Jan 2009

Problem 1: Spin $\frac{1}{2}$ particles (10 points)

Consider a system made up of spin 1/2 particles. If one measures the spin of the particles, one can only measure spin up or spin down. The general spin state of a spin 1/2 particle can be expressed as a two-element column matrix.

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

The spin matrices are:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) Can one simultaneously measure S_x, S_y and S_z? Explain your answer. (1 pt)

b) Can one simultaneously measure S^2 and S_z? Explain your answer. (1 pt)

c) Show S_z is Hermetian. (1 pt)

d) Calculate the normalized eigenvectors and eigenvalues of S_z. (2 pts)

Suppose a spin 1/2 particle is in the state

$$\chi = A\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

e) Normalize the state in order to determine A (1 pt)

f) If one measures S_z, what is the probability of getting -$\hbar/2$? (1 pt)

g) If one measures S_x, what is the probability of getting $+\hbar/2$? (2 pts)

h) What is the expectation value of S_y (1 pt)

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Quantum #1

a) Simultaneous measurements can only occur if two or more operators have the same eigenbasis. Said another way, if the commutator b/w two operators is 0, then they can be simultaneously measured. It is common knowledge that for the spin operators,

$$[S_i, S_j] = i\hbar \epsilon_{ijk} S_k$$

Thus S_x, S_y, and S_z <u>cannot</u> be measured simultaneously.

b) Similarly to part a, S^2 and S_z can only be measured simultaneously if $[S^2, S_z]=0$. Again, it is well known that $[S^2, S_i]=0$ where $i = \{x, y, z\}$. Thus S^2 and S_z <u>can</u> be measured simultaneously.

c) The condition of Hermitivity is $A = A^+$

$$\Rightarrow \quad S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$S_z^+ = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$> S_z$ is Hermitian

d) Begin by finding eigenvalues.

$\hookrightarrow$ b/c S_z is diagonalized eigenvalues are $\pm \hbar/2$

By similar logic, the corresponding eigenvectors are

$$\frac{\hbar}{2} : \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \frac{-\hbar}{2} : \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as they would be for any diagonalized 2x2 matrix

e) Normalization Condition: $1 = \langle \chi | \chi \rangle$

$$\hookrightarrow \quad 1 = A^2 \begin{bmatrix} 1+i & 2 \end{bmatrix} \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$
$$= A^2 (1+1 \quad +4)$$
$$= 6A^2$$
$$\hookrightarrow \quad A = \sqrt{\frac{1}{6}}$$

#1 (cont.)

f) $$P(S_z = \tfrac{\hbar}{2}) = |\langle S_z = -\tfrac{\hbar}{2} | \chi \rangle|^2$$

$$= \left| \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{6}} \right|^2$$

$$= \frac{1}{6} \cdot |2|^2$$

$$= \frac{2}{3}$$

g) * We must first find eigenvectors of S_x using $S_x \vec{v} = \lambda \vec{v}$

$$\begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \hbar/2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad \frac{\hbar}{2} x_2 = \frac{\hbar}{2} x_1 \quad \rightarrow \quad \vec{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{\hbar}{2} x_1 = \frac{\hbar}{2} x_1$$

$$P(S_x = \tfrac{\hbar}{2}) = |\langle S_x = \tfrac{\hbar}{2} | \chi \rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \frac{1}{\sqrt{6}} \right|^2$$

$$= \frac{1}{12} |3+i|^2$$

$$= \frac{1}{12} (9+1)$$

$$= \frac{10}{12} = \frac{5}{6}$$

h) $$\langle S_y \rangle = \langle \chi | S_y | \chi \rangle$$

$$= \frac{1}{6} \begin{bmatrix} 1-i & 2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix} \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1-i & 2 \end{bmatrix} \begin{bmatrix} -i\hbar \\ -\frac{\hbar}{2} + \frac{i\hbar}{2} \end{bmatrix}$$

$$= \frac{1}{6} \left(-i\hbar - \hbar \quad - \hbar + i\hbar \right)$$

$$= -\hbar/3$$

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Problem 2: A two-state system (10 points)

We can approximate the ammonia molecule NH_3 by a simple two-state system. The three H nuclei are in a plane, and the N nucleus is at a fixed distance either above or below the plane of the H's. Each is approximately a stationary state with some energy E_0. But there is a small amplitude for transition from up to down. Thus the total Hamiltonian is $H = H_0 + H_1$, where

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix} \quad \text{and} \quad H_1 = \begin{pmatrix} 0 & -A \\ -A & 0 \end{pmatrix}$$

with $|A| \ll |E_0|$.

