Quantum Mechanics
Qualifying Exam - January 2016

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write on only one side of the paper for your solutions.
- Write your alias on the top of every page of your solutions.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

Possibly useful formulas:

Spin Operator

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

In spherical coordinates,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi. \tag{2}$$

Harmonic oscillator wave functions

$u_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}$

$u_1(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega x^2}{2\hbar}}$

Problem 1: Clebsh-Gordon coefficients (10 pts)

A system of two particles with spins $s_1 = \frac{3}{2}$ and $s_2 = \frac{1}{2}$ is described by the Hamiltonian

$$\mathcal{H} = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2$$

with α a constant and $\mathbf{S}_i$ $(i = 1, 2)$ is the spin operator of the i-th particle.

a) What are the allowed values for the quantum numbers of the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$? (2 Points)

b) Calculate the energy levels of the Hamiltonian. (2 Points)

c) Let us define the basis of eigenstates of the $\mathbf{S}_1^2$, $\mathbf{S}_2^2$, S_{1z}, S_{2z} operators, $|s_1 s_2; m_1 m_2\rangle$, where m_1 and m_2 are the quantum numbers of the projection operators S_{1z} and S_{2z} respectively. The system at time $t = 0$ is initially in the state

$$\left| s_1 s_2; \frac{1}{2}, \frac{1}{2} \right\rangle .$$

Find the state of the system at times $t > 0$. (4 Points)

d) Assuming the initial state above, what is the probability of finding the system in the state

$$\left| s_1 s_2; \frac{3}{2}, -\frac{1}{2} \right\rangle$$

at $t > 0$? (2 Points)

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Quantum #1

a) The allowed S values are:

$$|S_1 - S_2| < S < S_1 + S_2$$
$$\left|\tfrac{3}{2} - \tfrac{1}{2}\right| < S < \tfrac{3}{2} + \tfrac{1}{2}$$
$$\hookrightarrow S = 1, 2$$

b) $H = \alpha S_1 \cdot S_2$

$^{\times}$ Remember $S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2$

$$\hookrightarrow S_1 \cdot S_2 = \tfrac{1}{2}(S^2 - S_1^2 - S_2^2)$$

$$H|S_1, S_2; S, S_z\rangle = \frac{\alpha}{2}(S^2 - S_1^2 - S_2^2)|3/2, 1/2; 2, S_z\rangle = \frac{\alpha\hbar^2}{2}\left(2(2+1) - \tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right)|\,\rangle$$

or

$$\frac{\alpha}{2}(S^2 - S_1^2 - S_2^2)|3/2, 1/2; 1, S_z\rangle = \frac{\alpha\hbar^2}{2}\left(1(1+1) - \tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right)|\,\rangle$$

$\Rightarrow$ Our energy states are $E = \frac{\alpha\hbar^2}{2}\left(6 - \frac{15}{4} - \frac{3}{4}\right) = \frac{3\alpha\hbar^2}{2}$ $S = 2$

$E = \frac{\alpha\hbar^2}{2}\left(2 - \frac{15}{4} - \frac{3}{4}\right) = -\frac{\alpha\hbar^2}{2}$ $S = 1$

c) $S_1 \cdot S_2 = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}$

$$= \tfrac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+}) + S_{1z}S_{2z}$$

To determine $|S_1, S_2; \tfrac{1}{2}, \tfrac{1}{2}\rangle$ in basis of H, we must start in max S state $|S_1, S_2; S, S_z\rangle$ and lower to appropriate state

$$|2,2\rangle = |3/2, 1/2\rangle$$

$$S_-|2,2\rangle = \sqrt{(2+2)(2-2+1)}\,|2,1\rangle$$
$$= 2|2,1\rangle$$

$$S_-|3/2, 1/2\rangle = S_{1-}|3/2, 1/2\rangle + S_{2-}|3/2, 1/2\rangle$$
$$= \sqrt{(\tfrac{3}{2}+\tfrac{3}{2})(\tfrac{3}{2}-\tfrac{3}{2}+1)}\,|1/2, 1/2\rangle + \sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\,|3/2, -1/2\rangle$$
$$= \sqrt{3}\,|1/2, 1/2\rangle + |3/2, -1/2\rangle$$

#1(cont.)

