Quantum Mechanics
Qualifying Exam - January 2018

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write on only one side of the paper for your solutions.
- Write your alias on the top of every page of your solutions.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

Possibly useful formulas:

Spin Operator

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In spherical coordinates,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi.$$

$$\nabla^2\psi(\vec{r}) = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi(\vec{r}) - \frac{L^2}{\hbar^2 r^2}\psi(\vec{r})$$

where L^2 is the usual angular momentum operator.

Clebsch-Gordon coefficients

1/2 × 1/2		1 +1	1 0	0 0	1 -1
+1/2	+1/2	1			
+1/2	-1/2		1/2	1/2	
-1/2	+1/2		1/2	-1/2	
-1/2	-1/2				1

Spherical Harmonics:

$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$

$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$

$Y_{1,\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta$

$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$

$Y_{2,\pm 1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\phi}\cos\theta\sin\theta$

$Y_{2,\pm 2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}e^{\pm 2i\phi}\sin^2\theta$

8-10

Problem 1: Matrix Mechanics (10 points):

Consider a particle with mass m and one continuous degree of freedom (spatial coordinate z with associated momentum operator $\hat{p}_z = -\imath\hbar\frac{d}{dz}$) and two discrete internal (pseudo-) spin states described by the Hamiltonian operator $\hat{H}$:

$$\hat{H} = \frac{\hat{p}_z^2}{2m}\hat{I} + \frac{\hbar k_{so}}{m}\hat{\sigma}_z\hat{p}_z + \frac{\Omega}{2}\hat{\sigma}_x. \quad (1)$$

Here, $\hat{I}$ is the identity operator in spin space, k_{so} and Ω are constants, and $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are the usual Pauli spin operators for a spin-1/2 particle. Different from what you might be used to, the Hamiltonian $\hat{H}$, Eq. (1), couples the spin and spatial degrees of freedom.

✓ a) (1 pt) What are the units of k_{so} and Ω? Explain your answer.

✓ b) (1 pt) Choose a convenient basis that spans the spin space and express the Hamiltonian operator $\hat{H}$ in this spin basis (you should obtain a 2×2 matrix). Explain your reasoning.

? c) (1 pt) Show that the operator $\hat{p}_z$ commutes with every element of the 2×2 matrix obtained in b).

✓ d) (3 pts) (Use your results from parts b) and c) to determine the eigen energies $E(p_z)$ of $\hat{H}$. Here, p_z is not an operator but a number.

✓ e) (1 pt) What happens to the eigen energies in the large p_z limit?

? f) (3 pts) Plot the eigen energies obtained in d) as a function of p_z for:
2? i) vanishing Ω
ii) large Ω
iii) small Ω

Explain what the terms "large" and "small" mean in this context, i.e., identify the quantity that Ω needs to be compared with in both cases.

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Quantum #1

a) All terms in the Hamiltonian must have units of energy $\frac{\text{mass}\cdot\text{distance}^2}{\text{time}^2}$

$$\Rightarrow [\Omega] = \frac{\text{mass}\cdot\text{distance}^2}{\text{time}^2}$$

$$[k_{so}] = \frac{1}{\text{distance}}$$

b) We want to use the $|\uparrow\rangle$, $|\downarrow\rangle$ basis

$$\Rightarrow H = \begin{bmatrix} \frac{P_z^2}{2m} & 0 \\ 0 & \frac{P_z^2}{2m} \end{bmatrix} + \begin{bmatrix} \frac{\hbar k_{so}}{m} P_z & 0 \\ 0 & -\frac{\hbar k_{so} P_z}{m} \end{bmatrix} + \begin{bmatrix} 0 & \frac{\Omega}{2} \\ \frac{\Omega}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{P_z^2}{2m} + \frac{\hbar k_{so} P_z}{m} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \frac{P_z^2}{2m} - \frac{\hbar k_{so} P_z}{2m} \end{bmatrix}$$

c) $[P_z, P_z] = 0$ identically. Therefore $[P_z, P_z^2] = 0$ by application of $[A, BC]$ where all 3 terms are P_z. Also $[P_z, \sigma_i] = 0$ b/c linear momentum and spin exist in different Hilbert spaces

d) We can determine the eigenenergies by solving the eigenvalue equation $\det(H - \lambda \mathbb{I}) = 0$

$$\begin{vmatrix} \frac{P_z^2}{2m} + \frac{\hbar k_{so} P_z}{m} - \lambda & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \frac{P_z^2}{2m} - \frac{\hbar k_{so} P_z}{m} - \lambda \end{vmatrix} = 0$$

$$\hookrightarrow 0 = \left(\frac{P_z^2}{2m} + \frac{\hbar k_{so} P_z}{m} - \lambda\right)\left(\frac{P_z^2}{2m} - \frac{\hbar k_{so} P_z}{m} - \lambda\right) - \frac{\Omega^2}{4}$$

$$= \frac{P_z^4}{4m^2} + \cancel{\frac{\hbar k_{so} P_z^3}{2m^2}} - \lambda\frac{P_z^2}{2m} - \cancel{\frac{\hbar k_{so} P_z^3}{2m^2}} - \frac{\hbar^2 k_{so}^2 P_z^2}{m^2} - \cancel{\frac{\hbar k_{so} P_z \lambda}{m}} - \lambda\frac{P_z^2}{2m}$$
$$+ \cancel{\frac{\hbar k_{so} p_z \lambda}{m}} + \lambda^2 - \frac{\Omega^2}{4}$$

#1 (cont.)

d) $$0 = \lambda^2 - \frac{p_z^2}{2m}\lambda + \left(\frac{p_z^4}{4m^2} - \frac{\Omega^2}{4}\right)$$

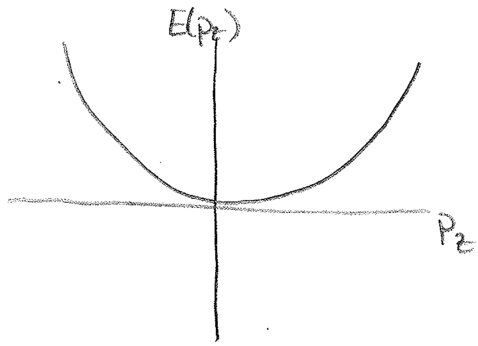
$$\hookrightarrow \lambda = \frac{\frac{p_z^2}{2m} \pm \sqrt{\frac{p_z^4}{m^2} - 4\left(\frac{p_z^4}{4m^2} - \frac{\Omega^2}{4}\right)}}{2}$$

$$= \frac{p_z^2}{4m} \pm \frac{\Omega}{2}$$

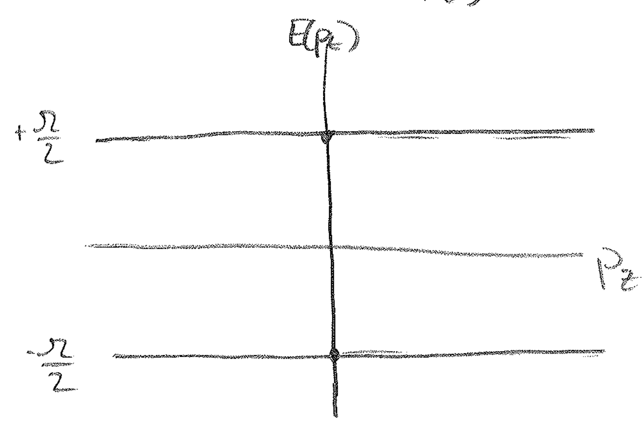
e) In the limit where p_z is large

$\lambda_\pm \approx \frac{p_z^2}{4m}$, eigenenergies will approach infinity as p increases