(a) Find the exact eigenvalues of H. (*1 points*)

(b) Now suppose the molecule is in an electric field that distinguishes the two states. The new Hamiltonian is $H = H_0 + H_1 + H_2$, where

$$H_2 = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

Find the new exact energy levels. (*1 points*)

(c) Apply perturbation theory and find the energy levels to the lowest non-vanishing order for $\epsilon_i \ll |A|$. Compare the results to the exact answer in (b). (*4 points*)

(d) Apply perturbation theory and find the energy levels to the lowest non-vanishing order for $\epsilon_i \gg |A|$. Compare the results to the exact answer in (b). (*4 points*)

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Quantum #2

$$H_0 = \begin{bmatrix} E_0 & 0 \\ 0 & E_0 \end{bmatrix} \qquad H_1 = \begin{bmatrix} 0 & -A \\ -A & 0 \end{bmatrix}$$

$$H = H_0 + H_1 = \begin{bmatrix} E_0 & -A \\ -A & E_0 \end{bmatrix}$$

a) Using the eigenvalue equation $\det(H - \lambda \mathbb{I}) = 0$

$$\begin{vmatrix} E_0-\lambda & -A \\ -A & E_0-\lambda \end{vmatrix} = 0 = (E_0-\lambda)^2 - (-A)^2$$

$$= E_0^2 - 2\lambda E_0 + \lambda^2 - A^2$$

$$= \lambda^2 - 2E_0\lambda + (E_0^2 - A^2)$$

$$\hookrightarrow \lambda = \frac{2E_0 \pm \sqrt{4E_0^2 - 4(1)(E_0^2 - A^2)}}{2}$$

$$= \frac{2E_0 \pm \sqrt{\cancel{4E_0^2} - \cancel{4E_0^2} + 4A^2}}{2}$$

$$= E_0 \pm A$$

b) $H_2 = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix}$

$$H = H_0 + H_1 + H_2 = \begin{bmatrix} E_0 + \epsilon_1 & -A \\ -A & E_0 + \epsilon_2 \end{bmatrix}$$

Again, as above:

$$\begin{vmatrix} E_0+\epsilon_1-\lambda & -A \\ -A & E_0+\epsilon_2-\lambda \end{vmatrix} = 0 = (E_0+\epsilon_1-\lambda)(E_0+\epsilon_2-\lambda) - (-A)^2$$

$$= E_0^2 + E_0\epsilon_2 - E_0\lambda + \epsilon_1 E_0 + \epsilon_1\epsilon_2 - \lambda\epsilon_1 - \lambda E_0 - \epsilon_2\lambda + \lambda^2 - A^2$$

$$= \lambda^2 - (2E_0 + \epsilon_1 + \epsilon_2)\lambda + (E_0^2 + E_0[\epsilon_1 + \epsilon_2] + \epsilon_1\epsilon_2 - A^2)$$

$$\hookrightarrow \lambda = \frac{2E_0 + \epsilon_1 + \epsilon_2 \pm \sqrt{4E_0^2 - 4(1)(E_0^2 + E_0[\epsilon_1 + \epsilon_2] + \epsilon_1\epsilon_2 - A^2)}}{2}$$

$$= \frac{2E_0 + \epsilon_1 + \epsilon_2 \pm \sqrt{4A^2 - 4E_0(\epsilon_1 + \epsilon_2) - 4\epsilon_1\epsilon_2}}{2}$$

#2 (cont.)

c) Assume H_2 is a perturbation on $H = H_0 + H_1$. Therefore, use non-degenerate perturbation theory, and we must solve for eigenvectors of $H = H_0 + H_1$

$\Rightarrow$ Using the eigenvector equation $H\vec{a} = \lambda \vec{a}$

$$\begin{bmatrix} E_0 & -A \\ -A & E_0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} E_0 a_1 - A a_2 &= \lambda a_1 \\ -A a_1 + E_0 a_2 &= \lambda a_2 \end{aligned}$$

