S2013

Problem 4: Operator Solutions to the Harmonic Oscillator

Consider the Harmonic Oscillator Hamiltonian in one dimension:

$$H_{ho} = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2 \tag{1}$$

To simplify this problem, define the new observables:

$$p = \sqrt{\frac{1}{m\hbar\omega}}P \;, \quad q = \sqrt{\frac{m\omega}{\hbar}}X \tag{2}$$

This gives the dimensionless Hamiltonian,

$$H = \frac{1}{\hbar\omega}H_{ho} = \frac{1}{2}\left(p^2 + q^2\right) \tag{3}$$

(a) [1 pt] Calculate the commutation relation for these new variables, $[q, p]$. Be sure to show your work.

(b) [1 pt] Define the non-Hermitian operators $a = \frac{1}{\sqrt{2}}(q + ip)$, $\quad a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$ and the Hermitian operator $n = a^\dagger a$. Compute $[a, a^\dagger]$, $[n, a^\dagger]$, and $[n, a]$

(c) [1 pt] Write the dimensionless Hamiltonian H in terms of a and $a^\dagger$. Write the dimensionless Hamiltonian H in terms of n.

(d) [3 pts] Define the eigenvalues and eigenvectors of n as:

$$n|\lambda\rangle = \lambda|\lambda\rangle. \tag{4}$$

and assume that these eigenvectors form a complete set.

Show that

$$\begin{aligned} a^\dagger|\lambda\rangle &= A|\lambda+1\rangle \\ a|\lambda\rangle &= B|\lambda-1\rangle \end{aligned} \tag{5}$$

Determine the normalization constants A and B.

(e) [2 pts.] Show that $n = a^\dagger a$ must have non-negative eigenvalues, $\lambda \geq 0$. Explain why this implies that there must be a state where $a|0\rangle = 0$ and that the eigenvalues of n must be non-negative integers.

(f) [2 pts.] Write the definition for the state $|0\rangle$

$$a|0\rangle = 0 \tag{6}$$

as a differential equation, in q, for the ground state wavefunction of H. Solve this expression for the normalized ground state wavefunction.

(a)

We know $[X, P] = i\hbar$. So

$$
\begin{aligned}
[q,p] &= qp - pq \\
&= \sqrt{\frac{m\omega}{\hbar}}X\sqrt{\frac{1}{m\hbar\omega}}P - \sqrt{\frac{1}{m\hbar\omega}}P\sqrt{\frac{m\omega}{\hbar}}X \\
&= \frac{1}{\hbar}(XP - PX) \\
&= \frac{1}{\hbar}[X,P] \\
&= \frac{1}{\hbar}(i\hbar)
\end{aligned}
$$

$$\boxed{[q,p] = i}$$

(b)

$$
\begin{aligned}
[a,a^+] &= aa^+ - a^+a \\
&= \tfrac{1}{\sqrt{2}}(q+ip)\tfrac{1}{\sqrt{2}}(q-ip) - \tfrac{1}{\sqrt{2}}(q-ip)\tfrac{1}{\sqrt{2}}(q+ip) \\
&= \tfrac{1}{2}(q^2 - iqp + ipq + p^2) - \tfrac{1}{2}(q^2 + iqp - ipq + p^2) \\
&= \frac{i}{2}(pq - qp) - \frac{i}{2}(qp - pq) \\
&= \frac{-i}{2}(qp - pq) - \frac{i}{2}(qp - pq) \\
&= -i(qp - pq) \\
&= -i[q,p] \\
&= -i(i)
\end{aligned}
$$

$$\boxed{[a,a^+] = 1}$$

(b), cont'd...

$$[n, a^+] = [a^+a, a^+]$$
$$= a^+aa^+ - a^+a^+a$$

We know

$$[a, a^+] = aa^+ - a^+a = 1$$
$$aa^+ = a^+a + 1$$

So

$$[n, a^+] = a^+(a^+a + 1) - a^+a^+a$$
$$= a^+a^+a + a^+ - a^+a^+a$$
$$\boxed{[n, a^+] = a^+}$$

$$[n, a] = [a^+a, a]$$
$$= a^+aa - aa^+a$$
$$= a^+aa - (a^+a + 1)a$$
$$= a^+aa - a^+aa - a$$
$$\boxed{[n, a] = -a}$$

