PROBLEM 2: Harmonic Oscillator with Two Particles

Consider a Hamiltonian for two non-interacting particles:

$$\begin{aligned} H(1,2) &= \frac{P_1^2}{2m} + \frac{1}{2}m\omega_1^2 X_1^2 + \frac{P_2^2}{2m} + \frac{1}{2}m\omega_2^2 X_2^2 \\ &= H_1 + H_2 \end{aligned}$$

where $\omega_2 = 2\omega_1 = 2\omega$.

Defining the raising and lowering operators:

$$\begin{aligned} a_n &= \frac{1}{\sqrt{2}}(\bar{X}_n + i\bar{P}_n) \\ a_n^\dagger &= \frac{1}{\sqrt{2}}(\bar{X}_n - i\bar{P}_n) \end{aligned}$$

where $n = 1, 2$ and

$$\begin{aligned} \bar{X}_n &= \left(\frac{m\omega_n}{\hbar}\right)^{1/2} X_n \\ \bar{P}_n &= \left(\frac{1}{\hbar m\omega_n}\right)^{1/2} P_n \end{aligned}$$

such that $[a_m, a_n^\dagger] = \delta_{mn}, m, n = 1, 2$.

Answer the following questions:

~~(a)~~ [2 points] Write the Hamiltonian in terms of raising and lowering operators.

~~(b)~~ [2 points] Write the eigenvector $|\psi_{n_1,n_2}\rangle$ in terms of the ground state $\psi_{0,0}\rangle = |\phi_{n_1=0}\rangle|\phi_{n_2=0}\rangle$ where $|\phi_{n_1}\rangle$ is the eigenvector for particle 1, i.e.,

$$H_1|\phi_{n_1}\rangle = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1|\phi_{n_1}\rangle$$

and similarly for particle 2.

~~(c)~~ [1 points] Write a formula for the energy levels of this oscillator, E_n with n defined in terms of n_1 and n_2.

(d) [1 points] Determine a formula for the degeneracy, g_n , of an energy level E_n.

~~(e)~~ [2 points] Using your results from part (d) determine the degeneracy g_n for the energy, $E = 15/2\hbar\omega$ and list all the eigenfunctions $|\psi_{n_1,n_2}\rangle$ that have this energy.

~~(f)~~ [2 points] Determine ΔX_1, the uncertainty in X_1 for the state $|\psi_{n_1=1,n_2=2}\rangle$ using raising and lowering operators. Discuss the dependence of ΔX_1, on the frequency ω_1 and explain why it makes sense physically.

(a)

Our Hamiltonian is

$$H(1,2) = \frac{P_1^2}{2m} + \frac{1}{2} m\omega_1^2 X_1^2 + \frac{P_2^2}{2m} + \frac{1}{2} m\omega_2^2 X_2^2$$

where $\omega_2 = 2\omega_1 = 2\omega$. We know

$$a_1 = \frac{1}{\sqrt{2}}\left(\overline{X}_1 + i\overline{P}_1\right), \quad a_1^\dagger = \frac{1}{\sqrt{2}}\left(\overline{X}_1 - i\overline{P}_1\right)$$

$$a_2 = \frac{1}{\sqrt{2}}\left(\overline{X}_2 + i\overline{P}_2\right), \quad a_2^\dagger = \frac{1}{\sqrt{2}}\left(\overline{X}_2 - i\overline{P}_2\right)$$

where

$$\overline{X}_1 = \left(\frac{m\omega_1}{\hbar}\right)^{1/2} X_1, \quad \overline{P}_1 = \left(\frac{1}{\hbar m\omega_1}\right)^{1/2} P_1$$

$$\overline{X}_2 = \left(\frac{m\omega_2}{\hbar}\right)^{1/2} X_2, \quad \overline{P}_2 = \left(\frac{1}{\hbar m\omega_2}\right)^{1/2} P_2 .$$

We have

$$\sqrt{2}\, a_1 = \overline{X}_1 + i\overline{P}_1$$

$$\overline{X}_i = \sqrt{2}\, a_1 - i\overline{P}_1 ,$$

so

$$\sqrt{2}\, a_1^\dagger = \left(\sqrt{2}\, a_1 - i\overline{P}_1\right) - i\overline{P}_1$$

$$\sqrt{2}\left(a_1^\dagger - a_1\right) = -2i\overline{P}_1$$

$$\overline{P}_1 = \frac{i\left(a_1^\dagger - a_1\right)}{\sqrt{2}}$$

(a), cont'd...

