PROBLEM 5: Addition of angular momenta

Consider an electron. We know its orbital angular momentum $\ell = 1$ and the z component $m = 1/2$ of its total angular momentum j.

a) What are the possible values of j? (2 Points).

b) Write down the kets $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$ in terms of products of spin and orbital angular momentum states (3 Points)

c) Calculate the expectation value of the spin operator $\mathbf{S}$ in the state $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$. Consider all possible values of j. (3 Points).

d) The magnetic dipole moment of the electron is

$$\boldsymbol{\mu} = \frac{e}{2m_e c}(\mathbf{L} + 2\mathbf{S}),$$

with $\mathbf{L}$ the orbital angular momentum operator, e the electron charge, m_e the mass and c the speed of light. Calculate the expectation value of $\boldsymbol{\mu}$ in the states $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$. (2 Points)

Raising and lowering angular momentum operators:

$$J_{\pm}|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

Aug 2014

Quantum #5

a) Given an electron w/ $l=1$, $m=\frac{1}{2}$, we know that

$$|l-m| \leq j \leq |l+s|$$

$$|1-\tfrac{1}{2}| \leq j \leq |1+\tfrac{1}{2}|$$

$$\tfrac{1}{2} \leq j \leq \tfrac{3}{2}$$

$$\hookrightarrow j = \{\tfrac{1}{2}, \tfrac{3}{2}\}$$

b) We must use Clebsh-Gordon coefficients, we start in the highest state and use the lowering operator

$$|l=1, m=\tfrac{1}{2}; \tfrac{3}{2}, \tfrac{3}{2}\rangle = |l=1, m_\ell=1\rangle \otimes |s=\tfrac{1}{2}, m_s=\tfrac{1}{2}\rangle$$

$$J_-|l=1, m=\tfrac{1}{2}; \tfrac{3}{2}, \tfrac{3}{2}\rangle = \hbar\sqrt{(\tfrac{3}{2}+\tfrac{3}{2})(\tfrac{3}{2}-\tfrac{3}{2}+1)}\ |1, \tfrac{1}{2}; \tfrac{3}{2}, \tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{3}$$

$$J_-|1,1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle = J_-^{l}|1,1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle + |1,1\rangle \otimes J_-^{s}|\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= \sqrt{2}\hbar|1,0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle + \hbar|1,1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$|1, \tfrac{1}{2}; \tfrac{3}{2}, \tfrac{1}{2}\rangle = \frac{\sqrt{2}}{\sqrt{3}}\left[|1,0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle\right] + \frac{1}{\sqrt{3}}\left[|1,1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right]$$

To determine the $|1, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\rangle$ state, we use the orthogonality condition

$$\langle 1, \tfrac{1}{2}; \tfrac{3}{2}, \tfrac{1}{2} | 1, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle = 0$$

letting $|1, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\rangle = A\left[|1,0\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle\right] + B\left[|1,1\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle\right]$

where $A^2 + B^2 = 1$

$$\hookrightarrow 0 = A\cdot\sqrt{\tfrac{2}{3}} + B\sqrt{\tfrac{1}{3}}$$

$$-B\sqrt{\tfrac{1}{3}} = A\sqrt{\tfrac{2}{3}} \Rightarrow A = \frac{-B}{\sqrt{2}}$$

$$1 = \frac{B^2}{2} + B^2 \Rightarrow B = \sqrt{\tfrac{2}{3}},\ A = \frac{-1}{\sqrt{3}}$$

$$\hookrightarrow |1, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle = \frac{-1}{\sqrt{3}}|1,0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|1,1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

#5 (cont.)

c)