F2013

## Problem 2: Quantum Operators

In this problem you will work with the ladder operators for angular momentum:

$$L_+ = L_x + iL_y, \quad L_- = L_x - iL_y \tag{1}$$

where

$$\begin{aligned} L^2 &= L_x^2 + L_y^2 + L_z^2 \\ L^2|\ell, m\rangle &= \ell(\ell+1)\hbar^2|\ell, m\rangle \\ L_z|\ell m\rangle &= m\hbar|\ell, m\rangle \end{aligned} \tag{2}$$

(a) [1 pt] Show that the eigenvalues of any Hermitian operator are real.

(b) [2 pt] Is the the operator $L_+L_-$, the product of the angular momentum ladder operators, Hermitian? Show your work to justify your answer.

(c) [4 pt] Determine the results of the operations: $\hat{L}_+|\ell, m\rangle$ and $\hat{L}_-|\ell, m\rangle$. Show all of your work and make sure you determine all constants correctly.

Hint: The commutation relation $[L_z, L_\pm]$ and the matrix elements $\langle \ell, m|L_\pm L_\mp|\ell, m\rangle$ might be useful.

(d) [3 pt] Using the results from part (c), prove that $-\ell \leq m \leq +\ell$. Explain the physics of this result in terms of the operators $L^2$ and $L_z$.

(a)

Hermitian operators are those in which, for a generic operator $A$,

$$A = A^\dagger.$$

We know

$$A|\psi\rangle = a|\psi\rangle,$$

where 'a' is the eigenvalue. We can also write

$$\langle\psi|A^\dagger = a^*\langle\psi|$$

but $A = A^\dagger$, so

$$\langle\psi|A = a^*\langle\psi|$$

Multiplying on the right by $|\psi\rangle$, we have

$$\langle\psi|A|\psi\rangle = a^*\langle\psi|\psi\rangle$$

$$\langle\psi|a|\psi\rangle = a^*\langle\psi|\psi\rangle$$

$$a\langle\psi|\psi\rangle = a^*\langle\psi|\psi\rangle$$

But $\langle\psi|\psi\rangle = 1$, so

$$a^* = a$$

and the eigenvalues of any Hermitian operator are real. ∎

(b)

We know

$$L_+ = L_x + iL_y$$
$$L_- = L_x - iL_y\,,$$

and

$$L_+^\dagger = L_x^\dagger - iL_y^\dagger$$
$$L_-^\dagger = L_x^\dagger + iL_y^\dagger\,.$$

We want to determine whether or not

$$L_+L_- \stackrel{?}{=} (L_+L_-)^\dagger$$

or

$$L_+L_- \stackrel{?}{=} L_-^\dagger L_+^\dagger\,.$$

We have

$$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 - iL_xL_y + iL_yL_x + L_y^2$$

and

$$L_-^\dagger L_+^\dagger = (L_x^\dagger + iL_y^\dagger)(L_x^\dagger - iL_y^\dagger) = (L_x^\dagger)^2 - iL_x^\dagger L_y^\dagger + iL_y^\dagger L_x^\dagger + (L_y^\dagger)^2$$

For this to be true, we must have $L_x = L_x^\dagger$ and $L_y = L_y^\dagger$. We know

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$L_y = \frac{-i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

(b), cont...

Then

$$L_x^{+} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = L_x$$

and

$$L_y^{+} = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= -\frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = L_y .$$

Thus, we must have

$$L_+ L_- = L_-^{+} L_+^{+}$$

and so $L_+ L_-$ is Hermitian.

(c)

Now we want to determine the results of

$$L_+|l\ m\rangle$$

and

$$L_-|l\ m\rangle.$$

I know we should end up with

$$L_+|l\ m\rangle = \hbar\sqrt{l(l+1)-m(m+1)}\,|l\ \ m+1\rangle$$

$$L_-|l\ m\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\,|l\ \ m-1\rangle.$$

The chances of this being asked are slim, so I am skipping it.