S2016

Problem 3: Identical particles (10 pts)

Two non-interacting particles of mass m are trapped in a 1-dimensional infinite box of length L situated between $x = 0$ and $x = L$. (In the cases you are considering fermions, assume them to all be spin up.)

(a) [1 points] Write down the single particle energy eigenvalues and wavefunctions.

(b) [1 points] Write down the energy eigenvalues and wavefunctions for two distinguishable particles. Label the states by n_1 for particle 1 and n_2 for particle 2.

(c) [2 points] An energy measurement of the *two identical particle* system yields $E = \hbar^2\pi^2/mL^2$. Write down the state vector/wave function of the system.

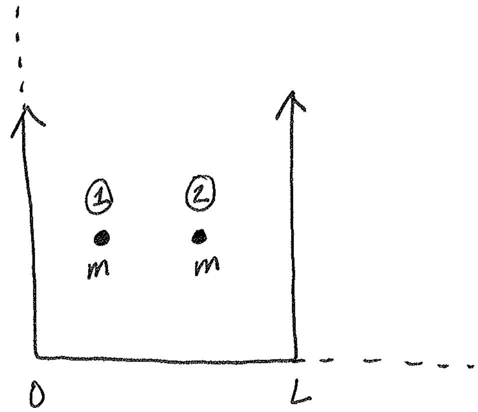
Sakurai, Ch 7

(d) [2 points] Suppose instead the energy of the two identical particle system is measured to be $E = 5\hbar^2\pi^2/mL^2$. What is the wave function? *Hint: there are two possibilities.*

(e) [2 points] Show that the fermion state you found in part (d) is an eigenfunction of the Hamiltonian, with the appropriate eigenvalue.

(f) [1 points] Write down the wavefunction for two identical spin-up fermions in the $n_1 = 2$ and $n_2 = 2$ state.

(g) [1 points] If instead you had three spin-up particles in the orthonormal states Ψ_1,Ψ_2, and Ψ_3, construct the three particle state for identical fermions.



(a) In general, the time-independent Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + (V-E)\right]\psi = 0 \,,$$

in one dimension. In the case of the infinite square well, we know

$$V = \begin{cases} \infty \,, & x<0 \text{ or } x>L \\ 0 \,, & 0 \le x \le L \end{cases} \,.$$

So our equation becomes

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E\right]\psi = 0$$

or

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \,.$$

Rearranging,

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \,.$$

Letting

$$k = \sqrt{\frac{2mE}{\hbar^2}} \,,$$

we have

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

(a), cont'd...

This is just the equation of a simple harmonic oscillator, so we know

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

Our boundary condition requires that

$$\psi(0) = \psi(L) = 0 \qquad \text{(continuity)}.$$

Then

$$\psi(0) = A\sin(k(0)) + B\cos(k(0)) = 0$$
$$B = 0$$

and

$$\psi(L) = A\sin(k(L)) = 0,$$

which implies

$$kL = n\pi \qquad (n = 1, 2, 3, \ldots)$$

or

$$k = \frac{n\pi}{L}.$$

So our general wavefunction is

$$\psi(x) = A\sin\left(\frac{n\pi x}{L}\right).$$

We need to normalize this function to determine A. Then

$$\int_0^L |\psi(x)|^2\, dx = 1$$

(a), cont'd...

$$\int_0^L A^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

$$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

$$A^2\left[\frac{L}{2} - \frac{L\sin(2\pi n)}{4\pi n}\right] = 1$$

but $\sin(2\pi n) = 0$ for all n, so

$$\frac{A^2 L}{2} = 1$$

$$A = \sqrt{\frac{2}{L}}.$$

So our single particle wave-function for particles ① and ② are

$$\boxed{\begin{aligned} \psi_1(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{n_1 \pi x}{L}\right) \\ \psi_2(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{n_2 \pi x}{L}\right) \end{aligned}}$$

We defined

$$k = \sqrt{\frac{2mE}{\hbar^2}},$$

but also

$$k = \frac{n\pi}{L}.$$

(a), cont'd...

So

$$\sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{L}$$

$$\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{L^2}$$

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

Then our single particle energies are

$$\boxed{\begin{aligned} E_1 &= \frac{n_1^2\pi^2\hbar^2}{2mL^2} \\ E_2 &= \frac{n_2^2\pi^2\hbar^2}{2mL^2} \end{aligned}}.$$

(b) I already did this in part (a).

(c) Since we are dealing with identical, non-interacting fermions, we know that it is impossible for the particles to be in the same state thanks to Pauli's Exclusion Principle. In general, the wave function is given by

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)\right],$$

which is antisymmetric, as expected. We also know that the energy is simply

$$E = E_1 + E_2$$

$$= \frac{(n_1^2 + n_2)^2 \pi^2 \hbar^2}{2mL^2}.$$

In order to have

$$E = \frac{\hbar^2 \pi^2}{mL^2},$$

we must have $n_1 = n_2 = 1$. Based on ψ above, this leaves us with

$$\boxed{\psi = 0}.$$

This is to be expected due to the Pauli Exclusion Principle.