(c)

We know

$$H = \frac{1}{2}(p^2 + q^2)$$

and

$$a = \frac{1}{\sqrt{2}}(q + ip)$$

$$a^+ = \frac{1}{\sqrt{2}}(q - ip).$$

So

$$\sqrt{2}\,a = q + ip$$

and

$$\sqrt{2}\,a^+ = q - ip$$

$$q = \sqrt{2}\,a^+ + ip.$$

Then

$$\sqrt{2}\,a = (\sqrt{2}\,a^+ + ip) + ip$$

$$\sqrt{2}\,(a - a^+) = 2ip$$

$$p = -\frac{i(a - a^+)}{\sqrt{2}}. \qquad (1)$$

Similarly,

$$ip = q - \sqrt{2}\,a^+$$

so

$$\sqrt{2}\,a = q + (q - \sqrt{2}\,a^+)$$

$$\sqrt{2}\,(a + a^+) = 2q$$

$$q = \frac{a + a^+}{\sqrt{2}}. \qquad (2)$$

(c), cont'd...

Plugging ① and ② into H, we get

$$H = \frac{1}{2}\left[\left(-\frac{i(a-a^+)}{\sqrt{2}}\right)^2 + \left(\frac{a+a^+}{\sqrt{2}}\right)^2\right]$$

$$= \frac{1}{4}\left[-(\cancel{aa} - aa^+ - a^+a + \cancel{a^+a^+}) + (\cancel{aa} + aa^+ + a^+a + \cancel{a^+a^+})\right]$$

$$= \frac{1}{4}\left[2aa^+ + 2a^+a\right]$$

$$\boxed{H = \frac{1}{2}\left(aa^+ + a^+a\right)}$$

We know

$$[a, a^+] = aa^+ - a^+a = 1$$

$$a a^+ = a^+a + 1$$

so

$$H = \frac{1}{2}\left[(a^+a + 1) + a^+a\right]$$

$$= \frac{1}{2}\left[2a^+a + 1\right]$$

$$\boxed{H = a^+a + 1/2}$$

$$\boxed{H = n + 1/2}$$

(d)

We know

$$n|\lambda\rangle = \lambda|\lambda\rangle .$$

So

$$\begin{aligned} na|\lambda\rangle &= \left([n,a] + an\right)|\lambda\rangle \\ &= (-a + an)|\lambda\rangle \\ &= (\lambda - 1)a|\lambda\rangle \end{aligned}$$

and so we must have

$$a|\lambda\rangle = b|\lambda-1\rangle .$$

Then

$$\begin{aligned} \langle\lambda|a^{+}a|\lambda\rangle &= b^2 \langle\lambda-1|\lambda-1\rangle \\ \langle\lambda|n|\lambda\rangle &= b^2 \\ \langle\lambda|\lambda|\lambda\rangle &= b^2 \\ \lambda\langle\lambda|\lambda\rangle &= b^2 \\ b &= \sqrt{\lambda} \end{aligned}$$

and therefore,

$$\boxed{a|\lambda\rangle = \sqrt{\lambda}\,|\lambda-1\rangle}$$

(d), cont'd...

Similarly,

$$\begin{aligned} n a^{+} |\lambda\rangle &= \left([n, a^{+}] + a^{+} n\right)|\lambda\rangle \\ &= (a^{+} + a^{+} n)|\lambda\rangle \\ &= (\lambda + 1) a^{+} |\lambda\rangle \end{aligned}$$

and so we must have

$$a^{+}|\lambda\rangle = A|\lambda + 1\rangle .$$

Then

$$\langle \lambda | a a^{+} | \lambda \rangle = A^{2} \langle \lambda + 1 | \lambda + 1 \rangle .$$

