F 2007

PROBLEM 3

A particle of mass m has a potential energy represented by two one-dimensional harmonic oscillator potentials centered at $\pm a$:

$$V(x) = \frac{1}{2}K(x-a)^2 + \frac{1}{2}K(x+a)^2$$

[a] (3 pts) What are the eigenvalues of the particle given this potential V? You may derive this result from first principles or deduce the result from the well known eigenvalues of a particle moving in a single harmonic-oscillator potential.

[b] (3 pts) The normalized ground-state eignefunction of the particle is given by

$$\phi(x) = \frac{1}{\pi^{1/4}\Delta^{1/2}} \exp\left(-\frac{x^2}{2\Delta^2}\right)$$

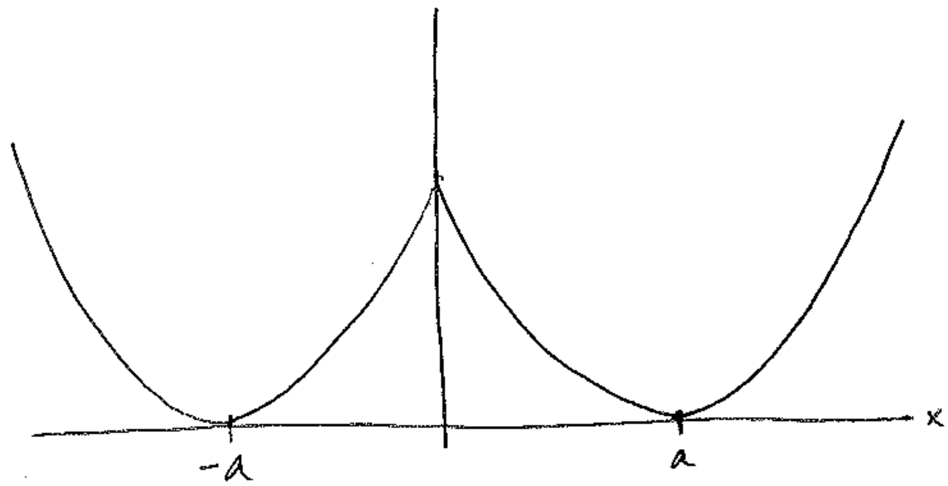
Use Schrodinger's equation to determine the constant Δ in terms of K, m, and fundamental constants.

[c] (4 pt) The potential well at $x = -a$ suddenly disappears, leaving the particle in a new potential

$$U(x) = \frac{1}{2}K(x-a)^2$$

Suppose that *before the sudden change,* the particle was in the ground state of the double-well potential $V(x)$. Derive an expression for the probability that after the sudden change the particle will be in the ground state of the single well potential $U(x)$. Express your answer in terms of a and Δ.

(a)



We have

$$V(x) = \frac{1}{2}K(x-a)^2 + \frac{1}{2}K(x+a)^2$$
$$= \frac{1}{2}K\left[(x-a)^2 + (x+a)^2\right]$$
$$= \frac{1}{2}K(2x^2+2a^2)$$
$$= m\omega^2(x^2+a^2).$$

So our Hamiltonian is

$$H = \frac{p^2}{2m} + m\omega^2(x^2+a^2).$$

We know

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\,x - \frac{i}{\sqrt{2\hbar m\omega}}\,p$$
$$a = \sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i}{\sqrt{2\hbar m\omega}}\,p,$$

so

$$p = -i\sqrt{\frac{\hbar m\omega}{2}}\,(a-a^+)$$
$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^+)$$

(a), cont'd...

To avoid confusion with the operators, let $x = a = \alpha$.
Plugging in p and x, we have

$$H = \frac{1}{2m}\left(-i\sqrt{\frac{\hbar m\omega}{2}}(a-a^+)\right)^2 + m\omega^2\left[\left(\sqrt{\frac{\hbar}{2m\omega}}(a+a^+)\right)^2 + \alpha^2\right]$$

$$= -\frac{\hbar\omega}{4}(a-a^+)^2 + \frac{\hbar\omega}{2}(a+a^+)^2 + m\omega^2\alpha^2$$

$$= -\frac{\hbar\omega}{4}(aa - aa^+ - a^+a + a^+a^+) + \frac{\hbar\omega}{2}(aa + aa^+ + a^+a + a^+a^+) + m\omega^2\alpha^2$$

$$= \frac{\hbar\omega}{4}(aa + a^+a^+) + \frac{3\hbar\omega}{4}(aa^+ + a^+a) + m\omega^2\alpha^2.$$