c) $\Rightarrow |2,1\rangle = \sqrt{3/4}\,|1/2, 1/2\rangle + \frac{1}{\sqrt{4}}|3/2, -1/2\rangle$

* Note: The $|1,1\rangle$ state is also a linear combination of $|1/2, 1/2\rangle$ and $|3/2, -1/2\rangle$

$$|1,1\rangle = \frac{-1}{\sqrt{4}}|1/2, 1/2\rangle + \frac{\sqrt{3}}{\sqrt{4}}|3/2, -1/2\rangle$$

$$-\sqrt{3}\,|2,1\rangle + |1,1\rangle = -|1/2, 1/2\rangle$$

$$\Downarrow$$

$$|1/2, 1/2\rangle = \sqrt{\tfrac{3}{4}}|2,1\rangle - \sqrt{\tfrac{1}{4}}|1,1\rangle$$

$$|\Psi(t)\rangle = U(t,t_0)|\Psi(0)\rangle, \quad U(t,t_0) = \exp\left[-\frac{iHt}{\hbar}\right]$$

$$\hookrightarrow |1/2, 1/2\,(t)\rangle = \sqrt{\tfrac{3}{4}}\exp\left[\frac{-i3\alpha\hbar t}{2}\right]|2,1\rangle - \sqrt{\tfrac{1}{4}}\exp\left[\frac{i\alpha\hbar t}{2}\right]|1,1\rangle$$

d) $|\langle 3/2, -1/2 \,|\, 1/2, 1/2\,(t)\rangle|^2$

$$|3/2, -1/2\rangle = \sqrt{\tfrac{1}{4}}|2,1\rangle + \sqrt{\tfrac{3}{4}}|1,1\rangle$$

$$\left|\left[\sqrt{\tfrac{3}{4}}\langle 1,1| + \sqrt{\tfrac{1}{4}}\langle 2,1|\right]\left[\sqrt{\tfrac{3}{4}}\exp\left[\frac{-i3\alpha\hbar t}{2}\right]|2,1\rangle - \sqrt{\tfrac{1}{4}}\exp\left[\frac{i\alpha\hbar t}{2}\right]|1,1\rangle\right]\right|^2$$

$$\left|\frac{\sqrt{3}}{4}\exp\left[\frac{i\alpha\hbar t}{2}\right] + \frac{\sqrt{3}}{4}\exp\left[\frac{-i3\alpha\hbar t}{2}\right]\right|^2$$

$$\frac{3}{16}\left(\exp\left[\frac{-i\alpha\hbar t}{2}\right] - \exp\left[\frac{i3\alpha\hbar t}{2}\right]\right)\left(\exp\left[\frac{i\alpha\hbar t}{2}\right) - \exp\left[\frac{-i3\alpha\hbar t}{2}\right]\right)$$

$$\frac{3}{16}\left(2 - \exp[2i\alpha\hbar t] - \exp[-2i\alpha\hbar t]\right)$$

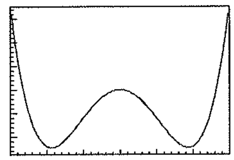

Figure 1: U(x)

Problem 2: Perturbation to a Harmonic Oscillator (10 pts)

Consider a particle of mass, m, moving in a 1-dimensional potential (see Figure 1)

$$U(x) = \lambda x^4 - kx^2.$$

λ and k are positive, and $\lambda \ll \frac{(k^{3/2}m^{1/2})}{4\hbar}$. Approximate the potential near the minima by a simple harmonic oscillator. Here are some useful integrals:

$$\int_{-\infty}^{\infty} x^4 e^{-A(x-a)^2} dx = \frac{1}{4A^{5/2}}(3 + 4a^2 A(3 + a^2 A))\sqrt{\pi}, \text{ for A} > 0$$

$$\int_{-\infty}^{\infty} x^4 e^{-A(x-a)^2} e^{-A(x+a)^2} dx = \frac{3}{16A^{5/2}} e^{-2a^2 A} \sqrt{\frac{\pi}{2}}, \text{ for A} > 0$$

a. Sketch the wavefunctions of the state $|\psi_R\rangle$ which is defined as the state when the particle is found at $x > 0$ and the state $|\psi_L\rangle$ which is the state when the particle is found at $x < 0$. Only consider the lowest energy states near the minima. **(2 Points)**

b. Since the potential is invariant under reflection about the origin, the stationary states must be eigenstates of the parity operator. Express the ground-state and first excited state wavefunctions in terms of $|\psi_R\rangle$ and $|\psi_L\rangle$. **(2 Points)**

c. Estimate the energies of the 2 lowest states using the approximations already described. Hint: use the space representation of the harmonic oscillator wavefunctions and carry out the integrals to find the perturbed energies. **(6 Points)**