(d) Again, we know that, in general,

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)\right]$$

and

$$E = \frac{(n_1^2 + n_2^2)\,\pi^2\hbar^2}{2mL^2}.$$

If

$$E = \frac{5\hbar^2\pi^2}{mL^2},$$

then we can have either

$$n_1 = 1 \text{ and } n_2 = 3$$

or

$$n_1 = 3 \text{ and } n_2 = 1.$$

Then our wavefunction can be either

$$\boxed{\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_3(x_2) - \psi_1(x_2)\psi_3(x_1)\right]}$$

or

$$\boxed{\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_3(x_1)\psi_1(x_2) - \psi_3(x_2)\psi_1(x_1)\right]}.$$

Simplifying...

$$\boxed{\begin{aligned}\psi(x_1, x_2) &= \frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) - \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right] \\ \psi(x_1, x_2) &= \frac{\sqrt{2}}{L}\left[\sin\left(\frac{3\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right) - \sin\left(\frac{3\pi x_2}{L}\right)\sin\left(\frac{\pi x_1}{L}\right)\right]\end{aligned}}.$$

(e) If the state in part (d) is an eigenfunction of the Hamiltonian, we should have

$$H|\psi\rangle = E|\psi\rangle.$$

The Hamiltonian is given by

$$H\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_2^2}.$$

We want to show that

$$\psi(x_1,x_2) = \frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) - \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right]$$

satisfies the first equation above. We have

$$\frac{\partial\psi}{\partial x_1} = \frac{\sqrt{2}}{L}\left[\frac{\pi}{L}\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) - \frac{3\pi}{L}\sin\left(\frac{\pi x_2}{L}\right)\cos\left(\frac{3\pi x_1}{L}\right)\right]$$

$$\frac{\partial^2\psi}{\partial x_1^2} = \frac{\sqrt{2}}{L}\left[-\frac{\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \frac{9\pi^2}{L^2}\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right]$$

and

$$\frac{\partial\psi}{\partial x_2} = \frac{\sqrt{2}}{L}\left[\frac{3\pi}{L}\sin\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{3\pi x_2}{L}\right) - \frac{\pi}{L}\cos\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right]$$

$$\frac{\partial^2\psi}{\partial x_2^2} = \frac{\sqrt{2}}{L}\left[-\frac{9\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \frac{\pi^2}{L^2}\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right].$$

Then

$$H\psi = -\frac{\hbar^2}{2m}\left[\frac{\sqrt{2}}{L}\left(-\frac{\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \frac{9\pi^2}{L^2}\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right) - \frac{9\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \frac{\pi^2}{L^2}\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right)\right]$$

$$= -\frac{\hbar^2}{2m}\left[\frac{\sqrt{2}}{L}\left(-\frac{10\pi^2}{L^2}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) + \frac{10\pi^2}{L^2}\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right)\right]$$

(e), cont'd ...

Simplifying, we have

$$H\psi = \frac{10\pi^2\hbar^2}{2mL^2}\left[\frac{\sqrt{2}}{L}\left(\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) - \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right)\right]$$

$$= \frac{5\hbar^2\pi^2}{mL^2}\psi$$

and indeed, we get

$$E = \frac{5\hbar^2\pi^2}{mL^2}$$

as expected and so $\psi = \frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) - \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right]$ is an eigenfunction of the Hamiltonian.

(f) If $n_1 = 2$ and $n_2 = 2$, then the wavefunction is

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_2(x_1)\psi_2(x_2) - \psi_2(x_2)\psi_2(x_1)\right]$$

$$= \frac{\sqrt{2}}{L}\left[\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{2\pi x_2}{L}\right)\sin\left(\frac{2\pi x_1}{L}\right)\right]$$

$$\boxed{\psi(x_1, x_2) = 0}$$

(g) We know that the resulting wavefunction must be antisymmetric. Then we will have something along the lines of

$$\boxed{\begin{aligned}\psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}\Big[&\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)\psi_{n_3}(x_3) \\ &- \psi_{n_1}(x_3)\psi_{n_2}(x_2)\psi_{n_3}(x_1) - \psi_{n_1}(x_1)\psi_{n_2}(x_3)\psi_{n_3}(x_2) \\ &+ \psi_{n_1}(x_2)\psi_{n_2}(x_3)\psi_{n_3}(x_1) + \psi_{n_1}(x_3)\psi_{n_2}(x_1)\psi_{n_3}(x_2)\Big]\end{aligned}}$$