* for $\lambda = E_0 + A$

$$E_0 a_1 - A a_2 = E_0 a_1 + A a_1$$

$$-A a_2 = A a_1$$

$$-a_2 = a_1$$

$$\hookrightarrow \vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$$

* for $\lambda = E_0 - A$

$$E_0 a_1 - A a_2 = E_0 a_1 - A a_1$$

$$-A a_2 = -A a_1$$

$$a_2 = a_1$$

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad |E_0 + A\rangle = \langle 1, -1 \rangle \cdot \frac{1}{\sqrt{2}}$$

$$|E_0 - A\rangle = \langle 1, 1 \rangle \cdot \frac{1}{\sqrt{2}}$$

* Dot product verifies orthogonality

$$\frac{1}{2}\left[(1 \cdot 1) + (1 \cdot -1)\right] = 0$$

In general $\Delta E_n^{(1)} = \langle n^{(0)} | V | n^{(0)} \rangle$

$$\hookrightarrow \Delta E_1^{(1)} = \langle E+A | H_2 | E+A \rangle$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ -\epsilon_2 \end{bmatrix}$$

$$= \frac{1}{2}(\epsilon_1 + \epsilon_2)$$

$$\hookrightarrow \Delta E_2^{(1)} = \langle E-A | H_2 | E-A \rangle$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}(\epsilon_1 + \epsilon_2)$$

*If $\epsilon_i \ll A$, our exact energies are

$$E \approx \frac{2E_0 + \epsilon_1 + \epsilon_2 \pm \sqrt{4A^2 - \cancelto{0}{4E_0(\epsilon_1 + \epsilon_2)} - 4\epsilon \ldots}}{2}$$

$$= E_0 \pm A + \epsilon_1 + \epsilon_2$$

which matches what we get from perturbation theory

#2 (cont.)

d) If $\epsilon_i >> |A|$, then $H = H_0 + H_2$ and H_1 is our perturbation

↳ $\lambda = E_0 + \epsilon_i$
$\vec{a} = \langle 1, 0\rangle$ and $\langle 0, 1\rangle$

$\Rightarrow |E_0 + \epsilon_1\rangle = \langle 1, 0\rangle$ * Dot product verifies orthogonality
$|E_0 + \epsilon_2\rangle = \langle 0, 1\rangle$

Again as in part c

$$\Delta E_n^{(1)} = \langle n^{(0)} | V | n^{(0)}\rangle$$

$$\Delta E_1^{(1)} = \langle E_0 + \epsilon_1 | H_1 | E_0 + \epsilon_1\rangle = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & -A \\ -A & 0\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix}\cdot\frac{1}{2} = \frac{1}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 \\ -A\end{bmatrix} = 0$$

$$\Delta E_2^{(1)} = \langle E_0 + \epsilon_2 | H_1 | E_0 + \epsilon_2\rangle = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0 & -A \\ -A & 0\end{bmatrix}\begin{bmatrix}0 \\ 1\end{bmatrix}\cdot\frac{1}{2} = \frac{1}{2}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}-A \\ 0\end{bmatrix} = 0$$

*We must proceed to $\Delta E_n^{(2)}$ which requires $|n^{(1)}\rangle$

$$|n^{(1)}\rangle = \sum_{k\neq n}' \frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}} |k^{(0)}\rangle$$

* remember that n and k refer to Energy eigen values

$$↳ |(E_0 + \epsilon_1)^{(1)}\rangle = \frac{\langle E_0 + \epsilon_2 | H_1 | E_0 + \epsilon_1\rangle}{(E_0 + \epsilon_1) - (E_0 + \epsilon_2)} |E_0 + \epsilon_2\rangle = \frac{-A}{\epsilon_1 - \epsilon_2} |E_0 + \epsilon_2\rangle$$

$$↳ |(E_0 + \epsilon_2)^{(1)}\rangle = \frac{\langle E_0 + \epsilon_1 | H_1 | E_0 + \epsilon_2\rangle}{(E_0 + \epsilon_2) - (E_0 + \epsilon_1)} |E_0 + \epsilon_1\rangle = \frac{-A}{\epsilon_2 - \epsilon_1} |E_0 + \epsilon_1\rangle$$

*In general, our second order correction formula is: $\Delta E_n^{(2)} = \langle n^{(0)} | V | n^{(1)}\rangle = \sum_{k\neq n}' \frac{|V_{kn}|^2}{E_n^{(0)} - E_k^{(0)}}$

$$\Rightarrow \Delta E_1^{(2)} = \frac{A^2}{\epsilon_1 - \epsilon_2} \qquad \Delta E_2^{(2)} = \frac{A^2}{\epsilon_2 - \epsilon_1}$$