We can also say

$$i\bar{P}_1 = \sqrt{2}a_1 - \bar{X}_1 ,$$

so

$$\sqrt{2}a_1^+ = \bar{X}_1 - (\sqrt{2}a_1 - \bar{X}_1)$$

$$\sqrt{2}(a_1^+ + a_1) = 2\bar{X}_1$$

$$\bar{X}_1 = \frac{a_1^+ + a_1}{\sqrt{2}} .$$

Similarly, we get

$$\bar{P}_2 = \frac{i(a_2^+ - a_2)}{\sqrt{2}}$$

$$\bar{X}_2 = \frac{a_2^+ + a_2}{\sqrt{2}} .$$

Putting these in terms of X_1, X_2, P_1, and P_2...

$$\bar{P}_1 = \frac{i(a_1^+ - a_1)}{\sqrt{2}} = \left(\frac{1}{\hbar m\omega_1}\right)^{1/2} P_1$$

$$P_1 = i\left(\frac{\hbar m\omega_1}{2}\right)^{1/2}(a_1^+ - a_1) \qquad (1)$$

$$\bar{X}_1 = \frac{a_1^+ + a_1}{\sqrt{2}} = \left(\frac{m\omega_1}{\hbar}\right)^{1/2} X_1$$

$$X_1 = \left(\frac{\hbar}{2m\omega_1}\right)^{1/2}(a_1^+ + a_1) \qquad (2)$$

(a), cont'd...

and similarly,

$$P_2 = i\left(\frac{\hbar m\omega_2}{2}\right)^{1/2}\left(a_2^+ - a_2\right) \qquad (3)$$

$$X_2 = \left(\frac{\hbar}{2m\omega_2}\right)^{1/2}\left(a_2^+ + a_2\right) \qquad (4)$$

We have

$$\begin{aligned}
H(1,2) &= \frac{1}{2m}\left[i\left(\frac{\hbar m\omega_1}{2}\right)^{1/2}(a_1^+ - a_1)\right]^2 + \frac{1}{2}m\omega_1^2\left[\left(\frac{\hbar}{2m\omega_1}\right)^{1/2}(a_1^+ + a_1)\right]^2 \\
&\quad + \frac{1}{2m}\left[i\left(\frac{\hbar m\omega_2}{2}\right)^{1/2}(a_2^+ - a_2)\right]^2 + \frac{1}{2}m\omega_2^2\left[\left(\frac{\hbar}{2m\omega_2}\right)^{1/2}(a_2^+ + a_2)\right]^2 \\
&= \frac{1}{2m}\left[-\left(\frac{\hbar m\omega_1}{2}\right)(a_1^+ - a_1)^2\right] + \frac{1}{2}m\omega_1^2\left[\frac{\hbar}{2m\omega_1}(a_1^+ + a_1)^2\right] \\
&\quad + \frac{1}{2m}\left[-\left(\frac{\hbar m\omega_2}{2}\right)(a_2^+ - a_2)^2\right] + \frac{1}{2}m\omega_2^2\left[\frac{\hbar}{2m\omega_2}(a_2^+ + a_2)^2\right] \\
&= \frac{1}{2m}\left[-\left(\frac{\hbar m\omega}{2}\right)(a_1^+ - a_1)^2\right] + \frac{1}{2}m\omega^2\left[\frac{\hbar}{2m\omega}(a_1^+ + a_1)^2\right] \\
&\quad + \frac{1}{2m}\left[-\left(\frac{\hbar m(2\omega)}{2}\right)(a_2^+ - a_2)^2\right] + \frac{1}{2}m(2\omega)^2\left[\frac{\hbar}{2m(2\omega)}(a_2^+ + a_2)^2\right] \\
&= -\frac{\hbar\omega}{4}(a_1^+ - a_1)^2 + \frac{\hbar\omega}{4}(a_1^+ + a_1)^2 - \frac{\hbar\omega}{2}(a_2^+ - a_2)^2 + \frac{\hbar\omega}{2}(a_2^+ + a_2)^2 \\
&= -\frac{\hbar\omega}{4}\left[(a_1^+ - a_1)^2 - (a_1^+ + a_1)^2\right] - \frac{\hbar\omega}{2}\left[(a_2^+ - a_2)^2 - (a_2^+ + a_2)^2\right] \\
&= -\frac{\hbar\omega}{4}\left[\cancel{a_1^+a_1^+} - a_1^+a_1 - a_1a_1^+ + \cancel{a_1a_1} - \cancel{a_1^+a_1^+} - a_1^+a_1 - a_1a_1^+ - \cancel{a_1a_1}\right] \\
&\quad - \frac{\hbar\omega}{2}\left[\cancel{a_2^+a_2^+} - a_2^+a_2 - a_2a_2^+ + \cancel{a_2a_2} - \cancel{a_2^+a_2^+} - a_2^+a_2 - a_2a_2^+ - \cancel{a_2a_2}\right] \\
&= -\frac{\hbar\omega}{4}\left[-2a_1^+a_1 - 2a_1a_1^+\right] - \frac{\hbar\omega}{2}\left[-2a_2^+a_2 - 2a_2a_2^+\right] \\
&= \frac{\hbar\omega}{2}\left[a_1^+a_1 + a_1a_1^+\right] + \hbar\omega\left[a_2^+a_2 + a_2a_2^+\right]
\end{aligned}$$

