F2012

PROBLEM 2: Generalized Uncertainty Principle

Consider the spin 1/2 operator

$$\mathbf{S} = \frac{\hbar}{2}\vec{\sigma},$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli matrices, which are defined in the basis of the S_z operator eigenvectors,

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) Compute the commutator $[S_i, S_j]$, with $i, j = x, y, z$. [2 Points]

(b) Compute the expectation values $\langle(\delta S_x)^2\rangle$ and $\langle(\delta S_y)^2\rangle$ for the state

$$|\alpha\rangle = \cos(\alpha)|+\rangle + \sin(\alpha)|-\rangle,$$

where $\delta\mathbf{S} = \mathbf{S} - \langle\mathbf{S}\rangle$. Show explicitly that the relation

$$\langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle \geq \frac{1}{4}\left|\langle[S_x, S_y]\rangle\right|^2$$

is satisfied. What does it physically mean? [4 Points]

(c) Find the states that maximize and minimize the product $\langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle$. Interpret the results. [2 Points]

(d) Suppose one performs an experiment which filters the $+\hbar/2$ eigenstate of the S_z operator from the initially prepared state $|\alpha\rangle$. Then the S_x component of the spin is measured. Compute the expectation value of this measurement in the state $|\alpha\rangle$. [2 Points]

(a)

We want to compute $[S_i, S_j]$ for $i,j = x,y,z$.
We know that

$$[S_i, S_j] = i\hbar \, \varepsilon_{ijk} S_k ,$$

where

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{even permutations} \\ -1, & \text{odd permutations} \\ 0, & \text{otherwise} \end{cases}.$$

So

$$[S_x, S_y] = i\hbar \, \varepsilon_{xyz} S_z \Rightarrow [S_x, S_y] = i\hbar S_z$$

$$\boxed{[S_x, S_y] = \frac{i\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$[S_x, S_z] = i\hbar \, \varepsilon_{xzy} S_y \Rightarrow [S_x, S_z] = -i\hbar S_y$$

$$\boxed{[S_x, S_z] = -\frac{i\hbar^2}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

$$[S_y, S_z] = i\hbar \, \varepsilon_{yzx} S_x \Rightarrow [S_y, S_z] = i\hbar S_x$$

$$\boxed{[S_y, S_z] = \frac{i\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

(a), cont'd...

$$[S_y, S_x] = i\hbar\,\varepsilon_{yxz} S_z \Rightarrow [S_y, S_x] = -i\hbar\, S_z$$

$$\boxed{[S_y, S_x] = -\frac{i\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$[S_z, S_x] = i\hbar\,\varepsilon_{zxy} S_y \Rightarrow [S_z, S_x] = i\hbar\, S_y$$

$$\boxed{[S_z, S_x] = \frac{i\hbar^2}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

$$[S_z, S_y] = i\hbar\,\varepsilon_{zyx} S_x \Rightarrow [S_z, S_y] = -i\hbar\, S_x$$

$$\boxed{[S_z, S_y] = -\frac{i\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

(b)

Let

$$|\alpha\rangle = \cos(\alpha)|+\rangle + \sin(\alpha)|-\rangle$$
$$= \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}.$$

We want to calculate $\langle(\delta S_x)^2\rangle$ and $\langle(\delta S_y)^2\rangle$, where

$$\delta S = S - \langle S\rangle.$$

$\underline{\langle(\delta S_x)^2\rangle}$

$$\delta S_x = S_x - \langle S_x\rangle,$$

so

$$\langle S_x\rangle = \frac{\hbar}{2}\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$
$$= \frac{\hbar}{2}\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$$
$$= \frac{\hbar}{2}\left[\sin\alpha\cos\alpha + \sin\alpha\cos\alpha\right]$$
$$= \hbar\sin\alpha\cos\alpha$$

But we know

$$\langle(\delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2, \qquad \star$$

so

$$\langle S_x^2\rangle = \frac{\hbar^2}{4}\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= (\cos^2\alpha + \sin^2\alpha)\frac{\hbar^2}{4}$$
$$= \hbar^2/4$$

(b), cont'd...

Then

$$\langle (\delta S_x)^2 \rangle = \frac{\hbar^2}{4} - (\hbar \sin\alpha \cos\alpha)^2$$

$$\boxed{\langle (\delta S_x)^2 \rangle = \frac{\hbar^2}{4} - \hbar^2 \sin^2\alpha \cos^2\alpha}$$

$\underline{\langle (\delta S_y)^2 \rangle}$

$$\langle (\delta S_y)^2 \rangle = \langle S_y^2 \rangle - \langle S_y \rangle^2,$$

So

$$\langle S_y \rangle = \frac{\hbar}{2} \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} -i\sin\alpha \\ i\cos\alpha \end{pmatrix}$$

$$= \frac{i\hbar}{2} (-\sin\alpha \cdot \cos\alpha + \sin\alpha \cdot \cos\alpha)$$

$$= 0$$

$$\langle S_y^2 \rangle = \frac{\hbar^2}{4} \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= \frac{\hbar^2}{4}$$

So

$$\boxed{\langle (\delta S_y)^2 \rangle = \frac{\hbar^2}{4}}$$

(b), cont'd...