Problem 3: Identical particles (10 pts)

Two non-interacting particles of mass m are trapped in a 1-dimensional infinite box of length L situated between $x = 0$ and $x = L$. (In the cases you are considering fermions, assume them to all be spin up.)

(a) [1 points] Write down the single particle energy eigenvalues and wavefunctions.

(b) [1 points] Write down the energy eigenvalues and wavefunctions for two distinguishable particles. Label the states by n_1 for particle 1 and n_2 for particle 2.

(c) [2 points] An energy measurement of the *two identical particle* system yields $E = \hbar^2\pi^2/mL^2$. Write down the state vector/wave function of the system.

(d) [2 points] Suppose instead the energy of the two identical particle system is measured to be $E = 5\hbar^2\pi^2/mL^2$. What is the wave function? *Hint: there are two possibilities.*

(e) [2 points] Show that the fermion state you found in part (d) is an eigenfunction of the Hamiltonian, with the appropriate eigenvalue.

(f) [1 points] Write down the wavefunction for two identical spin-up fermions in the $n_1 = 2$ and $n_2 = 2$ state.

(g) [1 points] If instead you had three particles in the orthonormal states Ψ_1, Ψ_2, and Ψ_3, construct the three particle state for identical fermions.

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Quantum #3

a) For an infinite well b/w 0 and L, our solution is:

$$\psi_n(x) = \sqrt{\tfrac{2}{L}} \sin\left(\tfrac{n\pi x}{L}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

for any single particle

b) Assuming the particles are distinguishable, we simply use the above equations

$$\psi_{n_1} = \sqrt{\tfrac{2}{L}} \sin\left(\tfrac{n_1\pi x}{L}\right) \qquad E_{n_1} = \frac{n_1^2\pi^2\hbar^2}{2mL^2}$$

$$\psi_{n_2} = \sqrt{\tfrac{2}{L}} \sin\left(\tfrac{n_2\pi x}{L}\right) \qquad E_{n_2} = \frac{n_2^2\pi^2\hbar^2}{2mL^2}$$

c) Considering two identical spin-up fermions, the exclusion principle prevents both particles from being in the same state. Since the only combination that yields $E_{n_1 n_2} = \frac{(n_1^2+n_2^2)\pi^2\hbar^2}{2mL^2} = \frac{\pi^2\hbar^2}{mL^2}$ is $n_1 = n_2 = 1$, this state is disallowed by the exclusion principle, therefore

$$\psi_{1,1} = 0$$

d) If $E_{n_1,n_2} = \frac{5\pi^2\hbar^2}{mL^2}$, our possible configurations are $n_1=1, n_2=3$; $n_1=3, n_2=1$

Our general wavefunction for identical fermions is:

$$\psi_{n_1,n_2} = \frac{1}{\sqrt{2}}\left(\psi_{n_1}(x_1)\,\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\,\psi_{n_2}(x_1)\right)$$

This yields the following potential wave functions:

$$\psi_{13} = \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\,\psi_2(x_2) - \psi_1(x_2)\,\psi_2(x_1)\right)$$

$$= \frac{\sqrt{2}}{L}\left(\sin\left(\tfrac{\pi x_1}{L}\right)\sin\left(\tfrac{3\pi x_2}{L}\right) - \sin\left(\tfrac{\pi x_2}{L}\right)\sin\left(\tfrac{3\pi x_1}{L}\right)\right)$$

$$\psi_{31} = \frac{\sqrt{2}}{L}\left(\sin\left(\tfrac{3\pi x_1}{L}\right)\sin\left(\tfrac{\pi x_2}{L}\right) - \sin\left(\tfrac{3\pi x_2}{L}\right)\sin\left(\tfrac{\pi x_1}{L}\right)\right)$$

#3 (cont.)

e) We know that $H\psi_n = E_n\psi_n$ where $H = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$

