Quantum Mechanics
Qualifying Exam–January 2012

Notes and Instructions:

- There are **6** problems and **7** pages.
- Be sure to write your alias at the top of every page.
- Number each page with the problem number, and page number of your solution (e.g. "Problem 3, p. 1/4" is the first page of a four page solution to problem 3).
- **You must show all your work.**

Possibly useful formulas:

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-dimensional simple harmonic oscillator operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

$$P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$Y_0^0(\theta,\varphi) = \tfrac{1}{\sqrt{4\pi}}$$

$$Y_2^2(\theta,\varphi) = \tfrac{5}{\sqrt{96\pi}}\, 3\sin^2\theta\, e^{2i\varphi}$$

$$Y_2^1(\theta,\varphi) = -\tfrac{5}{\sqrt{24\pi}}\, 3\sin\theta\cos\theta\, e^{i\varphi}$$

$$Y_1^1(\theta,\varphi) = -\tfrac{3}{\sqrt{8\pi}}\sin\theta\, e^{i\varphi} \qquad Y_2^0(\theta,\varphi) = \tfrac{5}{\sqrt{4\pi}}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

$$Y_1^0(\theta,\varphi) = \tfrac{3}{\sqrt{4\pi}}\cos\theta \qquad Y_2^{-1}(\theta,\varphi) = \tfrac{5}{\sqrt{24\pi}}\, 3\sin\theta\cos\theta\, e^{-i\varphi}$$

$$Y_1^{-1}(\theta,\varphi) = \tfrac{3}{\sqrt{8\pi}}\sin\theta\, e^{-i\varphi} \qquad Y_2^{-2}(\theta,\varphi) = \tfrac{5}{\sqrt{96\pi}}\, 3\sin^2\theta\, e^{-2i\varphi}$$

In spherical coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

PROBLEM 1: Stationary States

For a quantum system with a time independent Hamiltonian ($\mathbf{H}$), the wave function ($\Psi(x,t)$) is a linear combination of stationary state solutions ($\Psi_n(x,t)$) to the Schrödinger equation:

$$\Psi_n(x,t) = u_n(x)\exp\left(-iE_n t/\hbar\right)$$

where $u_n(x)$ are eigenfunctions of the Hamiltonian

$$\mathbf{H}u_n(x) = E_n u_n(x)$$

and they form a complete orthonormal basis.

(a) Evaluate the uncertainty in the energy for a system in a stationary state with the wave function $\Psi(x,t) = \Psi_n(x,t)$. [Show all work.] (2 Points)

(b) Derive the time evolution operator $\mathbf{U}(t, t_0)$ in terms of the Hamiltonian ($\mathbf{H}$), and apply it to a stationary state $\Psi_n(x, t_0 = 0)$. Describe the change in the stationary state. (2 Points)

Now consider a particle that starts out in a normalized wave function

$$\Psi(x,0) = c_1 u_1(x) + c_2 u_2(x)$$

where the $u_n(x)$ are real eigenfunctions of the Hamiltonian and c_n are real.

(c) Determine an expression for the wave function $\Psi(x,t)$ at subsequent times. (2 Points)

(d) Evaluate the probability density and describe its motion in time. (3 Points)

(e) Determine the uncertainty in the energy ΔE with $\Delta t = \tau$ that is the period of oscillation in (d). (1 Points)

PROBLEM 2: Dirac Notation in Quantum Mechanics

Consider the kets $|a_n\rangle$ as the eigenstates of an observable operator $\mathbf{A}$

$$\mathbf{A}|a_n\rangle = a_n|a_n\rangle\,.$$

Assume that $|a_n\rangle$ form a discrete orthonormal basis in the vector space. Define an operator $U(m,n)$ as

$$U(m,n) = |a_m\rangle\langle a_n|.$$

(a) Show that $U(m,n)$ is an Hermitian operator. Calculate the commutator $[A, U(m,n)]$. [2 Points]

(b) For a generic operator with matrix elements $B_{mn} = \langle a_m|B|a_n\rangle$, show that

$$B = \sum_{mn} B_{mn} U(m,n).$$

[2 Points]

(c) Assume the Hamiltonian of a three-level system

$$\mathbf{H} = H_{12}U(1,2) + H_{21}U(2,1) + H_{23}U(2,3) + H_{32}U(3,2)$$

where $H_{12} = H_{23}$, and $H_{21} = H_{32}$ are complex numbers with dimension of energy. Find the eigenvectors and the eigenvalues of the Hamiltonian in the $|a_n\rangle$ basis. [4 Points]

(d) Assuming the Hamiltonian above, and $n = 1,2,3$, find the condition where the observable operator A is time independent. [2 Points]

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Quantum #2

a) * The condition for Hermiticity is $A^{+} = A$

$$U(m,n)^{+} = \left[|a_m\rangle\langle a_n| \right]^{+}$$
$$= |a_m\rangle^{+}\langle a_n|^{+}$$
$$= |a_n\rangle\langle a_m|$$
$$= U(n,m) \quad \text{as expected}$$

