Quantum Mechanics
Qualifying Exam - August 2017

*Notes and Instructions*

- There are 6 problems. Attempt them all as partial credit will be given.
- Write on only one side of the paper for your solutions.
- Write your alias on the top of every page of your solutions.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

**Possibly useful formulas:**

Spin Operator

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In spherical coordinates,

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial \psi}{\partial r}\right)$$

$$\nabla^2\psi = \cancel{\frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi.$$

Harmonic oscillator wave functions

$$u_0(x) = (\tfrac{m\omega}{\pi\hbar})^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$u_1(x) = (\tfrac{m\omega}{\pi\hbar})^{1/4}\sqrt{\tfrac{2m\omega}{\hbar}}\, x e^{-\frac{m\omega x^2}{2\hbar}}$$

Spherical Harmonics:

$$Y_{0,0}(\theta,\phi) = \tfrac{1}{\sqrt{4\pi}} \qquad Y_{2,0}(\theta,\phi) = \sqrt{\tfrac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$Y_{1,0}(\theta,\phi) = \sqrt{\tfrac{3}{4\pi}}\cos\theta \qquad Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\tfrac{15}{8\pi}}e^{\pm i\phi}\cos\theta\sin\theta$$

$$Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\tfrac{3}{8\pi}}e^{\pm i\phi}\sin\theta \qquad Y_{2,\pm2}(\theta,\phi) = \sqrt{\tfrac{15}{32\pi}}e^{\pm 2i\phi}\sin^2\theta$$

## Problem 1: Periodic Perturbation (10 Points):

Consider a two-level system under a periodic perturbation, $V(t) = V_0 e^{i\omega t}$, where $V_0$ is real. Take the time dependent amplitude for the lower state $|a\rangle$ to be $a(t)$ and the upper state $|b\rangle$ to be $b(t)$. Take the energy of the upper level to be at $\hbar\omega_0$ and the lower level to be at 0.

a. Find differential equations for the time-dependent probability amplitudes to be in the upper state $b(t)$ and the amplitude to be in the lower state $a(t)$. **(3 Points)**

b. Solve the equations you obtained in (a.) for the initial conditions $a(0) = 1$ and $b(0) = 0$. These initial conditions correspond to the system starting in the ground state. Take $\Delta = \omega - \omega_0 = 0$. Use the following unitary transformation to simplify the Hamiltonian you used in (a.) to solve for the time dependent wavefunction:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\omega t} \end{pmatrix}$$

**(3 Points)**

c. Using your result in (b.) find the probability for the system to be in $|b\rangle$. **(2 Points)**

d. Sketch the probability as a function of time that you found in (c.) and interpret the result. **(2 Points)**

## Problem 2: WKB approximation (10 Points):

The one-dimensional Schrodinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

can be rewritten as

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\cancel{2m}\,\hbar^2}\psi\,,$$

where

$$p(x) \equiv \sqrt{2m[E - V(x)]}\,.$$

The wave function $\psi(x)$ is often expressed as $\psi(x) = A(x)e^{i\phi(x)}$ where $A(x)$ is the amplitude and $\phi(x)$ is the phase. Both $A(x)$ and $\phi(x)$ can be real.

(a) Show that the amplitude is $A = \frac{C}{\sqrt{\phi'}}$ where $C$ is a constant and prime is the derivative with respect to $x$. (2 points)

(b) (3 points) Let us assume that $A''/A \ll (\phi')^2$ and $A''/A \ll p^2/\hbar^2$. Show that the wave function in the WKB approximation is

$$\psi(x) \simeq \frac{C}{\sqrt{p(x)}}e^{\pm\frac{i}{\hbar}\int p(x)dx}\,.$$

In parts (c)–(e), the potential energy of the one-dimensional harmonic oscillator is

$$V(x) = \frac{1}{2}m\omega^2x^2\,.$$

(c) Find the classical turning points $x_1 < x_2$ for an energy $E$. (1 points)

(d) Evaluate the phase $\phi$ in terms of $E$ and $\omega$ with the WKB method. (3 points)

(e) Apply the eigenvalue condition $\phi = \left(n + \frac{1}{2}\right)\pi\hbar$ and find energy eigenvalues $E_n$. (1 points)

