F 2013

Problem 1: 1D Square Wells

(a) [1 pt] Consider an electron confined to an infinitely deep 1D well with walls at $x = 0$ and $x = L$. In the ground state, the electron has an energy of 2.5 eV (the bottom of the well is defined as $V = 0$). What is the width of the well?

(b) [1 pt] A proton is confined to an infinite 1D square well of width 10 fm. What is the wavelength (or frequency) of a photon emitted when the proton undergoes a transition from the first excited state to the ground state of the well?

(c) [2 pt] Sketch the probability density as a function of x for the first 3 energy eigenstates for an electron in an infinite well of width L. Describe qualitatively (or draw) how the probability densities for these states will differ (from the infinite well case) for a square well with an infinite potential barrier at $x = 0$ and a finite potential barrier at $x = L$.

(d) [2 pt] Consider an electron in the nth energy eigenstate of an infinitely deep well with walls at $x = 0$ and $x = L$. Calculate the probability that the electron will be measured between $x = 0$ and $x = \epsilon\ L$, with $0 < \epsilon < 1$. Your answer should be a function of both n and ϵ.

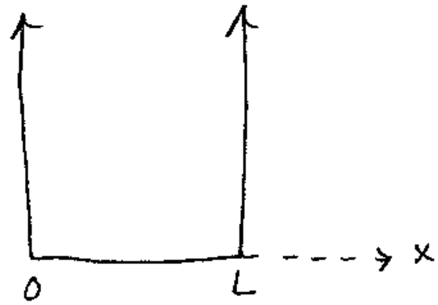
Give a physical explanation for your solution as $n \to \infty$.

(e) [2 pt] The electron is in the ground state of the infinite well when the wall at $x = L$ is very suddenly moved to $x = 2L$. What is the probability that the electron will be found in the ground state of the expanded box?

(f) [1 pt] What energy eigenstate in the expanded box will have the highest probability of being occupied by the electron? What is this probability? Hint: You should be able to determine this result without doing an integral, but you should explain your answer.

(g) [1 pt] Suppose the electron is in the ground state of the infinitely deep well when the walls are suddenly removed completely. Write down an expression for the probability distribution for the momentum of the freed electron. Setup but do not solve the integral.

(a)



The energy, in general, is given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

In the ground state, $n=1$. So we have

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2} = 2.5\,eV.$$

We know $m_e = 0.511\ MeV/c^2$ and $\hbar = 6.582\times10^{-16}\ eV\cdot s$. but we can also use $\hbar c = 0.197\ eV\cdot\mu m$. So we can rearrange to solve for L, the width of the well.

$$2mL^2 = \frac{\pi^2\hbar^2}{2.5[eV]}$$

$$L^2 = \frac{\pi^2\hbar^2}{5m_e\,[eV]}$$

$$= \frac{\pi^2(\hbar c)^2}{5(0.511\times10^6[eV])[eV]}$$

$$= \frac{\pi^2(0.197\,[eV\cdot\mu m])^2}{2.56\times10^6\,[eV^2]}$$

$$= \frac{(3.14)^2\,0.0388\,[eV^2\cdot\mu m^2]}{2.56\times10^6\,[eV^2]}$$

$$= \frac{0.385\ eV^2\cdot\mu m^2}{2.56\times10^6\ eV^2}$$

(a), cont'd...

$$L^2 = 1.5 \times 10^{-7}\ \mu m^2$$
$$= 15 \times 10^{-8}\ \mu m^2$$
$$L = \sqrt{15} \times 10^{-4}\ \mu m$$
$$= \sqrt{3} \cdot \sqrt{5} \times 10^{-4}\ \mu m$$
$$\approx (1.7)(2.2) \times 10^{-4}\ \mu m$$
$$L \approx 3.7 \times 10^{-4}\ \mu m$$

or

$$\boxed{L \approx 0.37\ nm}$$

Note that this is approximate since a calculator has not been used.

