F2008

## Problem 4: The Infinite Square Well: (10 Points)

A single particle is in a one dimensional infinite well whose potential $V(x)$ is given by:

$$V(x) = \begin{cases} 0, & \text{if -L} \le \text{x} \le \text{L} \\ \infty, & \text{otherwise} \end{cases}$$

a. Find the allowed energies ($E_n$) and the normalized eigenfunctions ($\Phi_n(x)$) to Schrodinger's Equation for this potential. Show all your work. **(2 Points)**

Assume the particle is in the ground state and a position measurement of the particle is made. Since any measuring apparatus has a finite resolution, the exact location of the particle cannot be detemined. We therefore only know the location of the particle within some resolution $\epsilon$. After making the position measurement the wave function $\Psi(x)$ is:

$$\Psi(x) = \frac{1}{\sqrt{\epsilon}} \qquad |x| < \frac{\epsilon}{2}$$

$$\Psi(x) = 0 \qquad |x| > \frac{\epsilon}{2}$$

b. What is the probability that the particle has energy $E_n$? **(2 Points)**

c. If $\epsilon = 2L$, we know that the particle is somewhere in the box. What is the probability that the particle is in the ground state? **(1 Point)**

? d. Before the position measurement we knew the particle was in the box and in the ground state. If after the measurement and $\epsilon = 2L$ we know that the particle is in the box, why is probability that the particle is in the ground state not 1? **(1 Point)**

For parts e), f) and g) now assume that the particle is in the potential $V(x)$

$$V(x) = \begin{cases} 0, & \text{if -L} \le \text{x} \le \text{L} \\ \infty, & \text{otherwise} \end{cases}$$

and in the ground state. The position of the walls are quickly increased to

$$V(x) = \begin{cases} 0, & \text{if -L'} \le \text{x} \le \text{L'} \\ \infty, & \text{otherwise} \end{cases}$$

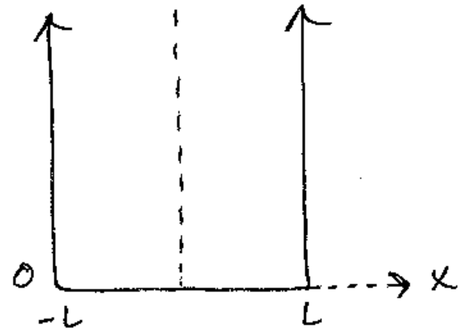
where $|L'| > |L|$

e. After the expansion, what is the probability that the particle has energy $E_n$? You do not need to solve the integral. **(2 Points)**

f. Before the walls of the potential are increased, does $|\Psi(x,t)|^2$ (where $\Psi(x,t)$ is a solution to Schrodinger's equation before the expansion) have any time dependance? Explain **(1 Point)**

g. After the position of the walls are increased to $L'$, does $|\Psi(x,t)|^2$ (where $\Psi(x,t)$ is a solution to Schrodinger's equation after the expansion) have any time dependance? Explain. **(1 Point)**

(a)




Since $V=0$ between $-L < x < L$, Schrödinger's equation is

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi,$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}.$$

The general solution is

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

We know from the boundary conditions that

$$\psi(-L) = \psi(L) = 0.$$

So

$$\psi(L) = A\sin(kL) + B\cos(kL) = 0.$$

Let

$$kL = n\frac{\pi}{2}.$$

(a), cont'd...

When $n = 1, 3, 5, \ldots$ we must have $A = 0$ and so

$$\psi(x) = B\cos(kx)$$

When $n = 2, 4, 6, \ldots$ we must have $B = 0$ and so

$$\psi(x) = A\sin(kx).$$

Normalizing...

$$B^2 \int_{-L}^{L} \cos^2(kx)\,dx = 1$$

$$B^2 \int_{-L}^{L} \frac{1}{2}\left(1 + \cos(2kx)\right)dx = 1$$

$$\frac{B^2}{2}\left[x + \frac{1}{2k}\sin(2kx)\right]\Big|_{-L}^{L} = 1$$

$$\frac{B^2}{2}\left[x + \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]\Big|_{-L}^{L} = 1$$

$$\frac{B^2}{2}(2L) = 1$$

$$B = \sqrt{\frac{1}{L}}.$$

We also have $A = \sqrt{\frac{1}{L}}$ since $\sin^2(kx) = \frac{1}{2}\left(1 - \cos(2kx)\right)$. Then

$$\boxed{\psi_n(x) = \begin{cases} \sqrt{\frac{1}{L}}\cos\left(\frac{n\pi x}{2L}\right), & n = 1, 3, 5, \ldots \\ \sqrt{\frac{1}{L}}\sin\left(\frac{n\pi x}{2L}\right), & n = 2, 4, 6, \ldots \end{cases}}$$

and

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{2L}$$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{8mL^2}}.$$

(b)

The probability is, assuming odd $n$,

$$P = \left| \int_{-\varepsilon/2}^{\varepsilon/2} \left(\sqrt{\tfrac{1}{L}} \cos\left(\tfrac{n\pi x}{2L}\right)\right)\left(\tfrac{1}{\sqrt{\varepsilon}}\right) dx \right|^2$$

$$= \frac{1}{\varepsilon L} \left| \int_{-\varepsilon/2}^{\varepsilon/2} \cos\left(\frac{n\pi x}{2L}\right) dx \right|^2$$

$$= \frac{1}{\varepsilon L} \left| \frac{2L}{n\pi} \sin\left(\frac{n\pi x}{2L}\right) \Big|_{-\varepsilon/2}^{\varepsilon/2} \right|^2$$

$$= \frac{4L^2}{n^2\pi^2 \varepsilon L} \left(2 \sin\left(\frac{n\pi(\varepsilon/2)}{2L}\right)\right)^2$$

$$\boxed{P = \frac{16L}{n^2\pi^2\varepsilon} \sin^2\left(\frac{n\pi\varepsilon}{4L}\right)}$$

(c) If $\varepsilon = 2L$ and $n=1$, then our probability is

$$P = \frac{16L}{(1)^2\pi^2(2L)} \sin^2\left(\frac{(1)\pi(2L)}{4L}\right)$$

$$\boxed{P = \frac{8}{\pi^2}}$$

(d) By measuring the position, the system was perturbed so we don't know for certain that the particle is in the ground state.

(e)

Our new wavefunction in this expanded potential is

$$\psi_{n'}(x) = \begin{cases} \sqrt{\frac{1}{L'}}\cos\left(\frac{n\pi x}{2L'}\right), & n' = 1,3,5,\ldots \\ \sqrt{\frac{1}{L'}}\sin\left(\frac{n\pi x}{2L'}\right), & n' = 2,4,6,\ldots, \end{cases}$$

where $|L'| > |L|$. The probability that the particle has energy $E_n$ is (assuming odd $n$ again)

$$P = \left| \int_{-L}^{L} \left(\sqrt{\frac{1}{L}}\cos\left(\frac{n\pi x}{2L}\right)\right)\left(\sqrt{\frac{1}{L'}}\cos\left(\frac{n'\pi x}{2L'}\right)\right)dx \right|^2$$

$$\boxed{P = \frac{1}{LL'}\left| \int_{-L}^{L} \cos\left(\frac{n\pi x}{2L}\right)\cos\left(\frac{n'\pi x}{2L'}\right) dx \right|^2}$$

(f) No, $|\Psi(x,t)|^2$ does not have any time dependence because we are dealing with stationary states.

(g) We are still working with stationary states, so $|\Psi(x,t)|^2$ does not have any time dependence after the expansion.