$\Rightarrow$ For the ψ_{13} state;

$$\frac{\partial^2}{\partial x_1^2}\psi_{13} = \frac{\sqrt{2}}{L}\left(\frac{-\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \left(\frac{3\pi}{L}\right)^2\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right)$$

$$\frac{\partial^2}{\partial x_2^2}\psi_{13} = \frac{\sqrt{2}}{L}\left(-\left(\frac{3\pi}{L}\right)^2\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \left(\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right)$$

$$\hookrightarrow H\psi_{13} = \frac{-\hbar^2}{2m}\left(\frac{\sqrt{2}}{L}\left[\frac{-\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \frac{9\pi^2}{L^2}\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right.\right.$$

$$\left.\left. + \frac{\sqrt{2}}{L}\left[-\frac{9\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \frac{\pi^2}{L^2}\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right)\right)$$

$$= \frac{\hbar^2 10\pi^2}{2mL^2}\left(\frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right]\right)$$

$$= \frac{10\hbar^2\pi^2}{2mL^2}\psi_{13} \checkmark$$

* A similar process will reach the same conclusion for the ψ_{31} state

f) The $n_1 = n_2 = 2$ state is disallowed by the exclusion principle

$\hookrightarrow \psi_{22} = 0$

g) My guess is this follows something like a cyclic permutation

$$\Rightarrow \psi_{n_1 n_2 n_3} = \frac{1}{\sqrt{6}}\left[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) - \psi_{n_1}(x_2)\psi_{n_2}(x_3)\psi_{n_3}(x_1)\right.$$

$$- \psi_{n_1}(x_3)\psi_{n_2}(x_1)\psi_{n_3}(x_2) + \psi_{n_1}(x_3)\psi_{n_2}(x_2)\psi_{n_3}(x_1)$$

$$\left. + \psi_{n_1}(x_2)\psi_{n_2}(x_1)\psi_{n_3}(x_3) + \psi_{n_1}(x_1)\psi_{n_2}(x_3)\psi_{n_3}(x_2)\right]$$

Problem 4: Matrix Mechanics (10 pts)

Consider a system governed by a Hamiltonian H, with an observable C. The Hamiltonian is represented in the $|e_i\rangle$ basis as:

$$H = \hbar\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{Where } |e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvalues and eigenvectors of H are

$$|E_1 = -\hbar\omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, |E_2 = \hbar\omega, 1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, |E_2 = \hbar\omega, 2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let C be represented in the $|e_i\rangle$ basis as

$$C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

At t=0, the system is in the state: $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$

a) At time t=0, the observable C is measured. What results are possible and with what probabilities? (2 pts)

b) Determine the representation of the time evolution operator $U(t, t_0 = 0)$ in the $|e_i\rangle$ representation. (2 pts)

c) Determine $|\Psi(t)\rangle$ in the $|e_i\rangle$ basis. (2 pts)

d) If C is measured at some later time t, what results are possible and with what probabilities? (2 pts)

e) Are your probabilities time dependent or time independent? Explain (2 pts)

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Quantum #4

a) * Read question as: Starting in $|\Psi(t=0)\rangle$, what is the probability of obtaining each eigenvalue of C

* Determine eigenvalues

$$|C - \lambda \mathbb{I}| = 0$$

$$\Rightarrow \quad 0 = \begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix}$$

$$0 = -\lambda\left[(1-\lambda)(-\lambda) - 0\right] - 0\left[0(-\lambda) - 0(2)\right] + 2\left[0(0) - (1-\lambda)(2)\right]$$

$$= (-\lambda)^2(1-\lambda) - 4(1-\lambda)$$

$$= (1-\lambda)\left[\lambda^2 - 4\right]$$

$$= (1-\lambda)(\lambda+2)(\lambda-2)$$

$$\hookrightarrow \lambda = 1, 2, -2$$

* Determine eigenvectors

$$C\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\hookrightarrow \quad 2x_3 = \lambda x_1$$

$$x_2 = \lambda x_2$$

$$2x_1 = \lambda x_3$$

* for $\lambda = 1$

$$2x_3 = x_1$$
$$x_2 = x_2$$
$$2x_1 = x_3$$
$$\Rightarrow \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