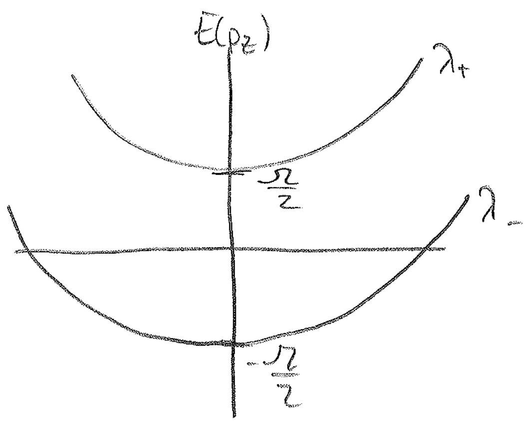
f) * For vanishing Ω



* For large Ω ($\Omega \gg p_z^2$)



* For small Ω ($\Omega \approx p_z^2$)



$S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2$

5-7 ?

Problem 2: Perturbation with 2 spins (10 points):

Let $\vec{S_1}$ and $\vec{S_2}$ be the spin operators of two spin-1/2 particles. Then $\vec{S} = \vec{S_1} + \vec{S_2}$ is the spin operator for this two-particle system.

✓ a) (2 pts) Consider the Hamiltonian

$$\hat{H}_0 = \alpha(\hat{S}_x^2 + \hat{S}_y^2 - \hat{S}_z^2)/\hbar^2 \quad \alpha : \text{ real constant greater than 0}$$

Determine the Energy eigenvalues and degeneracies for this Hamiltonian.

✓ b) (4 pts) Consider a perturbation to the above Hamiltonian:

$$\hat{H}_1 = \lambda(\hat{S_{1x}} - \hat{S_{2x}}) \quad \lambda : \text{ real constant greater than 0.}$$

Calculate the new energies and degeneracies to first-order in perturbation theory.

? c) (3 pts) Now consider an unperturbed Hamiltonian

$$\hat{H}_0 = -A(\hat{S}_{1z} + \hat{S}_{2z}) \quad A : \text{ real constant greater than 0}$$

with a perturbing Hamiltonian of the form

$$\hat{H}_1 = B(\hat{S}_{1x}\hat{S}_{2x} - \hat{S}_{1y}\hat{S}_{2y}) \quad B : \text{ real constant greater than 0}$$

by means of perturbation theory, calculate the ground state energy of $\hat{H}_0$ and calculate the first and second order shifts of the ground state energy of $\hat{H}_0$ as a consequence of the perturbation $\hat{H}_1$.

? d) (1 pt) The exact ground state energy for $\hat{H}_0 + \hat{H}_1$ found in part c) is

$$E_{ground} = -\frac{\hbar}{2}\sqrt{4A^2 + B^2\hbar^2}$$

Compare your results from c) to the exact energy. What conditions on A and B are required so that your results from c) and d) agree?

$S_1^2 = S_{1x}^2 + S_{1y}^2 + S_{1z}^2$

$S_2^2 = S_{2x}^2 + S_{2y}^2 + S_{2z}^2$

$S^2 = S_x^2 + S_y^2 + S_z^2 - 2 S_1 \cdot S_2$

$S_x^2 + S_y^2 = S^2 - S_z^2 + 2 S_1 \cdot S_2$

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Quantum #2

a) $H_0 = \frac{\alpha}{\hbar^2}(S_x^2 + S_y^2 - S_z^2)$

$= \frac{\alpha}{\hbar^2}(S^2 - 2S_z^2)$

$S^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle$

$S_z^2 |j,m\rangle = m^2\hbar^2 |j,m\rangle$

* We write our states in the $|s_1, s_2; j, m\rangle$ basis

$|s_1 - s_2| \leq j \leq s_1 + s_2$

$0 \leq j \leq 1$

$>$ Possible $|j,m\rangle$ states: $|0,0\rangle$, $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$

$H_0|0,0\rangle = \frac{\alpha}{\hbar^2}(S^2 - 2S_z^2)|0,0\rangle$

$= \frac{\alpha^2}{\hbar^2}(0(0+1) - 2(0^2)\hbar^2$

$= 0$

$H_0|1,1\rangle = \frac{\alpha}{\hbar^2}(S^2 - 2S_z^2 |1,1\rangle$

$= \frac{\alpha}{\hbar^2}(1(1+1) - 2(1)^2)\hbar^2$

$= 0$

$H_0|1,0\rangle = \frac{\alpha}{\hbar^2}(S^2 - 2S_z^2)|1,0\rangle$

$= \frac{\alpha}{\hbar^2}(1(1+1) - 2(0)^2)\hbar^2$

$= 2\alpha$

$H_0|1,-1\rangle = \frac{\alpha}{\hbar^2}(S^2 - 2S_z^2)|1,-1\rangle$

$= \frac{\alpha}{\hbar^2}(1(1+1) - 2(-1)^2)\hbar^2$

$= 0$

$\Rightarrow$ Triplet state at $E = 0$, $|0,0\rangle, |1,1\rangle, |1,-1\rangle$

Singlet state at $E = 2\alpha$; $|1,0\rangle$

#2 (cont.)

b) $H_1 = \lambda(S_{1x} - S_{2x})$

* Given $S_\pm = S_x \pm i S_y$, we can rewrite our perturbation as:

$$S_{1x} = (S_{1+} - S_{1-}) \cdot \frac{1}{2}$$

$$S_{2x} = (S_{2+} - S_{2-}) \cdot \frac{1}{2}$$

$$\hookrightarrow H_1 = \frac{\lambda}{2}(S_{1+} - S_{1-} - S_{2+} + S_{2-})$$

* Note: Using this form of the operator requires rewriting our states in the $|s_1, s_2; s_{1z}, s_{2z}\rangle$ basis.

