S2012

PROBLEM 6: Stationary Perturbation Theory

Let us consider the Hamiltonian $\mathbf{H}$ for a harmonic oscillator with a charged particle in a constant electric field (E):

$$\begin{aligned} \mathbf{H} &= \mathbf{H}_0 + \mathbf{H}_1 \\ \mathbf{H}_0 &= \frac{\mathbf{P}^2}{2m} + \frac{1}{2}k\mathbf{X}^2 \quad \text{and} \\ \mathbf{H}_1 &= \lambda \mathbf{X} \end{aligned}$$

where $\lambda = qE$ and q is the electric charge.

The non-perturbed Hamiltonian has the following eigenvalue equation

$$\mathbf{H}_0|n\rangle = E_n^{(0)}|n^{(0)}\rangle\,, \quad E_n^{(0)} = \hbar\omega(n+\frac{1}{2}) \quad \text{and} \quad \omega = \sqrt{k/m}\,.$$

(a) Apply perturbation theory and determine the first order energy $E_n^{(1)}$. [2 Points]

(b) Apply perturbation theory and evaluate the second order energy $E_n^{(2)}$. [3 Points]

(c) Solve this problem exactly and find the energy E_n. [3 Points]

(d) Determine the eigenvector to the first order $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle$. [2 Points]

(a)

We have

$$H' = \lambda x ,$$

so

$$\begin{aligned} E^{(1)} &= \langle n | H' | n \rangle \\ &= \langle n | \lambda x | n \rangle \\ &= \lambda \langle n | \sqrt{\tfrac{\hbar}{2m\omega}} (a + a^{+}) | n \rangle \\ &= \lambda \sqrt{\tfrac{\hbar}{2m\omega}} \left(\sqrt{n} \langle n | n-1 \rangle + \sqrt{n+1} \langle n | n+1 \rangle \right) \end{aligned}$$

$$\boxed{E^{(1)} = 0} .$$

(b)

We know

$$E^{(2)} = \sum_{m \neq n} \frac{|\langle m | H' | n \rangle|^2}{E_n^{(0)} - E_m^{(0)}} ,$$

So

$$\begin{aligned} E^{(2)} &= \frac{\lambda^2 \hbar}{2m\omega} \sum_{m \neq n} \frac{|\langle m | a + a^{+} | n \rangle|^2}{E_n^{(0)} - E_m^{(0)}} \\ &= \frac{\lambda^2 \hbar}{2m\omega} \sum_{m \neq n} \frac{|\langle m | n-1 \rangle \sqrt{n} + \langle m | n+1 \rangle \sqrt{n+1}|^2}{E_n^{(0)} - E_m^{(0)}} \end{aligned}$$

Cross terms will cancel, so

$$\begin{aligned} E^{(2)} &= \frac{\lambda^2 \hbar}{2m\omega} \sum_{m \neq n} \frac{n \langle m | n-1 \rangle^2 + (n+1) \langle m | n+1 \rangle^2}{\hbar\omega(n+\frac{1}{2}) - \hbar\omega(m+\frac{1}{2})} \\ &= \frac{\lambda^2}{2m\omega^2} \left[\sum_{m \neq n} \frac{n \delta_{m,n-1}^2}{n-m} + \sum_{m \neq n} \frac{(n+1) \delta_{m,n+1}^2}{n-m} \right] \end{aligned}$$

(b), cont'd ...

$$E^{(2)} = \frac{\lambda^2}{2m\omega^2}\left[\frac{n}{n-(n-1)} + \frac{n+1}{n-(n+1)}\right]$$

$$= \frac{\lambda^2}{2m\omega^2}\left(n-(n+1)\right)$$

$$\boxed{E^{(2)} = -\frac{\lambda^2}{2m\omega^2}}.$$

(c) Our Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x. \qquad (1)$$

We want to put this in the form

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x-c)^2 - a,$$

where c and a are constants. Expanding,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x^2 - 2cx + c^2) - a$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 - m\omega^2 cx + \frac{1}{2}m\omega^2c^2 - a.$$

Taking our original Hamiltonian in (1), we can add and subtract the constant, 'a'. This gives us

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \lambda x + a - a.$$

Then we can see

$$\lambda = -m\omega^2 c$$

or

$$c = -\frac{\lambda}{m\omega^2},$$

(c), cont'd...

and

$$a = \frac{1}{2} m\omega^2 c^2$$
$$= \frac{1}{2} m\omega^2 \left(-\frac{\lambda}{m\omega^2}\right)^2$$
$$= \frac{1}{2} m\omega^2 \cdot \frac{\lambda^2}{m^2\omega^4}$$
$$= \frac{\lambda^2}{2m\omega^2}.$$

Then our Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \left(x + \frac{\lambda}{m\omega^2}\right)^2 - \frac{\lambda^2}{2m\omega^2}.$$

Let $x' = x + \frac{\lambda}{m\omega^2}$. Then this becomes

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x'^2 - \frac{\lambda^2}{2m\omega^2}.$$

The Schrödinger equation tells us

$$H|n\rangle = E|n\rangle.$$

So

$$H|n\rangle = \left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x'^2 - \frac{\lambda^2}{2m\omega^2}\right)|n\rangle$$
$$= \left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x'^2\right)|n\rangle - \frac{\lambda^2}{2m\omega^2}|n\rangle$$
$$= \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle - \frac{\lambda^2}{2m\omega^2}|n\rangle$$
$$= \left[\hbar\omega\left(n + \frac{1}{2}\right) - \frac{\lambda^2}{2m\omega^2}\right]|n\rangle$$

Thus,

$$\boxed{E_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{\lambda^2}{2m\omega^2}}$$

(d)

We know

$$|n^{(1)}\rangle = \sum_{m\neq n} \frac{\langle m|H'|n\rangle}{E_n^{(0)} - E_m^{(0)}} |m^{(0)}\rangle$$

$$= \frac{\lambda}{\sqrt{2m\hbar\omega^3}} \sum_{m\neq n} \frac{\langle m|a+a^\dagger|n\rangle}{n-m} |m\rangle$$

$$= \frac{\lambda}{\sqrt{2m\hbar\omega^3}} \sum_{m\neq n} \frac{\sqrt{n}\,\langle m|n-1\rangle + \sqrt{n+1}\,\langle m|n+1\rangle}{n-m} |m\rangle$$

$$= \frac{\lambda}{\sqrt{2m\hbar\omega^3}} \left[\sum_{m\neq n} \frac{\sqrt{n}\,\delta_{m,n-1}}{n-m} |m\rangle + \sum_{m\neq n} \frac{\sqrt{n+1}\,\delta_{m,n+1}}{n-m} |m\rangle\right]$$

$$= \frac{\lambda}{\sqrt{2m\hbar\omega^3}} \left(\frac{\sqrt{n}}{n-(n-1)} |n-1\rangle + \frac{\sqrt{n+1}}{n-(n+1)} |n+1\rangle\right)$$

$$= \frac{\lambda}{\sqrt{2m\hbar\omega^3}} \left(\sqrt{n}\,|n-1\rangle - \sqrt{n+1}\,|n+1\rangle\right)$$

Then to first order,

$$\boxed{|n\rangle = |n^{(0)}\rangle + \frac{\lambda}{\sqrt{2m\hbar\omega^3}}\sqrt{n}\,|n^{(0)}-1\rangle - \frac{\lambda}{\sqrt{2m\hbar\omega^3}}\sqrt{n+1}\,|n^{(0)}+1\rangle}$$