S2012

PROBLEM 3: Harmonic Oscillator

A particle of mass m is under the influence of the following potential

$$V(x) = V_0\sqrt{A^2 + x^2}$$

where V_0 and A are constants. For small displacements $x \ll A$ this potential can be approximated by a simple harmonic oscillator.

(a) Determine the lowest energy this particle can have in terms of $\hbar$, m, V_0 and a for $x \ll A$. (2 Points)

Now consider the Hamiltonian describing the true one-dimensional harmonic oscillator

$$\mathbf{H} = \frac{\mathbf{P}^2}{2m} + \frac{1}{2}k\mathbf{X}^2$$

with eigenstates

$$\mathbf{H}|n\rangle = E_n|n\rangle \quad n = 0, 1, 2, \cdots .$$

(b) Using commutation relations, calculate the equations of motion for $\mathbf{P}$ and $\mathbf{X}$ in the Heisenberg picture. (Find $\dot{X}$ and $\dot{P}$.) (2 Points)

(c) Solve for $P(t)$ and $X(t)$ in terms of $P(0)$ and $X(0)$ and show that $[X(t), X(0)] \neq 0$ for $t \neq 0$. (2 Points)

A harmonic oscillator system is known to be in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle)$$

where $|0\rangle$ and $|3\rangle$ are the normalized ground state and the third excited state of the harmonic oscillator respectively.

(d) What is the value of $n > 0$ for the first non-zero value of $\langle X^n\rangle$ with the state vector $|psi\rangle$? (2 Points)

(e) What is the expectation value $\langle X^3\rangle$ with the state vector $|\psi\rangle$? (2 Points)

(a) Skipped

(b) In general, for an operator A, we know

$$\frac{d}{dt}A = \frac{i}{\hbar}[H,A].$$

So

$$\begin{aligned}
\frac{dX}{dt} &= \frac{i}{\hbar}[H,X] \\
&= \frac{i}{\hbar}\left[\frac{P^2}{2m} + \frac{1}{2}kX^2, X\right] \\
&= \frac{i}{\hbar}\left(\left[\frac{P^2}{2m}, X\right] + \left[\frac{kX^2}{2}, X\right]\right) \\
&= \frac{i}{\hbar}\left(\frac{1}{2m}[P^2, X] + 0\right) \\
&= \frac{i}{\hbar}\left(-\frac{1}{2m}[X,P^2]\right) \\
&= \frac{-i}{2\hbar m}\left([X,P]P + P[X,P]\right) \\
&= \frac{-i}{2\hbar m}(2i\hbar P)
\end{aligned}$$

$$\boxed{\dot{X} = \frac{P}{m}}$$

(b), cont'd...

$$\frac{dP}{dt} = \frac{i}{\hbar}[H, P]$$

$$= \frac{i}{\hbar}\left[\frac{P^2}{2m} + \frac{1}{2}kX^2, P\right]$$

$$= \frac{i}{\hbar}\left(\left[\frac{P^2}{2m}, P\right] + \left[\frac{1}{2}kX^2, P\right]\right)$$

$$= \frac{i}{\hbar}\left(0 + \frac{1}{2}k[X^2, P]\right)$$

$$= -\frac{ik}{2\hbar}[P, X^2]$$

$$= -\frac{ik}{2\hbar}\left([P, X]X + X[P, X]\right)$$

$$= \frac{ik}{2\hbar}(2i\hbar X)$$

$$\boxed{\frac{dP}{dt} = -kX}$$