* The equation that governs first order perturbation theory is:

$$\Delta E^{(1)} = \langle n^{(0)} | H_1 | n^{(0)} \rangle$$

* For the $|1,1\rangle$ state:

$$|1,1\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$\Delta E^{(1)} = \langle \tfrac{1}{2}, \tfrac{1}{2} | \left(\frac{\lambda}{2}[S_{1+} - S_{1-} - S_{2+} + S_{2-}]\right) | \tfrac{1}{2}, \tfrac{1}{2} \rangle$$

$$= 0$$

$\hookrightarrow$ S_{i+} operators go to unallowed states

S_{i-} operators push state to an orthogonal state, resulting in inner product of 0

* For the $|1,-1\rangle$ state:

$$|1,-1\rangle = |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$\Delta E^{(1)} = \langle -\tfrac{1}{2}, -\tfrac{1}{2} | \frac{\lambda}{2}(S_{1+} - S_{1-} - S_{2+} + S_{2-}) | -\tfrac{1}{2}, -\tfrac{1}{2} \rangle$$

$$= 0$$

$\hookrightarrow$ S_{i-} operators go to unallowed state

S_{i+} operators push state to an orthogonal state, resulting in inner product of 0

#2 (cont.)

b) * For the $|1,0\rangle$ state:

$$|1,0\rangle = \frac{1}{\sqrt{2}}|{-1/2}, 1/2\rangle + \frac{1}{\sqrt{2}}|1/2, {-1/2}\rangle$$

$$\Delta E^{(1)} = \frac{1}{\sqrt{2}}\left(\langle 1/2, -1/2| + \langle -1/2, 1/2|\right)\left(\frac{\lambda}{2}[S_{1+} - S_{1-} - S_{2+} + S_{2-}]\right)\left(|1/2,-1/2\rangle + |-1/2, 1/2\rangle\right)$$

$$= \frac{\lambda}{4}\Big[\langle 1/2,-1/2|\overset{0}{\cancel{S_{1+}}} - \overset{0}{\cancel{S_{1-}}} - \overset{0}{\cancel{S_{2+}}} + \overset{0}{\cancel{S_{2-}}}|1/2,-1/2\rangle + \langle 1/2,-1/2|\overset{0}{\cancel{S_{1+}}} - \overset{0}{\cancel{S_{1-}}} - \overset{0}{\cancel{S_{2+}}} + \overset{0}{\cancel{S_{2-}}}|-1/2,1/2\rangle$$

$$+ \langle -1/2, 1/2|\overset{0}{\cancel{S_{1+}}} - \overset{0}{\cancel{S_{1-}}} - \overset{0}{\cancel{S_{2+}}} + \overset{0}{\cancel{S_{2-}}}|1/2,-1/2\rangle + \langle -1/2, 1/2|\overset{0}{\cancel{S_{1+}}} - \overset{0}{\cancel{S_{1-}}} - \overset{0}{\cancel{S_{2+}}} + \overset{0}{\cancel{S_{2-}}}|-1/2,1/2\rangle\Big]$$

$$= 0$$

↳ All operators either push states to unallowed values of $S_{iz} = \pm 3/2$ or to $|\pm 1/2, \pm 1/2\rangle$ states which are orthogonal to our known state.

* For the $|0,0\rangle$ state:

$$|0,0\rangle = \frac{1}{\sqrt{2}}|1/2, -1/2\rangle - \frac{1}{\sqrt{2}}|-1/2, 1/2\rangle$$

$\Delta E^{(1)} = 0$ by same logic as for $|1,0\rangle$ state

⟹ Second order pertubation theory needed to break this degenerecy

c) $H_0 = -A(S_{1z} + S_{2z})$

* We want our states in the $|S_{1z}, S_{2z}\rangle$ basis

$$H_0|1/2, 1/2\rangle = -A(S_{1z}+S_{2z})|1/2,1/2\rangle = -A(1/2 + 1/2) = -A\hbar \leftarrow \boxed{\text{Ground State}}$$

$$H_0|-1/2, -1/2\rangle = -A(S_{1z}+S_{2z})|-1/2,-1/2\rangle = -A(-1/2 - 1/2) = A\hbar$$

$$H_0|1/2, -1/2\rangle = -A(S_{1z}+S_{2z})|1/2,-1/2\rangle = -A(1/2 - 1/2) = 0$$

$$H_0|-1/2, 1/2\rangle = -A(S_{1z}+S_{2z})|-1/2,1/2\rangle = -A(-1/2 + 1/2) = 0$$

#2 (cont.)

c) $H_1 = B(S_{1x}S_{2x} - S_{1y}S_{2y})$

$= \frac{B}{2}(S_{1+}S_{2+} + S_{1-}S_{2-})$

* The equation for the first order energy correction is:

$$\Delta E^{(1)} = \langle n^{(0)}|H_1|n^{(0)}\rangle$$

$$= \langle \tfrac{1}{2}, \tfrac{1}{2}| \frac{B}{2}(S_{1+}S_{2+} + S_{1-}S_{2-})|\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= \frac{B}{2}\Big[\underbrace{\cancel{\langle \tfrac{1}{2}, \tfrac{1}{2}| S_{1+}S_{2+}}}_{\text{unallowed state}} |\tfrac{1}{2}, \tfrac{1}{2}\rangle \xrightarrow{} 0 + \underbrace{\langle \tfrac{1}{2}, \tfrac{1}{2}| \cancel{S_{1-}S_{2-}}}_{\text{orthogonality}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle \xrightarrow{} 0$$

$$= 0$$

* The equation governing second order perturbation energy corrections is:

$$\Delta E^{(2)} = \sum_{k\neq n} \frac{|\langle k|H_1|n\rangle|^2}{E_n - E_k}$$

* our only non-zero component will be $|k\rangle = |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$$= \frac{|\langle -\tfrac{1}{2}, -\tfrac{1}{2}| \frac{B}{2}(S_{1+}S_{2+} + S_{1-}S_{2-})|\tfrac{1}{2}, \tfrac{1}{2}\rangle|^2}{-A - A}$$

$$= \frac{B^2}{8A^2}\left|\left(\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\right)^2\right|^2$$

$$= \frac{B^2}{8A}|(1)(1)|^2$$

$$= \frac{B^2\hbar^2}{8A^2}$$

d) $A \in \mathbb{R}$, $B = 0$

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Problem 3: Infinite Well (10 points):

Assume that a particle is placed in a one dimensional infinitely deep square well potential of width $L = 1$, which has the analytic form

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \leq x \leq 1 \\ \infty & x > 1. \end{cases}$$

a) (2 pts) Calculate the eigenfunctions and eigenvalues for this potential.

b) (1 pt) Sketch the ground state wave function and the first 2 excited states

c) (2 pts) Assume that a particle is placed in the potential well in the state given by the following wavefunction at $t = 0$

$$\psi(x,0) = \sqrt{\frac{8}{13}}\sin(\pi x) + \sqrt{\frac{72}{13}}\sin(\pi x)\cos(\pi x).$$

Calculate the probability that the particle is in each of the following eigenstates: the ground state, the first excited state, in any state greater than the first excited state.

d) (1 pt) Calculate the expectation value of the energy.

e) (2 pts) Calculate the expectation value of the position operator for the initial state that is given in c).

f) (2 pts) The energy of the particle is measured and is found to be in the ground state. The wall located at $x = 1$ is quickly moved to $x = 2$. What is the probability that the energy is found equal to that of the ground state?