#2 (cont.)

d) This gives us energies of: $E_0 + \epsilon_1 + \dfrac{A^2}{\epsilon_1 - \epsilon_2} \approx E_1$

$E_0 + \epsilon_2 + \dfrac{A^2}{\epsilon_2 - \epsilon_1} \approx E_2$

* Returning to our exact solution

$$E_i = \frac{2E_0 + \epsilon_1 + \epsilon_2 \pm \sqrt{\cancel{4A^2}^{\approx 0} - 4E_0(\epsilon_1 + \epsilon_2) - 4\epsilon_1\epsilon_2}}{2}$$

$$= E_0 + \frac{\epsilon_1 + \epsilon_2}{2} \pm \sqrt{E_0(\epsilon_1 + \epsilon_2) - \epsilon_1\epsilon_2}$$

$$= E_0 + \frac{\epsilon_1 + \epsilon_2}{2} \mp E_0(\epsilon_1 + \epsilon_2)\sqrt{1 + \frac{\epsilon_1\epsilon_2}{E_0(\epsilon_1 + \epsilon_2)}}$$

$$= E_0 + \frac{\epsilon_1 + \epsilon_2}{2} \mp E_0(\epsilon_1 + \epsilon_2)\left[1 + \frac{1}{2}\left(\frac{\epsilon_1\epsilon_2}{E_0(\epsilon_1 + \epsilon_2)}\right) \overset{\text{ignore}}{\cancel{- \frac{1}{8}\left(\frac{\epsilon_1\epsilon_2}{E_0(\epsilon_1 + \epsilon_2)^2}\right)}} + \dots\right]$$

$$= E_0 + \frac{\epsilon_1 + \epsilon_2}{2} \mp \left[E_0(\epsilon_1 + \epsilon_2) + \frac{1}{2}\epsilon_1\epsilon_2\right]$$

↳

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Problem 3: 2-d potential (10 points)

A particle of mass m is confined by two impenetrable parallel walls at $x = \pm a$ to move on a two-dimensional strip defined by

$$-a < x < a$$
$$-\infty < y < \infty$$

The wave function for this system can be expressed as the product of two functions: one that depends only on the spatial co-ordinates (x and y), and one that depends only on time t.

a) Use the separation of variables technique to find the time dependent function. (2 points)

b) The part of the wave function that depends only on spatial co-ordinates can be expressed as the product of two functions: one that depends only on x and one that depends only on y. Use the separation of variables technique to find these two functions. (3 points)

c) What is the minimum energy of the particle that measurement can yield? (2 points)

d) Suppose that two additional walls are inserted at $y = \pm a$. Can a measurement of the particle's energy yield the value $3\pi^2\hbar^2/8ma^2$ Explain your answer. (3 points)

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Problem 4: Angular momentum (10 points)

A $|jm\rangle = |1,0\rangle$ state scatters from a $|jm\rangle = |\frac{1}{2},\frac{1}{2}\rangle$ state via a $|jm\rangle = |\frac{1}{2},\frac{1}{2}\rangle$ resonance.

a) Relate the highest weight (highest possible m) states in the total j basis to the highest weight states in the direct product basis for this system of $\frac{1}{2} \otimes 1$. (1 pt)

b) Acting on the highest weight states with lowering operators, give an expansion of each total-j state in terms of direct product states and their Clebsch-Gordon co-efficients. (5 pts)
Hint: $J_\pm|jm\rangle = \hbar[(j \mp m)(j \pm m + 1)]^{1/2}|j, m \pm 1\rangle$

c) How often do the above-mentioned spin states scatter elastically, and how often do they scatter inelastically? (4 pts)

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Problem 5: Measurement and Probability (10 points)5

Consider the following two observables, H and C, whose representation in the unit basis $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$ is:

$$H = \hbar\omega \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

where:

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Assume that at time t=0 the ensemble of particles is in the state:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$$

The eigenvalues of H are given by $\lambda = 2, 1, -1$ with normalized eigenvectors given by $(1,1,1)/\sqrt{3}$, $(1,0,-1)/\sqrt{2}$ and $(1,-2,1)/\sqrt{6}$ respectively.

