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## Problem 1: Step Potential (10 points)

1

Consider the potential  $V(x)$

$$V(x) = \begin{cases} 0, & x \leq 0 \\ -V, & x > 0 \end{cases}$$

A particle of mass  $m$  and kinetic energy  $E$  approaches the step from  $x < 0$ .

- a) Write the solution to Schrodinger's equation for  $x < 0$ . (1 pt)
- b) Write the solution to Schrodinger's equation for  $x > 0$ . (1 pt)
- c) Sketch the wave function for  $x < 0$  as well as  $x > 0$ . Making sure to describe how the amplitude and frequency of the wave function changes. (1 pt)
- d) What is the probability that particle will reflect back if  $E = V/8$ ? (2 pts)
- e) What is the probability that the particle will be transmitted if  $E = V/8$ . (2 pts)  
(Determine the transmission probability directly by using the flow of probability current and do not simply use  $T = 1 - R$ )
- f) Show that  $T + R = 1$ . What does this mean physically? (1 pt)
- g) If instead the particle approached the step from  $x > 0$ , how do your answers to parts a), b), d) and e) change? (2 pts)

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## Problem 2: Variational Method (10 points)<sup>2</sup>

Let us consider the hydrogen atom without spin. The Hamiltonian is

$$H = \frac{P^2}{2m} - \frac{C}{r} . \quad (1)$$

Since the ground state is an  $S$  state the wave function must be spherically symmetrical. Suppose you could not solve this problem exactly. Estimate the ground state wave function with a Gaussian:

$$\psi(\vec{r}) = N e^{-r^2/b^2} .$$

- a) Compute the normalization constant  $N$  so that  $\psi(\vec{r})$  is correctly normalized. (2 pts)
- b) Evaluate the expectation value of  $H$  in this state. (3 pts)
- c) Find the best estimate for  $E_0$  by applying the variational method. (4 pts)
- d) The true ground state energy is

$$E_0 = -\frac{1}{2}(C^2 m) .$$

How much does your estimate in (c) differ from the correct answer? (1 pt)

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### Problem 3: Artificial Atoms (10 points) <sup>3</sup>

Modern techniques in nanotechnology research can create artificial atoms, man-made structures that confine electrons like real atoms but with properties that can be engineered. In this problem, consider a 2D atom (electrons tightly bound in the z-direction) with a parabolic potential in the x- and y-directions. The Hamiltonian is:

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2}{2} (x^2 + y^2). \quad (1)$$

**Note:** In solving this problem, you might want to use the standard operators:

$$a_x = \frac{1}{\sqrt{2}} \left( \frac{x}{\lambda} + i\frac{\lambda}{\hbar} p_x \right), \quad a_y = \frac{1}{\sqrt{2}} \left( \frac{y}{\lambda} + i\frac{\lambda}{\hbar} p_y \right) \quad (2)$$

and their Hermitian conjugates, where  $\lambda = \sqrt{\frac{\hbar}{m\omega}}$ .

- a) What are the eigenenergies of this atom? What are the degeneracies of these energy levels? If the separation between adjacent levels is 20 meV (0.02 eV), approximately how large are the low-energy electron states in the atom (the radius)? (2 pts)
- b) If the atom is put in a constant electric field, the Hamiltonian  $H_0$  is perturbed by a potential:

$$H_1 = -eE_1x \quad (3)$$

where  $E_1$  is a constant (the electric field). Prove that to first order in the field, the energy levels of the atom do not change. (2 pts)

- c) Next the atom is placed in a more complex field to study its properties. The new potential is:

$$H_2 = \frac{C_2}{\lambda^2} xy \quad (4)$$

To first order in  $C_2$ , what are the new eigenenergies of what were the first three energy levels of  $H_0$ ? Show your work. (4 pts)

- d) If a different perturbing potential:

$$H_3 = \frac{C_3}{\lambda^2} x^2 \quad (5)$$

is applied (rather than  $H_2$ ), how would your answers to part (c) change? No computations should be necessary to answer this question. (2 pts)

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## Problem 4: 3-d central-force problem (10 points)

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A particle of mass  $m$  and spin  $s = 0$  has a short-range potential energy  $V(r)$ . The particle is in a stationary state with Hamiltonian eigenfunction

$$\psi_E(\mathbf{r}) = N \frac{1}{r} (e^{-\alpha r} - e^{-\beta r}), \quad (6)$$

where  $N$  is a normalization constant (which you need not determine), and  $\alpha$  and  $\beta$  are real numbers such that  $\beta > \alpha$ .

