Quantum Mechanics
Qualifying Exam - January 2014

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write your alias (not your name) on the top of every page of your solutions.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, Page 2 of 4, is the second of four pages for the solution to problem 3.)
- You must show all your work to receive full credit.

Possibly useful formulas:

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Laplacian in spherical coordinates

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2} r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi.$$

One dimensional simple harmonic oscillator operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \qquad P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$\begin{aligned}
Y_0^0(\theta,\phi) &= \frac{1}{\sqrt{4\pi}} \\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_1^{\pm 1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta \mathrm{e}^{\pm i\phi} \\
Y_2^0(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \\
Y_2^{\pm 1}(\theta,\phi) &= \mp\sqrt{\frac{15}{8\pi}}(\sin\theta\cos\theta)\,\mathbf{e}^{\pm i\phi} \\
Y_2^{\pm 2}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\sin^2\theta\,\mathbf{e}^{\pm 2i\phi}
\end{aligned}$$

PROBLEM 1: Rigid Rotator

A free molecule of NaCl can be approximated as a dumbell, or rigid rotator. Attach a reference frame to its center of mass, with z-axis oriented in an arbitrary direction. The Hamiltonian can be taken to be $H = \frac{L^2}{2I}$ where $\vec{L}$ is angular momentum and I is the (fixed) moment of inertia.

a) Write the Schroedinger equation for the molecule. (1 Point)

b) What are the energy eigenvalues? (2 points)

c) What are the steady-state eigenfunctions? (2 points)

d) Sketch an energy level diagram for the rotator. Note any possible degeneracies. (2 points)

e) The rotator, with electric dipole moment $\vec{D}$ oriented along the dumbell symmetry axis, is placed in an electric field $\vec{E} = E\hat{z}$. The dipole interaction is $H_D = -\vec{D} \cdot \vec{E}$. What is the first order perturbative correction to the lowest energy level? (3 points)

PROBLEM 2: Particle in a Box

A particle of mass m is in the ground state of a one dimension box of length L. At $t = 0$, the box suddenly expands *symmetrically* to *three* times its size, leaving the wavefunction of the particle undisturbed. Assume the particle was in the ground state before the expansion.

a) Solve the Schrodinger equation and calculate the eigenenergies and eigenfunctions in the box before and *after* the expansion (show all your work). (3 Points)

b) What is the probability of finding the particle in the ground state immediately after the expansion? (4 Points)

c) Compute the wave function of the particle $\psi(x,t)$ for $t \geq 0$. Hint: express your answer as a superposition of eigenstates. (3 Points)

Hint: $\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \, \cos\theta \cos(q\theta) = \frac{2}{1-q^2} \cos\left(q\frac{\pi}{2}\right)$,

$\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \, \cos\theta \sin(q\theta) = 0.$

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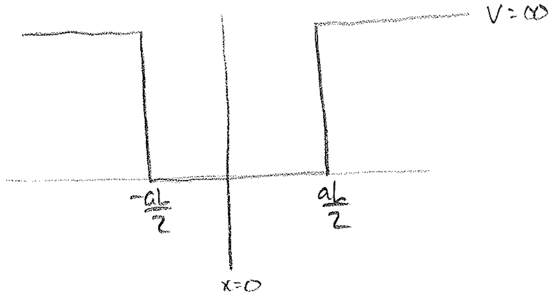
Quantum #2

* Due to symmetric expansion, we choose our edges to by symmetric about $x=0$

a)



$H\psi = E\psi$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

$$= Ae^{ikx} + Be^{-ikx}$$

$$= A\sin(kx) + B\cos(kx)$$

* Solving for our boundary conditions, we know $\psi\left(\frac{-aL}{2}\right) = 0 = \psi\left(\frac{aL}{2}\right)$

$$\pm\frac{kaL}{2} = \frac{n\pi}{2} \quad \Rightarrow \quad k = \frac{n\pi}{aL}$$

↳ if n = even, $\sin(kL) = 0 \rightarrow B = 0$

n = odd, $\cos(kL) = 0 \rightarrow A = 0$

$$\Rightarrow \psi(x) = \begin{cases} A\sin\left(\frac{n\pi x}{aL}\right) & n \text{ even} \\ B\cos\left(\frac{n\pi x}{aL}\right) & n \text{ odd} \end{cases}$$