(a), cont'd...

We know

$$[a_1, a_1^+] = a_1 a_1^+ - a_1^+ a_1 = 1$$
$$a_1 a_1^+ = 1 + a_1^+ a_1$$

and

$$[a_2, a_2^+] = a_2 a_2^+ - a_2^+ a_2 = 1$$
$$a_2 a_2^+ = 1 + a_2^+ a_2 ,$$

so

$$H = \frac{\hbar\omega}{2}\left[a_1^+ a_1 + (1 + a_1^+ a_1)\right] + \hbar\omega\left[a_2^+ a_2 + (1 + a_2^+ a_2)\right]$$
$$= \frac{\hbar\omega}{2}\left[2a_1^+ a_1 + 1\right] + \hbar\omega\left[2a_2^+ a_2 + 1\right]$$
$$H = \hbar\omega\left[a_1^+ a_1 + 1/2\right] + 2\hbar\omega\left[a_2^+ a_2 + 1/2\right]$$

or

$$\boxed{H = \hbar\omega\left(a_1^+ a_1 + 2a_2^+ a_2 + 3/2\right)}$$

(b)

We have

$$H_1 |\phi_{n_1}\rangle = \left(n_1 + \tfrac{1}{2}\right)\hbar\omega_1 |\phi_{n_1}\rangle$$

$$H_2 |\phi_{n_2}\rangle = \left(n_2 + \tfrac{1}{2}\right)\hbar\omega_2 |\phi_{n_2}\rangle.$$

We want to write $|\psi_{n_1,n_2}\rangle$ in terms of $|\psi_{00}\rangle = |\phi_0\rangle|\phi_0\rangle$.
We know

$$|\phi_{n_1}\rangle = \left[\frac{(a_1^+)^{n_1}}{\sqrt{n_1!}}\right]|\phi_{n_1=0}\rangle$$

$$|\phi_{n_2}\rangle = \left[\frac{(a_2^+)^{n_2}}{\sqrt{n_2!}}\right]|\phi_{n_2=0}\rangle.$$

Then

$$\begin{aligned}|\psi_{n_1 n_2}\rangle &= |\phi_{n_1}\rangle|\phi_{n_2}\rangle \\ &= \left[\frac{(a_1^+)^{n_1}}{\sqrt{n_1!}}\right]|\phi_{n_1=0}\rangle\left[\frac{(a_2^+)^{n_2}}{\sqrt{n_2!}}\right]|\phi_{n_2=0}\rangle \\ &= \frac{(a_1^+)^{n_1}(a_2^+)^{n_2}}{\sqrt{n_1!\,n_2!}}|\phi_{n_1=0}\rangle|\phi_{n_2=0}\rangle\end{aligned}$$

$$\boxed{|\psi_{n_1 n_2}\rangle = \frac{(a_1^+)^{n_1}(a_2^+)^{n_2}}{\sqrt{n_1!\,n_2!}}|\psi_{00}\rangle}$$

