S 2008

Problem 2: The Harmonic Oscillator (10 Points):

The normalized wave functions for the one-dimensional quantum harmonic oscillator can be written as,

$$\Psi_n(x) = \left(\frac{\sqrt{\alpha}}{2^n n! \sqrt{\pi}}\right)^{1/2} e^{-\alpha x^2/2} H_n(\sqrt{\alpha} x),$$

where n is the principle quantum number of the oscillator, H_n is the n^{th} order Hermite polynomial, $\alpha = \omega m/\hbar$, ω is the oscillator frequency, and m is its mass. The following equations may be useful,

$$H_{n+1}(q) + 2nH_{n-1}(q) - 2qH_n(q) = 0$$

$$\frac{dH_n(q)}{dq} = 2nH_{n-1}(q)$$

and

$$\begin{aligned} \langle H_n | q H_{n+1} \rangle &= 2^n (n+1)! \sqrt{\pi} \\ \langle H_n | q H_n \rangle &= 0 \\ \langle H_n | q H_{n-1} \rangle &= 2^{n-1} n! \sqrt{\pi} \end{aligned}$$

(a) 1. Calculate the expectation value of x and x^2 for the n^{th} state of the harmonic oscillator, where x is the position. **(2 Points)**

(b) 2. Calculate the expectation value of p and p^2 for the n^{th} state of the harmonic oscillator, where p is the momentum. **(2 Points)**

(c) 3. Calculate Δx and Δp for the n^{th} state. What is the uncertainty product $(\Delta x \Delta p)$ for the oscillator? **(2 Points)**

(d) 4. Calculate the expectation value of the kinetic energy and the potential energy of the n^{th} state of the oscillator. Show that the sum of the expectation value of the kinetic and potential energies are equal to the total energy of the n^{th} state. **(2 Points)**

(e) 5. How does the uncertainty principle relate to the fact that the energy is not zero in the ground state? Explain and interpret your answer to receive credit. **(2 Points)**

(a)

This is much simpler if we work with raising and lowering operators, and use Dirac notation.

We know

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{+})$$
$$p = -i\sqrt{\frac{\hbar m\omega}{2}}\,(a - a^{+}).$$

We also know

$$a^{+}|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle$$
$$a|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle.$$

So...

$$\langle X\rangle = \langle\psi_n|x|\psi_n\rangle$$
$$= \langle\psi_n|\sqrt{\frac{\hbar}{2m\omega}}(a+a^{+})|\psi_n\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle\psi_n|a|\psi_n\rangle + \langle\psi_n|a^{+}|\psi_n\rangle\right]$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\langle\psi_n|\psi_{n-1}\rangle + \sqrt{n+1}\,\langle\psi_n|\psi_{n+1}\rangle\right]$$

$$\boxed{\langle X\rangle = 0}$$

$$\langle X^2\rangle = \langle\psi_n|X^2|\psi_n\rangle$$
$$= \frac{\hbar}{2m\omega}\left[\langle\psi_n|aa + aa^{+} + a^{+}a + a^{+}a^{+}|\psi_n\rangle\right]$$
$$= \frac{\hbar}{2m\omega}\left[\langle\psi_n|aa|\psi_n\rangle + \langle\psi_n|aa^{+}|\psi_n\rangle + \langle\psi_n|a^{+}a|\psi_n\rangle + \langle\psi_n|a^{+}a^{+}|\psi_n\rangle\right]$$
$$= \frac{\hbar}{2m\omega}\left[\sqrt{n(n-1)}\,\langle\psi_n|\psi_{n-2}\rangle + (n+1)\langle\psi_n|\psi_n\rangle + n\,\langle\psi_n|\psi_n\rangle + \sqrt{(n+1)(n+2)}\,\langle\psi_n|\psi_{n+2}\rangle\right]$$

$$\boxed{\langle X^2\rangle = \frac{\hbar}{2m\omega}(2n+1)}$$

(b)

