F2013

Problem 6: Harmonic Oscillators in 1D

A quantum harmonic oscillator is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \tag{1}$$

where p is momentum, x is position, m is mass, and ω is the oscillation frequency.

The Hamiltonian has the usual eigenstates and energies:

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle, \quad n = 0, 1, 2, ... \tag{2}$$

Let the system be perturbed by a potential in the form $V = Ax^2$ where A is a real constant.

(a) [2 pt] What is the change in the energy of the unperturbed eigenstates $|n\rangle$ to first order in A? Show your work.

(b) [2 pt] If the perturbation is time-dependent, $V(t) = A(t)x^2$, it can cause transitions between the harmonic oscillator states. To study these transitions, it is helpful to use the time-dependent expansion:

$$|\psi(t)\rangle = \sum_{n'} c_{n'}(t) e^{-\frac{i}{\hbar}E_{n'}t}|n'\rangle \tag{3}$$

The $c_{n'}(t)$ are time-dependent probability amplitudes for the states $|n'\rangle$ and the energies $E_{n'}$ are the unperturbed eigenenergies. Use the Schroedinger equation to show that the expansion amplitudes satisfy a set of coupled equations:

$$i\hbar\frac{\partial}{\partial t}c_n(t) = \sum_{n'} c_{n'}(t) e^{-\frac{i}{\hbar}(E_{n'}-E_n)t}\langle n|V(t)|n'\rangle \tag{4}$$

(c) [3 pt] Consider the case where the oscillator starts at time $t = 0$ in the ground state, $c_n(t = 0) = \delta_{n,0}$. Use the result from (b) to write down the time dependence of the excited state probability amplitudes to first order in V, $c_n^{(1)}(t)$, $n > 0$. This will be an integral equation, as we have not yet defined $A(t)$.

Show that, to first order, there is a transition only to the $n = 2$ excited state.

(d) [3 pt] Finally, consider a time dependent perturbation with $A(t)$ of the form

$$A(t) = Ae^{-i\Omega t}e^{-\Gamma t} \tag{5}$$

Ω and Γ being real and positive.

Compute the probability that the $n = 2$ state is populated for $t \to \infty$, and explain the dependence of your result on Ω and Γ.

Note: In this problem, it is useful to use

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - i\frac{\lambda}{\hbar}p\right), \quad a = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} + i\frac{\lambda}{\hbar}p\right) \tag{6}$$

where $\lambda = \sqrt{\frac{\hbar}{m\omega}}$ is the length scale in the problem.

You do not need to derive the properties of these two operators, but you should state the results you are using.

(a)

We know the first order correction is given by

$$E^{(1)} = \langle n^{(0)} | V' | n^{(0)} \rangle,$$

where $V' = Ax^2$, the perturbation, and $|n^{(0)}\rangle$ is the unperturbed state. So

$$\begin{aligned}
E^{(1)} &= \langle n^{(0)} | Ax^2 | n^{(0)} \rangle \\
&= \langle n^{(0)} | A\left(\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\right)^2 | n^{(0)} \rangle \\
&= \frac{A\hbar}{2m\omega} \langle n^{(0)} | aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger | n^{(0)} \rangle \\
&= \frac{A\hbar}{2m\omega} \Big[\sqrt{n(n-1)} \langle n | n-2 \rangle + (n+1) \langle n | n \rangle \\
&\qquad + n \langle n | n \rangle + \sqrt{(n+1)(n+2)} \langle n | n+2 \rangle \Big] \\
&= \frac{A\hbar}{2m\omega}(n+1+n) \\
&= \frac{A\hbar}{2m\omega}(2n+1)
\end{aligned}$$

and our change in the energy of the unperturbed states to first order is

$$\boxed{E^{(1)} = \frac{A\hbar}{m\omega}\left(n + \tfrac{1}{2}\right)}$$

(b)

It is easier to work in the interaction picture. We have

$$|\psi(t)\rangle_I = e^{iH_0 t/\hbar} |\psi(t)\rangle_S$$

where

$$e^{iH_0 t/\hbar} |\psi(t)\rangle_S = e^{iH_0 t/\hbar} |n\rangle$$
$$= e^{iE_n t/\hbar} |n\rangle .$$

So we can rewrite our expansion as

$$|\psi(t)\rangle_I = \sum_n c_n(t) e^{-\frac{i}{\hbar}E_n t} e^{iE_n t/\hbar} |n\rangle$$
$$= \sum_n c_n(t) |n\rangle .$$

Our interaction picture Schrödinger equation is

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = V_I |\psi(t)\rangle$$

where

$$V_I = e^{iH_0 t/\hbar} V' e^{-iH_0 t/\hbar}$$

and

$$V' = A(t) x^2 .$$

Then

$$i\hbar \frac{\partial}{\partial t} \sum_n c_n(t) |n\rangle = e^{iH_0 t/\hbar} V' e^{-iH_0 t/\hbar} \sum_n c_n(t) |n\rangle$$

$$i\hbar \sum_n \dot{c}_n(t) |n\rangle = \sum_n c_n(t) e^{iH_0 t/\hbar} V' e^{-iH_0 t/\hbar} |n\rangle$$

(b), cont'd...