(c)

The total energy is given by

$$E_n = E_1 + E_2$$

$$= \hbar\omega_1\left(n_1 + \frac{1}{2}\right) + \hbar\omega_2\left(n_2 + \frac{1}{2}\right)$$

$$= \hbar\omega\left(n_1 + \frac{1}{2}\right) + \hbar(2\omega)\left(n_2 + \frac{1}{2}\right)$$

$$= \hbar\omega\left(n_1 + \frac{1}{2} + 2n_2 + 1\right)$$

$$\boxed{E_n = \hbar\omega\left(n + 3/2\right)}$$

where

$$\boxed{n = n_1 + 2n_2}.$$

(d)

We have

$$E = \frac{15}{2}\hbar\omega,$$

so

$$n + \frac{3}{2} = \frac{15}{2}$$

$$n = 6$$

$$n_1 + 2n_2 = 6.$$

Possible eigenfunctions:

$|\psi_{0,3}\rangle$ $|\psi_{4,1}\rangle$

$|\psi_{6,0}\rangle$

$|\psi_{2,2}\rangle$

So the degeneracy is $\boxed{4}$.

(f)

In general, we know

$$\Delta X_1 = \sqrt{\langle X_1^2\rangle - \langle X_1\rangle^2}$$

we have

$$\begin{aligned}
\langle X_1^2\rangle &= \langle \psi_{1,2} | X_1^2 | \psi_{1,2}\rangle \\
&= \langle \psi_{1,2} | \tfrac{\hbar}{2m\omega_1}(a_1^+ + a_1)^2 | \psi_{1,2}\rangle \\
&= \frac{\hbar}{2m\omega_1}\langle \psi_{1,2} | a_1^+ a_1^+ + a_1^+ a_1 + a_1 a_1^+ + a_1 a_1 | \psi_{1,2}\rangle \\
&= \frac{\hbar}{2m\omega_1}\Big[\langle \psi_{1,2}|a_1^+ a_1^+|\psi_{1,2}\rangle + \langle \psi_{1,2}|a_1 a_1|\psi_{1,2}\rangle + \\
&\qquad \langle \psi_{1,2}|a_1^+ a_1 + a_1 a_1^+|\psi_{1,2}\rangle\Big] \\
&= \frac{\hbar}{2m\omega_1}\Big[\langle \psi_{1,2}|a_1^+ a_1|\psi_{1,2}\rangle + \langle \psi_{1,2}|a_1 a_1^+|\psi_{1,2}\rangle\Big] \\
&= \frac{\hbar}{2m\omega_1}\Big[\sqrt{1}\langle \psi_{1,2}|a_1^+|\psi_{0,2}\rangle + \sqrt{1+1}\langle \psi_{1,2}|a_1|\psi_{2,2}\rangle\Big] \\
&= \frac{\hbar}{2m\omega_1}\Big[1\langle \psi_{1,2}|\psi_{1,2}\rangle + 2\langle \psi_{1,2}|\psi_{1,2}\rangle\Big] \\
&= \frac{3\hbar}{2m\omega_1}
\end{aligned}$$

and

$$\begin{aligned}
\langle X_1\rangle &= \langle \psi_{1,2}|X_1|\psi_{1,2}\rangle \\
&= \left(\frac{\hbar}{2m\omega_1}\right)^{1/2}\langle \psi_{1,2}|a_1^+ + a_1|\psi_{1,2}\rangle \\
&= \left(\frac{\hbar}{2m\omega_1}\right)^{1/2}\Big[\sqrt{2}\langle \psi_{1,2}|\psi_{2,2}\rangle + \langle \psi_{1,2}|\psi_{0,2}\rangle\Big] \\
&= 0
\end{aligned}$$

So $\boxed{\Delta X_1 = \left(\frac{3\hbar}{2m\omega_1}\right)^{1/2}}$. It makes sense that the uncertainty in X_1 depends on ω_1 since ω_1 describes how quickly the particle oscillates.