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# Quantum #2

a) To show $A = \frac{C}{\sqrt{\varphi'}}$, assuming $\psi(x) = A(x)e^{i\varphi(x)}$, we substitute $\psi$ into the rewritten Schrödinger eqn: $\frac{d^2\psi}{dx^2} = \frac{-p^2}{\hbar^2}\psi$, $p = \sqrt{2m(E-V(x))}$

$$\hookrightarrow \frac{d\psi}{dx} = A'e^{i\varphi(x)} + Ai\varphi' e^{i\varphi(x)}$$

$$\frac{d^2\psi}{dx^2} = A''e^{i\varphi(x)} + iA'\varphi' e^{i\varphi(x)} + iA'\varphi' e^{i\varphi(x)} + iA\varphi'' e^{i\varphi(x)} - A(\varphi')^2 e^{i\varphi(x)}$$

$$e^{i\varphi(x)} \cdot \left[A'' - A(\varphi')^2 + i\left[2A'\varphi' + A\varphi''\right]\right] = \frac{-p^2}{\hbar^2}A(x)e^{i\varphi(x)}$$

Real: $\frac{-p^2}{\hbar^2}A = A'' - A(\varphi')^2$

Imaginary: $0 = 2A'\varphi' + A\varphi''$

$$0 = \frac{d}{dx}(A^2\varphi')$$

$\boxed{\begin{array}{l} \frac{d}{dx}(A^2\varphi') = 2AA'\varphi' + A^2\varphi'' = 0 \\ \text{*divide by } A \\ 2A'\varphi' + A\varphi'' = 0 \end{array}}$

$$\hookrightarrow C^2 = A^2\varphi'$$

$$\hookrightarrow A = \frac{C}{\sqrt{\varphi'}} \checkmark$$

b) Assuming $A''/A \ll (\varphi')^2$ and $A''/A \ll p^2/\hbar^2$, we can now use the real equation from part A to show $\psi(x) \approx \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar}\int p(x)\,dx}$

$$\frac{-p^2}{\hbar^2}A = A'' - A(\varphi')^2$$

$$\frac{p^2}{\hbar^2}\frac{\cancel{C}}{\cancel{\sqrt{\varphi'}}} = \frac{\cancel{C}}{\cancel{\sqrt{\varphi'}}}(\varphi')^2$$

$$\varphi' = \pm\frac{p}{\hbar} \Rightarrow \varphi = \pm\frac{1}{\hbar}\int_0^x p(x)\,dx$$

$$\hookrightarrow \psi(x) = Ae^{i\varphi(x)}$$

$$= \frac{C}{\sqrt{\varphi'}} \exp\left[\frac{\pm i}{\hbar}\int p(x)\,dx\right]$$

#2 (cont.)

c) The classical turning points occur when $E = V(x)$

$\hookrightarrow E = \frac{1}{2} m \omega^2 x^2$

$\hookrightarrow x = \pm \sqrt{\frac{2E}{m}}\, \omega$

d) Note: In the region where $E < V$, $p(x)$ is imaginary

$E > V$, $p(x)$ is real

$E = V$ $p(x)$ is $0$

* If $p(x) = 0$, $\psi(x) = 0$

* If $p(x)$ is real $(E < V)$

$$\psi(x) = \int_{x_1}^{x_2} p(x)\,dx$$

$$= \int_{x_1}^{x_2} \left[2m\left(E - \tfrac{1}{2} m\omega^2 x^2\right)\right]^{1/2} dx$$

$$= \int_{x_1}^{x_2} \left(2mE - m^2\omega^2 x^2\right)^{1/2} dx$$

*let $a = \sqrt{2mE}$, $u = m\omega x$

$$= \int_{x_1}^{0} \sqrt{a^2 - u^2}\,\frac{du}{m\omega} + \int_0^{x_2} \sqrt{a^2 - u^2}\, du/m\omega$$

$$= \int_0^{x_2} \frac{1}{m\omega}\sqrt{a^2 - u^2}\,du - \int_0^{x_1} \frac{1}{m\omega}\sqrt{a^2 - u^2}\,du$$

$$= \frac{1}{m\omega}\left[\frac{\pi x_2^2}{4} - \frac{\pi x_1^2}{4}\right]$$

$$= \frac{\pi}{4m\omega}\left[x_2^2 - x_1^2\right]$$

$$= \frac{\pi}{4m\omega^2}$$

See Griffiths QM ex. 8.4

$\frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 0$ $\frac{1}{10}\begin{bmatrix}1 & 3\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix} = \frac{-8}{10} \cdot 1$

