F2008

Problem 3: The Harmonic Oscillator(10 Points)

A one dimensional harmonic oscillator has a potential given by

$$V(x) = m\omega^2 x^2/2.$$

where ω is the oscillator frequency and m is its mass. Derive all results.

a. Write the Schrodinger equation for a single particle in a one dimensional harmonic oscillator potential. **(1 Point)**

b. Consider the raising and lowering operators

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\hbar\omega}}$$

and

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\hbar\omega}},$$

respectively, where p is the momentum operator. If Ψ_E is an eigenvector of the Hamiltonian with energy eigenvalue E, find the energy eigenvalues of $a^\dagger\Psi_E$ and $a\Psi_E$. (You may need to use the fact that $[x,p] = i\hbar$). **(2 Points)**

c. Using the raising and lowering operators find the energy eigenvalues for a single particle in a one dimensional harmonic oscillator potential. **(2 Points)**

d. Find the normalized ground state wave function. **(2 Points)**

e. The harmonic oscillator models a particle attached to an ideal spring. If the spring can only be stretched, and not compressed, so that $V = \infty$ for $x < 0$, what will be the energy levels of this system? **(3 Points)**

(a)

In general, the Schrödinger equation is given by

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x) = E\Psi(x).$$

We can write this in terms of the momentum as

$$\left[\frac{p^2}{2m} + V(x)\right]\Psi(x) = E\Psi(x).$$

If $V(x) = \frac{1}{2}m\omega^2 x^2$, we have

$$\boxed{\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right]\Psi(x) = E\Psi(x)}$$

(b)

We are told

$$H|\Psi_E\rangle = E|\Psi_E\rangle$$

and we want to determine A and B, where

$$H\left(a^\dagger|\Psi_E\rangle\right) = A\left(a^\dagger|\Psi_E\rangle\right)$$
$$H\left(a|\Psi_E\rangle\right) = B\left(a|\Psi_E\rangle\right)$$

and

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

We begin by writing H in terms of $a^\dagger$ and a.

(b), cont'd...

We have

$$\sqrt{2m\hbar\omega}\, a^{+} = m\omega x - ip$$

and

$$\sqrt{2m\hbar\omega}\, a = m\omega x + ip .$$

Then

$$m\omega x = \sqrt{2m\hbar\omega}\, a^{+} + ip$$

and

$$\sqrt{2m\hbar\omega}\, a = \sqrt{2m\hbar\omega}\, a^{+} + 2ip$$

$$2ip = \sqrt{2m\hbar\omega}\,(a - a^{+})$$

$$p = -i\sqrt{\frac{m\hbar\omega}{2}}\,(a - a^{+}).$$

Similarly,

$$ip = m\omega x - \sqrt{2m\hbar\omega}\, a^{+}$$

and

$$\sqrt{2m\hbar\omega}\, a = 2m\omega x - \sqrt{2m\hbar\omega}\, a^{+}$$

$$2m\omega x = \sqrt{2m\hbar\omega}\,(a + a^{+})$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{+}).$$

Plugging into our Hamiltonian,

$$H = \frac{1}{2m}\left(-\frac{m\hbar\omega}{2}(a-a^{+})^{2}\right) + \frac{1}{2}m\omega^{2}\left(\frac{\hbar}{2m\omega}(a+a^{+})^{2}\right)$$

$$= -\frac{\hbar\omega}{4}(a-a^{+})^{2} + \frac{\hbar\omega}{4}(a+a^{+})^{2}$$

$$= \frac{\hbar\omega}{2}(a^{+}a + aa^{+}).$$

But

$$[a, a^{+}] = aa^{+} - a^{+}a = 1 .$$

(b), cont'd...

So

$$H = \frac{\hbar\omega}{2}(a^\dagger a + a^\dagger a + 1)$$

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}).$$

Now we want to determine $[H, a^\dagger]$ and $[H, a]$. So

$$\begin{aligned}
[H, a^\dagger] &= [\hbar\omega(a^\dagger a + \tfrac{1}{2}), a^\dagger] \\
&= \hbar\omega(a^\dagger a + \tfrac{1}{2})a^\dagger - a^\dagger\hbar\omega(a^\dagger a + \tfrac{1}{2}) \\
&= \hbar\omega(a^\dagger a a^\dagger + \tfrac{1}{2}a^\dagger) - \hbar\omega(a^\dagger a^\dagger a + \tfrac{1}{2}a^\dagger) \\
&= \hbar\omega(a^\dagger a a^\dagger - a^\dagger a^\dagger a) \\
&= \hbar\omega[a^\dagger(a^\dagger a + 1) - a^\dagger a^\dagger a] \\
&= \hbar\omega a^\dagger
\end{aligned}$$

