Quantum Mechanics
Qualifying Exam–January 2010

Notes and Instructions:

- There are **6** problems and **7** pages.
- Be sure to write your alias at the top of every page.
- Number each page with the problem number, and page number of your solution (e.g. "Problem 3, p. 1/4" is the first page of a four page solution to problem 3).
- **You must show all your work.**

Possibly useful formulas:

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-dimensional simple harmonic oscillator operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

$$P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \qquad Y_2^2(\theta,\varphi) = \frac{5}{\sqrt{96\pi}}\,3\sin^2\theta\, e^{2i\varphi}$$

$$Y_2^1(\theta,\varphi) = -\frac{5}{\sqrt{24\pi}}\,3\sin\theta\cos\theta\, e^{i\varphi}$$

$$Y_1^1(\theta,\varphi) = -\frac{3}{\sqrt{8\pi}}\sin\theta\, e^{i\varphi} \qquad Y_2^0(\theta,\varphi) = \frac{5}{\sqrt{4\pi}}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

$$Y_1^0(\theta,\varphi) = \frac{3}{\sqrt{4\pi}}\cos\theta \qquad Y_2^{-1}(\theta,\varphi) = \frac{5}{\sqrt{24\pi}}\,3\sin\theta\cos\theta\, e^{-i\varphi}$$

$$Y_1^{-1}(\theta,\varphi) = \frac{3}{\sqrt{8\pi}}\sin\theta\, e^{-i\varphi} \qquad Y_2^{-2}(\theta,\varphi) = \frac{5}{\sqrt{96\pi}}\,3\sin^2\theta\, e^{-2i\varphi}$$

Angular momentum raising and lowering operators:

$$\hat{L}_\pm = (\hat{L}_x \pm i\,\hat{L}_y)$$

PROBLEM 1: The Delta-Function Potential

Let us consider a single particle of mass m moving in one dimension with the Hamiltonian

$$H = T + V(x) \, ,$$

where the kinetic energy is

$$T = \frac{P^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \, ,$$

the potential energy is

$$V(x) = -V_0 \, \delta(x) \, ,$$

and $\delta(x)$ is the Dirac delta function.

(a) [2 points] Find an expression for the discontinuity of the derivative of the wave function at $x = 0$.

(b) [3 points] Find the ground state wave function.

(c) [2 points] Find the ground state energy.

(d) [3 points] Find the expectation value for the kinetic energy, $\langle T \rangle$.

PROBLEM 2: Hydrogenic Atoms with One Electron

In terms of the first Bohr radius, $a_0 \equiv \hbar/(c\alpha m_e)$, where α is the fine-structure constant, the ground-state eigenfunction of a hydrogen atom is

$$\psi_{1,0,0}(r, \theta, \varphi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}.$$

(a) [3 points] Evaluate the probability of finding an electron in the ground-state of a hydrogen atom in the classically forbidden region. The classically forbidden region is the region of space where the classical kinetic energy is negative.

(b) [4 points] For the ground state, evaluate the uncertainty in the Cartesian coordinate x and the uncertainty in the corresponding component of the linear momentum, p_x. *Hint: You need not use the explicit form of the operator for the linear momentum to evaluate* Δp_x.

(c) [3 points] Show explicitly that the product of your uncertainties, $\Delta x \, \Delta p_x$, is consistent with the Heisenberg uncertainty principle.

PROBLEM 3: Time-Dependent Perturbation Theory

Consider a non-relativistic particle of mass m and charge q with the potential energy:

$$V(x) = \frac{1}{2}\, k\, X^2$$

A homogeneous electric field $\mathcal{E}(t)$ directed along the x-axis is switched on at time $t = 0$. This causes a perturbation of the form

$$H' = -q\, X\, \mathcal{E}(t)$$

where $\mathcal{E}(t)$ has the form

$$\mathcal{E}(t) = \mathcal{E}_o e^{-t/\tau}$$

where $\mathcal{E}_o$ and τ are constants.

The particle is in the ground state at time $t \leq 0$. This problem will deal with calculating the probability that it will be found in an excited state as $t \longrightarrow \infty$.