## Problem 3: Two-State Problem (10 Points):

$\begin{bmatrix}-3 & 1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = 0$

Consider a two-state quantum system. In the orthonormal and complete set of basis vectors $|1\rangle$ and $|2\rangle$, the Hamiltonian operator for the system is represented by $(\omega > 0)$

$$\hat{H} = 10\hbar\omega|1\rangle\langle 1| - 3\hbar\omega|1\rangle\langle 2| - 3\hbar\omega|2\rangle\langle 1| + 2\hbar\omega|2\rangle\langle 2|$$

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Consider another complete and orthonormal basis $|\alpha\rangle$, $|\beta\rangle$, such that $\hat{H}|\alpha\rangle = E_1|\alpha\rangle$, and $\hat{H}|\beta\rangle = E_2|\beta\rangle$ (with $E_1 < E_2$). Let the action of operator $\hat{A}$ on the $|\alpha\rangle$, $|\beta\rangle$ basis vectors be given as

$$\hat{A}|\alpha\rangle = 2ia_0|\beta\rangle$$
$$\hat{A}|\beta\rangle = -2ia_0|\alpha\rangle - 3a_0|\beta\rangle$$

where $a_0 > 0$ is real.

✓a) Find the eigenvalues and eigenvectors of H in the $|1\rangle, |2\rangle$ basis (1 pt).

✓b) Find the eigenvalues and eigenvectors of $\hat{A}$ in the $|\alpha\rangle, |\beta\rangle$ basis (1 pt).

Suppose a measurement of $\hat{A}$ is carried out at t=0 on an arbitrary state and the largest possible value is obtained.

✓c) Calculate the probability P(t) that another measurement made at time t will yield the value as the one measured at t=0. (2 pts)

✓d) Calculate the time dependence of the expectation value $\langle\hat{A}\rangle$. What is the minimum value of $\langle\hat{A}\rangle$? At what time is the minimum value first achieved? (3 pts)

Now suppose that the average value obtained from a large number of measurements of $\hat{A}$ on identical quantum systems at a given time is $-a_0/4$.

e) (3 pts) Construct the most general normalized state vector (just before the measurement of $\hat{A}$) for your system consistent with this information in Dirac notation using the $|\alpha\rangle$, $|\beta\rangle$ basis. Express your answer as

$$|\Psi\rangle = C|\alpha\rangle + D|\beta\rangle$$

$\begin{bmatrix}1 & \frac{-i}{2}\end{bmatrix}\begin{bmatrix}1\\-\frac{2}{i}\end{bmatrix}$

$1 \cdot 1 + \frac{-i}{2} \cdot \frac{-2}{i}$

$\begin{bmatrix}1 & \frac{1}{2i}\end{bmatrix}\begin{bmatrix}1\\-\frac{2}{i}\end{bmatrix}$



$1 \cdot 1 + \frac{1}{2i} \cdot \frac{-2}{i} = \frac{-1}{i^2}$

$1 \cdot 1 + (2i)(-2i)$

$1 + 4(-i^2) =$

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# Quantum #3

a) $H = 10\hbar\omega|1\rangle\langle 1| - 3\hbar\omega|1\rangle\langle 2| - 3\hbar\omega|2\rangle\langle 1| + 2\hbar\omega|2\rangle\langle 2|$

$$\doteq \begin{array}{c c} & \begin{array}{cc} |1\rangle & |2\rangle \end{array} \\ \begin{array}{c} \langle 1| \\ \langle 2| \end{array} & \begin{bmatrix} 10\hbar\omega & -3\hbar\omega \\ -3\hbar\omega & 2\hbar\omega \end{bmatrix} \end{array}$$

