Quantum Mechanics
Qualifying Exam - January 2013

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write your alias on the top of every page of your solutions
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

Possibly useful formulas:

Spin Operator

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

In spherical coordinates,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi. \tag{2}$$

Problem 1: Bound States and Scattering for a Delta-Function Well

Consider a delta-fuction for a 1-D system,

$$V(x) = -g\ \delta(x) \tag{1}$$

where $g > 0$. We will consider the states of a particle of mass m interacting with this potential for both $E < 0$ and $E > 0$.

This potential has a single bound state $E_b < 0$.

(a) [1 pt] Explain why the bound state wavefunction for the particle will have the form $\Psi(x) = ce^{-|x|/\lambda}$. (You don't need to solve for anything to answer this question.)

(b) [2 pts] Derive the boundary conditions for $\Psi(x)$ and $\partial_x \Psi(x)$ at $x = 0$.

(c) [1 pt] Using the boundary conditions at $x = 0$, determine the value of λ.

(d) [1 pts] What is the energy of the bound state, E_b? What is the normalization constant c?

(e) [2 pts] What is the uncertainty in position, Δx for the particle in this bound state?

(f) [2 pts] Next consider a scattering state for this particle with energy $E > 0$

$$\begin{aligned} \Psi(x) &= e^{ikx} + ae^{-ikx}, \quad x < 0 \\ &= be^{ikx}, \quad x > 0 \end{aligned} \tag{2}$$

For this state, $E = \frac{\hbar^2 k^2}{2m}$

Using the boundary conditions you found in part (b), determine a and b, and the transmission and reflection coefficients for this scattering state.

Problem 2: Born Approximation

In the Born approximation, the scattering amplitude for a particle of mass m elastically scattering from a potential $V(\vec{r})$ is given by

$$f(\theta, \phi) \simeq -\frac{m}{2\pi\hbar^2} \int e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} V(\vec{r}) d^3r \tag{1}$$

and where $\hbar\vec{k}$ is the incoming momentum, $\hbar\vec{k}'$ is outgoing momentum, θ is the scattering angle measured from the incoming momentum, and ϕ is an azimuthal angle about the incoming momentum.

The scattering cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2. \tag{2}$$

(a) [2 pts] Define $\vec{\kappa} \equiv \vec{k}' - \vec{k}$. Show that the magnitude $|\vec{\kappa}| = 2k\sin(\theta/2)$ for elastic scattering.

(b) [6 pts] Find $\frac{d\sigma}{d\Omega}$ for the Yukawa potential: $V(r) = \beta\frac{e^{-\mu r}}{r}$

(c) [2 pts] Why does the cross section get larger as μ gets smaller? What is the scattering cross section the limit as $\mu \to 0$? What physical problem does this correspond to in the $\mu \to 0$ limit?

Problem 3: Spin Measurements and Uncertainty

Define the operator $S_\alpha = \vec{S} \cdot \hat{n}_\alpha$ where $\vec{S}$ is the vector spin operator and $\hat{n}_\alpha$ is a unit vector in the $x - z$ plane that makes an angle α with the z-axis. So $\hat{n}_\alpha = \hat{z}$ for $\alpha = 0$ and $\hat{n}_\alpha = \hat{x}$ for $\alpha = \pi/2$.

Consider a spin 1/2 system initially prepared to be in the eigenstate of S_α with eigenvalue $+\hbar/2$,

$$S_\alpha|\alpha, +\rangle = \frac{\hbar}{2}|\alpha, +\rangle \tag{1}$$

(a) [3 pts] Compute the eigenstates of S_α in the basis of the S_z operator, $|0, \pm\rangle \equiv |\pm\rangle$.

(b) [2 pts] If the spin is in the state $|\alpha, +\rangle$ and S_x is measured, what is the probability of measuring $-\hbar/2$?

(c) [3 pts] Compute the expectation value $\langle(\delta S_x)^2\rangle$ for the state $|\alpha, +\rangle$, where $\delta S_x = S_x - \langle S_x\rangle$.

If one measures S_x, what are the values of α that minimize the uncertainty of the measurement for the state $|\alpha, +\rangle$? Interpret the physical meaning of those states.

