F2014

PROBLEM 1: Stationary and Non-Statonary States

Consider a quantum system whose particles are in the following state:

$$\Psi(x,t) = \frac{1}{\sqrt{8}}\psi_1(x)\mathrm{e}^{-iE_1t/\hbar} - i\sqrt{\frac{3}{8}}\psi_3(x)\mathrm{e}^{-iE_3t/\hbar} + \frac{1}{\sqrt{2}}\psi_5(x)\mathrm{e}^{-iE_5t/\hbar}, \tag{1}$$

where $\psi_n(x)$, $n = 1, 2, 3\ldots$ are stationary states of the Hamiltonian governing the system,

$$H\psi_n(x) = E_n\psi(x).$$

Answer the following questions:

a) Do you expect $\langle x\rangle$, $\langle x^2\rangle$ and $\langle E\rangle$ to be time dependent or time independent? Discuss briefly, but do not calculate. (2 Points)

b) Is the uncertainty ΔE positive, negative or zero? Is ΔE time dependent or time independent? Again, discuss briefly but do not calculate. (2 Points)

c) Is $\Psi(t)$ above a solution of the time dependent Schroedinger equation? Demonstrate. (2 Points)

d) If the stationary states $\psi_1(x)$, $\psi_3(x)$ and $\psi_5(x)$ are eigenstates of the harmonic oscillator, will any of your answers to part a) change? Justify. (2 Points)

e) Now assume the particles are in the state

$$\Psi(x,t) = \psi_3(x)\mathrm{e}^{-iE_3t/\hbar}.$$

Answer parts a) and b) for this state. (2 Points)

(a)

We know that if an observable commutes with the Hamiltonian, the expectation value of the observable will not change over time.

So I would expect $\langle x \rangle$ and $\langle x^2 \rangle$ to both change over time since $[H, x] \neq 0$ and $[H, x^2] \neq 0$, but I would expect $\langle E \rangle$ to remain constant over time since $[H, E] = 0$.

(b) In general,

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}.$$

This clearly can't be negative. It is also not zero because we expect nonzero results for $\langle E^2 \rangle$ and $\langle E \rangle$ since the wavefunction is a linear combination of stationary states. I don't expect $\langle E^2 \rangle = \langle E \rangle^2$ for this reason, either.

(c)

The time-dependent Schrödinger equation is given by

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H |\Psi(t)\rangle.$$

We know

$$H|\psi_n(x)\rangle = E_n |\psi_n(x)\rangle,$$

so

$$H|\Psi(t)\rangle = H\left[\frac{1}{\sqrt{8}}|\psi_1(x)\rangle e^{-iE_1 t/\hbar} - i\sqrt{\frac{3}{8}}|\psi_3(x)\rangle e^{-iE_3 t/\hbar} + \frac{1}{\sqrt{2}}|\psi_5(x)\rangle e^{-iE_5 t/\hbar}\right]$$

(c), cont'd...

$$H|\psi(t)\rangle = \frac{E_1}{\sqrt{8}}|\psi_1(x)\rangle e^{-iE_1t/\hbar} - iE_3\sqrt{\frac{3}{8}}|\psi_3(x)\rangle e^{-iE_3t/\hbar} + \frac{E_5}{\sqrt{2}}|\psi_5(x)\rangle e^{-iE_5t/\hbar}$$

We also have

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\left[-\frac{iE_1}{\hbar\sqrt{8}}|\psi_1(x)\rangle e^{-\frac{iE_1t}{\hbar}} - \frac{E_3}{\hbar}\sqrt{\frac{3}{8}}|\psi_3(x)\rangle e^{-\frac{iE_3t}{\hbar}} - \frac{iE_5}{\hbar\sqrt{2}}|\psi_5(x)\rangle e^{-iE_5t/\hbar}\right]$$

$$= \frac{E_1}{\sqrt{8}}|\psi_1(x)\rangle e^{-\frac{iE_1t}{\hbar}} - iE_3\sqrt{\frac{3}{8}}|\psi_3(x)\rangle e^{-\frac{iE_3t}{\hbar}} + \frac{E_5}{\sqrt{2}}|\psi_5(x)\rangle e^{-iE_5t/\hbar}$$

Thus, we indeed have

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

and so the TDSE is satisfied

(d) If ψ_1, ψ_3, and ψ_5 are eigenstates of the harmonic oscillator, I would still expect $\langle E\rangle$ to be time-independent since E still commutes with the Hamiltonian. I would also still expect $\langle x^2\rangle$ to be time-dependent because $[H, x^2] \neq 0$. However, I now expect $\langle x\rangle$ to be time-independent because $\langle x\rangle = 0$. Applying raising and lowering operators cannot yield inner products that do not go to zero because they raise or lower the ket by one level and all of our states are two levels apart. $\left(x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\right)$

(e)

Since this is a stationary state, we know $\langle E \rangle$ is time-independent. We also know that $\langle x \rangle$ is time-independent because the time dependence of the state cancels out in the integral. By the same logic, $\langle x^2 \rangle$ is now time-independent, as well.

The expectation value of the energy is a positive value and when squared, should be equal to $\langle E^2 \rangle$. Thus, I expect $\Delta E = 0$.
This makes sense because this is a stationary state and we know the energy.