* for $\lambda = 2$

$$2x_3 = 2x_1$$
$$x_2 = 2x_2$$
$$2x_1 = 2x_3$$
$$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

* for $\lambda = -2$

$$2x_3 = -2x_1$$
$$x_2 = -2x_2$$
$$2x_1 = -2x_3$$
$$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

#4 (cont.)

a) * Rewriting $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle)$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(|\lambda=1\rangle + \frac{1}{2}\left[|\lambda=2\rangle + |\lambda=-2\rangle\right]\right)$$

* Probabilities of form
$|\langle i|C|\Psi(t=0)\rangle|^2$

$$C|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|\lambda=1\rangle + \frac{1}{2}(2|\lambda=2\rangle - 2|\lambda=-2\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}|\lambda=1\rangle + \frac{1}{\sqrt{2}}|\lambda=-2\rangle - \frac{1}{\sqrt{2}}|\lambda=-2\rangle$$

$$\Rightarrow |\langle\lambda=1|C|\Psi(t=0)\rangle|^2 = \frac{1}{2}$$

$$|\langle\lambda=+2|C|\Psi(t=0)\rangle|^2 = \frac{1}{4}$$

$$|\langle\lambda=-2|C|\Psi(t=0)\rangle|^2 = \frac{1}{4}$$

b) $U(t, t_0=0) = e^{-iHt/\hbar}$

$$= e^{-i\omega t}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_3| + |e_3\rangle\langle e_2|)$$

c) $|\Psi(t)\rangle = U(t,t_0=0)|\Psi(t=0)\rangle$

$$= \frac{1}{\sqrt{2}}\left[e^{-i\omega t}|e_1\rangle + e^{-i\omega t}|e_3\rangle\right]$$

d) * Rewriting $|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega t}(|e_1\rangle + e_3\rangle)$

$$= \frac{1}{\sqrt{2}}e^{-i\omega t}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$= e^{-i\omega t}|\lambda=2\rangle$$

Problem 5: Magnetic Moments and Spin (10 pts)

Consider a spin 1/2 particle with a magnetic moment. We can write the interaction between the spin and an external magnetic field using the Hamiltonian:

$$H = -\gamma \vec{B} \cdot \vec{S} \tag{1}$$

where $\vec{B}$ is the external field, $\vec{S}$ is the spin operator for the particle, and γ is a real positive constant. In this problem, use the usual basis states that are eigenstates of S_z

$$S_z \chi_{\pm} = \pm \frac{\hbar}{2} \chi_{\pm}, \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

For this problem, assume the magnetic field lies in the x-z plane:

$$\vec{B} = B_x \hat{e}_x + B_z \hat{e}_z \tag{3}$$

(a) [1 pt] Solve for the eigenenergies for the Hamiltonian, showing your work. Explain the physics of your results.

(b) [2 pts] Any state of the spin can be written in the $\chi_{\pm}$ basis as:

$$\Psi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \tag{4}$$

Using the Hamiltonian, derive the first-order coupled differential equations that give the time dependence for $\alpha(t)$ and $\beta(t)$. In other words, derive the equations for $\dot{\alpha}(t)$ and $\dot{\beta}(t)$.

(c) [2 pts] Show that you can re-write your results from part (b) as two uncoupled second-order differential equations:

$$\begin{aligned} \ddot{\alpha}(t) &= -\frac{\gamma^2 B_T^2}{4} \alpha(t) \\ \ddot{\beta}(t) &= -\frac{\gamma^2 B_T^2}{4} \beta(t) \end{aligned} \tag{5}$$

where $B_T = \sqrt{B_x^2 + B_z^2}$ is the magnitude of the total magnetic field. How is this result related to what you found in part (a)?

Of course, the solutions to these equations are:

$$\begin{aligned} \alpha(t) &= C_1 \cos(\omega t) + C_2 \sin(\omega t) \\ \beta(t) &= C_3 \cos(\omega t) + C_4 \sin(\omega t) \end{aligned} \tag{6}$$

with $\omega = \frac{\gamma B_T}{2}$.

(d) [3 pts] Consider the situation where the spin is in the spin-up S_z state χ_+ at time $t = 0$. Using the boundary conditions at time $t = 0$, determine the values for the constants C_1, C_2, C_3, C_4 that will solve for the time-dependence of the state. Remember that the equations in part (c) are second-order, so you need two boundary conditions at $t = 0$ for each.