Using the eigenvalue equation $\det(H - \lambda\mathbb{I}) = 0$

$$\begin{vmatrix} 10\hbar\omega - \lambda & -3\hbar\omega \\ -3\hbar\omega & 2\hbar\omega - \lambda \end{vmatrix} = 0 = (10\hbar\omega - \lambda)(2\hbar\omega - \lambda) - (-3\hbar\omega)^2$$

$$= 20\hbar^2\omega^2 - 12\hbar\omega\lambda + \lambda^2 - 9\hbar^2\omega^2$$

$$= \lambda^2 - 12\hbar\omega\lambda + 11\hbar^2\omega^2$$

$$= (\lambda - \hbar\omega)(\lambda - 11\hbar\omega)$$

$\hookrightarrow \lambda_1 = \hbar\omega$

$\lambda_2 = 11\hbar\omega$

Using the eigenvector equation $H\vec{a} = \lambda\vec{a}$

$$\begin{bmatrix} 10\hbar\omega & -3\hbar\omega \\ -3\hbar\omega & 2\hbar\omega \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \begin{array}{l} 10\hbar\omega a_1 - 3\hbar\omega a_2 = \lambda a_1 \\ -3\hbar\omega a_1 + 2\hbar\omega a_2 = \lambda a_2 \end{array}$$

* if $\lambda = \hbar\omega$

$10a_1 - 3a_2 = a_1$

$-3a_1 + 2a_2 = a_2$

$\hookrightarrow a_1 = \frac{1}{3}a_2$

$|\lambda = \hbar\omega\rangle = \langle 1, 3\rangle \frac{1}{\sqrt{10}}$

* if $\lambda = 11\hbar\omega$

$10a_1 - 3a_2 = 11a_1$

$-3a_1 + 2a_2 = 11a_2$

$\hookrightarrow -3a_2 = a_1$

$|\lambda = 11\hbar\omega\rangle = \langle -3, 1\rangle \cdot \frac{1}{\sqrt{10}}$

* Dot product verifies orthogonality $\frac{1}{10}(1\cdot -3 + 3\cdot 1) = 0$

b) $A|\alpha\rangle = 2ia_0|\beta\rangle$

$A|\beta\rangle = -2ia_0|\alpha\rangle - 3a_0|\beta\rangle$

$$\Rightarrow A \doteq \begin{bmatrix} 0 & 2ia_0 \\ -2ia_0 & -3a_0 \end{bmatrix}$$

#3 (cont.)

b) Similarly to part a:

$$\begin{vmatrix} 0-\lambda & 2ia_0 \\ -2ia_0 & -3a_0-\lambda \end{vmatrix} = 0 = -\lambda(-3a_0-\lambda) - (2ia_0)(-2ia_0)$$

$$= \lambda^2 + 3a_0\lambda - 4a_0^2$$

$$= (\lambda+4a_0)(\lambda-a_0)$$

$\Rightarrow \lambda = -4a_0, +a_0$

Using the eigenvector equation:

$$\begin{bmatrix} 0 & 2ia_0 \\ -2ia_0 & -3a_0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \begin{array}{l} 2ia_0a_2 = \lambda a_1 \\ -2ia_0a_1 - 3a_0a_2 = \lambda a_2 \end{array}$$

* if $\lambda = 4a_0$

$2ia_0a_2 = -4a_0a_1$

$-2ia_0a_1 - 3a_0a_2 = -4a_0a_2$

$\hookrightarrow ia_2 = -2a_1$

$-2ia_1 = -a_2$

$\Rightarrow |\lambda = -4a_0\rangle = \langle -i, 2\rangle$

* if $\lambda = a_0$

$2ia_0a_2 = a_0a_1$

$-2ia_0a_1 - 3a_0a_2 = a_0a_2$

$\hookrightarrow 2ia_2 = a_1$

$-ia_1 = 2a_2$

$\Rightarrow |\lambda = a_0\rangle = \langle 2i, 1\rangle$

c) To obtain the largest possible value of $A$ at $t=0$, $|\psi\rangle = |\lambda = a_0\rangle$

But since $U(t,t_0) = \exp[-iHt/\hbar]$, we must convert $|\lambda = a_0\rangle$ to the basis of the Hamiltonian.