* Checking normalization

$$1 = A^2\int_{-\frac{aL}{2}}^{aL/2} \sin^2\left(\frac{n\pi x}{aL}\right)dx$$

$$= \frac{A^2}{2}\int_{-aL/2}^{aL/2}\left(1 - \cos\left(\frac{2n\pi x}{aL}\right)\right)dx$$

$$= \frac{A^2}{2}\left[x - \frac{aL}{2n\pi}\sin\left(\frac{2n\pi x}{aL}\right)\right]\Bigg|_{-aL/2}^{aL/2}$$

#2 (cont.)

a) $1 = \frac{A^2}{2}(aL - 0)$ b/c n = even, sin → 0

$A^2 = \frac{2}{aL} \rightarrow A = \sqrt{\frac{2}{aL}}$

$1 = B^2 \int_{-aL/2}^{aL/2} \cos^2\left(\frac{n\pi x}{aL}\right) dx$

$1 = \frac{B^2}{2} \int_{-aL/2}^{aL/2} \left(1 + \cos\left(\frac{2n\pi x}{aL}\right)\right) dx$

$1 = \frac{B^2}{2} \left[x + \sin\left(\frac{2n\pi x}{aL}\right) \cdot \frac{La}{2n\pi} \right] \Big|_{-aL/2}^{aL/2}$

$1 = \frac{B^2}{2}[aL + 0]$ b/c n = odd, cos → 0

$B^2 = \frac{2}{aL} \rightarrow B = \sqrt{\frac{2}{aL}}$

$$\Rightarrow \psi(x) = \begin{cases} \sqrt{\frac{2}{aL}} \sin\left(\frac{n\pi x}{aL}\right) & n = \text{even} \\ \sqrt{\frac{2}{aL}} \cos\left(\frac{n\pi x}{aL}\right) & n = \text{odd} \end{cases}$$

* To determine energies

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{aL} \Rightarrow E = \frac{n^2\pi^2\hbar^2}{2ma^2L^2}$$

* Pre-expansion, $a = 1$

$$\hookrightarrow \psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & n = \text{even} \\ \sqrt{2/L} \cos\left(\frac{n\pi x}{L}\right) & n = \text{odd} \end{cases} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

* Post-expansion, $a = 3$

$$\hookrightarrow \psi_m(x) = \begin{cases} \sqrt{2/3L} \sin\left(\frac{n\pi x}{3L}\right) & n = \text{even} \\ \sqrt{2/3L} \cos\left(\frac{n\pi x}{3L}\right) & n = \text{odd} \end{cases} \qquad E_m = \frac{m^2\pi^2\hbar^2}{18mL^2}$$

#2 (cont.)

b) * Both before and after expansion, the ground state corresponds to n=1

$$P = \left| \langle \psi_{m=1} | \psi_{n=1} \rangle \right|^2$$

$$= \left| \int \psi^*_{m=1} \psi_{n=1} \, dx \right|^2$$

$$= \left| \int_{-L/2}^{L/2} \cos\left(\frac{\pi x}{3L}\right) \cos\left(\frac{\pi x}{L}\right) dx \cdot \frac{2}{L\sqrt{3}} \right.$$

$$= \left| \frac{1}{L\sqrt{3}} \int_{-L/2}^{L/2} \cos\left(\frac{\pi x}{3L} - \frac{\pi x}{L}\right) + \cos\left(\frac{\pi x}{3L} + \frac{\pi x}{L}\right) dx \right|^2$$

$$= \left| \frac{1}{L\sqrt{3}} \int_{-L/2}^{L/2} \cos\left(-\frac{2\pi x}{3L}\right) + \cos\left(\frac{4\pi x}{3L}\right) dx \right|^2$$

$$= \left| \frac{1}{L\sqrt{3}} \left[\frac{3L}{-2\pi} \sin\left(\frac{-2\pi x}{3L}\right) + \frac{3L}{4\pi} \sin\left(\frac{4\pi x}{3L}\right) \right] \Big|_{-L/2}^{L/2} \right|^2$$

