Quantum Mechanics
Qualifying Exam–August 2012

Notes and Instructions:

- There are **6** problems and **7** pages.
- Be sure to write your alias at the top of every page.
- Number each page with the problem number, and page number of your solution (e.g. "Problem 3, p. 1/4" is the first page of a four page solution to problem 3).
- **You must show all your work.**

Possibly useful formulas:

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-dimensional simple harmonic oscillator operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

$$P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$Y_0^0(\theta,\varphi) = \tfrac{1}{\sqrt{4\pi}} \qquad Y_2^2(\theta,\varphi) = \tfrac{5}{\sqrt{96\pi}}\, 3\sin^2\theta\, e^{2i\varphi}$$

$$Y_2^1(\theta,\varphi) = -\tfrac{5}{\sqrt{24\pi}}\, 3\sin\theta\cos\theta\, e^{i\varphi}$$

$$Y_1^1(\theta,\varphi) = -\tfrac{3}{\sqrt{8\pi}}\sin\theta\, e^{i\varphi} \qquad Y_2^0(\theta,\varphi) = \tfrac{5}{\sqrt{4\pi}}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

$$Y_1^0(\theta,\varphi) = \tfrac{3}{\sqrt{4\pi}}\cos\theta \qquad Y_2^{-1}(\theta,\varphi) = \tfrac{5}{\sqrt{24\pi}}\, 3\sin\theta\cos\theta\, e^{-i\varphi}$$

$$Y_1^{-1}(\theta,\varphi) = \tfrac{3}{\sqrt{8\pi}}\sin\theta\, e^{-i\varphi} \qquad Y_2^{-2}(\theta,\varphi) = \tfrac{5}{\sqrt{96\pi}}\, 3\sin^2\theta\, e^{-2i\varphi}$$

In spherical coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

PROBLEM 1: Eigenvalue Equation and Time Evolution

The Hamiltonian for a certain three-level system is represented by the matrix

$$H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix},$$

where a, b, and c are real numbers and $a - c \neq \pm b$. $\Rightarrow c \neq a \mp b$

(a) Find the eigenvalues E_n and normalized eigenvectors $|E_n\rangle, n = 1, 2, 3$ of H. [4 points]

(b) If the system starts out in the state

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

what is $|\psi(t)\rangle$? [3 points]

(c) If the system starts out in the state

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

what is $|\psi(t)\rangle$? [3 points]

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Quantum #1

a) $H = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{bmatrix}$

* To find eigenvalues, we solve characteristic equation

$$\det(H - \lambda \mathbb{I}) = 0$$

$$\begin{vmatrix} a-\lambda & 0 & b \\ 0 & c-\lambda & 0 \\ b & 0 & a-\lambda \end{vmatrix} = a-\lambda\left[(c-\lambda)(a-\lambda) - 0\right] - 0\left[0(a-\lambda) - 0(b)\right] + b\left[0 - b(c-\lambda)\right]$$

$$= (a-\lambda)^2(c-\lambda) - b^2(c-\lambda)$$

$$= (c-\lambda)\left[(a-\lambda^2) - b^2\right]$$

$$= (c-\lambda)(a-\lambda+b)(a-\lambda-b)$$

$$\hookrightarrow \boxed{\lambda = c,\ a-b,\ a+b}$$

* To find eigenvectors, we solve $H\vec{v} = \lambda\vec{v}$

$$\Rightarrow \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_3 \\ cx_2 \\ bx_1 + ax_3 \end{bmatrix}$$

* Case $\lambda = c$

$ax_1 + bx_3 = cx_1$

$cx_2 = cx_2$

$bx_1 + ax_3 = cx_3$

$\hookrightarrow$ $b = 0$

$c = 1 \neq a$

$\Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Case $\lambda = a-b$

$ax_1 + bx_3 = (a-b)x_1$

$cx_2 = (a-b)x_2$

$bx_1 + ax_3 = (a-b)x_3$

$b = 0,\ c \neq a$

$\hookrightarrow x_2 = 0$

$bx_3 = -bx_1$

$bx_1 = -bx_3$

$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Case $\lambda = a+b$

$ax_1 + bx_3 = (a+b)x_1$

$cx_2 = a+b\,x_2$

$bx_1 + ax_3 = (a+b)x_3$

$b = 0,\ c \neq a$

$\hookrightarrow x_2 = 0$

$bx_3 = bx_1$

$bx_1 = bx_3$

$v_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ +1 \end{bmatrix}$

#1 (cont)

b) $U(t,0) = \exp\left[-\frac{i}{\hbar} H t\right]$

$$\Rightarrow |\psi(t)\rangle = U(t,0)|\psi(0)\rangle$$

$$= e^{-iHt/\hbar}|\psi(0)\rangle$$

$*$ substituting $|\psi(0)\rangle = |c\rangle = c\begin{bmatrix}0\\1\\0\end{bmatrix}$