* To evaluate the commutator, we act it upon the state $|a_n\rangle$

$$\Rightarrow [A, U(m,n)]|a_n\rangle = AU(m,n) - U(m,n)A\,|a_n\rangle$$
$$= AU(m,n)|a_n\rangle - U(m,n)A|a_n\rangle$$
$$= A\,|a_m\rangle\langle a_n|a_n\rangle - |a_m\rangle\langle a_n|A|a_n\rangle$$
$$= A\,|a_m\rangle - a_n\,|a_m\rangle\langle a_n|a_n\rangle$$
$$= a_m|a_m\rangle - a_n|a_m\rangle$$
$$= (a_n - a_m)|a_n\rangle \quad \text{(indexes arbitrary, so we switch order)}$$
$$\hookrightarrow [A, U(m,n)] = a_n - a_m$$

b) We want to show: $B = \sum_{mn} B_{mn} U(m,n)$ where $B_{mn} = \langle a_m|B|a_n\rangle$

$$\hookrightarrow \sum_{mn} B_{mn} U(m,n) = \sum_{mn} \langle a_m|B|a_n\rangle |a_m\rangle\langle a_n|$$

* To determine matrix elements of B, we use completeness relation

$$B = \sum_{m,n} |a_m\rangle\langle a_m|B|a_n\rangle\langle a_n|$$
$$= \sum_{m,n} \langle a_m|B|a_n\rangle |a_m\rangle\langle a_n|$$

$$\Rightarrow B = \sum_{mn} B_{mn} U(m,n)$$

#2 (cont.)

c) Given $H = H_{12}U(1,2) + H_{21}U(2,1) + H_{23}U(2,3) + H_{32}U(3,2)$, we can rewrite H as:

$$H \doteq \begin{array}{c} \begin{array}{ccc} |a_1\rangle & |a_2\rangle & |a_3\rangle \end{array} \\ \begin{bmatrix} 0 & H_{12} & 0 \\ H_{21} & 0 & H_{23} \\ 0 & H_{32} & 0 \end{bmatrix} \end{array}$$

$$0 = \begin{vmatrix} -\lambda & H_{12} & 0 \\ H_{21} & -\lambda & H_{23} \\ 0 & H_{32} & -\lambda \end{vmatrix}.$$

$$0 = -\lambda(\lambda^2 - H_{23}H_{32}) - H_{12}(-H_{21}\lambda)$$

$$= -\lambda^3 + H_{23}H_{32}\lambda + H_{12}H_{21}\lambda$$

$$= -\lambda^3 + 2H_{12}H_{21}\lambda$$

$$= -\lambda(\lambda^2 - 2H_{12}H_{21})$$

$$\hookrightarrow \lambda = 0, +\sqrt{2H_{12}H_{21}}, -\sqrt{2H_{12}H_{21}}$$

* To find eigenvectors:

$$H\vec{v} = \lambda\vec{v}$$

$$\begin{bmatrix} 0 & H_{12} & 0 \\ H_{21} & 0 & H_{12} \\ 0 & H_{21} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{aligned} H_{12}x_2 &= \lambda x_1 \\ H_{21}x_1 + H_{12}x_3 &= \lambda x_2 \\ H_{21}x_2 &= \lambda x_3 \end{aligned}$$

* for $\lambda = 0$

$$H_{12}x_2 = 0$$

$$H_{21}x_1 + H_{12}x_3 = 0$$

$$H_{21}x_2 = 0$$

* for $\lambda = \sqrt{2H_{12}H_{21}}$

$$H_{12}x_2 = \sqrt{2H_{12}H_{21}}\,x_1$$

$$H_{21}x_1 + H_{12}x_3 = \sqrt{2H_{12}H_{21}}\,x_2$$

$$H_{21}x_2 = \sqrt{2H_{12}H_{21}}\,x_3$$

* for $\lambda = -\sqrt{2H_{12}H_{21}}$

$$H_{12}x_2 = -\sqrt{2H_{12}H_{21}}\,x_1$$

$$H_{21}x_1 + H_{12}x_3 = -\sqrt{2H_{12}H_{21}}\,x_2$$

$$H_{21}x_2 = -\sqrt{2H_{12}H_{21}}\,x_3$$

$$\Rightarrow \vec{v} = \begin{bmatrix} H_{12} \\ 0 \\ H_{21} \end{bmatrix}\frac{1}{\sqrt{H_{12}H_{21}}}$$

$$= \frac{1}{\sqrt{H_{12}H_{21}}}\left(H_{12}|a_1\rangle + H_{21}|a_3\rangle\right)$$

PROBLEM 3: Harmonic Oscillator

A particle of mass m is under the influence of the following potential

$$V(x) = V_0\sqrt{A^2 + x^2}$$

where V_0 and A are constants. For small displacements $x \ll A$ this potential can be approximated by a simple harmonic oscillator.