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Quantum #3

$$V(x) = \begin{cases} \infty & x<0 \\ 0 & 0 \le x \le 1 \\ \infty & x>1 \end{cases}$$

a) $H\psi = E\psi$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

* In the regions of infinite potential, $\psi = 0$

* For $0 \le x \le 1$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

$\hookrightarrow \psi = Ae^{ikx} + Be^{-ikx}$

$= A\sin(kx) + B\cos(kx)$ > Equivalent Answers

$\psi(0) = 0 \Rightarrow B = 0$

$\psi(1) = 0 = A\sin(k)$

$\hookrightarrow$ True if $k = n\pi$

$$\frac{\sqrt{2mE}}{\hbar} = n\pi \Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2m}$$

* Normalize wave function

$$1 = \int_0^1 |A|^2 \sin^2(n\pi x)\,dx$$

$$= |A|^2 \int_0^1 \frac{1}{2}(1-\cos(2n\pi x))\,dx$$

$$= |A|^2 \frac{1}{2}\left(x - \frac{1}{2n\pi}\sin(2n\pi x)\right)$$

$$= \frac{1}{2}|A|^2 \Rightarrow A = \sqrt{2}$$

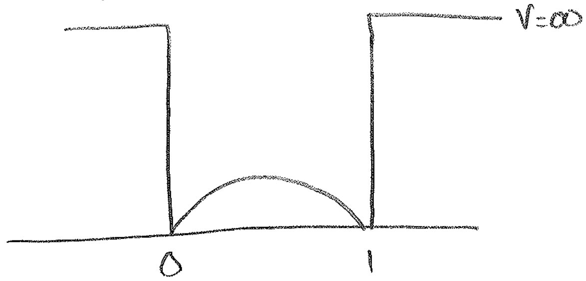
$$\psi_n = \sqrt{2}\sin(n\pi x)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2m}$$

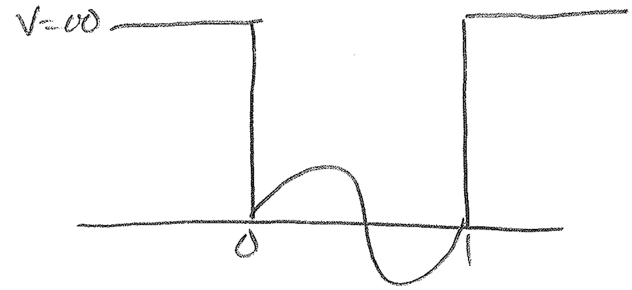
#3 (cont.)

b) Our first 3 wave functions are:

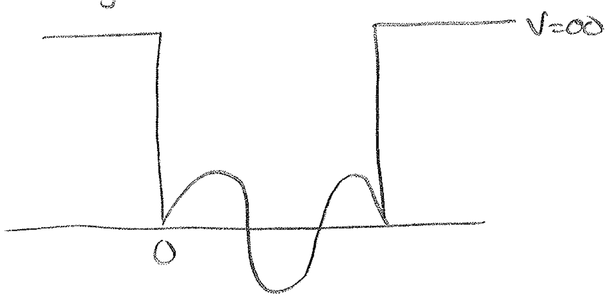
$\psi_1 = \sqrt{2}\sin(\pi x)$



$\psi_2 = \sqrt{2}\sin(2\pi x)$



$\psi_3 = \sqrt{2}\sin(3\pi x)$



c) Given $|\psi\rangle = \sqrt{\frac{8}{13}}\sin(\pi x) + \sqrt{\frac{72}{13}}\sin(\pi x)\cos(\pi x)$

$$P(\psi_1) = |\langle\psi_1|\psi\rangle|^2$$

$$= \left|\int_0^1 \sqrt{2}\sin(\pi x)\left[\sqrt{\tfrac{8}{13}}\sin(\pi x) + \sqrt{\tfrac{72}{13}}\sin(\pi x)\cos(\pi x)\right]dx\right|^2$$

$$= \left|\int_0^1 \sqrt{\tfrac{16}{13}}\sin^2(\pi x) + \sqrt{\tfrac{144}{13}}\sin^2(\pi x)\cos(\pi x)\,dx\right|^2$$

$$= \left|\int_0^1 \sqrt{\tfrac{16}{13}}\cdot\tfrac{1}{2}(1-\cos(2\pi x)) + \sqrt{\tfrac{144}{13}}\sin^2(\pi x)\,d(\sin(\pi x))\right|^2$$

$$= \left|\sqrt{\tfrac{16}{13}}\,\tfrac{1}{2}\left(x - \tfrac{1}{2\pi}\sin(2\pi x)\right)\Big|_0^1 + \sqrt{\tfrac{144}{13}}\sin^3(\pi x)\Big|_0^1\right|^2$$

$$= \left|\sqrt{\tfrac{16}{13}}\cdot\tfrac{1}{2}\right|^2$$

$$= \frac{16}{13}\cdot\frac{1}{4}$$

$$= \frac{4}{13}$$

#3 (cont.)

c) $P(\psi_2) = |\langle \psi_2 | \psi \rangle|^2$

$$= \left| \int_0^1 \sqrt{2} \sin(2\pi x) \left[\sqrt{\tfrac{8}{13}} \sin(\pi x) + \sqrt{\tfrac{72}{13}} \sin(\pi x)\cos(\pi x) \right] dx \right|^2$$

$$= \left| \int_0^1 \sqrt{\tfrac{16}{13}} \sin(2\pi x)\sin(\pi x) + \sqrt{\tfrac{144}{13}} \sin(2\pi x)\sin(\pi x)\cos(\pi x)\, dx \right|^2$$

$$= \left| \int_0^1 \sqrt{\tfrac{16}{13}} \sin(2\pi x)\sin(\pi x) + \sqrt{\tfrac{144}{13}} \cdot \tfrac{1}{2} \sin^2(2\pi x)\, dx \right|^2$$

$$= \left| \sqrt{\tfrac{16}{13}} \left[\frac{\sin(\pi x)}{2\pi} - \frac{\sin(3\pi x)}{6\pi} \right]\Big|_0^1 + \sqrt{\tfrac{36}{13}} \left[\frac{x}{2} - \frac{\sin(4\pi x)}{8\pi} \right]\Big|_0^1 \right|^2$$

$$= \left| \sqrt{\tfrac{16}{13}} \left[0-0-(0-0) \right] + \sqrt{\tfrac{36}{13}} \left[\tfrac{1}{2} - 0 - (0-0) \right] \right|^2$$

$$= \left| \sqrt{\tfrac{9}{13}} \right|^2$$

$$= \frac{9}{13}$$

$P(\psi_{3+}) = 1 - P(\psi_1) - P(\psi_2)$

$$= 1 - \frac{4}{13} - \frac{9}{13}$$

$$= 0$$

d) $\langle H \rangle = \langle \psi | H | \psi \rangle$

* Rewriting $|\psi\rangle$ as: $|\psi\rangle = \sqrt{\tfrac{8}{13}} \sin(\pi x) + \sqrt{\tfrac{18}{13}} \sin(2\pi x)$

$$= \sqrt{\tfrac{4}{13}} |\psi_1\rangle + \sqrt{\tfrac{9}{13}} |\psi_2\rangle$$

$$\Rightarrow \langle H \rangle = \left[\sqrt{\tfrac{4}{13}} \langle \psi_1 | + \sqrt{\tfrac{9}{13}} \langle \psi_2 | \right] H \left[\sqrt{\tfrac{4}{13}} |\psi_1\rangle + \sqrt{\tfrac{9}{13}} |\psi_2\rangle \right]$$