(d) [2 pts] Finally, define $\mathcal{P}_{x,+}$ to be the projection operator for the spin 1/2 state of S_x, $|\pi/2, +\rangle$. Compute the matrix element $\mathcal{P}_{x,+}$ in the initial state, $\langle +, \alpha|\mathcal{P}_{x,+}|\alpha, +\rangle$. Explain the behavior of the resultant expression as a function of the angle α.

Problem 4: Operator Solutions to the Harmonic Oscillator

Consider the Harmonic Oscillator Hamiltonian in one dimension:

$$H_{ho} = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2 \tag{1}$$

To simplify this problem, define the new observables:

$$p = \sqrt{\frac{1}{m\hbar\omega}}P \ , \quad q = \sqrt{\frac{m\omega}{\hbar}}X \tag{2}$$

This gives the dimensionless Hamiltonian,

$$H = \frac{1}{\hbar\omega}H_{ho} = \frac{1}{2}\left(p^2 + q^2\right) \tag{3}$$

(a) [1 pt] Calculate the commutation relation for these new variables, $[q, p]$. Be sure to show your work.

(b) [1 pt] Define the non-Hermitian operators $a = \frac{1}{\sqrt{2}}(q + ip)$, $\quad a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$ and the Hermitian operator $n = a^\dagger a$. Compute $[a, a^\dagger]$, $[n, a^\dagger]$, and $[n, a]$

(c) [1 pt] Write the dimensionless Hamiltonian H in terms of a and $a^\dagger$. Write the dimensionless Hamiltonian H in terms of n.

(d) [3 pts] Define the eigenvalues and eigenvectors of n as:

$$n|\lambda\rangle = \lambda|\lambda\rangle. \tag{4}$$

and assume that these eigenvectors form a complete set.

Show that

$$\begin{aligned} a^\dagger|\lambda\rangle &= A|\lambda + 1\rangle \\ a|\lambda\rangle &= B|\lambda - 1\rangle \end{aligned} \tag{5}$$

Determine the normalization constants A and B.

(e) [2 pts.] Show that $n = a^\dagger a$ must have non-negative eigenvalues, $\lambda \geq 0$. Explain why this implies that there must be a state where $a|0\rangle = 0$ and that the eigenvalues of n must be non-negative integers.

(f) [2 pts.] Write the definition for the state $|0\rangle$

$$a|0\rangle = 0 \tag{6}$$

as a differential equation, in q, for the ground state wavefunction of H. Solve this expression for the normalized ground state wavefunction.

Jan 2013

Quantum #4

a) * Remember $[x_i, p_j] = i\hbar\delta_{ij}$

$$\Rightarrow [q,p] = qp - pq$$

$$= \sqrt{\frac{m\omega}{\hbar}}\, x \sqrt{\frac{1}{m\hbar\omega}}\, p - \sqrt{\frac{1}{m\hbar\omega}}\, p \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$= \frac{1}{\hbar}(xp - px)$$

$$= \frac{1}{\hbar}[x,p]$$

$$= \frac{1}{\hbar} i\hbar$$

$$= i$$

b) $[a,a^\dagger] = aa^\dagger - a^\dagger a$

$$= \frac{1}{\sqrt{2}}(q+ip)\frac{1}{\sqrt{2}}(q-ip) - \frac{1}{\sqrt{2}}(q-ip)\frac{1}{\sqrt{2}}(q+ip)$$

$$= \frac{1}{2}(q^2 + ipq - iqp + p^2) - \frac{1}{2}(q^2 - ipq + iqp + p^2)$$

$$= \frac{i}{2}(pq - qp) - \frac{i}{2}(qp - pq)$$

$$= -i(pq + qp)$$

$$= -i([q,p])$$

$$= -i(i)$$

$$= 1$$

$$[n, a^\dagger] = [a^\dagger a, a^\dagger]$$

$$= a^\dagger a a^\dagger - a^\dagger a^\dagger a$$

$$= a^\dagger [aa^\dagger - a^\dagger a]$$

$$= a^\dagger [a, a^\dagger]$$

$$= a^\dagger$$

$$[n, a] = [a^\dagger a, a]$$

$$= a^\dagger a a - a a^\dagger a$$

$$= [a^\dagger a - a a^\dagger] a$$

$$= [a^\dagger, a] a$$

$$= -[a, a^\dagger] a$$

$$= -a$$

#4 (cont.)