But

$$[a, a^+] = 1$$
$$aa^+ - a^+a = 1$$
$$aa^+ = a^+a + 1$$

so

$$H = \frac{\hbar\omega}{4}(aa + a^+a^+) + \frac{3\hbar\omega}{4}(2a^+a + 1) + m\omega^2\alpha^2$$
$$H = \frac{\hbar\omega}{4}(aa + a^+a^+) + \frac{3\hbar\omega}{2}(a^+a + 1/2) + m\omega^2\alpha^2.$$

Then

$$H|n\rangle = \frac{\hbar\omega}{4}\left[\sqrt{n(n-1)}\,|n-2\rangle + \sqrt{(n+1)(n+2)}\,|n+2\rangle\right] + \frac{3\hbar\omega}{2}\left[n|n\rangle + \frac{1}{2}|n\rangle\right] + m\omega^2\alpha^2|n\rangle$$

and

$$\langle n|H|n\rangle = \frac{\hbar\omega}{4}\left[\sqrt{n(n-1)}\,\langle n|n-2\rangle + \sqrt{(n+1)(n+2)}\,\langle n|n+2\rangle\right] + \frac{3\hbar\omega}{2}\left[n\langle n|n\rangle + \frac{1}{2}\langle n|n\rangle\right] + m\omega^2\alpha^2\langle n|n\rangle$$

(a), cont'd...

$$\langle n|H|n\rangle = \frac{3\hbar\omega}{2}\left(n+\frac{1}{2}\right) + m\omega^2\alpha^2$$

Thus, our energy eigenvalues are

$$\boxed{E = \frac{3\hbar\omega}{2}\left(n+\frac{1}{2}\right) + m\omega^2 a^2}$$

(b)

We know

$$a|0\rangle = 0.$$

So

$$\left(\sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i}{\sqrt{2\hbar m\omega}}\,p\right)\psi_0 = 0$$

$$\left(\sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i}{\sqrt{2\hbar m\omega}}\left(-i\hbar\frac{d}{dx}\right)\right)\psi_0 = 0$$

$$\sqrt{\frac{\hbar}{2m\omega}}\,\frac{d\psi_0}{dx} = -\sqrt{\frac{m\omega}{2\hbar}}\,x\,\psi_0$$

$$\frac{d\psi_0}{dx} = -\sqrt{\frac{m\omega}{2\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\,x\,\psi_0$$

$$\frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}\,x\,\psi_0$$

and

$$\int \frac{1}{\psi_0}\,d\psi_0 = \int -\frac{m\omega}{\hbar}\,x\,dx$$

$$\ln\psi_0 = -\frac{m\omega}{2\hbar}x^2 + C_1$$

$$\psi_0 = C e^{-\frac{m\omega}{2\hbar}x^2}$$

Normalizing...

$$\int_{-\infty}^{\infty} |\psi_0|^2\,dx = 1$$

$$C^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}\,dx = 1$$

$$C^2\left(\sqrt{\pi\frac{\hbar}{m\omega}}\right) = 1$$

$$C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

(b), cont'd...

Then

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}.$$

We are given

$$\phi(x) = \frac{1}{\pi^{1/4}\Delta^{1/2}} \exp\left(-\frac{x^2}{2\Delta^2}\right).$$

So

$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} = \frac{1}{\pi^{1/4}\Delta^{1/2}}$$

$$\Delta^{1/2} = \frac{1}{\pi^{1/4}}\left(\frac{\pi\hbar}{m\omega}\right)^{1/4}$$

$$= \left(\frac{\hbar}{m\omega}\right)^{1/4}$$

$$\Delta = \left(\frac{\hbar}{m\omega}\right)^{1/2}.$$

This would give us

$$\phi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right),$$

which is what we expect. Thus,

$$\Delta = \left(\frac{\hbar}{m\omega}\right)^{1/2}$$

but $K = \omega^2 m$, so

$$\frac{\hbar}{m\omega} = \frac{\hbar}{m}\cdot\frac{1}{\sqrt{K/m}} = \hbar/\sqrt{mK}$$

and

$$\boxed{\Delta = \left(\frac{\hbar^2}{mK}\right)^{1/4}}$$

(c)

Let ψ_0^i be the wavefunction before the change, in the ground state, and let ψ_0^f be the ground state wave function after the change. Then, in general, our probability that the particle will be in the ground state after the change is

$$P = \left| \int_{-\infty}^{\infty} \psi_0^{*f} \cdot \psi_0^i \, dx \right|^2 .$$