$$= \left| \frac{\sqrt{3}}{\pi} \left[-\frac{1}{2}\left(\sin\left(-\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right) + \frac{1}{4}\left(\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{-2\pi}{3}\right)\right) \right] \right|^2$$

$$= \left| \frac{\sqrt{3}}{\pi} \left[\sin(\pi/3) + \frac{1}{2} \sin(2\pi/3) \right] \right|^2$$

$$= \left| \frac{\sqrt{3}}{\pi} \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) \right) \right|^2$$

$$= \left| \frac{\sqrt{3}}{\pi} \left(\frac{3\sqrt{3}}{4} \right) \right|^2$$

$$= \left| \frac{9}{4\pi} \right|^2$$

$$= \frac{81}{16\pi^2}$$

#2 (cont.)

c) $|\psi_n(t)\rangle = e^{-iHt/\hbar}|\psi_n\rangle$

* to write this as an expansion of eigenstates

$$\sum_m' |\psi_m\rangle\langle\psi_m|\psi_n\rangle = \sum_m' c_m |\psi_m\rangle$$

In integral form

$$c_m = \int \psi_m^* \psi_n \, dx$$

$$\hookrightarrow \psi_n = \sum_m' \left(\int \psi_m^* \psi_n \, dx\right) \psi_m(x)$$

$$\psi_n(t) = \sum_m' \int \psi_m^* \psi_n \, dx \, e^{-iHt/\hbar} \psi_m(x)$$

$$= \sum_m' \int \psi_m^* \psi_n \, dx \, e^{-iE_m t/\hbar} \psi_m(x)$$

PROBLEM 3: Matrix Mechanics

Let A, B and C be three ensembles that are represented in the orthonormal basis $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$,

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of A are doubly degenerated, $a = 1,\ 1,\ -1$, with eigenvectors

$$|a = 1, 1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad |a = 1, 2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |a = -1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The eigenvalues of C are also doubly degenerate, $c = 2,\ 1,\ 1$, with eigenvectors:

$$|c = 2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |c = 1, 1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |c = 1, 2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Assume that all particles in the ensemble are in the state $|\psi\rangle$,

$$|\psi\rangle = \frac{1}{2}|e_1\rangle - \frac{1}{2}|e_2\rangle + \frac{1}{\sqrt{2}}|e_3\rangle.$$

Answer the following questions:

a) Find the probability of measuring C and obtaining a value $c = 2$; then immediately measuring A and getting $a = 1$, *i.e.* find $P_{|\psi\rangle}(c = 2, a = 1)$. Identify the intermediate state $|\psi'\rangle$ after C is measured. (2 Points)

b) Now find the probability if those measurements are performed in the reverse order, *i.e.*, find $P_{|\psi\rangle}(a = 1, c = 2)$. Identify the intermediate state $|\psi''\rangle$ after A is measured. (2 Points)

c) Compare the results of parts a) and b) and explain why this happened. (1 Point)

d) If you are told that the eigenvalues of B are $b = -2, -2, 4$, justify whether or not the following 2 probabilities $P_{|\psi\rangle}(a = -1, b = 4)$ and $P_{|\psi\rangle}(b = 4, a = -1)$ will be equal (do NOT explicitly calculate the probabilities). Will the final states be the same or different? Explain. (2 Points)

e) Does $\{A, B\}$ constitute a complete set of commuting observables? Demonstrate explicitly. (3 Points)

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Quantum #3

a) * With the initial state $|\psi\rangle = \frac{1}{2}|e_1\rangle - \frac{1}{2}|e_2\rangle + \frac{1}{\sqrt{2}}|e_3\rangle$

$\Rightarrow$ Probability of measuring $C=2$

$\hookrightarrow$ $|\langle c=2|C|\psi\rangle|^2$

$\Rightarrow$ Immediately after, probability of measuring

$\hookrightarrow$ $|\langle a=1,1|A|c=2\rangle|^2 + |\langle a=1,2|A|c=2\rangle|$