c) We want to rewrite $H = \frac{1}{2}(p^2 + q^2)$ in terms of a and $a^\dagger$

$$\Rightarrow \sqrt{2}\, a = q + ip$$
$$\sqrt{2}\, a^\dagger = q - ip$$

$$\sqrt{2}(a + a^\dagger) = 2q \qquad \sqrt{2}(a - a^\dagger) = 2ip$$

$$\frac{1}{\sqrt{2}}(a + a^\dagger) = q \qquad \frac{-i}{\sqrt{2}}(a - a^\dagger) = p$$

$$\Rightarrow H = \frac{1}{2}\left[\left(\frac{-i}{\sqrt{2}}(a - a^\dagger)\right)^2 + \left(\frac{1}{\sqrt{2}}(a + a^\dagger)\right)^2\right]$$
$$= \frac{1}{2}\left[\frac{-1}{2}(aa - a^\dagger a - aa^\dagger + a^\dagger a^\dagger) + \frac{1}{2}(aa + a^\dagger a + aa^\dagger + a^\dagger a^\dagger)\right]$$
$$= \frac{1}{2}(a^\dagger a + aa^\dagger)$$
$$= \frac{1}{2}(n + aa^\dagger)$$
$$= \frac{1}{2}(n + 1 + a^\dagger a) \quad (\text{from } [a, a^\dagger] = 1)$$
$$= \frac{1}{2}(2n + 1)$$
$$= n + 1/2$$

d) * We must use the n-operator and its commutation relations to solve this problem

$$\Rightarrow a^\dagger|\lambda\rangle = A|\lambda + 1\rangle$$

$$\hookrightarrow n a^\dagger |\lambda\rangle = a^\dagger n + a^\dagger |\lambda\rangle$$
$$= (a^\dagger \lambda + a^\dagger)|\lambda\rangle$$
$$= a^\dagger(\lambda + 1)|\lambda\rangle$$
$$= (\lambda + 1)|\lambda\rangle$$

$$\Rightarrow \langle\lambda|a a^\dagger|\lambda\rangle = A^2\langle\lambda+1|\lambda+1\rangle$$
$$\langle\lambda|a^\dagger a + 1|\lambda\rangle = A^2$$
$$\langle\lambda|n + 1|\lambda\rangle = A^2$$
$$\lambda + 1 = A^2 \Rightarrow \boxed{A = \sqrt{\lambda + 1}}$$

#4 (cont.)

d) Similiarly,

$$n(a|\lambda\rangle) = an - a\,|\lambda\rangle$$
$$= a(n-1)\,|\lambda\rangle$$
$$= a(\lambda-1)|\lambda\rangle$$
$$= (\lambda-1)(a|\lambda\rangle)$$

$$\Rightarrow \langle\lambda|a^{\dagger}a|\lambda\rangle = B^2 \langle\lambda-1|\lambda-1\rangle$$
$$\langle\lambda|n|\lambda\rangle = B^2$$
$$\lambda = B^2 \rightarrow \boxed{\sqrt{\lambda} = B}$$

e)

#4 (cont.)

f) Given $a|0\rangle = 0$, where $a = \frac{1}{\sqrt{2}}(q + ip)$

$$\frac{1}{\sqrt{2}}(q + ip)|0\rangle = 0$$

$$\frac{1}{\sqrt{2}}\left(q + i\left(-i\hbar \frac{\partial}{\partial q}\right)\right)|0\rangle = 0$$

$$\frac{1}{\sqrt{2}}\left(q - \hbar \frac{\partial}{\partial q}\right)\psi_0 = 0$$