(e) [2 pt] Write down the time-dependent probabilities, $P_{\pm}$ of the spin being in the spin-up and spin-down S_z states. Show that your results are correct in the two cases where $B_x = 0$ and $B_z = 0$.

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Quantum #5

a) Given $H = -\gamma \vec{B} \cdot \vec{S}$

$$= -\gamma (B_x S_x + B_z S_z)$$

$$= -\gamma \left(B_x \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + B_z \hbar/2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

$$= -\gamma \frac{\hbar}{2} \begin{bmatrix} B_z & B_x \\ B_x & -B_z \end{bmatrix}$$

We can determine the energy eigenvalues by $\det(H - \lambda \mathbb{I}) = 0$

$$\begin{vmatrix} -\frac{\gamma\hbar B_z}{2} - \lambda & -\frac{\gamma\hbar B_x}{2} \\ -\frac{\gamma\hbar B_x}{2} & \frac{\gamma\hbar B_z}{2} - \lambda \end{vmatrix} = \left(-\frac{\gamma\hbar B_z}{2} - \lambda\right)\left(\frac{\gamma\hbar B_z}{2} - \lambda\right) - \frac{\gamma^2\hbar^2 B_x^2}{4}$$

$$= -\frac{\gamma^2\hbar^2 B_z^2}{4} - \cancel{\frac{\lambda\gamma\hbar B_z}{2}} + \cancel{\frac{\lambda\gamma\hbar B_z}{2}} + \lambda^2 - \frac{\gamma^2\hbar^2 B_x^2}{4}$$

$$0 = \lambda^2 - \frac{\gamma^2\hbar^2(B_z^2 + B_x^2)}{4}$$

$$\lambda^2 = \frac{\gamma^2\hbar^2}{4}(B_z^2 + B_x^2)$$

$$\lambda = \pm \frac{\gamma\hbar}{2}(B_z^2 + B_x^2)^{1/2}$$

b) The time-dependent Schrödinger eqn states:

$$i\hbar \frac{\partial}{\partial t} \Psi = H\Psi$$

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} = -\frac{\gamma\hbar}{2} \begin{bmatrix} B_z & B_x \\ B_x & -B_z \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix}$$

$$i \frac{\partial\alpha}{\partial t} = -\frac{\gamma}{2}(B_z \alpha(t) + B_x \beta(t))$$

$$i \frac{\partial\beta}{\partial t} = -\frac{\gamma}{2}(B_x \alpha(t) - B_z \beta(t))$$

or

$$\dot{\alpha}(t) = \frac{i\gamma}{2}(B_z \alpha(t) + B_x \beta(t))$$

$$\dot{\beta}(t) = \frac{i\gamma}{2}(B_x \alpha(t) - B_z \beta(t))$$

#5 (cont.)

c) To get 2nd order differential equations, we take another set of time derivatives

$$\ddot{\alpha}(t) = \frac{i\gamma}{2}\left(B_z\,\dot{\alpha}(t) + B_x\,\dot{\beta}(t)\right)$$

$$\ddot{\beta}(t) = \frac{i\gamma}{2}\left(B_x\,\dot{\alpha}(t) - B_z\,\dot{\beta}(t)\right)$$

Substituting our first order differential equations into the above equations yield:

$$\ddot{\alpha}(t) = \frac{-\gamma^2}{4}\left(B_z\left[B_z\alpha(t) + B_x\beta(t)\right] + B_x\left[B_x\alpha(t) - B_z\beta(t)\right]\right)$$

$$\ddot{\beta}(t) = \frac{-\gamma^2}{4}\left(B_x\left[B_z\alpha(t) + B_x\beta(t)\right] - B_z\left[B_x\alpha(t) - B_z\beta(t)\right]\right)$$

Simplifying and letting $B_T^2 = B_x^2 + B_z^2$

$$\ddot{\alpha}(t) = \frac{-\gamma^2 B_T^2}{4}\,\alpha(t)$$

$$\ddot{\beta}(t) = \frac{-\gamma^2 B_T^2}{4}\,\beta(t)$$

d) If we are in the spin-up state at $t=0$

$$1 = C_1\cos(\omega t) + C_2\sin(\omega t) = \alpha(t)$$

$$0 = C_3\cos(\omega t) + C_4\sin(\omega t) = \beta(t)$$

$\Rightarrow$ From this, we immediately determine $C_1 = 1$, $C_3 = 0$ b/c $\cos(\omega t) = 1$ at $t=0$