* State is $|c=2\rangle$ immediately after first measurement, need both terms above b/c $a=1$ is doubly degenerate eigenvalue

$$\Rightarrow P_{|\psi\rangle}(c=2, a=1) = \left(|\langle a=1,1|A|c=2\rangle|^2 + |\langle a=1,2|A|c=2\rangle|\right)^2 |\langle c=2|C|\psi\rangle|^2$$

$$= \left[\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)^2 + \left(\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)^2\right]\left(\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/2\\-1/2\\1/\sqrt{2}\end{bmatrix}\right)$$

$$= \left[\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)^2 + \left(\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)^2\right]\left(\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}1\\-1/2\\1/\sqrt{2}\end{bmatrix}\right)^2$$

$$= \left(\frac{1}{2} + 0\right)(1)$$

$$= \frac{1}{2}$$

b) * Proceeding in a similar manner as above:

$$P_{|\psi\rangle}(a=1, c=2) = \left(|\langle c=2|C|a=1,1\rangle|^2 + \langle a=1,1|A|\psi\rangle|^2\right) + \left(|\langle c=2|C|a=1,2\rangle|^2 |\langle a=1,2|A|\psi\rangle|^2\right)$$

$$= \left[\left(\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\frac{1}{\sqrt{2}}\right)^2\left(\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\frac{1}{\sqrt{2}}\right)^2\right] + \left[\left(\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right)^2\left(\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right.\right.$$

$$= \left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + (0)(0)\right]$$

$$= \frac{1}{4}$$

c) $$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&1&0\\2&0&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&2&0\\1&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$$

$\hookrightarrow$ non-comuting observables, thus order of observation matters

#1 (cont.)

d) Solving for the eigenvectors of B we see:

$B\vec{x} = \lambda\vec{x}$

Case $\lambda = 4$

$x_1 - 3x_2 = 4x_1 \rightarrow -x_2 = x_1$

$-3x_1 + x_2 = 4x_2 \rightarrow -x_1 = x_2$

$-2x_3 = 4x_3$

$\Rightarrow x_3 = 0$

$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$

Case $\lambda = -2$

$x_1 - 3x_2 = -2x_1$

$-3x_1 + x_2 = -2x_2$

$-2x_3 = -2x_3$

$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$

$\Rightarrow$ Same set of eigenvectors indicates commuting observables, and since both states under consideration have the same corresponding eigenvector, the eigenvalues are simultaneous. This will result in no difference in probability based on the order of observation and in both cases the particle will be in the same final state.

e) The criteria for $\{A, B\}$ to be a complete set of commuting observables is:

① All the observables commute in pairs

② If we specify the eigenvalues of all operators in the set, we identify a unique eigenvector in the Hilbert space

$[A,B] = AB - BA$

$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$= 0$ ✓ (Condition 1 satisfied)

$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = |a = 1, b = -2\rangle$ $\quad \vec{x} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = |a = -1, b = 4\rangle$ $\quad \vec{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = |a = -1, b = -2\rangle$

↳ Condition 2 satisfied

Problem 4: Clebsh-Gordon Coefficients

Consider a system with two distinguishable spinless particles with angular momentum $j_1 = 1$ and $j_2 = 1$. Suppose the system is prepared in a state with total angular momentum $j = 2$ and total angular momentum projection $m = m_1 + m_2 = 0$. The state in the total j basis $|j_1, j_2; j, m\rangle$ is

$$|\psi\rangle \equiv |1, 1; j = 2, m = 0\rangle.$$

a) Express $|\psi\rangle$ in terms of products of single particle states, namely in the direct product basis $|j_1 = 1, m_1\rangle|j_2 = 1, m_2\rangle$. (4 Points).

b) If the angular momentum projection of particle 1 is measured along the z direction, what is the probability of finding a non-zero result? (2 Points)

c) If $\mathbf{J}_i$ is the angular momentum operator of each particle ($i = 1, 2$), compute the expectation value of $\mathbf{J}_1 \cdot \mathbf{J}_2$ in the $|\psi\rangle$ state. (2 Points)

d) If the $|\psi\rangle$ state is rotated by an infinitesimal angle $\delta\theta$ around the x direction, compute the probability of measuring the $|1, 1; j = 2, m = 1\rangle$ state in leading order in $\delta\theta$. (2 Points)