$|\lambda_A = a_0\rangle = \frac{1}{\sqrt{5}}\langle 2i, 1\rangle$

Hamiltonian basis vectors: $|\lambda_H = \hbar\omega\rangle = \frac{1}{\sqrt{10}}\langle 1, 3\rangle$

$|\lambda_H = 11\hbar\omega\rangle = \frac{1}{\sqrt{10}}\langle -3, 1\rangle$

$\frac{2i}{\sqrt{5}} = a\frac{1}{\sqrt{10}} + b\frac{-3}{\sqrt{10}}$

$\frac{1}{\sqrt{5}} = a\frac{3}{\sqrt{10}} + b\frac{1}{\sqrt{10}}$

$2\sqrt{2}\,i = a - 3b$

$\sqrt{2} = 3a + b \Rightarrow b = \sqrt{2} - 3a$

$2\sqrt{2}\,i = a - 3(\sqrt{2} - 3a)$

$2\sqrt{2}\,i = 10a - 3\sqrt{2} \Rightarrow a = \frac{3\sqrt{2} - 2\sqrt{2}\,i}{10}$

$b = \frac{\sqrt{2} + 6\sqrt{2}\,i}{10}$

#3 (cont.)

c) $\Rightarrow |\lambda_A = a_0\rangle = \frac{3\sqrt{2} - 2\sqrt{2}i}{10} |\lambda_H = \hbar\omega\rangle + \frac{\sqrt{2} + 6\sqrt{2}i}{10} |\lambda_H = 11\hbar\omega\rangle$

$$|\lambda_A = a_0(t)\rangle = U(t, t_0) |\lambda_A = a_0\rangle$$

$$= \exp[-iHt/\hbar] \left[ \frac{3\sqrt{2} - 2\sqrt{2}i}{10} |\lambda_H = \hbar\omega\rangle + \frac{\sqrt{2} + 6\sqrt{2}i}{10} |\lambda_H = 11\hbar\omega\rangle \right]$$

$$= \exp[-i\omega t] \left( \frac{3\sqrt{2} - 2\sqrt{2}i}{10} \right) |\lambda_H = \hbar\omega\rangle + \exp[-11i\omega t] \left( \frac{\sqrt{2} + 6\sqrt{2}i}{10} \right) |\lambda_H = 11\hbar\omega\rangle$$

$$P(t) = \langle \lambda_a = a_0 | A | \lambda_a = a_0(t) \rangle$$

$$= \left[ \frac{3\sqrt{2} + 2\sqrt{2}i}{10} \langle \lambda_H = \hbar\omega | \right.$$

## Problem 4: Indistinguishable particles (10 Points):

Consider a system of two indistinguishable spin-1/2 particles.

?✓ a) Which of the following two-particle spin states are eigenstates of the operator of the scalar product $\hat{S}_1 \cdot \hat{S}_2$ of the spin vectors? What are their eigenvalues? (1 point)

- $|\uparrow\uparrow\rangle \equiv |\uparrow\rangle \otimes |\uparrow\rangle$
- $|\uparrow\downarrow\rangle \equiv |\uparrow\rangle \otimes |\downarrow\rangle$
- $|\downarrow\uparrow\rangle \equiv |\downarrow\rangle \otimes |\uparrow\rangle$
- $|\downarrow\downarrow\rangle \equiv |\downarrow\rangle \otimes |\downarrow\rangle$

$\left[ \frac{1-i}{\sqrt{2}} \quad 1 \right] \begin{bmatrix} 1 \\ -\frac{\sqrt{2}}{1+i} \end{bmatrix}$

$\frac{1-i}{\sqrt{2}} - \frac{\sqrt{2}}{1+i}\frac{(1-i)}{1-i} = 0$

?✓ b) Show that the states:
$|s_+\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ and $|s_-\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ are eigenstates of $\hat{S}_1 \cdot \hat{S}_2$. What are their eigenvalues? (1 point)

These two particles, separated by a distance $a$, interact with one another via the field of their magnetic dipole moments. This interaction is described by the Hamiltonian

$$\hat{H} = \frac{\mu_0}{4\pi a^3}\left(\hat{m}_{x,1}\hat{m}_{x,2} + \hat{m}_{y,1}\hat{m}_{y,2} - 2\hat{m}_{z,1}\hat{m}_{z,2}\right),$$

where $\hat{m}_j = \gamma \hat{S}_j$ and $\gamma$ is the gyromagnetic ratio of the particles.

