Quantum Mechanics
Qualifying Exam - August 2013
Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write your alias, the name you selected at the start of this test, on the top of every page of your solutions. *DO NOT* put your own name on your answer sheets.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

Possibly useful formulas:

Spin Operators

$$\vec{S}=\frac{\hbar}{2}\vec{\sigma},\quad \sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_z=\begin{pmatrix}1&0\\0&1\end{pmatrix} \tag{1}$$

Angular momentum operators in 3D obey

$$[L_i,L_j]=i\hbar\epsilon_{ijk}L_k \tag{2}$$

In spherical coordinates,

$$\nabla^2\psi=\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\psi+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi. \tag{3}$$

In cylindrical coordinates,

$$\nabla^2\psi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\psi\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\psi+\frac{\partial^2}{\partial z^2}\psi \tag{4}$$

Harmonic Oscillator States ($\beta=\sqrt{\frac{m\omega}{\hbar}}$),

$$\begin{aligned}\psi_n(x) &= \left(\frac{\beta^2}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}e^{-\beta^2x^2/2}H_n(\beta x)\\ H_0(x)=1,\quad H_1(x) &= 2x,\quad H_2(x)=4x^2-2,\quad H_3(x)=8x^3-12x\end{aligned} \tag{5}$$

Spherical Harmonics,

$$Y_0^0(\theta,\phi)=\frac{1}{\sqrt{4\pi}},\quad Y_1^0(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_1^{\pm1}(\theta,\phi)=\mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \tag{6}$$

Hydrogen Atom States (a_0 is the Bohr Radius),

$$\begin{aligned}\Psi_{n,\ell,m}(\vec{r}) &= R_{n,\ell}(r)Y_{\ell,m}(\theta,\phi)\\ R_{1,0}(r) &= \frac{2}{(a_0)^{3/2}}e^{-r/a_0}\\ R_{2,0}(r) &= \frac{2}{(2a_0)^{3/2}}\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}\\ R_{2,1}(r) &= \frac{1}{(2a_0)^{3/2}}\frac{r}{\sqrt{3}a_0}e^{-r/2a_0}\end{aligned} \tag{7}$$

Problem 1: 1D Square Wells

(a) [1 pt] Consider an electron confined to an infinitely deep 1D well with walls at $x = 0$ and $x = L$. In the ground state, the electron has an energy of 2.5 eV (the bottom of the well is defined as $V = 0$). What is the width of the well?

(b) [1 pt] A proton is confined to an infinite 1D square well of width 10 fm. What is the wavelength (or frequency) of a photon emitted when the proton undergoes a transition from the first excited state to the ground state of the well?

(c) [2 pt] Sketch the probability density as a function of x for the first 3 energy eigenstates for an electron in an infinite well of width L. Describe qualitatively (or draw) how the probability densities for these states will differ (from the infinite well case) for a square well with an infinite potential barrier at $x = 0$ and a finite potential barrier at $x = L$.

(d) [2 pt] Consider an electron in the *nth* energy eigenstate of an infinitely deep well with walls at $x = 0$ and $x = L$. Calculate the probability that the electron will be measured between $x = 0$ and $x = \epsilon\, L$, with $0 < \epsilon < 1$. Your answer should be a function of both n and ϵ.

Give a physical explanation for your solution as $n \to \infty$.

(e) [2 pt] The electron is in the ground state of the infinite well when the wall at $x = L$ is very suddenly moved to $x = 2L$. What is the probability that the electron will be found in the ground state of the expanded box?

(f) [1 pt] What energy eigenstate in the expanded box will have the highest probability of being occupied by the electron? What is this probability? Hint: You should be able to determine this result without doing an integral, but you should explain your answer.

(g) [1 pt] Suppose the electron is in the ground state of the infinitely deep well when the walls are suddenly removed completely. Write down an expression for the probability distribution for the momentum of the freed electron. Setup but do not solve the integral.