$$\langle p \rangle = \langle \psi_n | p | \psi_n \rangle$$
$$= \langle \psi_n | -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^+) | \psi_n \rangle$$
$$= -i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle \psi_n | a | \psi_n \rangle - \langle \psi_n | a^+ | \psi_n \rangle\right]$$
$$= -i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n}\langle \psi_n | \psi_{n-1} \rangle - \sqrt{n+1}\langle \psi_n | \psi_{n+1} \rangle\right]$$

$$\boxed{\langle p \rangle = 0}$$

$$\langle p^2 \rangle = \langle \psi_n | p^2 | \psi_n \rangle$$
$$= -\frac{\hbar m\omega}{2}\left[\langle \psi_n | aa - aa^+ - a^+a + a^+a^+ | \psi_n \rangle\right]$$
$$= -\frac{\hbar m\omega}{2}\left[\langle \psi_n | -aa^+ | \psi_n \rangle + \langle \psi_n | -a^+a | \psi_n \rangle\right]$$
$$= \frac{\hbar m\omega}{2}\left[(n+1)\langle \psi_n | \psi_n \rangle + n\langle \psi_n | \psi_n \rangle\right]$$

$$\boxed{\langle p^2 \rangle = \frac{\hbar m\omega}{2}(2n+1)}$$

(c)

Then since

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$
$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2},$$

we have

$$\boxed{\Delta x = \left[\frac{\hbar}{2m\omega}(2n+1)\right]^{1/2}}$$

$$\boxed{\Delta p = \left[\frac{\hbar m\omega}{2}(2n+1)\right]^{1/2}}$$

(c), cont'd...

Then we have

$$\Delta x \Delta p = \left[\frac{\hbar}{2m\omega}(2n+1)\right]^{1/2}\left[\frac{\hbar m\omega}{2}(2n+1)\right]^{1/2}$$

$$\boxed{\Delta x \Delta p = (2n+1)\frac{\hbar}{2}}$$

When $n=0$ (ground state), we get the familiar result

$$\Delta x \Delta p = \frac{\hbar}{2}.$$

(d)

The kinetic energy operator is

$$T = \frac{p^2}{2m}.$$

So

$$\langle T\rangle = \langle \psi_n | \frac{p^2}{2m} | \psi_n\rangle$$
$$= \frac{1}{2m}\langle \psi_n | p^2 | \psi_n\rangle$$
$$= \frac{1}{2m}\left(\frac{\hbar m\omega}{2}(2n+1)\right)$$

$$\boxed{\langle T\rangle = \frac{\hbar\omega}{4}(2n+1)}$$

(d), cont'd...

The potential energy operator is

$$V = \frac{1}{2} m\omega^2 x^2 .$$

So

$$\langle V \rangle = \langle \psi_n | \tfrac{1}{2} m\omega^2 x^2 | \psi_n \rangle$$
$$= \frac{1}{2} m\omega^2 \langle \psi_n | x^2 | \psi_n \rangle$$
$$= \frac{1}{2} m\omega^2 \left(\frac{\hbar}{2m\omega} (2n+1) \right)$$
$$\boxed{\langle V \rangle = \frac{\hbar\omega}{4} (2n+1)} .$$

Then

$$\langle T \rangle + \langle V \rangle = \frac{\hbar\omega}{4}(2n+1) + \frac{\hbar\omega}{4}(2n+1)$$
$$= \frac{\hbar\omega}{2}(2n+1)$$
$$= (n + 1/2)\hbar\omega$$

This is indeed the total energy of the n^{th} state.

(e)

The lowest (ground-state) energy is given by $E_0 = \frac{\hbar\omega}{2}$. This is the state of minimum uncertainty, with

$$\Delta x \Delta p = \hbar/2 .$$

(e), cont'd...

We know the total energy is

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Writing in terms of Δx, the uncertainty, we have

$$\Delta p = \frac{\hbar}{2\Delta x}$$

$$E = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2$$

The minimum occurs when

$$\frac{dE}{dx} = 0.$$

So

$$\frac{dE}{dx} = -\frac{\hbar^2}{4m(\Delta x)^3} + m\omega^2 \Delta x = 0$$

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}.$$

Plugging this into E, our minimum value allowed is

$$E_0 = \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}.$$

Thus, the uncertainty essentially requires that the energy is not zero in the ground state.