We know

$$[a, a^{+}] = a a^{+} - a^{+} a = 1$$
$$a a^{+} = a^{+} a + 1$$

so

$$\begin{aligned} \langle \lambda | (a^{+} a + 1) | \lambda \rangle &= A^{2} \\ \langle \lambda | a^{+} a | \lambda \rangle + \langle \lambda | 1 | \lambda \rangle &= A^{2} \\ \langle \lambda | n | \lambda \rangle + \langle \lambda | \lambda \rangle &= A^{2} \\ \langle \lambda | \lambda | \lambda \rangle + 1 &= A^{2} \\ \lambda \langle \lambda | \lambda \rangle + 1 &= A^{2} \\ \lambda + 1 &= A^{2} \\ A &= \sqrt{\lambda + 1} \end{aligned}$$

and

$$\boxed{a^{+}|\lambda\rangle = \sqrt{\lambda + 1}\,|\lambda + 1\rangle}$$

(e)

→ Let

$$n|\lambda\rangle = \lambda|\lambda\rangle.$$

Then consider $a|\lambda\rangle$. We know that the norm is

$$\begin{aligned}\langle\lambda|a^{+}a|\lambda\rangle &= \langle\lambda|n|\lambda\rangle \\ &= \langle\lambda|\lambda|\lambda\rangle \\ &= \lambda\langle\lambda|\lambda\rangle\end{aligned}$$

But the norm is always positive and we know $\langle\lambda|\lambda\rangle = 0$ or 1 (in this case, $\langle\lambda|\lambda\rangle = 1$), so we must have

$$\lambda \geq 0.$$

→ We know

$$a|\lambda\rangle = \sqrt{\lambda}\,|\lambda-1\rangle$$

Let $a|\lambda\rangle = 0$. Then

$$\begin{aligned}\langle\lambda|a^{+}a|\lambda\rangle &= \langle\lambda|n|\lambda\rangle \\ &= \lambda\langle\lambda|\lambda\rangle \\ &= 0\end{aligned}$$

This implies $\lambda = 0$. Now assume $\lambda = 0$. then we must have $\langle\lambda|a^{+}a|\lambda\rangle = 0$, and any eigenvector $|0\rangle$ of n with $\lambda = 0$ must obey

$$a|0\rangle = 0.$$

→ We know

$$a|\lambda\rangle = \sqrt{\lambda}\,|\lambda-1\rangle.$$

Assume λ^{*} is a non-integer such that $\lambda^{*} > 0$ and let m be the smallest integer larger than λ^{*}. Then

$$a^{m}|\lambda^{*}\rangle = c(\lambda^{*})|\lambda^{*}-m\rangle$$

But $m > \lambda^{*}$, so $(\lambda^{*}-m) < 0$, which is a contradiction. So the eigenvalues must be non-negative integers.

(f)

We know

$$a\psi_0 = 0.$$

Then

$$\frac{1}{\sqrt{2}}(q + ip)\psi_0 = 0$$

$$\frac{1}{\sqrt{2}}\left(q + i\left(-i\hbar\frac{d}{dq}\right)\right)\psi_0 = 0$$

$$q\psi_0 + \hbar\frac{d\psi_0}{dq} = 0$$

$$\boxed{\frac{d\psi_0}{dq} = -\frac{1}{\hbar}q\psi_0}.$$

Separating and integrating...

$$\int\frac{1}{\psi_0}\,d\psi_0 = \int -\frac{1}{\hbar}q\,dq$$

$$\ln\psi_0 = -\frac{1}{2\hbar}q^2 + C_1$$

$$\psi_0 = Ce^{-\frac{1}{2\hbar}q^2}$$

Normalizing...

$$\int_{-\infty}^{\infty}|\psi_0|^2\,dq = 1$$

$$C^2\int_{-\infty}^{\infty}e^{-\frac{1}{\hbar}q^2}\,dq = 1$$

$$C^2\left(\sqrt{\pi\hbar}\right) = 1$$

$$C = (\pi\hbar)^{-1/4}$$

and

$$\boxed{\psi_0(q) = (\pi\hbar)^{-1/4}e^{-\frac{1}{2\hbar}q^2}}$$