Problem 2: Quantum Operators

In this problem you will work with the ladder operators for angular momentum:

$$L_+ = L_x + iL_y, \quad L_- = L_x - iL_y \tag{1}$$

where

$$\begin{aligned} L^2 &= L_x^2 + L_y^2 + L_z^2 \\ L^2|\ell, m\rangle &= \ell(\ell+1)\hbar^2|\ell, m\rangle \\ L_z|\ell m\rangle &= m\hbar|\ell, m\rangle \end{aligned} \tag{2}$$

(a) [1 pt] Show that the eigenvalues of any Hermitian operator are real.

(b) [2 pt] Is the the operator L_+L_-, the product of the angular momentum ladder operators, Hermitian? Show your work to justify your answer.

(c) [4 pt] Determine the results of the operations: $\hat{L}_+|\ell, m\rangle$ and $\hat{L}_-|\ell, m\rangle$. Show all of your work and make sure you determine all constants correctly.

Hint: The commutation relation $[L_z, L_\pm]$ and the matrix elements $\langle \ell, m|L_\pm L_\mp|\ell, m\rangle$ might be useful.

(d) [3 pt] Using the results from part (c), prove that $-\ell \leq m \leq +\ell$. Explain the physics of this result in terms of the operators L^2 and L_z.

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Quantum #2

a) The condition for Hermiticity is $A = A^\dagger$ for an operator A

Using: $A|\lambda\rangle = a|\lambda\rangle$, it must be true that $\langle\lambda|A^\dagger = a^*\langle\lambda|$
$= \langle\lambda|A$

$$\Rightarrow \langle\lambda|A|\lambda\rangle = a^*\langle\lambda|\lambda\rangle$$
$$a\langle\lambda|\lambda\rangle = a^*\langle\lambda|\lambda\rangle$$
$$a = a^*$$, which is only true if $a \in \mathbb{R}$

b) $L_+ = L_x + iL_y$
$L_- = L_x - iL_y$
> We want $L_+L_- = (L_+L_-)^\dagger = L_-^\dagger L_+^\dagger$

$$L_+L_- = (L_x + iL_y)(L_x - iL_y)$$
$$= (L_x^2 + iL_yL_x - iL_xL_y + L_y^2)$$
$$= (L_x^2 + L_y^2 - i[L_x, L_y])$$
$$= (L_x^2 + L_y^2 - i(i\hbar L_z))$$
$$= (L_x^2 + L_y^2 + \hbar L_z)$$
$$= (L^2 - L_z^2 + \hbar L_z)$$

$$L_-^\dagger L_+^\dagger = (L_x - iL_y)^\dagger (L_x + iL_y)^\dagger$$
$$= (L_x^\dagger + iL_y^\dagger)(L_x^\dagger - iL_y^\dagger)$$

* but $L_x^\dagger = L_x$, $L_y^\dagger = L_y$ by their status as observables

$$= (L_x + iL_y)(L_x - iL_y)$$
$$= L^2 - L_z^2 + \hbar L_z$$

c) We must first determine $L_\pm|\ell, m\rangle$

$$L_z(L_\pm|\ell, m\rangle) = (L_\pm L_z \pm \hbar L_\pm)|\ell, m\rangle$$
$$= L_\pm(L_z \pm \hbar)|\ell, m\rangle$$
$$= (m \pm \hbar)(L_\pm|\ell, m\rangle)$$

↳ $L_\pm$ increments the z-states of angular momentum

#2 (cont.)