The probability that the particle makes a transition from an initial state i to a final state f is given by:

$$P_{fi}(t, t_o) = \frac{1}{\hbar^2} \left| \int_{t_o}^{t} dt' \, \langle \phi_f | H'(t') | \phi_i \rangle \, e^{i\omega_{fi} t'} \right|^2 .$$

where the particle originally is in state ϕ_i and finally in state ϕ_f.

(a) [2 points] In terms of known quantities, what is the value of ω_{fi} ?

(b) [2 points] How many excited states can the particle make a transition to?

(c) [6 points] Derive an expression for the probability that the particle will be found in any allowed excited state as $t \longrightarrow \infty$.

PROBLEM 4: Spin Physics

Spin-1/2 objects generally have magnetic moments that affect their energy levels and dynamics in magnetic fields. The interaction between the magnetic moment and a magnetic field, $\vec{B}$ can be written as:

$$H = -\mu\vec{S}\cdot\vec{B} \tag{1}$$

where $\vec{S}$ is the spin of the particle

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \tag{2}$$

where the σ_i's are Pauli matrices.

In this problem we'll be using as our basis the eigenstates of S_z,

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

with eigenvalues $\pm\frac{\hbar}{2}$.

(a) [1 point] If a particle is in the spin state $|+\rangle$, compute the expectation values of S_x, S_y, and S_z.

(b) [1 point] If a particle is in the spin state $|+\rangle$, what are the uncertainties of S_x, S_y, and S_z? ($\Delta S_i^2 = \langle S_i^2\rangle - \langle S_i\rangle^2$.) Explain the physics of your results in terms of the eigenvalues and measurement probabilities of the spin in the x, y, and z directions.

(c) [3 points] A large ensemble of particles are all prepared to be in the spin state $|+\rangle$ at time $t = 0$ when a magnetic field in the x-direction is switched on, $\vec{B} = B_0\hat{e}_x$. Solve for the time-dependent probabilities, $P_\pm(t)$, of measuring S_z to be $\pm\hbar/2$.

(d) [2 points] For the experiment described in part (c), what are the probabilities for measuring S_x to be $\pm\hbar/2$? Explain the differences between the results for S_z and S_x.

(e) [3 points] Consider the case where the magnetic field is $\vec{B} = \frac{B_0}{\sqrt{2}}(\hat{e}_x + \hat{e}_z)$. In this case what is the time-dependent probability of measuring S_z to be $+\hbar/2$?

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Quantum #4

a) $S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\quad S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\langle S_x \rangle = {}_z\langle + | S_x | + \rangle_z = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\langle S_y \rangle = {}_z\langle + | S_y | + \rangle_z = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\langle S_z \rangle = {}_z\langle + | S_z | + \rangle_z = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hbar/2$$

b) $\Delta S_i^2 = \langle S_i^2 \rangle - \langle S_i \rangle^2$

$$\langle S_x^2 \rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hbar^2/4$$

$$\langle S_y^2 \rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hbar^2/4$$

$$\langle S_z^2 \rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hbar^2/4$$

$$\Rightarrow \Delta S_x^2 = \frac{\hbar^2}{4} - (0)^2 = \hbar^2/4$$

$$\Delta S_y^2 = \frac{\hbar^2}{4} - (0)^2 = \frac{\hbar^2}{4}$$

$$\Delta S_z^2 = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2 = 0$$

* Because we are working in the S_z basis, there is no uncertainty b/c we know the state of the particle. However, the S_x and S_z eigenstates are linear combinations of the S_z states, thus there is some uncertainty as to which eigenvalue is preferred.

#4 (cont.)

c) $H = -\mu \vec{S} \cdot \vec{B}$ $\qquad |\psi(t=0)\rangle = |+\rangle_z$

$= -\mu(S_x B_0)$

* To act time evolution operator, convert to S_x basis

$$|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z)$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_z - |-\rangle_z) \quad > \quad |+\rangle_z = \frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x)$$

$$|\psi(t)\rangle = U(t, t_0=0)|\psi(t=0)\rangle$$

$$= e^{-iHt/\hbar}\left(\frac{1}{\sqrt{2}}[|+\rangle_x + |-\rangle_x]\right) \qquad E_\pm = \pm\frac{\mu_0 B\hbar}{2}$$