The eigenvalues of C are given by $\lambda = 1, 1, -1$ with normalized eigenvectors given by $(1,0,-1)/\sqrt{2}$, $(0,1,0)$ and $(1,0,1)/\sqrt{2}$ respectively.

a) What is the probability of measuring H and obtaining $E = \hbar\omega$? What state is the particle in after the measurement? (2 pts)

b) If one immediately measures C after the measurement of H in part b), what is the probability of obtaining $c = 1$? (1 pt)

c) What is the probability of measuring H first and getting $E = \hbar\omega$, then measuring C and getting $c = 1$, i.e. what is $P_{|\Psi(0)\rangle}(E = \hbar\omega, c = 1)$? (1 pt)

d) If the system is allowed to evolve in time after the measurement of H and before C is measured, will your answer to part c) change? Explain your reasoning. (1 pt)

e) With the ensemble of particles all in the original state: $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$, reverse the order of the above measurements and answer the same questions:

i) What is the probability of obtaining $c = 1$ if C is measured first? What state is the particle in after C is measured? (1 pt)

ii) If one immediately measures H after C is measured in part i), what is the probability of obtaining $E = \hbar\omega$? (1 pt) **(question continues on next page...)**

iii) What is the composite probability $P_{|\Psi(0)\rangle}(c = 1, E = \hbar\omega)$? (1 pt)

iv) If the system had been allowed to evolve in time after the measurement of C and before H is measured, would your answer to part ii) be different? Explain. (1 pt)

f) Are H and C compatible observables? Why?

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Quantum #5

$A^2 |\langle \psi | \varphi \rangle|^2$

Given! $H = \hbar\omega \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow$ $|\lambda_H = 2\rangle = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$

$|\lambda_H = 1\rangle = \frac{1}{\sqrt{2}} \langle 1, 0, -1 \rangle$

$|\lambda_H = -1\rangle = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle$

$C = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \Rightarrow$ $|\lambda_C = 1,1\rangle = \frac{1}{\sqrt{2}} \langle 1, 0, -1 \rangle$

$|\lambda_C = 1,2\rangle = \langle 0, 1, 0 \rangle$

$|\lambda_C = -1\rangle = \frac{1}{\sqrt{2}} \langle 1, 0, 1 \rangle$

$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}} \langle 1, 1, 0 \rangle$

a) $P(H=1) = |\langle \lambda_H = 1 | H | \psi(t=0)\rangle|^2$

$= \left| \frac{1}{\sqrt{2}} \langle 1, 0, -1 \rangle \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \right|^2$

$= \frac{1}{4} \left| \langle 1, 0, -1 \rangle \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right|^2$

$= \frac{1}{4} |1|^2 = \frac{1}{4}$

$\Rightarrow$ The state after measurement is $|\lambda_H = 1\rangle = \frac{1}{\sqrt{2}} \langle 1, 0, -1 \rangle$ automatically b/c eigenvalues of H are non-degenerate

b) * From above, we know our starting state is: $|\lambda_H = 1\rangle = \frac{1}{\sqrt{2}} \langle 1, 0, -1 \rangle$

$P(C=1) = |\langle \lambda_C = 1,1 | \lambda_H = 1\rangle|^2 + |\langle \lambda_C = 1,2 | \lambda_H = 1\rangle|^2$

$= \left| \frac{1}{\sqrt{2}} [1 \ 0 \ -1] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \right|^2 + \left| \cdot [0 \ 1 \ 0] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \right|^2$

$= \left| \frac{1}{2} \cdot 2 \right|^2 + \left| \frac{1}{\sqrt{2}} \cdot 0 \right|^2$

$= 1$

c) $P(H=1, C=1) = P(H=1) P(C=1)$

$= \frac{1}{4} \cdot 1$

$= \frac{1}{4}$

#5 (cont.)

d) Evolving the system in time after measuring H will have no impact on the measurement of C b/c the time evolution operator is a function of H and the eigenstates of H are thus stationary states

e) $P(c=1) = |\langle \lambda_c = 1, 1 | \psi(0)\rangle|^2 + |\langle \lambda_c = 1, 2 | \psi(0)\rangle|^2$

$= \left| \frac{1}{\sqrt{2}} [1\ 0\ -1] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \right|^2 + \left| [0\ 1\ 0] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \right|^2$

$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

* Our beginning state is either $|\lambda_c = 1, 1\rangle$ or $|\lambda_c = 1, 2\rangle$

