a^+ - aa^+ - a^+a + aa\right)$$

$$\Rightarrow H = \frac{1}{2m}\left(\frac{-\hbar m\omega}{2}\left[a^+a^+ - aa^+ - a^+a + aa\right]\right) + \frac{m\omega^2}{2}\left(\frac{\hbar}{2m\omega}\left[a^+a^+ + aa^+ + a^+a + aa\right]\right)$$

$$= -\frac{\hbar\omega}{4}\left(a^+a^+ - aa^+ - a^+a + aa\right) + \frac{\hbar\omega}{4}\left(a^+a^+ + aa^+ + a^+a + aa\right)$$

$$= \frac{\hbar\omega}{2}\left(aa^+ + a^+a\right)$$

$$= \frac{\hbar\omega}{2}\left(2a^+a + 1\right)$$

$$= \hbar\omega\left(a^+a + 1/2\right)$$

$$= \hbar\omega\left(N + 1/2\right) \checkmark$$

b) Prove by induction that $[a, (a^+)^n] = n(a^+)^{n-1}$ where $n \geq 1$, $n \in \mathbb{Z}^+$

Proof: We know that $[a, a^+] = 1$

Base Case: $n = 1$

$$[a, a^+] = 1(a^+)^{1-1}$$
$$= 1 \checkmark$$

Inductive Hypothesis: For any $n \geq 1$, $n \in \mathbb{Z}^+$, $[a, (a^+)^n] = n(a^+)^{n-1}$

We must now show that the n+1 case is true, ie

$$[a, (a^+)^{n+1}] = (n+1)(a^+)^n$$

#4 (cont.)

b) $$[a,(a^\dagger)^{n+1}] = [a,(a^\dagger)^n a^\dagger]$$
$$= [a,(a^\dagger)^n]a^\dagger + [a,a^\dagger](a^\dagger)^n$$
$$= n(a^\dagger)^{n-1}a^\dagger + 1(a^\dagger)^n$$
$$= (n+1)(a^\dagger)^n \checkmark$$

Because n is general, we have proved $[a,(a^\dagger)^n] = n(a^\dagger)^{n-1}$ for all $n \geq 1$, $n \in \mathbb{Z}^+$. QED

c) $$|f\rangle = \exp\left[\frac{-|f|^2}{2}\right]\exp[fa^\dagger]|0\rangle$$

$$a|f\rangle = a\exp\left[\frac{-|f|^2}{2}\right]\exp[fa^\dagger]|0\rangle$$
$$= \exp\left[\frac{-|f|^2}{2}\right] a \sum_n \frac{f^n(a^\dagger)^n}{n!}|0\rangle$$
$$= \exp\left[\frac{-|f|^2}{2}\right] \sum_n \frac{f^n}{n!}\left((a^\dagger)^n a + n(a^\dagger)^{n-1}\right)|0\rangle$$
$$= \exp\left[\frac{-|f|^2}{2}\right] \sum_n \frac{f^n}{n!} n (a^\dagger)^{n-1}|0\rangle$$
$$= f\exp\left[\frac{-|f|^2}{2}\right] \sum_{n-1} \frac{f^{n-1}(a^\dagger)^{n-1}}{(n-1)!}|0\rangle$$
$$= f\exp\left[\frac{-|f|^2}{2}\right]\exp[f(a^\dagger)^n]|0\rangle$$
$$= f|f\rangle \checkmark$$

d) $$\langle f|f\rangle \stackrel{?}{=} 1$$
$$= \langle 0|\exp\left[\frac{-|f|^2}{2}\right]\exp[f^* a^n]\exp\left[\frac{-|f|^2}{2}\right]\exp[f(a^\dagger)^n]|0\rangle$$
$$= \exp[-|f|^2]\langle 0|\exp[f^* a^n]\exp[f(a^\dagger)^n]|0\rangle$$
$$= \sum_{mn}\exp[-|f|^2]\langle 0|\frac{(f^*)^m a^m}{m!}\frac{f^n(a^\dagger)^n}{n!}|0\rangle$$
$$= \sum_{mn}\exp[-|f|^2]\langle m|\sqrt{m!}(f^*)^m f^n\sqrt{n!}|n\rangle$$

#4 (cont.)

d) $\langle f|f\rangle = \sum_{mn}' \exp\left[-\frac{|f|^2}{}\right] \sqrt{m!}\,(f^*)^m f^n \sqrt{n!}\, \delta_{mn} \frac{1}{n!}$

$= \exp\left[-\frac{|f|^2}{}\right] \sum_n' n!\,(|f|^2)^n \frac{1}{2(n!)}$

$= \exp\left[-\frac{|f|^2}{}\right] \sum_n' \frac{(|f|^2)^n}{n!}$

$= \exp\left[-\frac{|f|^2}{}\right] \exp\left[\frac{|f|^2}{}\right]$

$= 1$ ✓