$$q\psi_0 - \hbar \frac{\partial \psi_0}{\partial q} = 0$$

$$\Rightarrow \quad \frac{\partial \psi_0}{\partial q} = \frac{q}{\hbar}\psi_0$$

$$\int \frac{\partial \psi_0}{\psi_0} = \int \frac{q}{\hbar}\,\partial q$$

$$\ln(\psi_0) = -\frac{1}{2\hbar}q^2 + C$$

$$\psi_0 = \exp\left[\frac{-1}{2\hbar}q^2 + C\right]$$

$$= C\exp\left[\frac{-q^2}{2\hbar}\right]$$

*Checking our normalization

$$1 = c^2\int_{-\infty}^{\infty}\left|\exp\left[\frac{-q^2}{2\hbar}\right]\right|^2 dq$$

$$1 = C^2\int_{-\infty}^{\infty}\exp\left[-\frac{q^2}{\hbar}\right]dq$$

$$1 = C^2\sqrt{\pi\hbar}$$

$$\frac{1}{\sqrt{\pi\hbar}} = c^2$$

$$\hookrightarrow \quad C = \left(\frac{1}{\pi\hbar}\right)^{1/4}$$

$$\Rightarrow \psi_0 = \left(\frac{1}{\pi\hbar}\right)^{1/4}\exp\left[\frac{-q^2}{2\hbar}\right]$$

Problem 5: Perturbing a Square Well

Consider a particle of mass m in a 1D infinite square well of width a,

$$V(x) = 0, \quad 0 \leq x \leq a \qquad V(x) = \infty, \quad x < 0, \quad x > a. \tag{1}$$

(a) [2 pts] Derive the eigenfunctions and eigenenergies of the particle in this potential. Be sure to normalize the states.

(b) [2 pts] Show that if the well is perturbed by a potential $V'(x) = \alpha\, x$, the energy of all the unperturbed states shift by the same amount to first order in α. Find an expression for this energy shift. Give a physical explanation for why this perturbation results in an equal first-order energy shift for all states.

(c) [3 pts] Next, instead of the perturbing potential from part (b), the well is perturbed by a potential

$$V'(x) = V_0, \quad \frac{a}{2} - \delta \leq x \leq \frac{a}{2} + \delta \qquad V'(x) = 0, \quad x < \frac{a}{2} - \delta, \quad x > \frac{a}{2} + \delta \tag{2}$$

Compute the energy shift to first order in α for the unperturbed energy eigenstates $\psi_n(x)$. Explain the limit of this result as n, the unperturbed energy level, gets large.

(d) [2 pts.] What is the energy shift of the states $\psi_n(x)$ to first order in δ as $\delta \to 0$? (V_0 is constant.) Give a physical explanation of this result. Note: You should be able to answer this question even if you did not get a solution to part (c).

(e) [1 pt] What is the energy shift of the states $\psi_n(x)$ as $\delta \to \frac{a}{2}$? (V_0 is constant.) Give a physical explanation of this result. Note: You should again be able to answer this question even if you did not get a solution to part (c).

Problem 6: Spherical Square Well

Consider a spin 0 particle of mass m moving in a 3D square well, given by the potential

$$V(\vec{r}) = -V_0 \quad 0 \leq |\vec{r}| \leq a_0 \, , \qquad V(\vec{r}) = 0 \quad |\vec{r}| > a_0 \qquad (V_0 > 0). \tag{1}$$

In this problem we will only consider the bound states of this well, so that $-V_0 < E < 0$.

(a) [1 pt] Explain why we can write the eigenstates of this potential as

$$\Psi_{k,l,m} = f_{k,l}(r) Y_l^m(\theta, \phi). \tag{2}$$

(b) [2 pts] Defining the function $u_{k,l}(r) = r f_{k,l}(r)$, write the radial Schrödinger equation for $u_{k,l}(r)$.

(c) [2 pts] For $l = 0$, write the form for the function $u_{k,0}(r)$ in the regions $0 \leq r \leq a_0$ and $r \geq a_0$. Define any constants that you use.

(d) [3 pts] Using the boundary conditions on the function $u_{k,0}(r)$, derive an equation that gives the bound state energies for the $l = 0$ states. Hint: Considering that $f(r) = u(r)/r$, what is the boundary condition on u as $r \to 0$?

(e) [2 pts] For a fixed radius for the potential, a_0, calculate the minimum depth, $V_0 = V_{min}$, for the potential to have a bound state.