c) Given the above, we know: $J_\pm |l,m\rangle = c_\pm |l, m\pm\hbar\rangle$

$$\Rightarrow \langle l,m| L_+^\dagger L_+ |l,m\rangle = |c_+|^2 \langle l, m\cancel{+\hbar}|\cancel{l,m+\hbar}\rangle^{1}$$

$$\langle l,m| L^2 - L_z^2 - \hbar L_z |l,m\rangle = |c_+|^2 \qquad \text{(see part b for work)}$$

$$\hbar^2 l(l+1) - \hbar^2 m^2 - \hbar^2 m \langle l,m\cancel{|l,m}\rangle^{1} = |c_+|^2$$

$$\hookrightarrow |c_+|^2 = \hbar^2[l(l+1) - m^2 - m]$$

$$c_+ = \hbar\sqrt{(l-m)(l+m+1)}$$

Similarly for J_-

$$\Rightarrow \langle l,m| J_-^\dagger J_- |l,m\rangle = |c_-|^2 \langle l, m\cancel{-\hbar}|\cancel{l}, m-\hbar\rangle^{1}$$

$$\langle l,m| L^2 - L_z^2 + \hbar L_z |l,m\rangle = |c_-|^2$$

$$\hbar^2 l(l+1) - \hbar^2 m^2 + \hbar^2 m \langle l,m|l,m\rangle = |c_-|^2$$

$$\hookrightarrow |c_-|^2 = \hbar^2[l(l+1) - m^2 + m]$$

$$= \hbar\sqrt{(l+m)(l-m+1)}$$

$$\Rightarrow L_\pm |l,m\rangle = \hbar\sqrt{(l\mp m)(l\pm m+1)}\, |l, m\pm 1\rangle$$

d) This part of the problem is effectively asking us to find the extremum values therefore we act upon the max/min states

$$\Rightarrow L_+ |l, m_{max}\rangle = 0$$

$$L_- L_+ |l, m_{max}\rangle = 0$$

$$L^2 - L_z^2 - \hbar L_z |l, m_{max}\rangle = 0$$

$$\hbar^2 l(l+1) - \hbar^2 m^2 - \hbar^2 m |l, m_{max}\rangle = 0$$

* assuming a non-zero ket

$$l(l+1) = m(m+1) \Longrightarrow m_{max} = l$$

#2 (cont.)

d) $L_- |l, m_{min}\rangle = 0$

$L_+ L_- |l, m_{min}\rangle = 0$

$L^2 - L_z^2 + \hbar L_z |l, m_{min}\rangle = 0$

$\hbar^2 (l+1)l - \hbar^2 m_{min}^2 + \hbar^2 m_{min} |l, m_{min}\rangle = 0$

* assuming a non-zero ket

$l(l+1) = m_{min}(m_{min} - 1)$

$m_{max}(m_{max} + 1) = m_{min}(m_{min} - 1)$

$\hookrightarrow m_{max} = -m_{min}$

$\hookrightarrow m_{min} = -l$

$\Rightarrow m \in [-l, l]$

Physically, the z-state of angular momentum can only contain as much angular momentum as the overall angular momentum of the whole system

Problem 3: Barrier Scattering

Consider a particle of mass m in one dimension scattering off of a square barrier of width L:

$$\begin{aligned} V(x) &= 0, \quad x < 0 \\ V(x) &= V, \quad 0 < x < L, \quad V > 0 \\ V(x) &= 0, \quad x > L \end{aligned} \tag{1}$$

Assume the particle has an energy $E > V$ and is incoming from the left ($x < 0$).

Define the usual wavenumbers for this problem:

$$\frac{\hbar^2 k^2}{2m} = E, \quad \frac{\hbar^2 k'^2}{2m} = E - V \tag{2}$$

(a) [1 pt] Write down general expressions for the scattering wave function, the un-normalized eigenfunction of the scattering Hamiltonian, in the three regions, $x < 0$, $0 < x < L$, and $x > L$.

(b) [1 pt] Using the expressions from part (a), write down the boundary conditions on the scattering wave function. Explain the physics of each of these boundary conditions.

(c) [2 pt] Using your boundary conditions from part (b), show that

$$\frac{A}{E} = e^{ikL}\left(\cos k'L) - i\frac{k^2 + k'^2}{2kk'}\sin(k'L)\right) \tag{3}$$

where A is the amplitude of the incoming wave (from $x = -\infty$) and E is the amplitude of the outgoing wave (to $x = \infty$). Hint: We're not interested in the amplitude of the reflected wave.