(b)

The first excited state corresponds to $n=2$. For a transition $n=2 \rightarrow n=1$, we have

$$\Delta E = hf = \frac{\left((2)^2-(1)^2\right)\pi^2\hbar^2}{2mL^2}$$

$$= \frac{3\pi^2\hbar^2}{2mL^2}.$$

We want to determine the frequency of the transition. We know

$$\hbar = h/2\pi$$
$$h = 2\pi\hbar,$$

so

$$f = \frac{3\pi\hbar}{4mL^2}.$$

We are given $L = 10\,fm = 10\times10^{-15}\,m$. We know that $m_p \approx 938\ MeV/c^2$ and $\hbar c = 0.197\ eV\cdot\mu m$. So

$$f = \frac{3\pi(\hbar c)c}{4(938[MeV])L^2}$$

$$= \frac{3\pi\left(0.197[eV\cdot\mu m]\right)\left(3\times10^8[m\cdot s^{-1}]\right)}{4\left(938\times10^6[eV]\right)\left(10\times10^{-15}[m]\right)^2}$$

$$= \frac{3\pi\left(0.591\times10^2[eV\cdot m^2\cdot s^{-1}]\right)}{4\left(938\times10^6[eV]\right)\left(100\times10^{-30}[m^2]\right)}$$

$$= \frac{3\pi\left(0.591\times10^2[eV\cdot m^2\cdot s^{-1}]\right)}{4\left(938\times10^6[eV]\right)\left(1\times10^{-28}[m^2]\right)}$$

$$= \frac{3\pi\left(0.591\times10^2[eV\cdot m^2\cdot s^{-1}]\right)}{3752\times10^{-22}[eV\cdot m^2]}$$

$$= \frac{3\pi\left(0.591\times10^2[eV\cdot m^2\cdot s^{-1}]\right)}{3.752\times10^{-19}[eV\cdot m^2]}$$

(b), cont'd...

$$f = \frac{(9.42)(0.591)}{3.752} \times 10^{21}\,[s^{-1}]$$

$$= \frac{5.567}{3.752} \times 10^{21} \text{ Hz}$$

$$\approx \frac{5.6}{3.8} \times 10^{21} \text{ Hz}$$

$$\boxed{f \approx 1.4 \times 10^{21} \text{ Hz}}$$

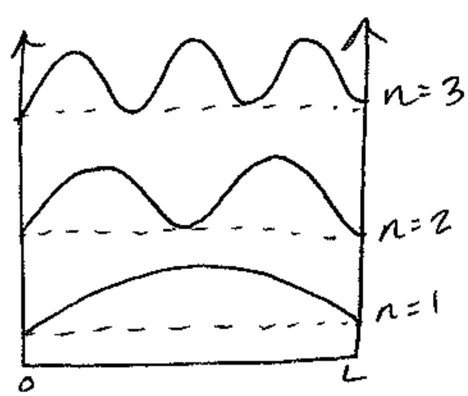
Again, this is very approximate since no calculator was used.

(c)

In general, the probability density is given by

$$\psi^* \psi .$$

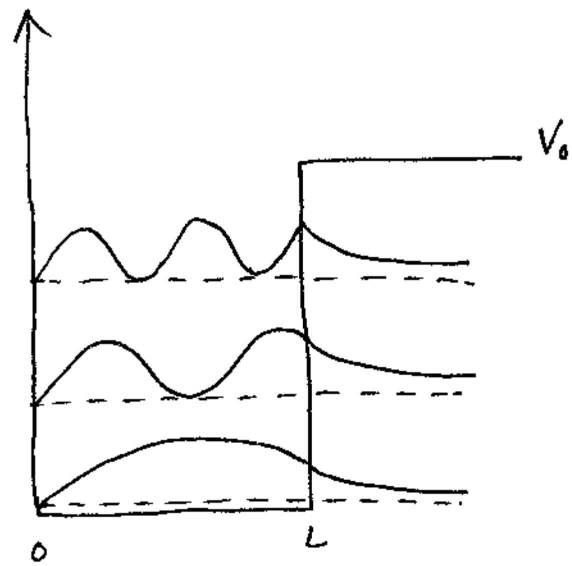
For $n=1$, $n=2$, and $n=3$, the first three eigenstates for an electron in an infinite well, we have the following:



(The vertical scaling is arbitrary.)