Our other condition comes from the first order differential equations

$\hookrightarrow$ $\dot{\alpha}(0) = \frac{i\gamma}{2}B_z$ $\qquad$ $\dot{\beta}(0) = \frac{i\gamma}{2}B_x$

$$\dot{\alpha}(t) = -\omega C_1\sin(\omega t) + \omega C_2\cos(\omega t) \rightarrow \dot{\alpha}(0) = \omega C_2$$

$$\dot{\beta}(t) = -\omega C_3\sin(\omega t) + \omega C_4\cos(\omega t) \rightarrow \dot{\beta}(0) = \omega C_4$$

$$\Rightarrow C_2 = \frac{i\gamma}{2\omega}B_z = \frac{iB_z}{B_T}$$

$$C_4 = \frac{i\gamma}{2\omega}B_x = \frac{iB_x}{B_T}$$

#5 (cont.)

e) We now know our time-dependent initial state

$$|\psi\rangle = \begin{bmatrix} \cos(\omega t) + \frac{iB_z}{B_T}\sin(\omega t) \\ \frac{iB_x}{B_T}\sin(\omega t) \end{bmatrix}$$

$$P_\pm = |\langle \chi_\pm | \psi(t) \rangle|^2$$

$$P_+ = |\langle \chi_+ | \psi(t) \rangle|^2$$

$$= \left| [1 \ 0] \begin{bmatrix} \cos(\omega t) + \frac{iB_z}{B_T}\sin(\omega t) \\ \frac{iB_x}{B_T}\sin(\omega t) \end{bmatrix} \right|^2$$

$$= \left| \cos(\omega t) + \frac{iB_z}{B_z}\sin(\omega t) \right|$$

$$= \cos^2(\omega t) + \cancel{\frac{iB_z}{B_T}\sin(\omega t)\cos(\omega t)} - \cancel{\frac{iB_z}{B_T}\sin(\omega t)\cos(\omega t)} + \frac{B_z^2}{B_T^2}\sin^2(\omega t)$$

$$= \cos^2(\omega t) + \frac{B_z^2}{B_T^2}\sin^2(\omega t)$$

$$P_- = |\langle \chi_- | \psi(t) \rangle|^2$$

$$= \left| [0 \ 1] \begin{bmatrix} \cos(\omega t) + \frac{iB_z}{B_T}\sin(\omega t) \\ \frac{iB_x}{B_T}\sin(\omega t) \end{bmatrix} \right|^2$$

$$= \frac{B_x^2}{B_T^2}\sin^2(\omega t)$$

$$P_+ + P_- = \cos(\omega t) + \frac{B_z^2}{B_T^2}\sin^2(\omega t) + \frac{B_x^2}{B_T^2}\sin^2(\omega t)$$

$$= \cos^2(\omega t) + \frac{\cancel{B_z^2 + B_x^2}}{\cancel{B_x^2 + B_z^2}} \sin^2(\omega t)$$

$$= \cos^2(\omega t) + \sin^2(\omega t)$$

$$= 1 \quad \Rightarrow \text{ Valid at all times}$$

* if $B_x = 0$

$P_+ = 1, \quad P_- = 0$

* if $B_z = 0$

$P_+ = \cos^2(\omega t)$

$P_- = \sin^2(\omega t)$

Problem 6: Electron in a Finite Square Well (10 pts)

Consider an electron of energy E incident from x=$-\infty$ on a symmetric one-dimensional square well of depth V_0 and width L.

$$V(x) = \begin{cases} 0, & \text{x < -L/2} \\ -V_0, & \text{-L/2 < x < L/2} \\ 0, & \text{x > L/2} \end{cases}$$

a) Write down the solutions to the time-independent Schrodinger Equation for this situation. There should be five integration constants (2 points)

b) Apply boundary conditions to find the probability that the electron is transmitted past the finite well (4 points)

c) For what values of E is there a 100% probability for transmission past the well? (2 points)

d) Consider a potential well with V_0 large enough for there to be two bound states. For this well, what is the smallest electron energy (E > 0) for which there is a 100% probability for transmission? Your answer will depend on V_0 and other parameters in the problem. (2 points)