(d) [3 pt] Solve for the transmission coefficient, T, for the barrier scattering. You may express this in terms of k, k', and L, but it will be useful for later parts of the question to write it in terms of E, V, L, and constants in the problem.

(e) [1 pt] What is the limit for the transmission coefficient T in the limit that $E \gg V$? Show your work and explain the physics of this result.

(f) [1 pt] There are energies where $T = 1$. What are these energies and the wavelength of the particle wave function? Give a physical argument of why the transmission coefficient is a maximum for these energies.

(g) [1 pt] What is the value for the transmission coefficient, T, in the limit that $E \to V$? Hint: To solve this you might define $\delta = E - V$.

Problem 4: Properties of the Hydrogen Atom

The wavefunctions for the ground state and first excited states of the hydrogen atom are given on the first page of this test.

(a) [2 pt] For the ground state of the hydrogen atom, determine the expectation value for the radial position of the electron, $\langle 1,0,0|r|1,0,0\rangle$.

(b) [3 pt] Define the radial probably density for the electron in a hydrogenic eigenstate: $P_{n,\ell,m}(r)dr$ as the probability of the electron being measured in the spherical shell between r and $r+dr$.

Write down expressions for $P_{1,0,0}(r)$ and $P_{2,1,1}(r)$, and sketch these as functions of r.

Hint: Remember that the integral of the probability density over r must be equal to one,

$$\int_0^\infty P_{n,\ell,m}(r)dr = 1 \tag{1}$$

(c) [3 pt] For the ground state of the hydrogen atom, determine the most probable radius for the electron. Compare your result to part (a) and explain the similarities and differences.

(d) [1 pt] What is the functional form for $P_{1,0,0}(r)$ in the limit as $r \to 0$? Explain your result considering that the ground state wavefunction is non-zero at $r = 0$.

(e) [1 pt] What are the functional forms of $P_{1,0,0}(r)$, $P_{2,1,1}(r)$, and $P_{200}(r)$ as $r \to 0$? Explain the similarities and differences.

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Quantum #4

a) $\psi_{n\ell m} = R_{n,r}(r)\, Y_\ell^m(\theta,\varphi)$

$$\psi_{100} = \frac{2}{\sqrt{a_0^3}} e^{-r/a_0} \frac{1}{\sqrt{4\pi}}$$

$$= \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

$$\langle r \rangle = \int dr^3 \frac{r}{\pi a_0^3} e^{-2r/a_0}$$

$$= \frac{4}{a_0^3}\int r^3 e^{-2r/a_0} dr$$

$$= \frac{4}{a_0^3} \frac{\Gamma(4)}{(2/a_0)^4}$$

$$= \frac{4\cdot 3!\, a_0^4}{2^4 a_0^3}$$

$$= \frac{3a_0}{2}$$

b) $P_{n\ell m}(r)\,dr = \int_r^{r+dr} \psi_{n\ell m}^* \psi_{n\ell m} \cdot 4\pi r^2 dr$

$$P_{100}(r)\,dr = \int_r^{r+dr} \frac{4}{a_0^3} e^{-2r/a_0} r^2 dr$$

$$\psi_{211} = \frac{1}{\sqrt{8a_0^3}} \frac{r}{\sqrt{3}\,a_0} e^{-r/2a_0} \cdot -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\varphi}$$

$$= -\frac{r}{8\sqrt{\pi a_0^5}} e^{-r/2a_0} e^{i\varphi} \sin\theta$$

$$P_{211} = \int_r^{r+dr}\int_0^{\pi} \frac{r^4 \cdot 2\pi}{64\pi a_0^5} e^{-r/a_0} \sin^4\theta\, dr\, d\theta$$

*Modified version of Sakurai 5.11

Problem 5: Two Level Systems

Consider the Hamiltonian for a two-state system:

$$H = \begin{pmatrix} \epsilon & \lambda\Delta \\ \lambda\Delta & -\epsilon \end{pmatrix} \tag{1}$$

where λ (a unitless parameter) determines the strength of the perturbation on the two-level system and ϵ and Δ are constants with the unit of energy.