(b), cont'd

For the half-infinite well, we expect exponential decay after the finite barrier if $E < V_0$. So



(d)

The wavefunction is given by

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right).$$

In general, the probability is given by

$$P_{a<x<b} = \int_a^b |\psi(x)|^2 dx.$$

The probability of measuring the electron between $x=0$ and $x = \varepsilon L$, where $0 < \varepsilon < 1$, is then

$$P = \int_0^{\varepsilon L} \left| \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right|^2 dx$$

$$= \frac{2}{L}\int_0^{\varepsilon L} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^{\varepsilon L} \frac{1}{2}\left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) dx$$

$$= \frac{1}{L}\int_0^{\varepsilon L} \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) dx$$

$$= \frac{1}{L}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]\Bigg|_0^{\varepsilon L}$$

$$= \frac{1}{L}\left[\varepsilon L - \frac{L}{2n\pi}\sin(2n\pi\varepsilon)\right]$$

$$\boxed{P = \varepsilon - \frac{\sin(2n\pi\varepsilon)}{2n\pi}}$$

(d), cont'd...

In general, we know

$$\lim_{x\to\infty} \frac{\sin(x)}{x} = 0,$$

so

$$\lim_{n\to\infty} P = \lim_{n\to\infty} \left[\varepsilon - \frac{\sin(2n\pi\varepsilon)}{2n\pi} \right]$$

$$\boxed{\lim_{n\to\infty} P = \varepsilon}$$

This is the classical result.

(e)

The new wave-function in this extended box is

$$\psi^{ext}(x) = \sqrt{\frac{1}{L}}\sin\left(\frac{n\pi x}{2L}\right).$$

In the ground state, we have

$$\psi^{ext}_{ground}(x) = \sqrt{\frac{1}{L}}\sin\left(\frac{\pi x}{2L}\right).$$

The probability that the electron will be found in the ground state of the expanded box is then

$$P = \left| \int_0^L \psi^{ext *}_{ground}(x)\,\psi_{ground}(x)\,dx \right|^2$$

$$= \left| \frac{\sqrt{2}}{L} \int_0^L \sin\left(\frac{\pi x}{2L}\right)\sin\left(\frac{\pi x}{L}\right)dx \right|^2$$

$$= \left| \frac{\sqrt{2}}{L} \int_0^L \frac{1}{2}\left[\cos\left(\frac{\pi x}{2L} - \frac{\pi x}{L}\right) - \cos\left(\frac{\pi x}{2L} + \frac{\pi x}{L}\right)\right]dx \right|^2$$

$$= \left| \frac{\sqrt{2}}{2L} \int_0^L \left[\cos\left(\frac{\pi x}{2L}\right) - \cos\left(\frac{3\pi x}{2L}\right)\right]dx \right|^2$$

$$= \left| \frac{\sqrt{2}}{2L} \left[\frac{2L}{\pi}\sin\left(\frac{\pi x}{2L}\right) - \frac{2L}{3\pi}\sin\left(\frac{3\pi x}{2L}\right)\right]\Bigg|_0^L \right|^2$$

$$= \left| \frac{\sqrt{2}}{2L} \left[\frac{2L}{\pi}\sin\left(\frac{\pi}{2}\right) - \frac{2L}{3\pi}\sin\left(\frac{3\pi}{2}\right)\right] \right|^2$$

$$= \left| \frac{\sqrt{2}}{2L} \left(\frac{2L}{\pi} + \frac{2L}{3\pi}\right) \right|^2$$

$$= \left| \frac{4\sqrt{2}}{3\pi} \right|^2$$

$$\boxed{P = \frac{32}{9\pi^2}}$$