Raising and lowering angular momentum operators:

$$J_\pm|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

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Quantum #4

* Two distinguishable spinless particles,

a) with both particles having $j_i = 1$, $m_i = \{-1, 0, 1\}$

$$\hookrightarrow |1,1, j=2, m=0\rangle = A|1, m_1=0\rangle \otimes |1, m_2=0\rangle + B|1, m_1=1\rangle \otimes |1, m_1=-1\rangle + C|1, m_1=-1\rangle \otimes |1, m_2=1\rangle$$

* To determine the values of A, B, C (the Clebsh-Gordon) coefficients, we must start in either the highest or lowest state for two distinguishable spinless particles and use raising/lowering operators to reach our known state

$$J_- = J_{1,-} + J_{2,-} = J_{1,-} \otimes \mathbb{I}_2 + \mathbb{I}_1 \otimes J_{2,-}$$

$$J_- |a,b\rangle = \hbar\sqrt{(j+m)(j-m+1)}\,|a, b-\hbar\rangle$$

* Our maximum state is: $|1,1,2,2\rangle = |1,1\rangle \otimes |1,1\rangle$, $\hbar = 1$

$$J_- |1,1;2,2\rangle = |1,1;2,1\rangle \cdot \sqrt{(2+2)(2-2+1)}$$

$$= 2|1,1;2,1\rangle$$

$$J_- |1,1\rangle \otimes |1,1\rangle = \sqrt{2}\,\hbar|1,0\rangle \otimes |1,1\rangle + \sqrt{2}\,|1,1\rangle \otimes |1,0\rangle$$

$$\Rightarrow |1,1;2,1\rangle = \frac{1}{\sqrt{2}}\left[|1,0\rangle \otimes |1,1\rangle\right] + \frac{1}{\sqrt{2}}\left[|1,1\rangle \otimes |1,0\rangle\right]$$

* Repeating

$$J_- |1,1;2,1\rangle = \sqrt{(2+1)(2-1+1)}\,|1,1;2,0\rangle$$

$$= \sqrt{6}\,|1,1;2,0\rangle$$

$$J_-\left[\frac{1}{\sqrt{2}}(|1,0\rangle \otimes |1,1\rangle) + \frac{1}{\sqrt{2}}(|1,1\rangle \otimes |1,0\rangle)\right] = \frac{1}{\sqrt{2}}\Big(\sqrt{(1+0)(1-0+1)}\,|1,-1\rangle \otimes |1,1\rangle + \sqrt{(1+1)(1-1+1)}\,|1,0\rangle \otimes |1,0\rangle + \sqrt{(1+1)(1-1+1)}\,|1,0\rangle \otimes |1,0\rangle + \sqrt{(1+0)(1-0+1)}\,|1,1\rangle \otimes |1,-1\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{2}\,|1,-1\rangle \otimes |1,1\rangle + 2\sqrt{2}\,|1,0\rangle \otimes |1,0\rangle + \sqrt{2}\,|1,1\rangle \otimes |1,-1\rangle\right)$$

#4 (cont.)

a) $\Rightarrow \quad |1,1;2,0\rangle = \frac{1}{\sqrt{6}}|1,-1\rangle \otimes |1,1\rangle + \sqrt{\frac{2}{3}}\,|1,0\rangle \otimes |1,0\rangle + \frac{1}{\sqrt{6}}|1,1\rangle \otimes |1,1\rangle$

b) $J_{z,1}|1,1;2,0\rangle = J_{z,1}\left[\frac{1}{\sqrt{6}}|1,-1\rangle \otimes |1,1\rangle + \sqrt{\frac{2}{3}}|1,0\rangle \otimes |1,0\rangle + \frac{1}{\sqrt{6}}|1,1\rangle \otimes |1,1\rangle\right]$

$$P(J_{z,1} \neq 0) = \langle 1,1;2,0|\,J_{z,1}\,|1,1;2,0\rangle$$

$$= \sum |c_n|^2$$

$$= \frac{1}{6}(-1) + \frac{1}{6}(1) + \frac{2}{3}(0)$$

$\hookrightarrow \frac{1}{3}$ overall, $\frac{1}{6}$ each for $J_{z,1} = \pm 1$

c) $J_1 \cdot J_2 = \left(J^2 - J_1^2 + J_2^2\right) \cdot \frac{1}{2}$

$\hookrightarrow$ From $J^2 = (J_1 + J_2)\cdot(J_1 + J_2)$

$J^2 = J_1^2 + 2J_1 \cdot J_2 + J_2^2$

$$\langle J_1 \cdot J_2 \rangle = \langle 1,1;2,0|\,J_1 \cdot J_2\,|1,1;2,0\rangle$$