The energy eigenvectors for the unperturbed Hamiltonian ($\lambda = 0$) are

$$\psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

(a) [2 pt] Solve for the energy eigenvalues E_1 and E_2 for the full Hamiltonian (for any λ). What is the functional form of the eigenenergies in the limits $\lambda \to 0$ and $\lambda \to \infty$?

(b) [2 pt] For the case that $\lambda|\Delta| \ll \epsilon$, solve for the energy eigenvalues to first order and second order in λ.

Compare these results with the exact results obtained in part (a) and show that they are in agreement.

(c) [1 pt] For the case that $\lambda|\Delta| \ll \epsilon$, what is the change in the unperturbed eigenstate ψ_+ to first order in λ?

(d) [2 pt] For the case that the unperturbed Hamiltonian is nearly degenerate, $\epsilon \ll \lambda|\Delta|$ show that the exact results obtained in part (a) agree with the results of applying first order degenerate perturbation theory with $\epsilon = 0$.

(e) [3 pts] For the case that $\epsilon \ll \lambda|\Delta|$, it would advantageous to use a different set of basis states to describe the system. Using basis states that are approximately eigenstates of the Hamiltonian for small ϵ, determine the Hamiltonian matrix in this new basis. Show that the exact solutions for the eigenenergies are the same as in part (a) in this basis.

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Quantum # 5

a) Given $H = \begin{bmatrix} \epsilon & \lambda\Delta \\ \lambda\Delta & -\epsilon \end{bmatrix}$ we want the energy eigenvalues

Using the eigenvalue equation $\det(H - a\mathbb{I}) = 0$

$$\begin{vmatrix} \epsilon - a & \lambda\Delta \\ \lambda\Delta & -\epsilon - a \end{vmatrix} = 0 = (\epsilon - a)(-\epsilon - a) - \lambda^2\Delta^2$$
$$= -\epsilon^2 + a^2 - \lambda^2\Delta^2$$

$$\hookrightarrow 0 = a^2 - [\lambda^2\Delta^2 + \epsilon^2]$$
$$0 = (a + \sqrt{\lambda^2\Delta^2 + \epsilon^2})(a - \sqrt{\lambda^2\Delta^2 + \epsilon^2})$$
$$\hookrightarrow E = \pm\sqrt{\lambda^2\Delta^2 + \epsilon^2}$$

* in the limit $\lambda \to 0$, $E = \pm\epsilon$

$\lambda \to \infty$ $E = \pm\lambda\Delta$ (assumes $\lambda^2\Delta^2 \gg \epsilon^2$)

b) Note $|\psi_+\rangle = \langle 1, 0 \rangle$ $E_+ = \epsilon$

$|\psi_-\rangle = \langle 0, 1 \rangle$ $E_- = -\epsilon$

$\Delta E_\pm^{(1)} = \langle \psi_\pm | V | \psi_\pm \rangle$ where $V = \begin{bmatrix} 0 & \lambda\Delta \\ \lambda\Delta & 0 \end{bmatrix}$

$$\Delta E_+^{(1)} = [1\ 0] \begin{bmatrix} 0 & \lambda\Delta \\ \lambda\Delta & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\hookrightarrow E_+ \approx \epsilon + \cancel{\lambda\Delta}$$

$$\Delta E_- = [0\ 1] \begin{bmatrix} 0 & \lambda\Delta \\ \lambda\Delta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$\hookrightarrow E_- \approx -\epsilon + \cancel{\lambda\Delta}$$

$$\Delta E_\pm^{(2)} = \frac{|V_{kn}|^2}{E_n - E_k} = \frac{\Delta^2\lambda^2}{\pm 2\epsilon}$$

#5 (cont.)

c) * Assuming $\lambda|\Delta| \ll \epsilon$

$$|\psi_+^{(1)}\rangle = \sum_{k\neq n}' \frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}} |\psi_k\rangle$$

$$= \frac{\Delta\lambda}{2\epsilon} |\psi_-^{(0)}\rangle$$

d)

Problem 6: Harmonic Oscillators in 1D

A quantum harmonic oscillator is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

where p is momentum, x is position, m is mass, and ω is the oscillation frequency.