?✓ c) Show that the anti-aligned states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are not eigenstates of the Hamiltonian. (1 point)

d) Derive the Hamiltonian in the basis of the anti-aligned states. (2 points)

e) What are the eigenvalues of this Hamiltonian? (1 point)

f) Find a unitary transformation matrix which diagonalizes the Hamiltonian. (2 points)

g) Use this transformation to diagonalize the Hamiltonian. (1 point)

h) What are the eigenstates of the Hamiltonian in this basis? (1 point)

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<u>Quantum #4</u>

a) Given two indistinguishable spin $1/2$ particles,

## Problem 5: Angular Momentum (10 Points):

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Suppose an electron is in a state described by the wave function

$$\psi = \frac{1}{\sqrt{4\pi}}(e^{i\phi}\sin\theta + \cos\theta)g(r)$$

where $\int_0^\infty |g(r)|^2 r^2 dr = 1$
and $\phi$, $\theta$ are the azimuth and polar angles respectively.

✓(a) Express $\psi$ in terms of spherical harmonics functions. (2 pts.)

✓(b) What are the possible results of a measurement of the z-component $L_z$ of the angular momentum of the electron in this state? (2 pts.)

✓(c) Determine if $\int |\psi|^2 \ d^3\vec{r} = 1$. (2 pts.)

✓(d) Use the result in (c) to find the probability of obtaining each of the possible results in part (b). (2 pts.)

✓(e) What is the expectation value of $L_z$? (2 pts.)

$\cos\varphi = \frac{1}{2} e^{i\varphi} + e^{-i\varphi}$

$\sin\varphi = \frac{1}{2i} e^{i\varphi} - e^{-i\varphi}$

$\cos\varphi - i\sin\varphi = e^{-i\varphi}$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

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# Quantum #5

a) $\psi = \frac{1}{\sqrt{4\pi}}\left(e^{i\varphi}\sin\theta + \cos\theta\right)g(r)$

* But we know $Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$

$$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$$

$$\hookrightarrow \psi = \frac{1}{\sqrt{4\pi}}\cdot\left(-\sqrt{\frac{8\pi}{3}}\,Y_{1,1} + \sqrt{\frac{4\pi}{3}}\,Y_{1,0}\right)g(r)$$

$$= \left(\frac{1}{\sqrt{3}}Y_{1,0} - \sqrt{\frac{2}{3}}\,Y_{1,1}\right)g(r)$$

* Check normalization

$$1 = \frac{1}{4\pi}A^2\int_0^\infty r^2dr\int_0^{2\pi}d\varphi\int_0^\pi d\theta\,|g(r)|^2\left(e^{-i\varphi}\sin\theta + \cos\theta\right)\left(e^{i\varphi}\sin\theta + \cos\theta\right)$$

$$1 = \frac{1}{4\pi}A^2\int_0^{2\pi}d\varphi\int_0^\pi \sin\theta\,d\theta\cdot \underbrace{\sin^2\theta + \cos^2\theta}_{1} + e^{-i\varphi}\sin\theta\cos\theta + e^{i\varphi}\sin\theta\cos\theta$$

$$1 = \frac{1}{4\pi}A^2\int_0^{2\pi}d\varphi\int_0^\pi\sin\theta\,d\theta\cdot\left(1 + \sin\theta\cos\theta\left(e^{-i\varphi} + e^{i\varphi}\right)\right)$$

$$1 = \frac{1}{4\pi}A^2\int_0^{2\pi}d\varphi\int_0^\pi \sin\theta + 2\sin^2\theta\cos\theta\cos\varphi\,d\theta$$

$$1 = \frac{1}{4\pi}A^2\int_0^{2\pi}d\varphi\left[-\cos\theta + \frac{2}{3}\sin^3\theta\cos\varphi\right]\Big|_0^\pi$$

$$1 = \frac{1}{4\pi}A^2\int_0^{2\pi}d\varphi\left[(1 + 0) - (-1 + 0\right.$$

$$1 = \frac{1}{2\pi}A^2\int_0^{2\pi}d\varphi$$

$$1 = A^2 \Rightarrow A = 1\ \checkmark$$

b) Rewriting our function in bra-ket notation

$$\psi = \frac{1}{\sqrt{3}}|1,0\rangle - \sqrt{\frac{2}{3}}|1,1\rangle$$

$$\hookrightarrow L_z|\psi\rangle = L_z\cdot\frac{1}{\sqrt{3}}|1,0\rangle - \sqrt{\frac{2}{3}}L_z|1,1\rangle$$