$$= e^{-iHt/\hbar}|c\rangle$$

$$= e^{-ict/\hbar}|c\rangle$$ (after taylor expansion to act out operator)

c) $|\psi(0)\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix} = \sqrt{2}\left(|a+b\rangle - |a-b\rangle\right)$ where $|a+b\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$, $|a-b\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$= \sqrt{2}$

$$\hookrightarrow |\psi(t)\rangle = U(t,0)|\psi(0)\rangle$$

$$= e^{-iHt/\hbar}\left(\sqrt{2}|a+b\rangle - \sqrt{2}|a-b\rangle\right)$$

$$= \sqrt{2}\left[e^{-i(a+b)t/\hbar}|a+b\rangle - e^{-i(a-b)t/\hbar}|a-b\rangle\right]$$ (taylor expand exponential to act operator as above)

PROBLEM 2: Generalized Uncertainty Principle

Consider the spin 1/2 operator

$$\mathbf{S} = \frac{\hbar}{2}\vec{\sigma},$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli matrices, which are defined in the basis of the S_z operator eigenvectors,

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) Compute the commutator $[S_i, S_j]$, with $i, j = x, y, z$. [2 Points]

(b) Compute the expectation values $\langle(\delta S_x)^2\rangle$ and $\langle(\delta S_y)^2\rangle$ for the state

$$|\alpha\rangle = \cos(\alpha)|+\rangle + \sin(\alpha)|-\rangle,$$

where $\delta\mathbf{S} = \mathbf{S} - \langle\mathbf{S}\rangle$. Show explicitly that the relation

$$\langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle \geq \frac{1}{4}\left|\langle[S_x, S_y]\rangle\right|^2$$

is satisfied. What does it physically mean? [4 Points]

(c) Find the states that maximize and minimize the product $\langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle$. Interpret the results. [2 Points]

(d) Suppose one performs an experiment which filters the $+\hbar/2$ eigenstate of the S_z operator from the initially prepared state $|\alpha\rangle$. Then the S_x component of the spin is measured. Compute the expectation value of this measurement in the state $|\alpha\rangle$. [2 Points]

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Quantum #2

a) It is well known that $[S_i, S_j] = i\hbar S_k$

$\Rightarrow S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad S_y = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Proof! $[S_x, S_y] = S_x S_y - S_y S_x$

* Continues as such for other pairs and can be proved quickly on exam

$$= \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\frac{\hbar^2}{4}$$

$$= \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\right)\frac{\hbar^2}{4}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$$

$$= \frac{i\hbar^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= i\hbar S_z$$

b) Using $|\alpha\rangle = \cos(\alpha)|+\rangle + \sin(\alpha)|-\rangle$, find $\langle(\delta S_x)^2\rangle \langle(\delta S_y)^2\rangle$

* Note! $\langle(\delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$

$$\Rightarrow \langle(\delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$$

$$= [\cos(\alpha)\ \sin(\alpha)]\begin{bmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{bmatrix}\begin{bmatrix} \cos(\alpha) \\ \sin\alpha \end{bmatrix} - \left([\cos(\alpha)\ \sin(\alpha)]\begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}\right)^2$$

$$= [\cos\alpha\ \sin\alpha]\begin{bmatrix} \hbar^2/4\cos\alpha \\ \hbar^2/4\sin\alpha \end{bmatrix} - \left([\cos\alpha\ \sin\alpha]\begin{bmatrix} \hbar/2\sin\alpha \\ \hbar/2\cos\alpha \end{bmatrix}\right)^2$$

$$= \hbar^2/4 - (\cdot\hbar\sin\alpha\cos\alpha)^2$$

$$= \hbar^2\left(\frac{1}{4} - \sin^2\alpha\cos^2\alpha\right)$$

$$= \frac{\hbar^2}{4}\left(1 - \frac{1-\cos(4\alpha)}{2}\right)$$

$$= \frac{\hbar^2}{4}\left(\frac{1}{2} - \cos(4\alpha)\right)$$

$$= \frac{\hbar^2}{8}(1-\cos(4\alpha))$$

#2 (cont.)