The Hamiltonian has the usual eigenstates and energies:

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle, \quad n = 0, 1, 2, ... \tag{2}$$

Let the system be perturbed by a potential in the form $V = Ax^2$ where A is a real constant.

(a) [2 pt] What is the change in the energy of the unperturbed eigenstates $|n\rangle$ to first order in A? Show your work.

(b) [2 pt] If the perturbation is time-dependent, $V(t) = A(t)x^2$, it can cause transitions between the harmonic oscillator states. To study these transitions, it is helpful to use the time-dependent expansion:

$$|\psi(t)\rangle = \sum_{n'} c_{n'}(t)e^{-\frac{i}{\hbar}E_{n'}t}|n'\rangle \tag{3}$$

The $c_{n'}(t)$ are time-dependent probability amplitudes for the states $|n'\rangle$ and the energies $E_{n'}$ are the unperturbed eigenenergies. Use the Schroedinger equation to show that the expansion amplitudes satisfy a set of coupled equations:

$$i\hbar\frac{\partial}{\partial t}c_n(t) = \sum_{n'} c_{n'}(t)e^{-\frac{i}{\hbar}(E_{n'}-E_n)t}\langle n|V(t)|n'\rangle \tag{4}$$

(c) [3 pt] Consider the case where the oscillator starts at time $t = 0$ in the ground state, $c_n(t = 0) = \delta_{n,0}$. Use the result from (b) to write down the time dependence of the excited state probability amplitudes to first order in V, $c_n^{(1)}(t)$, $n > 0$. This will be an integral equation, as we have not yet defined $A(t)$.

Show that, to first order, there is a transition only to the $n = 2$ excited state.

(d) [3 pt] Finally, consider a time dependent perturbation with $A(t)$ of the form

$$A(t) = Ae^{-i\Omega t}e^{-\Gamma t} \tag{5}$$

Ω and Γ being real and positive.

Compute the probability that the $n = 2$ state is populated for $t \to \infty$, and explain the dependence of your result on Ω and Γ.

Note: In this problem, it is useful to use

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - i\frac{\lambda}{\hbar}p\right), \quad a = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} + i\frac{\lambda}{\hbar}p\right) \tag{6}$$

where $\lambda = \sqrt{\frac{\hbar}{m\omega}}$ is the length scale in the problem.

You do not need to derive the properties of these two operators, but you should state the results you are using.

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Quantum #6

$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$ $\qquad$ $V' = Ax^2, \ A \in \mathbb{R}$

$H|n\rangle = \hbar\omega(n + 1/2)|n\rangle$

a) $V' = Ax^2$

* but we know the raising/lowering operators

$$a^+ = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - \frac{i\lambda}{\hbar}p\right)$$

$$a = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} + \frac{i\lambda}{\hbar}p\right)$$

$$a + a^+ = \frac{2x}{\sqrt{2}\lambda} \rightarrow x^2 = \frac{\lambda^2}{2}(a + a^+)^2$$

$$\Delta E_n^{(1)} = \langle n^{(0)}|V'|n^{(0)}\rangle$$

$$= A\langle n|\frac{\lambda^2}{2}(aa + aa^+ + a^+a + a^+a^+)|n\rangle$$

$$= \frac{A\lambda^2}{2}\left[\cancel{\langle n|\sqrt{n(n-1)}|n-2\rangle}^{\,0} + \langle n|(n+1)|n\rangle + \langle n|n|n\rangle + \cancel{\langle n|\sqrt{(n+1)(n+2)}|n+2\rangle}^{\,0}\right]$$

$$= \frac{A\lambda^2}{2}(2n+1)$$

$$= \frac{A\hbar}{2m\omega}(2n+1)$$

b)