10

Problem 3: Infinite Well (10 points):

Assume that a particle is placed in a one dimensional infinitely deep square well potential of width $L = 1$, which has the analytic form

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \leq x \leq 1 \\ \infty & x > 1. \end{cases}$$

a) (2 pts) Calculate the eigenfunctions and eigenvalues for this potential.

b) (1 pt) Sketch the ground state wave function and the first 2 excited states

c) (2 pts) Assume that a particle is placed in the potential well in the state given by the following wavefunction at $t = 0$

$$\psi(x,0) = \sqrt{\frac{8}{13}}\sin(\pi x) + \sqrt{\frac{72}{13}}\sin(\pi x)\cos(\pi x).$$

Calculate the probability that the particle is in each of the following eigenstates: the ground state, the first excited state, in any state greater than the first excited state.

d) (1 pt) Calculate the expectation value of the energy.

e) (2 pts) Calculate the expectation value of the position operator for the initial state that is given in c).

f) (2 pts) The energy of the particle is measured and is found to be in the ground state. The wall located at $x = 1$ is quickly moved to $x = 2$. What is the probability that the energy is found equal to that of the ground state?

Jan 2018

Quantum #3

$$V(x) = \begin{cases} \infty & x<0 \\ 0 & 0 \le x \le 1 \\ \infty & x>1 \end{cases}$$

a) $H\psi = E\psi$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

* In the regions of infinite potential, $\psi = 0$

* For $0 \le x \le 1$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\hookrightarrow \psi = Ae^{ikx} + Be^{-ikx}$$

$$= A\sin(kx) + B\cos(kx)$$

> Equivalent Answers

$\psi(0) = 0 \Rightarrow B = 0$

$\psi(1) = 0 = A\sin(k)$

$\hookrightarrow$ True if $k = n\pi$

$$\frac{\sqrt{2mE}}{\hbar} = n\pi \Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2m}$$

* Normalize wave function

$$1 = \int_0^1 |A|^2 \sin^2(n\pi x)\,dx$$

$$= |A|^2 \int_0^1 \frac{1}{2}(1-\cos(2n\pi x))\,dx$$

$$= |A|^2 \frac{1}{2}\left(x - \frac{1}{2n\pi}\sin(2n\pi x)\right)$$

$$= \frac{1}{2}|A|^2 \Rightarrow A = \sqrt{2}$$

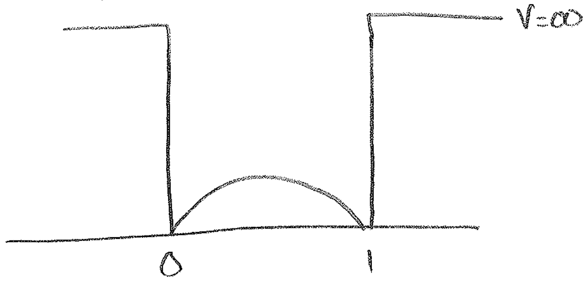
$$\psi_n = \sqrt{2}\sin(n\pi x)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2m}$$

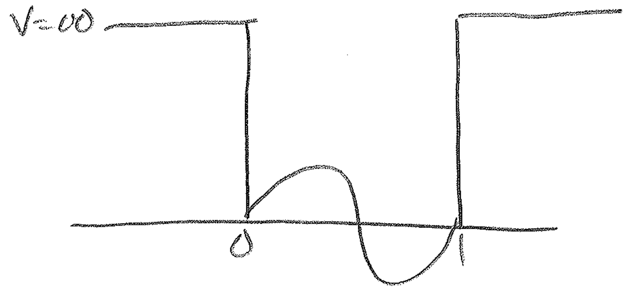
<u>#3 (cont.)</u>

b) Our first 3 wave functions are:

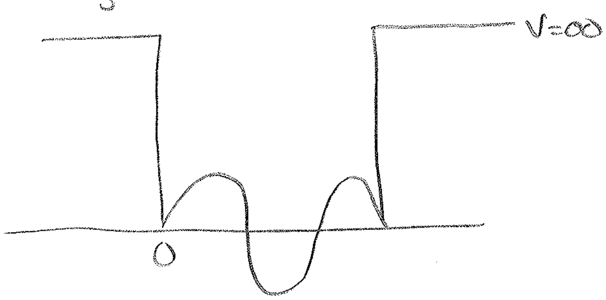
$\psi_1 = \sqrt{2}\sin(\pi x)$



$\psi_2 = \sqrt{2}\sin(2\pi x)$



$\psi_3 = \sqrt{2}\sin(3\pi x)$



c) Given $|\psi\rangle = \sqrt{\frac{8}{13}}\sin(\pi x) + \sqrt{\frac{72}{13}}\sin(\pi x)\cos(\pi x)$

$$P(\psi_1) = |\langle\psi_1|\psi\rangle|^2$$

$$= \left|\int_0^1 \sqrt{2}\sin(\pi x)\left[\sqrt{\tfrac{8}{13}}\sin(\pi x) + \sqrt{\tfrac{72}{13}}\sin(\pi x)\cos(\pi x)\right]dx\right|^2$$

$$= \left|\int_0^1 \sqrt{\tfrac{16}{13}}\sin^2(\pi x) + \sqrt{\tfrac{144}{13}}\sin^2(\pi x)\cos(\pi x)\,dx\right|^2$$

$$= \left|\int_0^1 \sqrt{\tfrac{16}{13}}\cdot\tfrac{1}{2}(1-\cos(2\pi x)) + \sqrt{\tfrac{144}{13}}\sin^2(\pi x)\,d(\sin(\pi x))\right|^2$$

$$= \left|\sqrt{\tfrac{16}{13}}\,\tfrac{1}{2}\left(x - \tfrac{1}{2\pi}\sin(2\pi x)\right)\Big|_0^1 + \sqrt{\tfrac{144}{13}}\sin^3(\pi x)\Big|_0^1\right|^2$$

$$= \left|\sqrt{\tfrac{16}{13}}\cdot\tfrac{1}{2}\right|^2$$

$$= \frac{16}{13}\cdot\frac{1}{4}$$

$$= \frac{4}{13}$$

#3 (cont.)

c) $P(\psi_2) = |\langle \psi_2 | \psi \rangle|^2$

$$= \left| \int_0^1 \sqrt{2}\sin(2\pi x)\left[\sqrt{\tfrac{8}{13}}\sin(\pi x) + \sqrt{\tfrac{72}{13}}\sin(\pi x)\cos(\pi x)\right] dx \right|^2$$

$$= \left| \int_0^1 \sqrt{\tfrac{16}{13}}\sin(2\pi x)\sin(\pi x) + \sqrt{\tfrac{144}{13}}\sin(2\pi x)\sin(\pi x)\cos(\pi x)\, dx \right|^2$$

$$= \left| \int_0^1 \sqrt{\tfrac{16}{13}}\sin(2\pi x)\sin(\pi x) + \sqrt{\tfrac{144}{13}}\cdot\tfrac{1}{2}\sin^2(2\pi x)\, dx \right|^2$$

$$= \left| \sqrt{\tfrac{16}{13}}\left[\frac{\sin(\pi x)}{2\pi} - \frac{\sin(3\pi x)}{6\pi}\right]\Big|_0^1 + \sqrt{\tfrac{36}{13}}\left[\frac{x}{2} - \frac{\sin(4\pi x)}{8\pi}\right]\Big|_0^1 \right|^2$$

$$= \left| \sqrt{\tfrac{16}{13}}\left[0-0-(0-0)\right] + \sqrt{\tfrac{36}{13}}\left[\tfrac{1}{2}-0-(0-0)\right] \right|^2$$

