PROBLEM 3: Harmonic Oscillator in 3 Dimensions

Consider a particle subject to a 3-dimensional harmonic oscillator potential:

$$\begin{aligned} H &= H_x + H_y + H_z \\ &= \frac{P_x^2}{2m} + \frac{1}{2}m\omega_x^2 X^2 + \frac{P_y^2}{2m} + \frac{1}{2}m\omega_y^2 Y^2 + \frac{P_z^2}{2m} + \frac{1}{2}m\omega_z^2 Z^2 \end{aligned}$$

where $\omega_x = \omega_y = \omega$ and $\omega_z = 2\omega$. The wave function is given by:

$$\Psi_{n_x,n_y,n_z}(x,y,z) = \Phi_{n_x}(x)\Phi_{n_y}(y)\Phi_{n_z}(z)$$

(a). For the ground state wave function, determine, ΔX, ΔP_x and their product $\Delta X \Delta P_x$ using Dirac notation and raising and lowering operators. (*3 points*)

(b). For the ground state, $\Psi_{0,0,0}$, do you expect
(i) ΔZ to be larger or smaller than ΔX?
(ii) $\Delta Z \Delta P_z$ to be larger or smaller than $\Delta X \Delta P_x$?
Explain your reasoning in each case. (*2 points*)

(c). Assume that at $t = 0$ the particle is in the state:

$$|\Psi(t=0)> = \frac{1}{\sqrt{6}}|\Psi_{1,0,0}> + \frac{1}{\sqrt{3}}|\Psi_{2,1,0}> + \frac{1}{2}|\Psi_{0,1,0}> + \frac{1}{2}|\Psi_{1,0,1}>$$

If one measures the total energy, E, what is the probability of obtaining $5\hbar\omega$? (*2 points*)

(d). Immediately after the measurement performed in part (c), what harmonic oscillator state (or superposition of states) is the system in? (*1 points*)

(e). Assume that at time $t = 0$ the measurement described in part (c) is performed and that the energy is found to be $E = 5\hbar\omega$. If the observable, X, is measured at a time $t > 0$, will its probability distribution be be time dependent or time independent? Explain your reasoning. (*2 points*)

(a)

Let's represent our state in Dirac notation,

$$|n\rangle = |n_x\, n_y\, n_z\rangle.$$

We know, in general, that

$$X = \sqrt{\frac{\hbar}{2m\omega_x}}\,(a_x + a_x^{+})$$

and

$$P_x = -i\sqrt{\frac{m\hbar\omega_x}{2}}\,(a_x - a_x^{+}).$$

So

$$\begin{aligned}
\langle X\rangle_n &= \langle n|X|n\rangle \\
&= \langle n|\sqrt{\frac{\hbar}{2m\omega_x}}\,(a_x + a_x^{+})|n\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega_x}}\left[\langle n|a_x|n\rangle + \langle n|a_x^{+}|n\rangle\right]
\end{aligned}$$

In the ground state, we have

$$|n\rangle = |0\rangle = |0\;0\;0\rangle.$$

So

$$\begin{aligned}
\langle X\rangle_0 &= \sqrt{\frac{\hbar}{2m\omega_x}}\left[0 + \langle 0|1\,0\,0\rangle\right] \\
&= 0.
\end{aligned}$$

(a), cont'd...

We also have

$$\langle X^2\rangle_0 = \frac{\hbar}{2m\omega_x}\left[\langle 0\,0\,0|a_xa_x + a_xa_x^\dagger + a_x^\dagger a_x + a_x^\dagger a_x^\dagger|0\,0\,0\rangle\right]$$

$$= \frac{\hbar}{2m\omega_x}\left[0 + 1\langle 0\,0\,0|0\,0\,0\rangle + 0 + \sqrt{2}\langle 0\,0\,0|2\,0\,0\rangle\right]$$

$$= \frac{\hbar}{2m\omega}$$

Thus,

$$\Delta X_0 = \sqrt{\langle X^2\rangle_0 - \langle X\rangle_0^2}$$

$$\boxed{\Delta X_0 = \left(\frac{\hbar}{2m\omega}\right)^{1/2}}.$$

We have

$$\langle P_x\rangle_0 = \langle 0|P_x|0\rangle$$

$$= \langle 0|-i\sqrt{\frac{m\hbar\omega_x}{2}}\,(a_x - a_x^\dagger)|0\rangle$$

$$= 0$$

and

$$\langle P_x^2\rangle_0 = \langle 0|P_x^2|0\rangle$$

$$= \langle 0|-\frac{m\hbar\omega_x}{2}(a_x - a_x^\dagger)^2|0\rangle$$

$$= -\frac{m\hbar\omega_x}{2}\langle 0|a_xa_x - a_xa_x^\dagger - a_x^\dagger a_x + a_x^\dagger a_x^\dagger|0\rangle$$

$$= -\frac{m\hbar\omega_x}{2}\left[0 - 1\langle 0|0\rangle - 0 + \sqrt{2}\langle 0|2\rangle\right]$$