* cross terms will cancel

$$= \frac{4}{13} \langle \psi_1 | H | \psi_1 \rangle + \frac{9}{13} \langle \psi_2 | H | \psi_2 \rangle$$

$$= \frac{1}{13} \left(\frac{4\pi^2\hbar^2}{2m} + \frac{36\pi^2\hbar^2}{2m} \right)$$

$$= \frac{20\pi^2\hbar^2}{13m}$$

#3 (cont.)

e) $\langle x \rangle = \langle \psi | x | \psi \rangle$

$$= \int_0^1 \left(\sqrt{\tfrac{8}{13}} \sin(\pi x) + \sqrt{\tfrac{18}{13}} \sin(2\pi x) \right) x \left(\sqrt{\tfrac{8}{13}} \sin(\pi x) + \sqrt{\tfrac{18}{13}} \sin(2\pi x) \right) dx$$

$$= \int_0^1 \tfrac{8}{13} x \sin^2(\pi x) + \tfrac{18}{13} x \sin^2(2\pi x)\, dx$$

$$= \frac{8}{13}\left[\frac{x^2}{4} - \frac{x\sin(2\pi x)}{4\pi} - \frac{\cos(2\pi x)}{8\pi^2} \right]\Big|_0^1 + \frac{18}{13}\left[\frac{x^2}{4} - \frac{x\sin(4\pi x)}{8\pi} - \frac{\cos(4\pi x)}{36\pi^2} \right]\Big|_0^1$$

$$= \frac{8}{13}\left[\left(\frac{1}{4} - 0 - \frac{1}{8\pi^2}\right) - \left(0 - 0 - \frac{1}{8\pi^2}\right) \right] + \frac{18}{13}\left[\frac{1}{4} - 0 - \frac{1}{36\pi^2} - \left(0 - 0 - \frac{1}{36\pi^2}\right) \right]$$

$$= \frac{1}{4} \cdot \frac{8+18}{13}$$

$$= \frac{26}{4 \cdot 13}$$

$$= \frac{1}{2}$$

f) Our general solution to a infinite well w/ walls at 0, L is

$$\psi_n = \sqrt{\tfrac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \psi_1' = \sin\left(\frac{\pi x}{2}\right)$$

$$P(\psi_1') = |\langle \psi_1' | \psi \rangle|^2$$

$$= \left| \int_0^1 \sin\left(\frac{\pi x}{2}\right) \cdot \sqrt{2} \sin(\pi x)\, dx \right|^2$$

$$= \left| \left(\frac{\sin\left(-\frac{\pi x}{2}\right)}{-\pi} - \frac{\sin\left(\frac{3\pi x}{2}\right)}{3\pi} \right)\Big|_0^1 \right|^2$$

$$= \left| \frac{-1}{-\pi} - \frac{-1}{3\pi} - (0 - 0) \right|$$

$$= \left| \frac{1}{\pi} + \frac{1}{3\pi} \right|^2$$

$$= \frac{16}{9\pi^2}$$

7

Problem 4: Coherent States and the Harmonic Oscillator (10 pts)

Consider a one dimensional harmonic oscillator with mass m and frequency ω

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \, x^2$$

The raising and lowering operators are useful for harmonic oscillator problems:

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - i\frac{p}{m\omega} \right) \qquad a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i\frac{p}{m\omega} \right) \quad .$$

✓ (a) (1 pt) Verify that the Hamiltonian can be recast to the form $H = \hbar\omega(N + \frac{1}{2})$, where $N = a^\dagger a$. Be sure to show your work.

(b) (3 pts) Prove by induction that

$$[a, (a^\dagger)^n] \;=\; n(a^\dagger)^{n-1}$$

$a a^{+n} - a^{+n} a = n(a^+)^{n-1}$

$a a^{+n} - n(a^+)^{n-1}$

where $n \geq 1$ denotes a positive integer.

✓ (c) (4 pts) Define a state

$$|f\rangle \;=\; e^{-|f|^2/2} \times e^{f a^\dagger} |0\rangle$$

where f is a complex number. This state is called a coherent state. Starting from your results in part (b) of this problem, show that

$$a|f\rangle \;=\; f|f\rangle \; .$$

✓ (d) (2 pts) Check that

$$\langle f|f\rangle \;=\; 1$$

If needed, you can use the fact that $(a^\dagger)^n|0\rangle \;=\; \sqrt{n!}|n\rangle$ for level n of the harmonic oscillator.

$(a+ib)(a-ib) = a^2+b^2$

Jan 2018

Quantum #4

a) $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

* Note: We can define x and p in terms of $a, a^\dagger$ as:

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a)$$

$$p = i\sqrt{\frac{\hbar m\omega}{2}}\,(a^\dagger - a)$$

$$\Rightarrow X^2 = \frac{\hbar}{2m\omega}\,(a^\dagger a^\dagger + aa^\dagger + a^\dagger a + aa)$$

$$p^2 = \frac{-\hbar m\omega}{2}\,(a^\dagger a^\dagger - aa^\dagger - a^\dagger a + aa)$$

$$\Rightarrow H = \frac{1}{2m}\left(\frac{-\hbar m\omega}{2}\left[a^\dagger a^\dagger - aa^\dagger - a^\dagger a + aa\right]\right) + \frac{m\omega^2}{2}\left(\frac{\hbar}{2m\omega}\left[a^\dagger a^\dagger + aa^\dagger + a^\dagger a + aa\right]\right)$$

$$= \frac{-\hbar\omega}{4}(a^\dagger a^\dagger - aa^\dagger - a^\dagger a + aa) + \frac{\hbar\omega}{4}(a^\dagger a^\dagger + aa^\dagger + a^\dagger a + aa)$$

$$= \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a)$$

$$= \frac{\hbar\omega}{2}(2a^\dagger a + 1)$$

$$= \hbar\omega(a^\dagger a + 1/2)$$

$$= \hbar\omega(N + 1/2) \checkmark$$

b) Prove by induction that $[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$ where $n \geq 1$, $n \in \mathbb{Z}^+$

Proof: We know that $[a, a^\dagger] = 1$

Base Case: $n = 1$

$$[a, a^\dagger] = 1(a^\dagger)^{1-1}$$
$$= 1 \checkmark$$

Inductive Hypothesis: For any $n \geq 1$, $n \in \mathbb{Z}^+$, $[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$