$$= \frac{1}{\sqrt{2}}\left(e^{-iE_+t/\hbar}|+\rangle_x + e^{-iE_-t/\hbar}|-\rangle_x\right)$$

$$P(S_z = +\hbar/2) = |{}_z\langle +|\psi(t)\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}[\langle +|_x + \langle -|_x]\frac{1}{\sqrt{2}}\, e^{-iE_+t/\hbar}|+\rangle_x + e^{-iE_-t/\hbar}|-\rangle_x\right.$$

$$= \frac{1}{4}\left|e^{iE_+t/\hbar} + e^{-iE_-t/\hbar}\right|^2$$

$$= \frac{1}{4}\left(2 + e^{-i(E_+ - E_i)t/\hbar} + e^{+i(E_+ - E_-)t/\hbar}\right)$$

$$= \frac{1}{4}\left(2 + 2\cos\left(\frac{\Delta E t}{\hbar}\right)\right), \quad \Delta E = E_+ - E_-$$

$$= \frac{1}{2}\left(1 + \cos\left(\frac{\Delta E t}{\hbar}\right)\right)$$

$$P(S_z = -\hbar/2) = 1 - P(S_z = +\hbar/2)$$

$$= 1 - \frac{1}{2}\left(1 + \cos\left(\frac{\Delta E t}{\hbar}\right)\right)$$

$$= \frac{1}{2}\left(1 - \cos\left(\frac{\Delta E t}{\hbar}\right)\right)$$

#4 (cont.)

d) * utilizing work done in part c;

$$P(S_x = \hbar/2) = |\,{}_x\langle + | \psi(t)\rangle|^2$$

$$= |\,{}_x\langle + | (\tfrac{1}{\sqrt{2}}[e^{-iE_+t/\hbar}|+\rangle_x + e^{-iE_-t/\hbar}|-\rangle_x])|^2$$

$$= \tfrac{1}{2}|e^{-iE_+t/\hbar}|^2$$

$$= \tfrac{1}{2}$$

$$P(S_x = -\hbar/2) = 1 - P(S_x = \hbar/2)$$

$$= \tfrac{1}{2}$$

* In this case, our final states are in the same eigenbasis as our original state, which is a linear combination of the two states where the coefficients evolve in time in a coupled manner. But since we are operating in only 1 eigenbasis, we must have time independent probabilities, as the basis states are stationary states.

e) With $\vec{B} = \frac{B_0}{\sqrt{2}}(\hat{e}_x + \hat{e}_z)$, our Hamiltonian now becomes $-\frac{\mu B_0 \hbar}{2\sqrt{2}}(S_x + S_z)$

* if $A = \frac{-\mu B_0 \hbar}{2\sqrt{2}}$

$$\Rightarrow H = A\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

* Determining eigenvalues + eigenvectors

$$0 = (1-\lambda)(-1-\lambda) - 1$$

$$= -1 + \cancel{\lambda} - \cancel{\lambda} + \lambda^2 - 1$$

$$= \lambda^2 - 2$$

$$\hookrightarrow \lambda = \pm\sqrt{2}$$

$$H\vec{v} = \lambda\vec{v} \rightarrow \quad x_1 + x_2 = \lambda x_1$$
$$x_1 - x_2 = \lambda x_2$$

* if $\lambda = +\sqrt{2}$

$$x_1 + x_2 = \sqrt{2}\,x_1$$
$$x_1 - x_2 = \sqrt{2}\,x_2$$
$$\hookrightarrow x_1 = 1+\sqrt{2},\ x_2 = 1$$
$$\vec{v}_1 = \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}$$
$$= |+\rangle_{xz}$$