$\Rightarrow P(H=1) = |\langle \lambda_H = 1 | \lambda_c = 1, 1\rangle|^2 + |\langle \lambda_H = 1 | \lambda_c = 1, 2\rangle|^2$

$= \left| \frac{1}{\sqrt{2}} [1\ 0\ -1] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{1}{\sqrt{2}} [1\ 0\ -1] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right|^2$

$= \left| \frac{1}{2} \cdot 2 \right|^2 + 0^2$

$= 1$

$P(c=1, H=1) = P(H=1)P(c=1)$

$= 1 \cdot 3/4$

$= 3/4$

* Allowing the system to evolve in time b/w measuring C and H (in that order) will result in a change in the probability of finding $E = \hbar\omega$ as the two possible eigenstates of $\lambda_c = 1$ not both eigenstates of H, thus they are non-stationary + will be changed after being acted upon by the time evolution operator

f) * Observables are compatible if $[A, B] = 0$, and also if they have a common, complete set of eigenvectors. Since H and C do not share the same eigenbasis, they are not compatible

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Problem 6: The hydrogen atom (10 points)

The figure below shows the radial function $R_{n,\ell}(r)$ for a stationary state of atomic hydrogen. The normalized Hamiltonian eigenfunction for this state, in atomic units, is

$$\psi_{n,\ell,m_\ell}(\boldsymbol{r}) = \frac{1}{81}\sqrt{\frac{2}{\pi}}(6-r)\ e^{-r/3}\ \cos\theta. \tag{1}$$

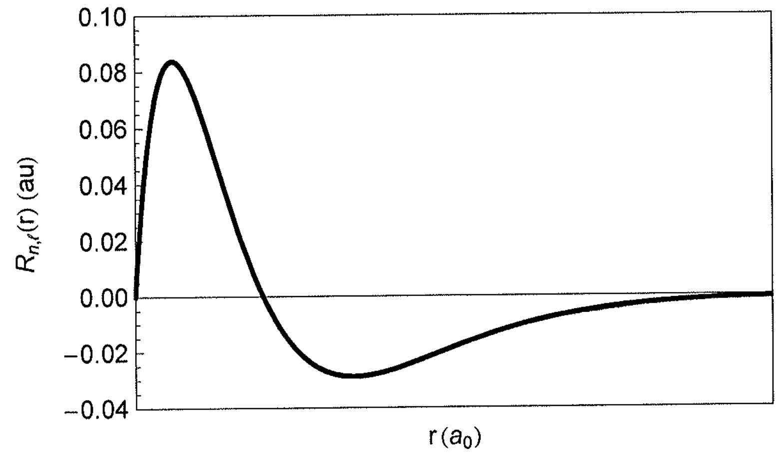

Figure 1: A radial function for a stationary state of atomic hydrogen.

1. **3 points.** What are the values of the quantum numbers n, ℓ, and m_ℓ for this state? To receive any credit, you must fully justify your answer.
2. **1 points.** What is the energy (in eV) of this state?
3. **2 points.** What are the mean value and uncertainty in r (in atomic units) for this state?
4. **2 points.** Calculate the value of r (in atomic units) at which a position measurement would be most likely to find the electron if the atom is in this state.
5. **2 points.** From Eq. 1, generate the normalized eigenfunction $\psi_{n,\ell,m_\ell+1}(\boldsymbol{r})$.

Hint:

$$\int_0^\infty e^{-2r/3} r^n\, dr = n! \left(\frac{3}{2}\right)^{n+1} \tag{2}$$

Hint: *The following table gives the orbital-angular-momentum operators in Cartesian and spherical coordinates.*

Component	Cartesian coordinates	Spherical coordinates
$\widehat{L}_x$	$-\imath\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)$	$\imath\hbar\left(\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\,\frac{\partial}{\partial\varphi}\right)$
$\widehat{L}_y$	$-\imath\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)$	$-\imath\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\sin\varphi\,\frac{\partial}{\partial\varphi}\right)$
$\widehat{L}_z$	$-\imath\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$	$-\imath\hbar\frac{\partial}{\partial\varphi}$
$\widehat{\boldsymbol{L}}^2$	$\widehat{L}_x^2+\widehat{L}_y^2+\widehat{L}_z^2$	$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$

Table 1: Components and square of the orbital angular momentum operator in Cartesian and spherical coordinates.