$$= \left| \sqrt{\tfrac{9}{13}} \right|^2$$

$$= \frac{9}{13}$$

$$P(\psi_{3+}) = 1 - P(\psi_1) - P(\psi_2)$$

$$= 1 - \frac{4}{13} - \frac{9}{13}$$

$$= 0$$

d) $\langle H \rangle = \langle \psi | H | \psi \rangle$

* Rewriting $|\psi\rangle$ as: $|\psi\rangle = \sqrt{\tfrac{8}{13}}\sin(\pi x) + \sqrt{\tfrac{18}{13}}\sin(2\pi x)$

$$= \sqrt{\tfrac{4}{13}}|\psi_1\rangle + \sqrt{\tfrac{9}{13}}|\psi_2\rangle$$

$$\Rightarrow \langle H \rangle = \left[\sqrt{\tfrac{4}{13}}\langle\psi_1| + \sqrt{\tfrac{9}{13}}\langle\psi_2|\right] H \left[\sqrt{\tfrac{4}{13}}|\psi_1\rangle + \sqrt{\tfrac{9}{13}}|\psi_2\rangle\right]$$

* cross terms will cancel

$$= \frac{4}{13}\langle\psi_1|H|\psi_1\rangle + \frac{9}{13}\langle\psi_2|H|\psi_2\rangle$$

$$= \frac{1}{13}\left(\frac{4\pi^2\hbar^2}{2m} + \frac{36\pi^2\hbar^2}{2m}\right)$$

$$= \frac{20\pi^2\hbar^2}{13m}$$

#3 (cont.)

e) $\langle x \rangle = \langle \psi | x | \psi \rangle$

$$= \int_0^1 \left(\sqrt{\tfrac{8}{13}} \sin(\pi x) + \sqrt{\tfrac{18}{13}} \sin(2\pi x)\right) x \left(\sqrt{\tfrac{8}{13}} \sin(\pi x) + \sqrt{\tfrac{18}{13}} \sin(2\pi x)\right) dx$$

$$= \int_0^1 \frac{8}{13} x \sin^2(\pi x) + \frac{18}{13} x \sin^2(2\pi x)\, dx$$

$$= \frac{8}{13}\left[\frac{x^2}{4} - \frac{x\sin(2\pi x)}{4\pi} - \frac{\cos(2\pi x)}{8\pi^2}\right]\Big|_0^1 + \frac{18}{13}\left[\frac{x^2}{4} - \frac{x\sin(4\pi x)}{8\pi} - \frac{\cos(4\pi x)}{36\pi^2}\right]\Big|_0^1$$

$$= \frac{8}{13}\left[\left(\frac{1}{4} - 0 - \frac{1}{8\pi^2}\right) - \left(0 - 0 - \frac{1}{8\pi^2}\right)\right] + \frac{18}{13}\left[\frac{1}{4} - 0 - \frac{1}{36\pi^2} - \left(0 - 0 - \frac{1}{36\pi^2}\right)\right]$$

$$= \frac{1}{4} \cdot \frac{8+18}{13}$$

$$= \frac{26}{4 \cdot 13}$$

$$= \frac{1}{2}$$

f) Our general solution to a infinite well w/ walls at $0, L$ is

$$\psi_n = \sqrt{\tfrac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \psi_1' = \sin\left(\frac{\pi x}{2}\right)$$

$$P(\psi_1') = |\langle \psi_1' | \psi \rangle|^2$$

$$= \left|\int_0^1 \sin\left(\frac{\pi x}{2}\right) \cdot \sqrt{2} \sin(\pi x)\, dx\right|^2$$

$$= \left|\left(\frac{\sin\left(-\frac{\pi x}{2}\right)}{-\pi} - \frac{\sin\left(\frac{3\pi x}{2}\right)}{3\pi}\right)\Big|_0^1\right|^2$$

$$= \left|\frac{-1}{-\pi} - \frac{-1}{3\pi} - (0 - 0)\right|$$

$$= \left|\frac{1}{\pi} + \frac{1}{3\pi}\right|^2$$

$$= \frac{16}{9\pi^2}$$