We must now show that the n+1 case is true, ie

$$[a, (a^\dagger)^{n+1}] = (n+1)(a^\dagger)^n$$

#4 (cont.)

b) $$[a,(a^\dagger)^{n+1}] = [a,(a^\dagger)^n a^\dagger]$$
$$= [a,(a^\dagger)^n]a^\dagger + [a,a^\dagger](a^\dagger)^n$$
$$= n(a^\dagger)^{n-1}a^\dagger + 1(a^\dagger)^n$$
$$= (n+1)(a^\dagger)^n \checkmark$$

Because n is general, we have proved $[a,(a^\dagger)^n] = n(a^\dagger)^{n-1}$ for all $n \geq 1$, $n \in \mathbb{Z}^+$. QED

c) $$|f\rangle = \exp\left[\frac{-|f|^2}{2}\right]\exp[fa^\dagger]|0\rangle$$

$$a|f\rangle = a\exp\left[\frac{-|f|^2}{2}\right]\exp[fa^\dagger]|0\rangle$$
$$= \exp\left[\frac{-|f|^2}{2}\right] a \sum_n \frac{f^n(a^\dagger)^n}{n!}|0\rangle$$
$$= \exp\left[\frac{-|f|^2}{2}\right] \sum_n \frac{f^n}{n!}\left((a^\dagger)^n a + n(a^\dagger)^{n-1}\right)|0\rangle$$
$$= \exp\left[\frac{-|f|^2}{2}\right] \sum_n \frac{f^n}{n!} n(a^\dagger)^{n-1}|0\rangle$$
$$= f\exp\left[\frac{-|f|^2}{2}\right] \sum_{n-1} \frac{f^{n-1}(a^\dagger)^{n-1}}{(n-1)!}|0\rangle$$
$$= f\exp\left[\frac{-|f|^2}{2}\right]\exp[f(a^\dagger)^n]|0\rangle$$
$$= f|f\rangle \checkmark$$

d) $$\langle f|f\rangle \stackrel{?}{=} 1$$
$$= \langle 0|\exp\left[\frac{-|f|^2}{2}\right]\exp[f^*a^n]\exp\left[\frac{-|f|^2}{2}\right]\exp[f(a^\dagger)^n]|0\rangle$$
$$= \exp[-|f|^2]\langle 0|\exp[f^*a^n]\exp[f(a^\dagger)^n]|0\rangle$$
$$= \sum_{mn}\exp[-|f|^2]\langle 0|\frac{(f^*)^m a^m}{m!}\frac{f^n(a^\dagger)^n}{n!}|0\rangle$$
$$= \sum_{mn}\exp[-|f|^2]\langle m|\sqrt{m!}(f^*)^m f^n\sqrt{n!}|n\rangle$$

#4 (cont.)

d) $\langle f|f\rangle = \sum_{mn}' \exp\left[\frac{-|f|^2}{2}\right] \sqrt{m!}\,(f^*)^m f^n \sqrt{n!}\, \delta_{mn} \frac{1}{n!}$

$= \exp\left[\frac{-|f|^2}{2}\right] \sum_n' n!\,(|f|^2)^n \frac{1}{2(n!)}$

$= \exp\left[\frac{-|f|^2}{2}\right] \sum_n' \frac{(|f|^2)^n}{n!}$

$= \exp\left[\frac{-|f|^2}{}\right] \exp\left[\frac{|f|^2}{}\right]$

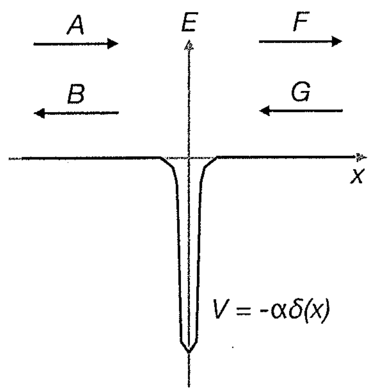
$= 1$ ✓

2

Problem 5: Transmission across delta functions(10 points):

a) (1 pt) Consider the potential $V(x) = -\alpha\delta(x)$. Show that the derivative of the wave function is discontinuous across the potential.
i.e $lim_{\epsilon\to 0}\left(\left(\frac{\partial\psi(x)}{\partial x}\right)_{x=\epsilon} - \left(\frac{\partial\psi(x)}{\partial x}\right)_{x=-\epsilon}\right) = -\frac{2m\alpha}{\hbar^2}\psi(0)$

b) (2 pts) A particle with $E > 0$ is incident on the delta function potential from $x < 0$. Determine the probability that the particle will be transmitted across the potential. Can the probability of transmission $= 1$?



c) (3 pts) One can define a transfer matrix M, which gives the amplitudes to the right of the potential in terms of those on the left.

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

Construct the M-matrix for scattering from a single delta-function potential at point a.

$$V(x) = -\alpha\delta(x-a)$$

d) (1 pt) Show that if you have a potential consisting of 2 isolated pieces, the M-matrix for the combination is the product of the two M-matrices for each section separately.

$$M = M_2 M_1$$

e) (3 pts) Now consider a double delta function potential

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)]$$

Determine the probability of transmission across the double delta function potential ($T = \frac{1}{|M_{22}|^2}$). Can the probability of transmission $= 1$?

Jan 2018

Quantum #5

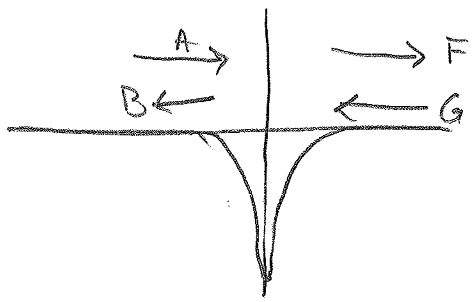
a) We can show this discontinuity by integrating the Schrödinger equation from $-\epsilon$ to ϵ

$$\Rightarrow \int_{-\epsilon}^{\epsilon} -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = \int_{-\epsilon}^{\epsilon} E\psi$$

$$\lim_{\epsilon\to 0} \frac{d\psi}{dx}\Big|_{-\epsilon}^{\epsilon} = \left[+E\cancelto{0}{\int_{-\epsilon}^{\epsilon}\psi} - \int_{-\epsilon}^{\epsilon} -\alpha\psi\delta(x)\,dx\right]\frac{-2m}{\hbar^2}$$

$$\frac{d\psi}{dx}\Big|_{\epsilon} - \frac{d\psi}{dx}\Big|_{-\epsilon} = \frac{-2m\alpha}{\hbar^2}\psi(0)$$

b)



Our boundary conditions are:

① $Ae^{ikx} + Be^{-ikx} = Fe^{ikx}$

② $-A(ik)e^{ikx} + B(ik)e^{-ikx} + F(ik)e^{ikx} = -\frac{2m\alpha}{\hbar^2}F$

Solving ① at $\psi(0)$ (ψ must be continuous)

$$A + B = F \iff B = F - A$$

Solving ② ($d\psi/dx$ discontinuous by $-\frac{2m\alpha}{\hbar^2}\psi(0)$)

$$(F - A + B)\,ik = -\frac{2m\alpha}{\hbar^2}F$$

$$F - A + (F - A) = \frac{i\,2m\alpha}{\hbar^2 k}F$$

$$F - A = \frac{im\alpha}{\hbar^2 k}F$$

$$F\left(1 - \frac{im\alpha}{\hbar^2 k}\right) = A$$

$$\frac{F}{A} = \frac{1}{1 - \frac{im\alpha}{\hbar^2 k}}$$

$$T = \left|\frac{F}{A}\right|^2 = \frac{1}{1 + \frac{m^2\alpha^2}{\hbar^4 k^2}} \Rightarrow T \neq 1 \text{ at any point}$$

#5 (cont.)

c) $$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

* We must use our boundary conditions to determine this matrix

① $Ae^{ikx} + Be^{-ikx} = Fe^{ikx} + Ge^{-ikx}$ (continuous ψ)

② $ik[Fe^{ikx} - Ge^{-ikx}] - ik[Ae^{ikx} - Be^{-ikx}] = \frac{-2m\alpha}{\hbar^2}(Ae^{ikx} + Be^{ikx})$ (discontinuous $\frac{d\psi}{dx}$)

* We can rewrite these as:

① $Ae^{2ikx} + B = Fe^{2ikx} + G$

② $Fe^{2ikx} - G = +Ae^{2ikx} - B + \frac{2im\alpha}{\hbar^2 k}(Ae^{2ikx} + B)$

* Adding ① and ② yields

$$2Fe^{2ikx} = 2Ae^{2ikx} + \frac{2im\alpha}{\hbar^2 k}(Ae^{2ikx} + B)$$

* let $\frac{m\alpha}{\hbar^2 k} = \beta$

↳ $F = (1 + i\beta)A + i\beta e^{-2ikx} B$

$\Rightarrow M_{11} = 1 + i\beta$

$M_{12} = i\beta e^{-2ikx}$

* Subtracting 2 from ①

$$2G = 2B - 2i\beta(Ae^{2ikx} + B)$$

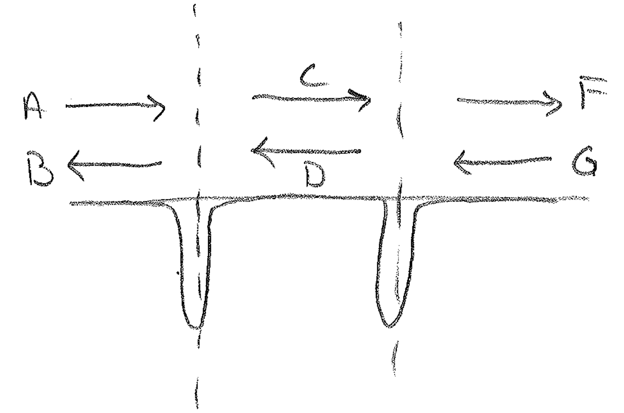
$$G = (1 - i\beta)B - i\beta e^{2ikx} A$$

$\Rightarrow M_{22} = 1 - i\beta$

$M_{21} = -i\beta e^{2ikx}$

#5 (cont.)

d) * For a double - δ potential



$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} F \\ G \end{bmatrix} = m' m \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} F \\ G \end{bmatrix} = \underline{M} \begin{bmatrix} A \\ B \end{bmatrix} \quad \text{where} \quad \underline{M} = m'm$$

e) $T = \left| \frac{1}{\underline{M}_{22}} \right|^2$

$$\begin{aligned} \underline{M}_{22} &= m'_{21} m_{12} + m'_{22} m_{22} \\ &= (-i\beta e^{2ikx})(i\beta e^{-2ikx}) + (1-i\beta)(1-i\beta) \\ &= \beta^2 + 1 - 2i\beta - \beta^2 \\ &= 1 - 2i\beta \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{|1-2i\beta|^2} \\ &= \frac{1}{1+4\beta^2} \end{aligned}$$

Problem 6: Hydrogenic Systems (10 pts):

(Note this problem is 2 pages and has 5 parts)

Consider the quantum system consisting of two charged particles interacting due to the Coulomb Potential:

$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} - \frac{qe^2}{|\vec{r}_1 - \vec{r}_2|}$$

$\vec{p}_1$ and $\vec{r}_1$ are the position and momentum of particle 1 with mass m_1. $\vec{p}_2$ and $\vec{r}_2$ are the position and momentum of particle 2 with mass m_2.

The charge of particle 1 is $-e$ and the charge of particle 2 is $+qe$, where q is an integer greater than or equal to 1.

a) (2 pts.) To solve this problem, you first want to convert to the center-of-mass and relative coordinates:

$$\vec{R} = \frac{m_1}{m_1 + m_2}\vec{r}_1 + \frac{m_2}{m_1 + m_2}\vec{r}_2 \ , \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

Derive the conjugate momenta to these spatial coordinates, $\vec{P}$ and $\vec{p}$, defined by:

$$[r_i, p_j] = i\hbar\delta_{i,j}, \quad [R_i, P_j] = i\hbar\delta_{i,j}, \quad [r_i, P_j] = [R_i, p_j] = 0$$

In these expressions, the subscripts indicate the vector components x, y, z, p_x, p_y, p_z, etc. Show your work.

Using these coordinates, the 2-particle Hamiltonian can be written:

$$H = \frac{\vec{P}^2}{2(m_1 + m_2)} + \frac{\vec{p}^{\,2}}{2\mu} - \frac{qe^2}{|\vec{r}|} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{1}$$

For the rest of the problem, assume $\vec{P} = 0$ (the center-of-mass reference frame).

b) (2 pts.) Define the wavefunction for the system as:

$$\Psi_{n,\ell,m}(\vec{r}) = \frac{u_{n,\ell}(r)}{r} Y_\ell^m(\theta, \phi)$$

where r, θ, ϕ are the usual spherical coordinates, and Y_ℓ^m the spherical harmonics.

Problem 6 continued

Show, in detail, that the Radial (Schrodinger) wave equation for the bound eigen-states, $u_{n,\ell}(r)$ can be written as:

$$\frac{\partial^2}{\partial r^2}u(r) - \frac{\ell(\ell+1)}{r^2}u(r) + \frac{2}{a_0}\frac{u(r)}{r} = \kappa_n^2 u(r)$$

What are a_0 and κ_n in terms of properties of the bound state system (μ, e, q, etc.)?

c) (3 pts.) Using the radial wave equation, determine the form of the function $u_{n,\ell}(r)$ in the limit as $r \to \infty$. How does $u_{n,\ell}(r)$ depend on the quantum number n for large values of r?

d) (2 pts.) In the limit that $r \to 0$, show that there are two possible solutions for $u_{n,\ell}(r)$, with the physical solution being $u_{n,\ell}(r) \propto r^{\ell+1}$. Do this for $\ell > 0$. (The $\ell = 0$ solution is a bit more complicated.)

e) (1 pt.) What are the ground-state energy and radius (Bohr radius) of the hydrogen-like system of a muon bound to an alpha particle?

