F2005

4. Perturbations of the Harmonic Oscillator

Consider a particle of mass m and charge q moving in a one dimensional harmonic oscillator potential and an electric field E. The Hamiltonian for this system is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + qEx.$$

Use the usual notation $|n\rangle$ for the states of the harmonic oscillator without an external field.

(a) (2 pts.) Using perturbation theory, show that there is no change in the eigenenergies of the system to first order in qE.

(b) (2 pts.) Calculate the change in the energy eigenstate $|n\rangle$ to second order in E.

(c) (1 pt) Show, by rewriting the Hamiltonian in equation (1), that the result that you found in (b) is exact to all orders in E

(d) (3 pts.) Next, let's consider what happens to the harmonic oscillator if the electric field is turned on abruptly at time $t = 0$.

Assume the oscillator is initially in its ground state, $|0\rangle$ when the field is turned on. Using an expansion of the time-dependent state:

$$|\Psi(t)\rangle = \sum_n c_n(t)|n\rangle, \qquad \text{interaction picture}$$

and the time-dependent Schrödinger Equation, derive an expression for $\frac{d}{dt}c_n(t)$ that is correct to first order in the electric field strength E.

Note: This expression should contain terms with the matrix elements of the form $\langle n|qEx|m\rangle$.

(e) (2 pts.) Calculate the probability of finding the oscillator in the first excited state at some time t assuming it was initially in the ground state and the field is turned on abruptly at $t = 0$,

$$\begin{aligned} E(t) &= 0, \quad t < 0 \\ E(t) &= E, \quad t > 0. \end{aligned}$$

Note: The problem is done most easily using the raising and lowering operators a, $a^\dagger$, and $n = a^\dagger a$, where $[a, a^\dagger] = 1$.

(a)

Our perturbing potential is

$$V' = qEx.$$

The first order correction is given by

$$\begin{aligned}
E^{(1)} &= \langle n|V'|n\rangle \\
&= \langle n|qEx|n\rangle \\
&= qE\langle n|\sqrt{\tfrac{\hbar}{2m\omega}}(a+a^\dagger)|n\rangle \\
&= qE\sqrt{\tfrac{\hbar}{2m\omega}}\left[\sqrt{n}\langle n|n-1\rangle + \sqrt{n+1}\langle n|n+1\rangle\right] \\
&= 0.
\end{aligned}$$

Thus, there is no change in the eigenenergies of the system to first order in qE.

(b)

The second order correction is

$$\begin{aligned}
E^{(2)} &= \sum_{m\neq n}\frac{|\langle m|V'|n\rangle|^2}{E_n^{(0)}-E_m^{(0)}} \\
&= \sum_{m\neq n}\frac{|\langle m|\,qE\sqrt{\tfrac{\hbar}{2m\omega}}(a+a^\dagger)|n\rangle|^2}{E_n^{(0)}-E_m^{(0)}} \\
&= \frac{q^2E^2\hbar}{2m\omega}\sum_{m\neq n}\frac{|\sqrt{n}\langle m|n-1\rangle + \sqrt{n+1}\langle m|n+1\rangle|^2}{\hbar\omega(n+\frac{1}{2})-\hbar\omega(m+\frac{1}{2})}
\end{aligned}$$

All of the cross-terms cancel when the numerator is squared.

(b), cont'd...

$$E^{(2)} = \frac{q^2 E^2}{2m\omega^2} \sum_{m \neq n} \frac{n \langle m|n-1\rangle^2 + (n+1)\langle m|n+1\rangle^2}{n-m}$$

$$= \frac{q^2 E^2}{2m\omega^2} \sum_{m \neq n} \frac{n \delta_{m,n-1}^2 + (n+1)\delta_{m,n+1}^2}{n-m}$$

$$= \frac{q^2 E^2}{2m\omega^2} \left[\sum_{m \neq n} \frac{n \delta_{m,n-1}^2}{n-m} + \sum_{m \neq n} \frac{(n+1)\delta_{m,n+1}^2}{n-m} \right]$$

$$= \frac{q^2 E^2}{2m\omega^2} \left(\frac{n}{n-(n-1)} + \frac{n+1}{n-(n+1)} \right)$$

$$= \frac{q^2 E^2}{2m\omega^2} \left(n - (n+1) \right)$$

$$\boxed{E^{(2)} = -\frac{q^2 E^2}{2m\omega^2}}$$

This is the change in the energy to second order in E.

(c)

We want to put our Hamiltonian in the form

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x-C)^2 - A,$$

where C and A are constants. Expanding,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x^2 - 2Cx + C^2) - A$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - Cm\omega^2 x + \frac{1}{2}C^2 m\omega^2 - A$$

Let $-Cm\omega^2 x = qEx$. Then

$$C = -\frac{qE}{m\omega^2}$$

and

$$\frac{1}{2}C^2 m\omega^2 = \frac{1}{2}\left(-\frac{qE}{m\omega^2}\right)^2 m\omega^2$$

$$= \frac{q^2E^2}{2m^2\omega^4}m\omega^2$$

$$= \frac{q^2E^2}{2m\omega^2}.$$

If we let

$$A = \frac{q^2E^2}{2m\omega^2},$$

then we end up with our original Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + qEx,$$

as expected.

