3X01U

Problem 5: Magnetic Moments and Spin (10 pts)

Consider a spin 1/2 particle with a magnetic moment. We can write the interaction between the spin and an external magnetic field using the Hamiltonian:

$$H = -\gamma \vec{B} \cdot \vec{S} \tag{1}$$

where $\vec{B}$ is the external field, $\vec{S}$ is the spin operator for the particle, and γ is a real positive constant. In this problem, use the usual basis states that are eigenstates of S_z

$$S_z \chi_{\pm} = \pm \frac{\hbar}{2} \chi_{\pm}, \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

For this problem, assume the magnetic field lies in the x-z plane:

$$\vec{B} = B_x \hat{e}_x + B_z \hat{e}_z \tag{3}$$

(a) [1 pt] Solve for the eigenenergies for the Hamiltonian, showing your work. Explain the physics of your results.

(b) [2 pts] Any state of the spin can be written in the $\chi_{\pm}$ basis as:

$$\Psi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \tag{4}$$

Using the Hamiltonian, derive the first-order coupled differential equations that give the time dependence for $\alpha(t)$ and $\beta(t)$. In other words, derive the equations for $\dot{\alpha}(t)$ and $\dot{\beta}(t)$.

(c) [2 pts] Show that you can re-write your results from part (b) as two uncoupled second-order differential equations:

$$\begin{aligned} \ddot{\alpha}(t) &= -\frac{\gamma^2 B_T^2}{4} \alpha(t) \\ \ddot{\beta}(t) &= -\frac{\gamma^2 B_T^2}{4} \beta(t) \end{aligned} \tag{5}$$

where $B_T = \sqrt{B_x^2 + B_z^2}$ is the magnitude of the total magnetic field. How is this result related to what you found in part (a)?

Of course, the solutions to these equations are:

$$\begin{aligned} \alpha(t) &= C_1 \cos(\omega t) + C_2 \sin(\omega t) \\ \beta(t) &= C_3 \cos(\omega t) + C_4 \sin(\omega t) \end{aligned} \tag{6}$$

with $\omega = \frac{\gamma B_T}{2}$.

(d) [3 pts] Consider the situation where the spin is in the spin-up S_z state χ_+ at time $t = 0$. Using the boundary conditions at time $t = 0$, determine the values for the constants C_1, C_2, C_3, C_4 that will solve for the time-dependence of the state. Remember that the equations in part (c) are second-order, so you need two boundary conditions at $t = 0$ for each.

(e) [2 pt] Write down the time-dependent probabilities, $P_{\pm}$ of the spin being in the spin-up and spin-down S_z states. Show that your results are correct in the two cases where $B_x = 0$ and $B_z = 0$.

(a)

We know

$$H = -\gamma \vec{B} \cdot \vec{S}$$

where

$$\vec{B} = B_x \hat{i} + B_z \hat{k}.$$

So

$$H = -\gamma \left[B_x S_x + B_z S_z \right]$$

$$= -\gamma \left[\frac{B_x \hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{B_z \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$= -\frac{\gamma \hbar}{2} \begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix}$$

Then we can find the eigenvalues:

$$\begin{vmatrix} -\frac{\gamma\hbar}{2} B_z - E_n & -\frac{\gamma\hbar}{2} B_x \\ -\frac{\gamma\hbar}{2} B_x & \frac{\gamma\hbar}{2} B_z - E_n \end{vmatrix} = 0$$

$$\left(-\frac{\gamma\hbar}{2} B_z - E_n\right)\left(\frac{\gamma\hbar}{2} B_z - E_n\right) - \left(-\frac{\gamma\hbar}{2} B_x\right)\left(-\frac{\gamma\hbar}{2} B_x\right) = 0$$

$$-\frac{\gamma^2\hbar^2}{4} B_z^2 + \frac{\gamma\hbar}{2} B_z E_n - \frac{\gamma\hbar}{2} B_z E_n + E_n^2 - \left(\frac{\gamma\hbar}{2} B_x\right)^2 = 0$$

$$E_n^2 - \frac{\gamma^2\hbar^2}{4} B_z^2 - \frac{\gamma^2\hbar^2}{4} B_x^2 = 0$$

(a), cont'd...

$$E_n^2 = \frac{\gamma^2\hbar^2}{4}\left(B_x^2 + B_z^2\right)$$

$$\boxed{E_n = \pm\frac{\gamma\hbar}{2}\left(B_x^2 + B_z^2\right)^{1/2}}$$

(b)

The time-dependent Schrödinger equation is given by

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi.$$

So

$$i\hbar\begin{pmatrix}\dot{\alpha}(t)\\ \dot{\beta}(t)\end{pmatrix} = -\frac{\gamma\hbar}{2}\begin{pmatrix}B_z & B_x\\ B_x & -B_z\end{pmatrix}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}$$

$$= -\frac{\gamma\hbar}{2}\begin{pmatrix}B_z\alpha(t) + B_x\beta(t)\\ B_x\alpha(t) - B_z\beta(t)\end{pmatrix}$$

and our expressions are

$$\boxed{\begin{aligned}\dot{\alpha}(t) &= \frac{i\gamma}{2}\left(B_z\alpha(t) + B_x\beta(t)\right)\\ \dot{\beta}(t) &= \frac{i\gamma}{2}\left(B_x\alpha(t) - B_z\beta(t)\right)\end{aligned}} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

(c)

Taking the derivative of Eqn. (1) from part (b), we have

$$\ddot{\alpha}(t) = \frac{i\gamma}{2}\left(B_z \dot{\alpha}(t) + B_x \dot{\beta}(t)\right).$$

Plugging $\dot{\alpha}(t)$ and $\dot{\beta}(t)$ back into this equation, we get

$$\ddot{\alpha}(t) = \frac{i\gamma}{2}\left[B_z\left(\frac{i\gamma}{2}\left(B_z\alpha(t) + B_x\beta(t)\right)\right) + B_x\left(\frac{i\gamma}{2}\left(B_x\alpha(t) - B_z\beta(t)\right)\right)\right]$$

$$= -\frac{\gamma^2}{4}\left[B_z^2\alpha(t) + B_xB_z\beta(t) + B_x^2\alpha(t) - B_xB_z\beta(t)\right]$$

$$\boxed{\ddot{\alpha}(t) = -\frac{\gamma^2}{4}B_T^2\,\alpha(t)}$$

where $B_T = \sqrt{B_x^2 + B_z^2}$. Similarly for $\ddot{\beta}(t)$...

$$\ddot{\beta}(t) = \frac{i\gamma}{2}\left(B_x\dot{\alpha}(t) - B_z\dot{\beta}(t)\right)$$

$$\ddot{\beta}(t) = \frac{i\gamma}{2}\left[B_x\left(\frac{i\gamma}{2}\left(B_z\alpha(t) + B_x\beta(t)\right)\right) - B_z\left(\frac{i\gamma}{2}\left(B_x\alpha(t) - B_z\beta(t)\right)\right)\right]$$

$$= -\frac{\gamma^2}{4}\left[B_xB_z\alpha(t) + B_x^2\beta(t) - B_xB_z\alpha(t) + B_z^2\beta(t)\right]$$

$$\boxed{\ddot{\beta}(t) = -\frac{\gamma^2}{4}B_T^2\,\beta(t)}.$$