Some potentially useful information:

- Fine structure constant - $\alpha = \frac{e^2}{\hbar c}$
- Bohr radius for a hydrogen atom - $a_B = \frac{\hbar}{\alpha m_e c}$
- Rydberg - $\frac{1}{2}\alpha^2 m_e c^2$
- Electron mass - $m_e c^2 = 0.51 MeV$
- Proton and Neutron mass - $m_N c^2 = 940 MeV$
- muon mass - $m_\mu c^2 = 106 MeV$.

$$a_0 = \frac{\hbar}{\frac{e^2}{\hbar c} m_e c}$$

$$= \frac{\hbar^2}{e^2 m_e}$$

Jan 2018

Quantum #6

a) $\vec{R} = \frac{m_1}{m_1+m_2}\vec{r}_1 + \frac{m_2}{m_1+m_2}\vec{r}_2$

$\vec{r} = \vec{r}_1 - \vec{r}_2$

$[r_i, p_j] = i\hbar\delta_{ij}$ $\quad [R_i, P_j] = i\hbar\delta_{ij}$ $\quad [r_i, P_j] = [R_i, p_j] = 0$

* Assume $\underline{P} = a\vec{p}_1 + b\vec{p}_2$

$\vec{p} = c\vec{p}_1 + d\vec{p}_2$

* Note: All commutators in expansion that are commutators like $[r_{1i}, p_{2j}] = 0$ b/c they act on different particles and thus automatically commute

$[r_i, p_j] = i\hbar\delta_{ij}$

$= c[r_{1i}, p_{1j}] - d[r_{2i}, p_{2j}]$

$= c\, i\hbar\delta_{ij} - d i\hbar\delta_{ij}$

$\hookrightarrow \boxed{c - d = 1}$

$[R_i, P_j] = \frac{am_1}{m_1+m_2}[r_{1i}, p_{1j}] + \frac{bm_2}{m_1+m_2}[r_{2i}, p_{2j}]$

$= i\hbar\delta_{ij}$

$= \frac{am_1}{m_1+m_2}(i\hbar\delta_{ij}) + \frac{bm_2}{m_1+m_2}(i\hbar\delta_{ij})$

$\hookrightarrow \boxed{\frac{am_1}{m_1+m_2} + \frac{bm_2}{m_1+m_2} = 1}$

$[r_i, P_j] = 0$

$= a[r_{1i}, p_{1j}] - b[r_{2i}, p_{2j}]$

$= a(i\hbar\delta_{ij}) - b(i\hbar\delta_{ij})$

$\hookrightarrow \boxed{a - b = 0}$

#6 (cont.)

a) $[R_i, p_j] = 0$

$$= \frac{m_1 c}{m_1+m_2}[r_{1i}, P_{1j}] + \frac{m_2 d}{m_1+m_2}[r_{2i}, P_{2j}]$$

$$= \left(\frac{m_1 c}{m_1+m_2} + \frac{m_2 d}{m_1+m_2}\right) i\hbar\,\delta_{ij}$$

$$\hookrightarrow \boxed{\frac{m_1 c}{m_1+m_2} + \frac{m_2 d}{m_1+m_2} = 0}$$

* Solving the 4 boxed equations yields:

$$a = b = 1$$

$$c = \frac{m_2}{m_1+m_2}$$

$$d = \frac{-m_1}{m_1+m_2}$$

$$\Rightarrow \quad \vec{P} = \vec{P}_1 + \vec{P}_2$$

$$\vec{p} = \frac{m_2}{m_1+m_2}\vec{P}_1 - \frac{m_1}{m_1+m_2}\vec{P}_2$$

b) Given $\psi_{n\ell m}(\vec{r}) = \frac{U_{n\ell}(r)}{r} Y_\ell^m(\theta,\varphi)$, the Schrödinger Eqn says:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi - \frac{L^2}{\hbar^2 r^2}\psi\right) - \frac{qe^2}{r}\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial^2 u(r)}{\partial r^2} Y_\ell^m - \frac{L^2}{\hbar^2 r^3} u(r) Y_\ell^m\right) - \frac{qe^2}{r^2} u Y_\ell^m = \frac{E}{r} u Y_\ell^m$$

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial^2 u(r)}{\partial r^2} - \frac{\ell(\ell+1)\hbar^2}{\hbar^2 r^3} u(r)\right) Y_\ell^m - \frac{qe^2}{r^2} u Y_\ell^m = \frac{E}{r} u Y_\ell^m$$

$$-\frac{\hbar^2}{2mr}\frac{\partial^2 u}{\partial r^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^3} u - \frac{qe^2}{r^2} u - \frac{E}{r} u = 0$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} + \left(\frac{\hbar^2 \ell(\ell+1)}{2mr^2} - \frac{qe^2}{r} - E\right) u = 0$$

#6 (cont.)

b) $$\frac{\partial^2 u}{\partial r^2} - \frac{l(l+1)}{r^2}u - \frac{2mqe^2}{\hbar^2 r}u + \frac{2mE}{\hbar^2}u = 0$$

$*$ let $a_0 = \dfrac{\hbar^2}{mqe^2}$, $\kappa_n^2 = \dfrac{-2mE}{\hbar^2}$

$$\frac{\partial^2 u}{\partial r^2} - \frac{l(l+1)}{r^2}u - \frac{2}{a_0}u = \kappa_n^2 u$$

c) In the limit $r \to \infty$

$$\frac{\partial^2 u}{\partial r^2} \approx \kappa_n^2 u$$

$\hookrightarrow u(r) \propto e^{-\kappa_n r} + \cancel{e^{+\kappa_n r}}^{\,0}$ b/c unphysical ($\to \infty$ as $r \to \infty$)

$*$ But since $\kappa_n^2 = \dfrac{1}{a_0 n}$

$$u(r) \propto e^{-r/a_0 n}$$

Therefore as $n \to \infty$, the function will approach 0 faster

d) In the limit $r \to 0$, our equation becomes since $l > 0$

$$\frac{\partial^2 u}{\partial r^2} - \frac{l(l+1)}{r^2}u \approx \kappa_n^2 u$$

$\hookrightarrow u(r) \propto r^{l+1} + \cancel{r^{-l}}^{\,0}$ b/c ψ will not be normalizable ($\psi \propto r^{-(l+1)}$)

$*$ To check our solutions, first with $u \propto r^{l+1}$

$$\frac{\partial^2 u}{\partial r^2} = l(l+1)\, r^{l-1} \quad , \quad \frac{l(l+1)}{r^2} r^{l+1} = l(l+1)\, r^{l-1} \quad \checkmark$$

$*$ Now for $u \propto r^{-l}$

$$\frac{\partial^2 u}{\partial r^2} = l(l+1)\, r^{-l-2} \quad , \quad \frac{l(l+1)}{r^2} r^{-l} = l(l+1)\, r^{-l-2} \quad \checkmark$$