The product of these two is

$$\langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle = \left(\frac{\hbar^2}{4} - \hbar^2\sin^2\alpha\cos^2\alpha\right)\left(\frac{\hbar^2}{4}\right)$$
$$= \frac{\hbar^4}{16} - \frac{\hbar^4}{4}\sin^2\alpha\cos^2\alpha$$

We know

$$[S_x, S_y] = \frac{i\hbar^2}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

so

$$\langle[S_x, S_y]\rangle = \frac{i\hbar^2}{2}\begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}$$
$$= \frac{i\hbar^2}{2}\left(\cos^2\alpha - \sin^2\alpha\right)$$

And squaring,

$$|\langle[S_x, S_y]\rangle|^2 = \frac{\hbar^4}{4}\left(\cos^2\alpha - \sin^2\alpha\right)^2$$
$$= \frac{\hbar^4}{4}\left(\cos^4\alpha - 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha\right).$$

Then

$$\frac{\hbar^4}{16} - \frac{\hbar^4}{4}\sin^2\alpha\cos^2\alpha \overset{?}{\geq} \frac{\hbar^4}{4}\left(\cos^4\alpha - 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha\right)$$
$$\frac{1}{4} - \sin^2\alpha\cos^2\alpha \overset{?}{\geq} \cos^4\alpha - 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha$$
$$\frac{1}{4} + \sin^2\alpha\cos^2\alpha \overset{?}{\geq} \cos^4\alpha + \sin^4\alpha$$

This does not satisfy the relation, but I'm not sure what went wrong.

(b), cont'd...

This is essentially an uncertainty relation that tells us there is no way to know precisely both S_x and S_y.

(c)

Choose α such that the product is maximized and minimized. Assuming my expression is

$$K = \langle (\delta S_x)^2 \rangle \langle (\delta S_y)^2 \rangle = \frac{\hbar^4}{16} - \frac{\hbar^4}{4} \sin^2\alpha \cos^2\alpha$$

$$\frac{dK}{d\alpha} = \frac{d}{d\alpha}\left(\frac{\hbar^4}{16} - \frac{\hbar^4}{4} \sin^2\alpha \left(1 - \sin^2\alpha\right)\right)$$

$$= \frac{d}{d\alpha}\left(\frac{\hbar^4}{16} + \frac{\hbar^4}{4} \sin^4\alpha - \frac{\hbar^4}{4} \sin^2\alpha \right)$$

$$= \frac{\hbar^4}{4}\left(4 \sin^3\alpha \cos\alpha\right) - \frac{\hbar^4}{4}\left(2 \sin\alpha \cos\alpha\right)$$

The maximums and minimums will occur where $\frac{dK}{d\alpha} = 0$. So

$$\frac{\hbar^4}{4}\left(4\sin^3\alpha \cos\alpha\right) = \frac{\hbar^4}{4}\left(2\sin\alpha\cos\alpha\right)$$

$$2\sin^2\alpha = 1$$

$$2(\sin\alpha)^2 = 1$$

$$\sin\alpha = \pm\frac{\sqrt{2}}{2} \Rightarrow \alpha = \pi/4, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$$

We can also see the pre-simplified equality is satisfied when $\alpha = 0, \pi, 2\pi, \ldots$ or even when $\alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

(c), cont'd...

Then for $\alpha = \pi/4 ..$, we have a minimum, and for $\alpha = 0 ..$ and $\alpha = \frac{\pi}{2}$, we have a maximum. Then the corresponding states are:

Minimum

$$|\alpha\rangle = \pm|+\rangle = |0\rangle$$
$$|\alpha\rangle = \pm|-\rangle = |\pi/2\rangle$$

Minimum when in the spin-up state or in the spin-down state.

Maximum

$$|\alpha\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle = |\pi/4\rangle$$

$$|\alpha\rangle = -\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle = |3\pi/4\rangle$$

$$|\alpha\rangle = -\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle = |5\pi/4\rangle$$

$$|\alpha\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle = |7\pi/4\rangle$$

These results tell us that the uncertainty is at a minimum when we are in just one spin state, which makes sense because the state is known.

The uncertainty is at a maximum when we have an even mixture of spin-up and spin-down states.

(d)

The wording here is strange, but I'm assuming it means that

$$|\alpha\rangle = \cos(\alpha)|+\rangle .$$

So

$$\langle S_x \rangle = \langle \alpha | S_x | \alpha \rangle$$

$$= \cos^2\alpha \langle + | S_x | + \rangle$$

$$= \cos^2\alpha \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \cos^2\alpha \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{\langle S_x \rangle = 0}$$

This makes sense because S_x and S_z are not compatible.