* if $\lambda = -\sqrt{2}$

$$x_1 + x_2 = -\sqrt{2}\,x_1$$
$$x_1 - x_2 = -\sqrt{2}\,x_2$$
$$\hookrightarrow x_1 = 1-\sqrt{2},\ x_2 = 1$$
$$\vec{v}_2 = \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}$$
$$= |-\rangle_{xz}$$

$$A^2[(1+\sqrt{2})^2 + 1] = 1$$

$$A^2(1 + 2\sqrt{2} + 2 + 1) = 1$$

$$A^2 = \frac{1}{4+2\sqrt{2}}\left(\frac{4-2\sqrt{2}}{4-2\sqrt{2}}\right)$$

$$A^2 = \frac{4-2\sqrt{2}}{16+4(2)} = \frac{4-2\sqrt{2}}{24}$$

$$= \left(\frac{2-\sqrt{2}}{12}\right)^{1/2}$$

$$A^2[(1-\sqrt{2})^2 + 1] = 1$$

$$A^2[1 - 2\sqrt{2} + 2 + 1] = 1$$

$$A^2 = \frac{1}{[4-2\sqrt{2}]}$$

$$A^2 = \frac{4+2\sqrt{2}}{16-4(2)}$$

$$A^2 = \frac{2+2\sqrt{2}}{4}$$

$$A = \left(\frac{2+2\sqrt{2}}{4}\right)^{1/2}$$

#4 (cont.)

e) * To act time evolution operator, convert $|\psi(t=0)\rangle$ to H-basis

$$|\psi(t=0)\rangle = |+\rangle_z = \left(\left(\frac{1}{4+2\sqrt{2}}\right)^{1/2} |+\rangle_{xz} + \left(\frac{1}{4-2\sqrt{2}}\right)^{1/2} |-\rangle_{xz} \right)$$

* Not worth time + effort

PROBLEM 5: Two Level System

Consider a quantum system that can be accurately approximated as having two energy levels $|+\rangle$ and $|-\rangle$ such that

$$H_0|\pm\rangle = \pm\epsilon|\pm\rangle\,,$$

where ϵ is energy.

When placed in an external field, the eigenstates of H_0 are mixed by another term in the total Hamiltonian

$$V|\pm\rangle = \delta|\mp\rangle\,.$$

For simplicity, we choose ϵ to be real.

(a) [1 points] Using the states $|+\rangle$ and $|-\rangle$ as your basis states, write down the matrix representations for the operators H_0 and V.

(b) [3 points] What will be the possible results if a measurement is made of the energy for the full Hamiltonian $H = H_0 + V$?

(c) [2 points] Experiments are performed that measure the transition energies between eigenstates. Without the external field ($\delta = 0$) it is found that the transition energy is 4 eV and with the external field ($\delta \neq 0$) the transition energy is 6 eV. What is the coupling between the states $|\pm\rangle$, δ, in this case?

(d) [2 points] We can write the eigenstates of the total Hamiltonian in terms of two energy levels $|\pm\rangle$ as

$$\begin{aligned} |1\rangle &= \cos(\theta_1)|+\rangle + \sin(\theta_1)|-\rangle \\ |2\rangle &= \cos(\theta_2)|+\rangle + \sin(\theta_2)|-\rangle\,. \end{aligned}$$

Letting $\delta/\epsilon = C$, solve for the angles θ_1 and θ_2 in terms of C.

(e) [2 points] Consider an experiment where the two-level system starts in the eigenstate of H_0 with eigenvalue $-\epsilon$. A very weak field is turned on so that $C \ll 1$. To the lowest order in C, what is the probability of measuring a positive energy for the system when $\delta \neq 0$?

PROBLEM 6: Hyperfine Splitting

The hyperfine splitting in hydrogen comes from a spin-spin interaction between the electron and the proton. The total Hamiltonian can be written as

$$H = \frac{P_p^2}{2m_p} + \frac{P_e^2}{2m_e} - \frac{e^2}{r} + H_{HF}$$

where $H_{HF} = A\vec{S}_e \cdot \vec{S}_p$, and A is a real constant.

(a) [1 points] What are the spin quantum numbers s and m_s of the electron?

(b) [1 points] What are the spin quantum numbers s and m_s of the proton?

(c) [1 points] What are the spin quantum numbers s and m_s of the combined electron-proton system?

(d) [5 points] Diagonalize H_{HF} in the total $\vec{S} = \vec{S}_e + \vec{S}_p$ basis and compute the energy eigenvalues.

(e) [2 points] Write an expression for the energy of a photon that would be emitted from a hyperfine transition in terms of A, $\hbar$, and any other relevant constants.