b) $\langle(\delta S_y)^2\rangle = \langle S_y^2\rangle - \langle S_y\rangle^2$

$$= \hbar^2/4 - \left(\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}0 & -\frac{i\hbar}{2}\\ \frac{i\hbar}{2} & 0\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}\right)^2$$

$$= \frac{\hbar^2}{4} - \left(\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}-\frac{i\hbar}{2}\sin\alpha\\ \frac{i\hbar}{2}\cos\alpha\end{bmatrix}\right)^2$$

$$= \frac{\hbar^2}{4} - \left(-\frac{i\hbar}{2}\sin\alpha\cos\alpha + \frac{i\hbar}{2}\cos\alpha\sin\alpha\right)^2$$

$$= \frac{\hbar^2}{4}$$

$$\frac{1}{4}|\langle[S_x, S_y]\rangle|^2 = \frac{1}{4}|\langle S_z\rangle|^2$$

$$= \frac{1}{4}\left|\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}\hbar/2 & 0\\ 0 & -\hbar/2\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}\right|^2$$

$$= \frac{1}{4}\left|\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}\hbar/2\cos\alpha\\ -\hbar/2\sin\alpha\end{bmatrix}\right|^2$$

$$= \frac{1}{4}\left|\frac{\hbar}{2}(\cos^2\alpha - \sin^2\alpha)\right|^2$$

$$= \frac{\hbar^2}{16}(\cos^4\alpha - 2\cos^2\alpha\sin^2\alpha + \sin^2\alpha)$$

$$\Rightarrow \langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle \overset{?}{\geq} \frac{1}{4}|\langle[S_x, S_y]\rangle|^2$$

$$\frac{\hbar^2}{4}\left(\frac{\hbar^2}{4} - \hbar^2\sin^2\alpha\cos^2\alpha\right) \overset{?}{\geq} \frac{\hbar^2}{16}(\cos^4\alpha - 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha)$$

$$\frac{\hbar^4}{16} - \frac{\hbar^4}{4}\sin^2\alpha\cos^2\alpha \overset{?}{\geq} \frac{\hbar^2}{16}(\cos^4\alpha + \sin^4\alpha) - \frac{\hbar^2}{8}\cos^2\alpha\sin^2\alpha$$

#2 (cont.)

c) $A = \langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle \Rightarrow$ max/min @ $\frac{dA}{d\alpha} = 0$

$$\Rightarrow 0 = \frac{dA}{d\alpha} = \frac{d}{d\alpha}\left(\frac{\hbar^4}{16} - \frac{\hbar^4}{4}\sin^2\alpha\cos^2\alpha\right)$$

$$0 = -\frac{\hbar^4}{4}\left(2\sin\alpha\cos^3\alpha - 2\cos\alpha\sin^3\alpha\right)$$ $\Rightarrow$ any multiple of $\pi/2$ also zero's function

$$0 = \frac{-\hbar^2}{2}\sin\alpha\cos^3\alpha + \frac{\hbar^2}{2}\cos\alpha\sin^3\alpha$$

$$\sin\alpha\cos^3\alpha = \cos^\alpha\sin^3\alpha$$

$$\cos^2\alpha = \sin^2\alpha$$

$$1-\sin^2\alpha = \sin^2\alpha$$

$$1 = 2\sin^2\alpha$$

$$\frac{1}{2} = \sin^2\alpha$$

$$\pm\frac{1}{\sqrt{2}} = \sin\alpha$$

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \alpha = n\frac{\pi}{4}, \quad n = 1, 3, 5 \ldots$$

$\Rightarrow$ minimums at multiples of $n\pi/2$, maximums at multiples of $n\pi/4$

↳ Minimal states: $|\alpha\rangle = |+\rangle$
$= |-\rangle$

↳ Maximal states: $|\alpha\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$
$= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$
$= \frac{1}{\sqrt{2}}(-|+\rangle + |-\rangle)$
$= \frac{1}{\sqrt{2}}(-|+\rangle - |-\rangle)$

d) $|\alpha\rangle = |+\rangle$ by virtue of the experiment

$$\Rightarrow \langle S_x\rangle = \langle +|S_x|+\rangle$$

$$= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$= 0$$

PROBLEM 3: Clebsch-Gordan Coefficients

Consider a system of 2 spin 1/2 particles, i.e. $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}$ where:

$$S_{1z}|s_1, m_{s1}\rangle = m_{s1}\hbar|s_1, m_{s1}\rangle$$

$$S_1^2|s_1, m_{s1}\rangle = s_1(s_1+1)\hbar^2|s_1, m_{s1}\rangle = 3/4\hbar^2|s_1, m_{s1}\rangle$$

and similarly for S_{2z} and S_2^2.

Initially, the 2 spin particles are uncoupled and subject to a Hamiltonian:

$$H_0 = \omega_1 S_{1z} + \omega_2 S_{2z}$$

The eigenvectors $|s_1, s_2; m_{s1}, m_{s2}\rangle$, for this Hamiltonian can be written in compact notation as: $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ where the + and – denote the sign of m_{s1} and m_{s2} respectively.

