S2013

Problem 3: Spin Measurements and Uncertainty

Define the operator $S_\alpha = \vec{S} \cdot \hat{n}_\alpha$ where $\vec{S}$ is the vector spin operator and $\hat{n}_\alpha$ is a unit vector in the $x-z$ plane that makes an angle α with the z-axis. So $\hat{n}_\alpha = \hat{z}$ for $\alpha = 0$ and $\hat{n}_\alpha = \hat{x}$ for $\alpha = \pi/2$.

Consider a spin 1/2 system initially prepared to be in the eigenstate of S_α with eigenvalue $+\hbar/2$,

$$S_\alpha|\alpha,+\rangle = \frac{\hbar}{2}|\alpha,+\rangle \tag{1}$$

(a) [3 pts] Compute the eigenstates of S_α in the basis of the S_z operator, $|0,\pm\rangle \equiv |\pm\rangle$.

(b) [2 pts] If the spin is in the state $|\alpha,+\rangle$ and S_x is measured, what is the probability of measuring $-\hbar/2$?

(c) [3 pts] Compute the expectation value $\langle(\delta S_x)^2\rangle$ for the state $|\alpha,+\rangle$, where $\delta S_x = S_x - \langle S_x\rangle$.

If one measures S_x, what are the values of α that minimize the uncertainty of the measurement for the state $|\alpha,+\rangle$? Interpret the physical meaning of those states.

(d) [2 pts] Finally, define $\mathcal{P}_{x,+}$ to be the projection operator for the spin 1/2 state of S_x, $|\pi/2,+\rangle$. Compute the matrix element $\mathcal{P}_{x,+}$ in the initial state, $\langle +,\alpha|\mathcal{P}_{x,+}|\alpha,+\rangle$. Explain the behavior of the resultant expression as a function of the angle α.

(a)

We know

$$\hat{n}_\alpha = \sin\alpha\,\hat{x} + \cos\alpha\,\hat{z}$$

since

$$\hat{n}_\alpha = \hat{z} \quad \text{for } \alpha = 0$$
$$\hat{n}_\alpha = \hat{x} \quad \text{for } \alpha = \pi/2.$$

Then

$$S_\alpha = \vec{S}\cdot\hat{n}_\alpha$$
$$= S_x \sin\alpha + S_z \cos\alpha$$

where

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$S_\alpha = \frac{\hbar\sin\alpha}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar\cos\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\hbar\cos\alpha}{2} & \frac{\hbar\sin\alpha}{2} \\ \frac{\hbar\sin\alpha}{2} & -\frac{\hbar\cos\alpha}{2} \end{pmatrix}$$

(a), cont'd...

Finding our eigenvalues...

$$\begin{vmatrix} \frac{\hbar\cos\alpha}{2} - \lambda & \frac{\hbar\sin\alpha}{2} \\ \frac{\hbar\sin\alpha}{2} & -\frac{\hbar\cos\alpha}{2} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{\hbar\cos\alpha}{2} - \lambda\right)\left(-\frac{\hbar\cos\alpha}{2} - \lambda\right) - \left(\frac{\hbar\sin\alpha}{2}\right)^2 = 0$$

$$-\left(\frac{\hbar\cos\alpha}{2}\right)^2 + \lambda^2 - \left(\frac{\hbar\sin\alpha}{2}\right)^2 = 0$$

$$\lambda^2 = \frac{\hbar^2}{4}\left(\sin^2\alpha + \cos^2\alpha\right)$$

$$\lambda = \pm\frac{\hbar}{2}$$

Now we must find the eigenvectors.

$|S = \hbar/2\rangle$

$$\begin{pmatrix} \frac{\hbar}{2}(\cos\alpha - 1) & \frac{\hbar}{2}\sin\alpha \\ \frac{\hbar}{2}\sin\alpha & -\frac{\hbar}{2}(\cos\alpha + 1) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We know $\alpha = 0$, so

$$\begin{pmatrix} 0 & 0 \\ 0 & -\hbar \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} c_1 = \text{anything} \\ c_2 = 0 \end{array}$$

Let $c_1 = 1$. Then

$$|+\rangle = \boxed{|S = \hbar/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \qquad \text{(spin up)}$$

(a), cont'd...

$|s = -\hbar/2\rangle$

$$\begin{pmatrix} \frac{\hbar}{2}(\cos\alpha + 1) & \frac{\hbar}{2}\sin\alpha \\ \frac{\hbar}{2}\sin\alpha & \frac{\hbar}{2}(1-\cos\alpha) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Again, $\alpha = 0$ so

$$\begin{pmatrix} \hbar & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad \begin{array}{l} c_1 = 0 \\ c_2 = \text{anything} \end{array}$$

Let $c_2 = 1$. Then

$$|-\rangle = \boxed{|s = -\hbar/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \qquad \text{(spin down)}$$

(b)

We want to determine

$$P_{|S=\hbar/2\rangle}\left(S_x \stackrel{-}{=} \hbar/2\right),$$

so we need to find $|S_x = -\hbar/2\rangle$ in the S_z basis. Then

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\hbar/2)^2 = 0$$

$$\lambda = \pm \hbar/2$$

Let $\lambda = -\hbar/2$. Then

$$\begin{pmatrix} -\hbar/2 & \hbar/2 \\ \hbar/2 & -\hbar/2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} -\frac{\hbar}{2}c_1 + \frac{\hbar}{2}c_2 = 0 \\ \frac{\hbar}{2}c_1 - \frac{\hbar}{2}c_2 = 0. \end{array}$$

Then we have $c_1 = c_2$. Let $c_1 = c_2 = 1$. Then

$$|S_x \stackrel{-}{=} \hbar/2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|S=\hbar/2\rangle + \frac{1}{\sqrt{2}}|S=-\hbar/2\rangle$$

(b), cont'd...

So our probability is

$$P_{|S=\hbar/2\rangle}(S_x=-\hbar/2) = \left|\langle S_x=-\hbar/2|S=\hbar/2\rangle\right|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle S=\hbar/2| + \frac{1}{\sqrt{2}}\langle S=-\hbar/2|\right)|S=\hbar/2\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\langle S=\hbar/2|S=\hbar/2\rangle + \frac{1}{\sqrt{2}}\langle S=-\hbar/2|S=\hbar/2\rangle\right|^2$$

$$\boxed{P_{|S=\hbar/2\rangle}(S_x=-\hbar/2) = \frac{1}{2}}.$$

(c)

We want to compute $\langle(\delta S_x)^2\rangle$ for the $|S=\hbar/2\rangle$ state, where $\delta S_x = S_x - \langle S_x\rangle$. So

$$\langle S_x\rangle = \langle S=\hbar/2|S_x|S=\hbar/2\rangle$$

$$= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & \hbar/2\\ \hbar/2 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0 & \hbar/2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= 0$$

So

$$\delta S_x = S_x = \begin{pmatrix}0 & \hbar/2\\ \hbar/2 & 0\end{pmatrix}$$

(c), cont...

Then we have

$$(\delta S_x)^2 = \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{pmatrix}$$

and

$$\langle (\delta S_x)^2 \rangle = \langle s = \hbar/2 | \, (\delta S_x)^2 \, | s = \hbar/2 \rangle$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \hbar^2/4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boxed{\langle (\delta S_x)^2 \rangle = \hbar^2/4}$$