$$\begin{aligned}
[H, a] &= [\hbar\omega(a^\dagger a + \tfrac{1}{2}), a] \\
&= \hbar\omega(a^\dagger a + \tfrac{1}{2})a - a\hbar\omega(a^\dagger a + \tfrac{1}{2}) \\
&= \hbar\omega(a^\dagger a a + \tfrac{1}{2}a) - \hbar\omega(a a^\dagger a + \tfrac{1}{2}a) \\
&= \hbar\omega(a^\dagger a a - a a^\dagger a) \\
&= \hbar\omega[a^\dagger a a - (a^\dagger a + 1)a] \\
&= \hbar\omega[a^\dagger a a - a^\dagger a a - a] \\
&= -\hbar\omega a
\end{aligned}$$

(b), cont'd...

Now we can determine A and B.

We have

$$Ha^+|\psi_E\rangle = ([H,a^+] + a^+H)|\psi_E\rangle$$
$$= (\hbar\omega a^+ + a^+H)|\psi_E\rangle$$

so

$$Ha^+|\psi_E\rangle = (E+\hbar\omega)a^+|\psi_E\rangle,$$

and

$$Ha|\psi_E\rangle = ([H,a] + aH)|\psi_E\rangle$$
$$= (-\hbar\omega a + aH)|\psi_E\rangle$$

so

$$Ha|\psi_E\rangle = (E-\hbar\omega)a|\psi_E\rangle.$$

Thus, the energy eigenvalue of $a^+\psi_E$ is

$$\boxed{A = E+\hbar\omega}$$

and the energy eigenvalue of $a\psi_E$ is

$$\boxed{B = E-\hbar\omega}.$$

(c)

Define $N = a^\dagger a$, where

$$N|\psi_n\rangle = n|\psi_n\rangle. \qquad (n = 0, 1, 2, \ldots)$$

Then

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

and

$$\begin{aligned} H|\psi_n\rangle &= \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)|\psi_n\rangle \\ &= \hbar\omega\left(N + \tfrac{1}{2}\right)|\psi_n\rangle \\ &= \hbar\omega\left(n + \tfrac{1}{2}\right)|\psi_n\rangle. \end{aligned}$$

Thus,

$$\boxed{E_n = \hbar\omega\left(n + \tfrac{1}{2}\right)}$$

(d)

We want to find $\psi_0(x)$. We know

$$a|\psi_0\rangle = 0.$$

So

$$\begin{aligned} a|\psi_0\rangle &= 0 \\ \left(\sqrt{\frac{m\omega}{2\hbar}}\,x + i\frac{p}{\sqrt{2m\hbar\omega}}\right)\psi_0 &= 0 \\ \left(\sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i}{\sqrt{2m\hbar\omega}}\left(-i\hbar\frac{d}{dx}\right)\right)\psi_0 &= 0 \\ \left(\sqrt{\frac{m\omega}{2\hbar}}\,x + \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right)\psi_0 &= 0 \end{aligned}$$

(d), cont'd...

$$\sqrt{\frac{\hbar}{2m\omega}}\,\frac{d\psi_0}{dx} = -\sqrt{\frac{m\omega}{2\hbar}}\,x\,\psi_0$$

$$\frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}\,x\,\psi_0$$

Separating and integrating...

$$\int \frac{1}{\psi_0}\,d\psi_0 = -\frac{m\omega}{\hbar}\int x\,dx$$

$$\ln\psi_0 = -\frac{m\omega}{2\hbar}x^2 + C_1$$

$$\psi_0(x) = C e^{-\frac{m\omega}{2\hbar}x^2}.$$

Normalizing...

$$\int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = 1$$

$$C^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}\,dx = 1$$

$$C^2\left(\sqrt{\pi\cdot\frac{\hbar}{m\omega}}\right) = 1$$

$$C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

and our ground state wavefunction is

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}.$$

(e)

Our new potential is

$$V(x) = \begin{cases} \infty \ , & x < 0 \\ \frac{1}{2} m\omega^2 x^2 , & x > 0 \ . \end{cases}$$

In general, our energy levels still have the form

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right)$$

in the region $x > 0$. However, we now have an extra boundary condition on our wavefunctions —

$$\psi(x=0) = 0 \ .$$

This condition imposes that our wavefunctions must be odd and cannot be even. Thus,

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega \quad , \ n = 1, 3, 5, \ldots}$$