* possible measurements are $L_z = 0, 1$

#5 (cont.)

c) See work from part a checking normalization

d) $\langle \psi | L_z | \psi \rangle = 0 \cdot \frac{1}{3} \cancelto{1}{\langle 1,0 | 1,0 \rangle} + \frac{2}{3} \cancelto{1}{\langle 1,1 | 1,1 \rangle} \cdot 1$ (Other terms ignored due to orthogonality)

↳ $L_z = 0$ $\frac{1}{3}$ of the time

$L_z = 1$ $\frac{2}{3}$ of the time

(Expectation value is weighted sum of possible measurements)

e) $\langle \psi | L_z | \psi \rangle = \frac{2}{3}$

# Problem 6: 3D Square Well (10 Points):

5?

Consider a particle of mass $m$ moving in a 3D spherical well given by the potential

$$V(\vec{r}) = -V_0 \quad 0 \le |\vec{r}| \le a_0, \qquad V(\vec{r}) = 0 \quad |\vec{r}| > a_0$$

where $V_0 > 0$ and $a_0 > 0$.
In this problem, only consider bound states in this well, so $-V_0 < E < 0$.

?✓ (a) (1 pt.) Show that the energy eigenstates for this potential can be written in the form:

$$\Psi_{k,\ell,m}(\vec{r}) = f_{k,\ell}(r) Y_\ell^m(\theta,\phi)$$

$r$, $\theta$, $\phi$ are the usual spherical coordinates. and $Y_\ell^m$ the spherical harmonics.

?✓ (b) (1 pt.) Defining the function $u_{k,\ell}(r) = r f_{k,\ell}(r)$, write the radial Schrodinger equation for $u_{k,\ell}(r)$.

?✓ (c) (2 pts.) Consider the zero angular momentum states, $\ell = 0$. Write down the functional form for the states $u_{k,0}(r)$ in the two regions, $0 \le r \le a_0$ and $r \ge a_0$. Define any constants that you use in these functions.

?✓ (d) (1 pt.) What are the boundary conditions on the functions $u_{k,0}(r)$ as $r \to 0$, at $r = a_0$, and as $r \to \infty$? Hint: Consider the function $f_{k,\ell}(r)$ as $r \to 0$.

(e) (2 pts.) Using your boundary conditions, derive an equation that can be solved to give the bound state energies for the $\ell = 0$ states.

(f) (2 pt.) For a fixed value of the radius of the well, $a_0$, calculate the minimum depth, $V_0 = V_{min}$ for the potential well to have a bound state.

(g) (1 pt.) give a physical reason why there is always a bound state in a symmetric 1D quantum square well, but not in the 3D well studied in this problem.

In spherical coordinates, ( $L^2$ is the usual angular momentum operator)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\psi\right)$$

$$\nabla^2\psi(\vec{r}) = \cancel{\frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi(\vec{r})} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi(\vec{r})}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi(\vec{r})$$

$$\nabla^2\psi(\vec{r}) = \cancel{\frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi(\vec{r})} - \frac{L^2}{\hbar^2 r^2}\psi(\vec{r})$$

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# Quantum #6

a) $V(r) = \begin{cases} -V_0 & 0 \leq r \leq a_0 \\ 0 & r > a_0 \end{cases}$

* Note: We only consider the bound region

$$\frac{-\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi$$

$$\hookrightarrow \frac{-\hbar^2}{2m}\left[ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 \psi}{\partial \varphi^2}\right)\right] + V(r)\psi = E\psi$$

* Assuming a solution of the form $\psi = f_{k,\ell}(r)\, Y_\ell^m(\theta,\varphi)$

$$\frac{-\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)\cdot Y + \frac{f}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{f}{r^2\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\varphi^2}\right)\right] + VfY = EfY$$