Contracting with the state $|m\rangle$, we have

$$i\hbar \sum_n \dot{c}_n(t) \langle m|n\rangle = \sum_n c_n(t) \langle m| e^{iH_0 t/\hbar} V' e^{-iH_0 t/\hbar} |n\rangle.$$

But

$$e^{iH_0 t/\hbar} |m\rangle = e^{iE_m t/\hbar} |m\rangle$$
$$e^{-iH_0 t/\hbar} |n\rangle = e^{-iE_n t/\hbar} |n\rangle,$$

so

$$i\hbar \sum_n \dot{c}_n(t) \delta_{mn} = \sum_n c_n(t) \langle m| e^{iE_m t/\hbar} V' e^{-iE_n t/\hbar} |n\rangle$$
$$= \sum_n c_n(t) e^{-i(E_n - E_m)t/\hbar} \langle m|V'|n\rangle$$
$$i\hbar \dot{c}_m(t) = \sum_n c_n(t) e^{-\frac{i}{\hbar}(E_n - E_m)t} \langle m|V'|n\rangle$$

or, in the case where we let $n = n'$ and $m = n$,

$$\boxed{i\hbar \frac{\partial}{\partial t} c_n(t) = \sum_{n'} c_{n'}(t) e^{-\frac{i}{\hbar}(E_{n'} - E_n)t} \langle n|V(t)|n'\rangle}$$

(c)

The excited state probability amplitude is given by

$$c_n^{(1)}(t) = \frac{-i}{\hbar}\int_0^t e^{-\frac{i}{\hbar}(E_0 - E_n)t} \langle n|V(t)|0\rangle\, dt.$$

To show that there is a transition only to the $n=2$ excited state, we look at the matrix elements of $V(t)$. So

$$\begin{aligned}
\langle n|V(t)|0\rangle &= \langle n|A(t)x^2|0\rangle \\
&= A(t)\langle n|x^2|0\rangle \\
&= \frac{A(t)\hbar}{2m\omega}\langle n|aa + aa^+ + a^+a + a^+a^+|0\rangle \\
&= \frac{A(t)\hbar}{2m\omega}\Big[\langle n|aa|0\rangle + \langle n|aa^+|0\rangle + \langle n|a^+a|0\rangle \\
&\qquad + \langle n|a^+a^+|0\rangle\Big] \\
&= \frac{A(t)\hbar}{2m\omega}\left[0 + \langle n|0\rangle + 0 + \sqrt{2}\langle n|2\rangle\right]
\end{aligned}$$

We obviously cannot have $|n\rangle = 0$ because a transition cannot occur within a single level. Thus, the only other option is that $|n\rangle = 2$, as expected.

(d)

Now we let

$$A(t) = Ae^{-i\Omega t}e^{-\Gamma t},$$

where Ω and Γ are constants. So

$$C_2^{(1)}(t) = -\frac{i}{\hbar}\int_0^\infty e^{-\frac{i}{\hbar}(E_0-E_2)t}\langle 2|Ae^{-i\Omega t}e^{-\Gamma t}x^2|0\rangle dt$$

$$= -\frac{i\sqrt{2}}{\hbar}\int_0^\infty e^{-\frac{i}{\hbar}(E_0-E_2)t}Ae^{-i\Omega t}e^{-\Gamma t}dt$$

$$= -\frac{iA\sqrt{2}}{\hbar}\int_0^\infty e^{-\frac{i}{\hbar}(E_0-E_2)t}e^{-i\Omega t}e^{-\Gamma t}dt$$

$$= -\frac{iA\sqrt{2}}{\hbar}\int_0^\infty e^{-\frac{i}{\hbar}(E_0-E_2-\Omega\hbar-i\Gamma\hbar)t}dt$$

$$= -\frac{iA\sqrt{2}}{\hbar}\cdot\left[-\frac{\hbar}{i(E_0-E_2-\Omega\hbar-i\Gamma\hbar)}e^{-\frac{i}{\hbar}(E_0-E_2-\Omega\hbar-i\Gamma\hbar)t}\right]\Bigg|_0^\infty$$

$$= -\frac{iA\sqrt{2}}{\hbar}\left[\frac{\hbar}{i(E_0-E_2-\Omega\hbar-i\Gamma\hbar)}\right]$$

$$C_2^{(1)}(t) = \frac{-A\sqrt{2}}{E_0-E_2-\Omega\hbar-i\Gamma\hbar}, \qquad (t\to\infty)$$

Then the probability that $|n=2\rangle$ is populated for $t\to\infty$ is

$$P_{|2\rangle}(t\to\infty) = |C_2^{(1)}|^2$$

$$= \left(\frac{-A\sqrt{2}}{E_0-E_2-\Omega\hbar-i\Gamma\hbar}\right)\left(\frac{-A\sqrt{2}}{E_0-E_2-\Omega\hbar+i\Gamma\hbar}\right)$$

$$\boxed{P_{|2\rangle}(t\to\infty) = \frac{2A^2}{E_0^2+E_2^2+\Omega^2\hbar^2-2(E_0E_2+E_0\Omega\hbar+E_2\Omega\hbar)+\Gamma^2\hbar^2}}$$