1. Is the orbital angular momentum of the particle sharp in this state? (That is, does  $L^2$  have zero uncertainty?) If not, explain why not. If so, justify your answer and give the value of  $L^2$  for this state. (4 pts)
2. What is the stationary-state energy of this state? (4 pts)
3. What is the potential energy  $V(r)$ ? (2 pts)

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## Problem 5: Quantum statistics (10 points) <sup>5</sup>

1. Write down the energy eigenvalues and wave functions for a particle of mass  $m$  in an infinite square well, with  $V = 0$  for  $-L/2 < x < L/2$  and  $V = \infty$  for  $|x| > L/2$ . (2 pts)
2. What is the ground state energy and wave-function if 2 identical non-interacting bosons are in the well? (4 pts)
3. What is the ground state energy and wave-function if 2 identical non-interacting spin-up fermions are in the well? (4 pts)

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# Quantum #5

a)  $E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$

$$\psi_n = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & n \text{ even} \\ \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right) & n \text{ odd} \end{cases}$$

b) For bosons, there is no exclusion principle rule to follow, thus multiple bosons can occupy the same state in a system simultaneously, but the wavefunction must be symmetric

$$\begin{aligned} \Rightarrow E_{\text{sys}} &= E_{1,1} + E_{2,1} \\ &= \frac{\pi^2 \hbar^2}{2m_1 L^2} + \frac{\pi^2 \hbar^2}{2m_2 L^2} \\ &= \frac{\pi^2 \hbar^2}{m L^2} \end{aligned}$$

$$\begin{aligned} \Rightarrow \psi_{\text{sys}} &= \frac{1}{\sqrt{2}} (\psi_{1,1}(x_1) \psi_{2,1}(x_2) + \psi_{1,1}(x_2) \psi_{2,1}(x_1)) \\ &= \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x_1}{L}\right) \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x_2}{L}\right) + \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x_2}{L}\right) \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x_1}{L}\right) \right] \\ &= \frac{4}{L\sqrt{2}} \left[ \cos\left(\frac{\pi x_1}{L}\right) \cos\left(\frac{\pi x_2}{L}\right) \right] \end{aligned}$$

\* Normalizing the above wavefunction

$$\begin{aligned} 1 &= \frac{8}{L^2} A^2 \int_{-L/2}^{L/2} dx_1 \cos^2\left(\frac{\pi x_1}{L}\right) \int_{-L/2}^{L/2} dx_2 \cos^2\left(\frac{\pi x_2}{L}\right) \\ &= \frac{8A^2}{L^2} \cdot \frac{L^2}{4} \\ &= 2A^2 \rightarrow A = \frac{1}{\sqrt{2}} \end{aligned}$$

$$\Rightarrow \psi_{\text{sys}} = \frac{2}{L} \cos\left(\frac{\pi x_1}{L}\right) \cos\left(\frac{\pi x_2}{L}\right)$$

c) For fermions, we must follow the exclusion principle, thus our two particles cannot occupy the same state simultaneously and the wavefunction must be antisymmetric

$$\begin{aligned} E_{\text{sys}} &= E_{1,1} + E_{2,2} \\ &= \frac{\pi^2 \hbar^2}{2mL^2} + \frac{4\pi^2 \hbar^2}{2mL^2} \\ &= \frac{5\pi^2 \hbar^2}{2mL^2} \end{aligned}$$

#5 (cont.)

$$\begin{aligned} c) \quad \psi_{\text{sys}} &= \frac{1}{\sqrt{2}} (\psi_1(x_1) \psi_2(x_2) - \psi_1(x_2) \psi_2(x_1)) \\ &= \frac{1}{\sqrt{2}} \left( \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{2\pi x_2}{L}\right) - \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x_2}{L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x_1}{L}\right) \right) \\ &= \frac{\sqrt{2}}{L} \left[ \cos\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{2\pi x_2}{L}\right) - \cos\left(\frac{\pi x_2}{L}\right) \sin\left(\frac{2\pi x_1}{L}\right) \right] \end{aligned}$$

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## Problem 6: Spin $\frac{1}{2}$ System (10 points)

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Consider a spin  $\frac{1}{2}$  particle in the state space  $E_s$ . This space can be spanned by the 2 eigenvectors of  $S_x$ ,  $S_y$ , or  $S_z$ , the components of the spin operator  $S = S_x\hat{i} + S_y\hat{j} + S_z\hat{k}$ . The matrix representation of  $S_x$ ,  $S_y$  and  $S_z$  in the eigenbasis  $|+\rangle_z$ ,  $|-\rangle_z$  of  $S_z$  are given below:

$$S_x = \hbar/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \hbar/2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \hbar/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where  $S_z|+\rangle_z = \hbar/2|+\rangle_z$  and  $S_z|-\rangle_z = -\hbar/2|-\rangle_z$ .