$$= \langle 1,1;2,0|\,\tfrac{1}{2}(J^2 - J_1^2 - J_2^2)\,|1,1;2,0\rangle$$

$$= \tfrac{1}{2}(2^2 - 1^2 - 1^2)$$

$$= \tfrac{1}{2}(4 - 1 - 1)$$

$$= 1$$

d)

PROBLEM 5: Zeeman Field

Consider the eight $n = 2$ states of Hydrogen. This problem is on the *strong* field Zeeman effect with spin-orbit interaction. Assume that the constant magnetic field B lies along the z-direction. The spin orbit coupling term is

$$H_{SO} = \frac{1}{2m_l^2 c^2} \frac{1}{r} \frac{\mathrm{d}V}{\mathrm{d}r} \mathbf{L} \cdot \mathbf{S},$$

where $V(r)$ is the Coulomb potential, c is the speed of light and m_l is the angular momentum projection quantum number. Remember:

$$\langle n, l, m_l | \frac{1}{r^3} | n, l, m_l \rangle = \frac{1}{a_0^3 n^3 l(l + \frac{1}{2})(l + 1)}$$

for $l \neq 0$.

a) Find a general expression for the energy due to the spin-orbit term in the physical limit of strong magnetic field, where the strong field Zeeman splitting expressions are valid. Express your answer in terms of the good quantum numbers in this problem. Recall that because of the strong magnetic field, the good quantum numbers in this regime are n, l, m_l and m_s and not j and m_j. (Hint: compute $\langle H_{SO} \rangle$ in the proper basis) (3 Points)

b) Explicitly write down the quantum numbers for all eight $n = 2$ states. Find the energy of each state under strong field Zeeman splitting. Express the energy of each state as the sum of 3 terms: the Bohr energy, the spin-orbit interaction, and the Zeeman contribution. (4 Points)

c) If you ignore the spin-orbit interaction, how many distinct energy levels are there and what are their degeneracies? (3 Points)

* Modified version of Sakurai 5.4

Problem 6: Perturbation Theory

An isotropic Harmonic oscillator in two dimensions has the Hamiltonian

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2),$$

where x and y are position operators in Cartesian coordinates x and y.

a) What is the energy of the *three* lowest energy levels and their respective degeneracies? (2 Points)

b) Consider a perturbative potential of the form:

$$V(x, y) = Am\omega^2 xy.$$

Compute the energy correction of the lowest level in the lowest order in perturbation theory where the result is non-zero. (3 Points)

c) Compute the energy splitting of the first excited energy level (which is degenerate), due to the perturbation. Compute the split ket states in terms of the original unperturbed kets. (3 Points)

d) Suppose that there are three indistinguishable spin 1/2 particles in the system. Compute the total energy of the ground state in first order in perturbation theory. (2 Points)

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Quantum #6

a) The energies of the isotropic oscillator are the sum of 2 1-D harmonic oscillators as the differential equation will be seperable by $\Psi = X(x)\,Y(y)$

↳ $E_n = (n + 1/2)\hbar\omega$ in 1-D SHO

$\Rightarrow E_n = (n_x + n_y + 1/2)\hbar\omega$ in 2-D isotropic HO

Our 3 lowest levels are: $\frac{1}{2}\hbar\omega$ – $n_x = n_y = 0$

$\frac{3}{2}\hbar\omega$ – $(n_x = 1,\ n_y = 0),\ (n_x = 0,\ n_y = 1)$

$\frac{5}{2}\hbar\omega$ – $(n_x = 2,\ n_y = 0),\ (n_x = 1,\ n_y = 1),\ (n_x = 0, n_y = 2)$

b) $V'(x) = Am\omega^2 xy$

* We want lowest level $(n_x = n_y = 0)$, non-zero energy perturbation

↳ $\Delta E^{(1)}_{0,0} = \langle \Psi_{00} | V' | \Psi_{00} \rangle$

$= \langle 0,0 | Am\omega^2 xy | 0,0 \rangle$

$= Am\omega^2 \langle 0,0 | xy | 0,0 \rangle$

* Note that x, y can be written in terms of raising/lowering operators where:

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$= Am\omega^2 \cdot \frac{\hbar}{2m\omega} \langle 0,0 | (a_x + a_x^+)(a_y + a_y^+) | 0,0 \rangle$

$= \frac{A\hbar\omega}{2} \langle 0,0 | a_x a_y + a_x^+ a_y + a_x a_y^+ + a_x^+ a_y^+ | 0,0 \rangle$

$= \frac{A\hbar\omega}{2}\left[0 \xcancel{\langle 0,0|0,-1\rangle}^{\,0} + 0\xcancel{\langle 0,0|1,-1\rangle}^{\,0} + 0\xcancel{\langle 0,0|-1,1\rangle}^{\,0} + 1\xcancel{\langle 0,0|1,1\rangle}^{\,0}\right]$

* Note: First 3 terms physically impossible

$= 0$

#6 (cont.)

b) $$\Delta E^{(2)}_{0,0} = \sum_{k\neq n} \frac{|\langle k|V'|n\rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$= \sum_{k\neq n}' \frac{|\langle k_x k_y | \frac{A\hbar\omega}{2} a_x^\dagger a_y^\dagger |0,0\rangle|^2}{\frac{1}{2}\hbar\omega - E_k^{(0)}}$$

$$= \sum_{k\neq n}' \frac{|\langle k_x k_y | \frac{A\hbar\omega}{2} |1,1\rangle|^2}{\frac{1}{2}\hbar\omega - E_k^{(0)}}$$

* Numerator $\neq 0$ only if $k_x = k_y = 1$ by orthogonality

$$= \frac{A^2\hbar^2\omega^2/4}{\frac{1}{2}\hbar\omega - \frac{3}{2}\hbar\omega}$$

$$= -\frac{A^2\hbar\omega}{8}$$

c) This problem can be achieved by diagonalizing the perturbation matrix for the first excited state

* Remember $E_1 = \frac{3}{2}\hbar\omega$, $(n_x = 1, n_y = 0)$ or $(n_x = 0, n_y = 1)$

$$V' = \begin{array}{c|cc} & |1,0\rangle & |0,1\rangle \\ \hline |1,0\rangle & 0 & 1 \\ |0,1\rangle & 1 & 0 \end{array} \cdot \frac{A\hbar\omega}{2}$$

$$xy = a_x a_y + a_x^\dagger a_y + a_x a_y^\dagger + a_x^\dagger a_y^\dagger$$

*To find the energy corrections, we find the eigenvalues

$$\begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0 = \lambda^2 - 1$$
$$= (\lambda+1)(\lambda-1)$$

$$\hookrightarrow \lambda = \pm 1$$

*To find the states that correspond to these energy corrections, we find the eigenvectors by $V\vec{a} = \lambda\vec{a}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \begin{matrix} a_2 = \lambda a_1 \\ a_1 = \lambda a_2 \end{matrix}$$

#6 (cont.)

c) * for $\lambda = 1$

$$\left.\begin{array}{l} a_2 = a_1 \\ a_1 = a_2 \end{array}\right\} \vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

* for $\lambda = -1$

$$\left.\begin{array}{l} a_2 = -a_1 \\ a_1 = -a_2 \end{array}\right\} \vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

Orthogonality check: $\vec{a}_1 \cdot \vec{a}_{-1} = 1 + -1 = 0$ ✓

$\hookrightarrow$ $\Delta E_1^{(1)} = \frac{A\hbar\omega}{2}$, $\quad |\psi\rangle = \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{\sqrt{2}}|0,1\rangle$

$\Delta E_2^{(1)} = -\frac{A\hbar\omega}{2}$, $\quad |\psi\rangle = \frac{1}{\sqrt{2}}|1,0\rangle - \frac{1}{\sqrt{2}}|0,1\rangle$

d)