Answer the following questions:

(a) Set up the matrix representation for H_0 in this uncoupled basis. [1 point]

Now add an interaction term: $A\vec{S}_1 \cdot \vec{S}_2$ to H_0:

$$H = H_0 + A\vec{S}_1 \cdot \vec{S}_2$$

(b) Determine the commutator : $[H, S_{1z}]$. Will the uncoupled basis be an eigenbasis for H? Explain. [2 points]

(c) Determine a coupled basis for this system: $|S, M\rangle$ where S is the value of the total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ and M is its component, i.e.

$$S^2|S, M\rangle = S(S+1)\hbar^2|S, M\rangle, \;\; S_z|S, M\rangle = M\hbar|S, M\rangle.$$

by setting up the matrix for $S^2 = (\vec{S}_1 + \vec{S}_2)^2$ in the uncoupled basis and diagonalizing it. List the eigenvectors of S^2 with the correct values of S and M i.e. as $|S, M\rangle$ states. [3 points]

(d) Identify the Clebsch-Gordan coefficients: $\langle s_1, s_2, m_{s1}, m_{s2}|S, M\rangle$ from the expansions you found in part c). Fill in values for all the quantum numbers in the Dirac braket for each Clebsch-Gordan coefficient and give the numerical value for all the Clebsch-Gordan coefficients you have found. There should be 6 Clebsch-Gordan coefficients. [4 points]

PROBLEM 4: Stationary Perturbation Theory

Suppose an electron is in orbit in the ground state about a tritium nucleus. The tritium nucleus suddenly undergoes beta decay, so that ${}^3_1H \rightarrow {}^3_2 He^+ + e^- + \bar{\nu}_e$.

(a) What are the orbital quantum numbers of the still-bound electron after the beta emission and why? [2 points]

(b) Estimate the probability that the orbital electron remains in the ground state after the beta emission. [6 points]

(c) What is the probability that the orbital electron is in an excited state after the beta emission? [2 points]

Helpful information: the radial wavefunction of the still-bound electron in the ground state is $R_{10} = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$, *which is similar to the wavefunction of the hydrogen atom.*

PROBLEM 5: Time Dependent Perturbation Theory

A particle of charge q, undergoing simple harmonic motion along the x-axis (1-D), is acted on by a time-dependent homogeneous electric field,

$$\vec{E}(t) = E_0 e^{-t^2/\tau^2}\hat{x}$$

where E_0 and τ are constants.

(a) What is the new interaction term in the Hamiltonian for the simple harmonic motion due to the specified electric field? [1 Point]

(b) If the oscillator is in its ground state at $t = -\infty$, find the probability that it will be in an excited state at $t = \infty$. Assume the interaction can be treated as a time-dependent perturbation. [3 Points]

(c) Consider the same charged particle linear harmonic oscillator as in (a). Assuming that dE/dt is small, and that at $t = -\infty$ the oscillator is in the ground state, use the adiabatic approximation to obtain the probability that the oscillator will be found in an excited state as $t \to \infty$. Compare your result with the one you obtained in (b). [3 Points]

(d) Again consider the charged particle harmonic oscillator but with a slightly different perturbation. For $t < 0$

$$H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2 .$$

For $t > 0$

$$H(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}k(x-a)^2 - ka^2$$

with

$$a = \frac{qE_0}{m\omega^2},$$

where $\omega = \sqrt{k/m}$. Show that in the weak coupling limit for $t > 0$ that the only *eigenstate of* H_0 which will be excited with any sizable probability is the first excited state, $\psi_1(x)$, and that the corresponding transition probability is

$$P_{10}(t) = \frac{2q^2E_0^2}{m\hbar\omega^3}\sin^2(\omega t/2).$$

Assume the perturbation is turned on suddenly (fast). [3 Points]

PROBLEM 6: Neutron Evolution

A polarized beam of neutrons with energy E_0 and spin projection along the positive z-axis enters abruptly at $t = 0$ a region where there is a uniform magnetic field $\vec{B}$. If we ignore the spatial degrees of freedom the Hamiltonian for the neutron interacting with the magnetic field is

$$H = -\vec{B} \cdot \vec{\mu}_n = 2\omega \hat{n} \cdot \vec{S}$$

where $\hat{n}$ is a unit vector in the direction of the magnetic field and $\omega = B\mu_n/\hbar$.

(a) **Hamiltonian:** Express $\hat{n}$ in spherical coordinates $\{\theta, \phi\}$ and then find an expression for $\hat{n} \cdot \vec{S}$. [2 points]

(b) **Time Evolution Operator:** Write down an explicit expression for the time-evolution operator in terms of $\{\theta, \phi, t\}$. [3 points]

(c) **Evolved State:** Find the state of the time evolved system for any time $t > 0$. [2 points]

(d) **Expectations:** Find the expectation value of the spin $\vec{S}$. [2 points]

(d) **A Special Case:** Determine and describe the motion for a system where $\vec{B} = B\hat{x}$ [1 point]