(a) Determine the lowest energy this particle can have in terms of $\hbar$, m, V_0 and a for $x \ll A$. (2 Points)

Now consider the Hamiltonian describing the true one-dimensional harmonic oscillator

$$\mathbf{H} = \frac{\mathbf{P}^2}{2m} + \frac{1}{2}k\mathbf{X}^2$$

with eigenstates

$$\mathbf{H}|n\rangle = E_n|n\rangle \quad n = 0, 1, 2, \cdots .$$

(b) Using commutation relations, calculate the equations of motion for $\mathbf{P}$ and $\mathbf{X}$ in the Heisenberg picture. (Find $\dot{X}$ and $\dot{P}$.) (2 Points)

(c) Solve for $P(t)$ and $X(t)$ in terms of $P(0)$ and $X(0)$ and show that $[X(t), X(0)] \neq 0$ for $t \neq 0$. (2 Points)

A harmonic oscillator system is known to be in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle)$$

where $|0\rangle$ and $|3\rangle$ are the normalized ground state and the third excited state of the harmonic oscillator respectively.

(d) What is the value of $n > 0$ for the first non-zero value of $\langle X^n\rangle$ with the state vector $|psi\rangle$? (2 Points)

(e) What is the expectation value $\langle X^3\rangle$ with the state vector $|\psi\rangle$? (2 Points)

PROBLEM 4: Angular Momentum

The hydrogen atom including hyperfine splitting can be described by a Hamiltonian

$$\mathbf{H} = \frac{\mathbf{P}_p^2}{2m_p} + \frac{\mathbf{P}_e^2}{2m_e} - \frac{e^2}{r} + \mathbf{H}_{HF}$$

where $\mathbf{H}_{HF} = A\vec{S}_p \cdot \vec{S}_e$ describes the spin-spin or hyperfine interaction and the total spin angular momentum is given by $\vec{S} = \vec{S}_p + \vec{S}_e$. The subscripts ($p$ and e) refer to proton and electron, respectively

(a) Write down the form of the spin-spin direct product state vectors. What are the "good", *i.e.* diagonal operators for this set of state vectors? [2 points]

(b) Write down the form of the "total-s" state vectors. What are the "good", *i.e.* diagonal operators for this set of state vectors? [2 points]

(c) Choosing an appropriate set of state vectors, calculate the H_{HF} energy eigenvalues, and the energy splitting due to the hyperfine interaction. [5 points]

(d) If the photon wavelength (λ) is 21 cm from the hyperfine transition, evaluate the constant A in H_{HF}. *Hint:* $\hbar c = 1.97 \times 10^{-5}$ $eV{\cdot}cm$. [1 point]

PROBLEM 5: Interaction Picture

There is a 3rd 'picture' in quantum mechanics in addition to the Schrödinger and Heisenberg pictures that is often used. This picture is called the interaction picture. The interaction picture is related to the Schrödinger picture through the following unitary transformation for a Hamiltonian, $H = H_0 + V$.

$$\Psi_I(x,t) = \mathbf{U}_0^{-1}\Psi_S(x,t)$$

where

$$\mathbf{U}_0 = e^{-\frac{i}{\hbar}(t-t_0)H_0}\,.$$

The Hamiltonian $\mathbf{H}_0$ is assumed to be time independent, V is considered to be small in comparison to $\mathbf{H}_0$, I denotes interaction picture and S denotes Schrödinger picture, t_0 is the time when the two pictures coincide (you can take this to be $t_0 = 0$) and t is the time from when the two pictures coincide.

(a) Use this information to find the equation, analogous to the Schrödinger equation, that gives the time evolution for Ψ_I. To receive full credit justify all steps. (4 Points)

(b) How are operators in the interaction picture ($\mathbf{\Omega}_I$) and the Schrödinger picture ($\mathbf{\Omega}_S$) related? (2 Points)

(c) These 2 pictures are related to each other through a unitary transformation. In general, what is a unitary transformation and what are the important quantities that a unitary transformation preserves? (3 Points)

(d) Why do you think this is called the interaction picture? Why is it useful? To receive credit you must explain how the name relates to the dynamics. (1 Points)

PROBLEM 6: Stationary Perturbation Theory

Let us consider the Hamiltonian $\mathbf{H}$ for a harmonic oscillator with a charged particle in a constant electric field (E):

$$\begin{aligned} \mathbf{H} &= \mathbf{H}_0 + \mathbf{H}_1 \\ \mathbf{H}_0 &= \frac{\mathbf{P}^2}{2m} + \frac{1}{2}k\mathbf{X}^2 \quad \text{and} \\ \mathbf{H}_1 &= \lambda \mathbf{X} \end{aligned}$$

where $\lambda = qE$ and q is the electric charge.

The non-perturbed Hamiltonian has the following eigenvalue equation

$$\mathbf{H}_0|n\rangle = E_n^{(0)}|n^{(0)}\rangle\,, \quad E_n^{(0)} = \hbar\omega(n+\frac{1}{2}) \quad \text{and} \quad \omega = \sqrt{k/m}\,.$$

(a) Apply perturbation theory and determine the first order energy $E_n^{(1)}$. [2 Points]

(b) Apply perturbation theory and evaluate the second order energy $E_n^{(2)}$. [3 Points]

(c) Solve this problem exactly and find the energy E_n. [3 Points]

(d) Determine the eigenvector to the first order $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle$. [2 Points]