(a), cont'd...

$$\langle P_x^2 \rangle_0 = \frac{m\hbar\omega}{2}$$

Then

$$\Delta P_x = \sqrt{\langle P_x^2 \rangle - \langle P_x \rangle^2}$$

$$\boxed{\Delta P_x = \left(\frac{m\hbar\omega}{2}\right)^{1/2}}$$

So

$$\Delta X \Delta P_x = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \left(\frac{m\hbar\omega}{2}\right)^{1/2}$$

$$\boxed{\Delta X \Delta P_x = \frac{\hbar}{2}}.$$

(b)

(i) I would expect ΔZ to be $\boxed{\text{smaller}}$ than ΔX since

$$\Delta Z \propto \left(\frac{1}{2\omega}\right)^{1/2},$$

whereas

$$\Delta X \propto \left(\frac{1}{\omega}\right)^{1/2}.$$

(ii) I would expect $\Delta Z \Delta P_z = \Delta X \Delta P_x$ because

$$\Delta Z \propto \left(\frac{1}{2\omega}\right)^{1/2}$$

$$\Delta P_z \propto (2\omega)^{1/2},$$

so the result is independent of ω_z.

(c)

If we have

$$E = 5\hbar\omega,$$

we must have

$$n_x = 2,\ n_y = 1,\ n_z = 0$$

or

$$n_x = 1,\ n_y = 0,\ n_z = 1$$

since

$$E = (n_x + n_y + 2n_z + 2)\hbar\omega.$$

Then

$$P(E = 5\hbar\omega) = \left|\frac{1}{\sqrt{3}}\right|^2 + \left|\frac{1}{2}\right|^2$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{4}{12} + \frac{3}{12}$$

$$\boxed{P(E = 5\hbar\omega) = \frac{7}{12}}.$$

(d) Immediately after the measurement in part (c), our state is

$$|\psi'\rangle = \frac{1}{\sqrt{3}}|\psi_{210}\rangle + \frac{1}{2}|\psi_{101}\rangle.$$

Normalizing...

$$A^2\left(\frac{1}{3} + \frac{1}{4}\right) = 1$$

$$A^2\left(\frac{7}{12}\right) = 1$$

$$A = \sqrt{\frac{12}{7}}$$

so

$$\boxed{|\psi'\rangle = \sqrt{\frac{4}{7}}|\psi_{210}\rangle + \sqrt{\frac{3}{7}}|\psi_{101}\rangle}.$$

(e)

Since we collapsed our wave-function into a superposition of states with the same eigenenergy, our wavefunction after the measurement represents a stationary state. Thus, I would expect the probability distribution of the observable X to be $\boxed{\text{time-independent}}$.

Mathematically, we have

$$\begin{aligned}
\psi(x,t) &= U(t=0)\,\psi(t=0) \\
&= \sqrt{\frac{4}{7}}\,|\psi_{210}\rangle e^{-i(5\hbar\omega)t/\hbar} + \sqrt{\frac{3}{7}}\,|\psi_{101}\rangle e^{-i(5\hbar\omega)t/\hbar} \\
&= \left[\sqrt{\frac{4}{7}}\,|\psi_{210}\rangle + \sqrt{\frac{3}{7}}\,|\psi_{101}\rangle\right] e^{-i(5\hbar\omega)t/\hbar}
\end{aligned}$$

So

$$\begin{aligned}
\langle X\rangle &= \langle\psi(x,t)|\,X\,|\psi(x,t)\rangle \\
&= e^{i(5\hbar\omega)t/\hbar}\left[\sqrt{\frac{4}{7}}\langle\psi_{210}| + \sqrt{\frac{3}{7}}\langle\psi_{101}|\right] X \left[\sqrt{\frac{4}{7}}\,|\psi_{210}\rangle + \sqrt{\frac{3}{7}}\,|\psi_{101}\rangle\right] e^{\frac{-i(5\hbar\omega)t}{\hbar}} \\
&= \left(\sqrt{\frac{4}{7}}\langle\psi_{210}| + \sqrt{\frac{3}{7}}\langle\psi_{101}|\right) X \left(\sqrt{\frac{4}{7}}|\psi_{210}\rangle + \sqrt{\frac{3}{7}}|\psi_{101}\rangle\right)
\end{aligned}$$

which is obviously time-independent.