* multiplying by $\frac{-2mr^2}{fY\hbar^2}$ yields

$$\frac{1}{f}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - \frac{2mr^2 f(V-E)}{\hbar^2}\right) + \frac{1}{Y}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right) = 0$$

* Note: Our assumption of a seperable solution works where the angular equation can be solved to find that $Y_\ell^m(\theta,\varphi)$ is the solution we expect (ie spherical harmonics)

b) Defining $u_{k,\ell}(r) = r f_{k,\ell}(r) \Rightarrow f_{k,\ell} = \frac{u_{k,\ell}(r)}{r}$, the Schrödinger eqn becomes:

$$\frac{1}{f}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - \frac{2mr^2 f(V-E)}{\hbar^2}\right) = \ell(\ell+1)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - \frac{2mr^2 f(V-E)}{\hbar^2} = \ell(\ell+1)f$$

$$\frac{\partial}{\partial r}\left(r^2\frac{d}{dr}\left(\frac{u}{r}\right)\right) - \frac{2mr\, u_{k,\ell}(r)(V-E)}{\hbar^2} = \ell(\ell+1)\frac{u(r)}{r}$$

$$\frac{\partial}{\partial r}\left(\cancel{r^2}\left[\frac{r\,du/dr - u}{\cancel{r^2}}\right]\right) - \frac{2mr\,u(V-E)}{\hbar^2} = \frac{\ell(\ell+1)u}{r}$$

$$\frac{\partial}{\partial r}\left(r\frac{du}{dr} - u\right) - \frac{2mr\,u(V-E)}{\hbar^2} = \frac{\ell(\ell+1)u}{r}$$

$$\left[\frac{du}{dr} + r\frac{d^2u}{dr^2} - \frac{du}{dr}\right] - \frac{2mru(V-E)}{\hbar^2} = \frac{\ell(\ell+1)u}{r}$$

$$\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + \left(V + \frac{\hbar^2\ell(\ell+1)}{2m}\right)u = Eu$$

#6 (cont)

c) If we only consider $l = 0$, our equation becomes:

$$\frac{-\hbar^2}{2m}\frac{d^2u}{dr} + Vu = Eu$$

$$\hookrightarrow \frac{d^2u}{dr^2} = \frac{-2mE-V}{\hbar^2} u$$

let $k = \frac{\sqrt{2m(E+V_0)}}{\hbar}$ where $0 \le r \le a_0$

$$\frac{d^2u}{dr^2} = -k^2 u \Rightarrow u = Ae^{ikr} + Be^{-ikr} = C\sin(kr) + D\cos(kr)$$

let $\kappa = \frac{\sqrt{-2mE}}{\hbar}$ where $r > a_0$

$$\frac{d^2u}{dr^2} = \kappa^2 u \Rightarrow u = Ae^{\kappa r} + Be^{-\kappa r}$$

d) Our function must go to 0 at $r=0$ and $r=\infty$

$$\hookrightarrow f = \frac{u}{r} = C\frac{\sin(kr)}{r} + D\frac{\cos(kr)}{r}$$

(Inside well)

$\sin(0) = 0$

$\cos(0) = 1 \Rightarrow D = 0$

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$$f = \frac{u}{r} = A\frac{1}{r}e^{\kappa r} + B\frac{1}{r}e^{-\kappa r}$$

$e^{\kappa\infty} = \infty \Rightarrow A = 0$

$e^{-\kappa\infty} = 0$

e) Therefore, inside the well (bound states), it must be true that

$$\sin(kr) = 0 \Rightarrow kr = n\pi$$

$$\frac{\sqrt{2m(E+V_0)}\,r}{\hbar} = n\pi$$

$$2m(E+V_0) = \frac{n^2\pi^2\hbar^2}{r^2}$$

$$E = \frac{n^2\pi^2\hbar^2}{2mr^2} - V_0$$

#6 (cont.)

f) Assuming $a_0$ is fixed, we need to find the minimum depth for a bound state

↳ $E > 0 \Rightarrow \frac{n^2\pi^2\hbar^2}{2mr^2} > V_{min}$

* potential error in definitions of k an 𝒦

$$\frac{n^2\pi^2\hbar^2}{2ma_0^2} = V_{min}$$