Assume that the state of the system at time  $t = 0$  is:  $|\Psi(0)\rangle = |-\rangle_z$ .

a) If the observable  $S_x$  is measured at time  $t = 0$ , what results can be found and with what probabilities? (1 pt)

Now assume that a magnetic field is applied in the  $x$  **direction**:  $\vec{B} = B_0\hat{i}$ . The original wave function  $|\Psi(0)\rangle = |-\rangle_z$  is allowed to evolve in time. The Hamiltonian governing the evolution is:

$$H_{spin} = \vec{S} \cdot \vec{B}$$

b) Set up the time evolution operator for this system,  $U(t, 0)$ . (1 pt)

c) Find  $|\Psi(t)\rangle$ , the wave function at a later time  $t$ . (1 pt)

d) At time  $t > 0$  after  $|\Psi(0)\rangle$  has evolved,  $S_x$  is measured. What is the probability of obtaining  $+\hbar/2$ ? Is your answer time dependent or time independent? Explain correctly for credit. (1 pt)

e) Now let  $|\Psi(0)\rangle$  evolve again and measure  $S_z$  at time  $t$ . Determine the probability of measuring  $S_z$  at time  $t$  and obtaining  $-\hbar/2$ . Is your answer time dependent or time independent? Explain correctly for credit. (1 pt)

f) Without explicitly finding the probabilities, discuss whether you expect the following probabilities to be equal or not. Give a brief explanation of your reasoning for any credit. The symbol  $P_{|\Psi(t)\rangle}(a, c)$  represents the probability of starting with an ensemble in the state  $|\Psi(t)\rangle$ , measuring  $A$  first and getting eigenvalue "a" and then measuring  $C$  and getting eigenvalue "c". Assume that the eigenvalues of  $H_{spin}$  are  $E_+$  and  $E_-$ . (1 pt)

i) Is  $P_{|\Psi(0)\rangle}(+\hbar/2 \text{ for } S_y, -\hbar/2 \text{ for } S_x) = P_{|\Psi(0)\rangle}(-\hbar/2 \text{ for } S_x, +\hbar/2 \text{ for } S_y)$ ? All measurements are taken at  $t = 0$ , i.e. the second measurement is taken immediately after the first measurement in each case. (1 pt)

ii) Is  $P_{|\Psi(0)\rangle}(E_+, -\hbar/2 \text{ for } S_x) = P_{|\Psi(0)\rangle}(-\hbar/2 \text{ for } S_x, E_+)$ ? The first measurement in each case is taken at  $t = 0$ ; the second measurement is taken immediately after the first measurement in each case. (1 pt)

iii) Is the probability  $P_{|\Psi(0)\rangle}(+\hbar/2 \text{ for } S_x \text{ at } t, -\hbar/2 \text{ for } S_y \text{ at } t')$  time dependent or time independent in regards to the time  $t$  of the first measurement? Same question for the time  $t'$  of the second measurement. Discuss your reasoning in each case. (2 pts)



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## Quantum #6

a) \* To operate  $S_x$  on  $|-\rangle_z$ , we must rewrite  $S_x$  as:

$$S_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (|-\rangle_z \langle +|_z + |+\rangle_z \langle -|_z) \frac{\hbar}{2}$$

$$\Rightarrow S_x |-\rangle_z = \frac{\hbar}{2} (|-\rangle_z \langle +|_z + |+\rangle_z \langle -|_z) |-\rangle_z \\ = \frac{\hbar}{2} |+\rangle_z$$

$$\hookrightarrow |+\rangle_z \text{ w/ } P=1$$

b)  $|\psi(t)\rangle = U(t, t_0=0) |\psi(0)\rangle$  where  $U(t, t_0=0) = \exp[-\frac{i}{\hbar} H t]$

$$H = \vec{S} \cdot \vec{B} = S_x B_0 \\ = \frac{B_0 \hbar}{2} (|-\rangle_z \langle +|_z + |+\rangle_z \langle -|_z)$$

c)  $\hookrightarrow |\psi(t)\rangle = e^{-iB_0 t/2} |+\rangle_z$

d)  $S_x |\psi(t)\rangle = S_x (e^{-iB_0 t/2} |+\rangle_z)$

$$= e^{-iB_0 t/2} |-\rangle_z \quad \text{w/ probability } P=1$$

The answer is time independent b/c there is only one allowed state. Alternatively,  $|c_n|^2$  determines the probability of that state.  $|e^{-iB_0 t/2}|^2 = 1$

e)  $S_z |\psi(t)\rangle = (|+\rangle_z \langle +|_z - |-\rangle_z \langle -|_z) e^{-iB_0 t/2} |+\rangle_z \\ = e^{-iB_0 t/2} |+\rangle_z$

$$P(S_z = -\hbar/2) = 0 \text{ by same logic as above}$$

f) i -  $S_y = \frac{\hbar}{2} (-|-\rangle_z \langle +|_z + |+\rangle_z \langle -|_z)$

$$\hookrightarrow \text{No, } [S_x, S_y] \neq 0, \text{ therefore they are not compatible observables}$$

ii - yes, b/c  $H$  is simply a multiple of  $S_x$ , therefore  $[S_x, H] = 0$  and the observables are compatible.

#6 (cont.)

f) iii - Both probabilities are time dependent