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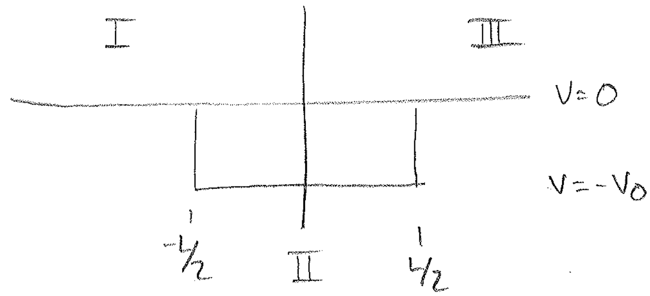
Quantum #6

a)



The time-independent Schrödinger equation states:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V\psi = E\psi$$

Regions I and III:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{-2mE}{\hbar^2}\psi$$

let $\kappa = \frac{\sqrt{-2mE}}{\hbar}$

$$\frac{d^2\psi}{dx^2} = \kappa^2\psi$$

$\hookrightarrow \psi_I = Ae^{\kappa x} + Be^{-\kappa x}$

$\psi_{III} = Fe^{\kappa x} + Ge^{-\kappa x}$

Region II:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{-2m(E+V_0)}{\hbar^2}\psi$$

let $k = \frac{\sqrt{2m(E+V_0)}}{\hbar}$

$$\frac{d^2\psi}{dx^2} = -k^2\psi \Rightarrow \psi_{II} = Ce^{ikx} + De^{-ikx}$$

#6 (cont.)

a) * Note: The above derivation assumes we have a bound state ($-V_0 < E < 0$)

If $E > 0$, then our wave functions become

$$\psi_I = Ae^{ikx} + Be^{-ikx} \qquad k_I = k_{III} = \frac{\sqrt{2mE}}{\hbar} = k$$

$$\psi_{II} = Ce^{ikx} + De^{ikx} \qquad k_{II} = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$$\psi_{III} = Fe^{ikx}$$ (Assume no incoming wave from left)

b) Our boundary conditions are that ψ and $\frac{d\psi}{dx}$ are continuous

↳ at $-L/2$:

$$Ae^{-ikL/2} + Be^{ikL/2} = C\sin\left(-k_{II}\tfrac{L}{2}\right) + D\cos\left(-\tfrac{k_{II}L}{2}\right)$$

$$ik_I\left(Ae^{-ik \cdot L/2} - Be^{ik \cdot L/2}\right) = k_{II}\left[C\cos\left(-k_{II}\tfrac{L}{2}\right) - D\sin\left(-\tfrac{k_{II}L}{2}\right)\right]$$

↳ at $L/2$:

$$C\sin\left(k_{II}\tfrac{L}{2}\right) + D\cos\left(k_{II}\tfrac{L}{2}\right) = Fe^{ikL/2}$$

$$k_{II}\left[C\cos\left(k_{II}\tfrac{L}{2}\right) - D\sin\left(k_{II}\tfrac{L}{2}\right)\right] = ikFe^{ikL/2}$$

* In the end, the transmission probability $T = \frac{|F|^2}{|A|^2}$

⇒ After using $\frac{L}{2}$ B.C's to eliminate C and D and substituting them into our $-\frac{L}{2}$ B.C, with waste of time algebra we find:

$$F = \frac{\exp[-ikL]}{\cos(k_{II}L) - i\frac{k^2+k_{II}^2}{2kk_{II}}\sin(k_{II}L)}A$$

$(-i)(i) = 1$

$$\Rightarrow T = \frac{1}{\left|\cos(k_{II}L) - i\frac{k^2+k_{II}^2}{2kk_{II}}\sin(k_{II}L)\right|^2}$$

$$= \frac{1}{\cos^2(k_{II}L) + \left(\frac{k^2+k_{II}^2}{2kk_{II}}\right)^2\sin^2(k_{II}L)}$$

#6 (cont.)

c) Perfect transmission will occur when $F = A$

$$\hookrightarrow E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} - V_0$$ (from Griffiths)

d) $$E = \frac{2\pi^2\hbar^2}{mL^2} - V_0$$ for 2 bound states