(c), cont'd...

So we have

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x + \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{2m\omega^2}$$

Let $x' = x + \frac{qE}{m\omega^2}$. Then

$$H' = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x'^2 - \frac{q^2E^2}{2m\omega^2}.$$

We know that for

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

we have

$$H\psi = E_n^{(0)}\psi$$
$$H|n\rangle = (n+\tfrac{1}{2})\hbar\omega|n\rangle.$$

So

$$H'\psi = E_n\psi$$
$$\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x'^2 - \frac{q^2E^2}{2m\omega^2}\right)|n\rangle = E_n|n\rangle$$
$$\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x'^2\right)|n\rangle - \frac{q^2E^2}{2m\omega^2}|n\rangle = E_n|n\rangle$$
$$\left(\hbar\omega(n+\tfrac{1}{2}) - \frac{q^2E^2}{2m\omega^2}\right)|n\rangle = E_n|n\rangle$$

and

$$E_n = \hbar\omega(n+\tfrac{1}{2}) - \frac{q^2E^2}{2m\omega^2}.$$

Thus, our result from (b) agrees with the result above and is exact to all orders in E.

(d)

In the interaction picture, we know

$$|\psi(t)\rangle = \sum_n c_n(t)|n\rangle$$

and

$$i\hbar \frac{\partial|\psi(t)\rangle}{\partial t} = V_I|\psi(t)\rangle,$$

where

$$V_I = e^{iH_0t/\hbar}\, V'\, e^{-iH_0t/\hbar}$$

with $V' = qEx$ in this case. Then

$$i\hbar \frac{\partial}{\partial t} \sum_n c_n(t)|n\rangle = e^{iH_0t/\hbar} V' e^{-iH_0t/\hbar} \sum_n c_n(t)|n\rangle$$
$$i\hbar \sum_n \dot{c}_n(t)|n\rangle = \sum_n c_n(t) e^{iH_0t/\hbar} V' e^{-iH_0t/\hbar}|n\rangle$$

but

$$e^{-iH_0t/\hbar}|n\rangle = e^{-iE_nt/\hbar}|n\rangle,$$

so

$$i\hbar \sum_n \dot{c}_n(t)|n\rangle = \sum_n c_n(t)\, e^{iH_0t/\hbar} V' e^{-iE_nt/\hbar}|n\rangle.$$

Contracting with an arbitrary state $|m\rangle$, we have

$$i\hbar \sum_n \dot{c}_n(t)\langle m|n\rangle = \sum_n c_n(t)\langle m|e^{iH_0t/\hbar} V' e^{-iE_nt/\hbar}|n\rangle.$$

but, again,

$$e^{iH_0t/\hbar}|m\rangle = e^{iE_mt/\hbar}|m\rangle,$$

so

$$i\hbar \sum_n \dot{c}_n(t)\,\delta_{mn} = \sum_n c_n(t)\langle m|e^{iE_mt/\hbar} V' e^{-iE_nt/\hbar}|n\rangle$$

(d), cont'd...

$$i\hbar \sum_n \dot{c}_n(t)\,\delta_{mn} = \sum_n c_n(t)\, e^{i(E_m-E_n)t/\hbar} \langle m|V'|n\rangle$$

$$\boxed{i\hbar\, \dot{c}_m(t) = \sum_n \langle m|qEx|n\rangle\, c_n(t)\, e^{i(E_m-E_n)t/\hbar}}.$$

(e)

We need to find an expresion for $c_m(t)$. The probability is the amplitude squared. So

$$\dot{c}_m(t) = -\frac{i}{\hbar} \sum_n \langle m|qEx|n\rangle\, c_n(t)\, e^{i(E_m-E_n)t/\hbar},$$

where $|n\rangle = |0\rangle$ and $|m\rangle = |1\rangle$. Then

$$\dot{c}_1(t) = -\frac{i}{\hbar} \sum_{n=0} \langle 1|qEx|0\rangle\, c_0(t)\, e^{i(E_1-E_0)t/\hbar}.$$

We know $c_0(t) = 1$, so integrating gives us

$$c_1^{(1)}(t) = -\frac{i}{\hbar} \int_{t_0}^{t} \langle 1|qEx|0\rangle\, e^{i(E_1-E_0)t/\hbar}\, dt$$

We have

$$\begin{aligned}
\langle 1|qEx|0\rangle &= qE \langle 1| \sqrt{\frac{\hbar}{2m\omega}} (a+a^\dagger) |0\rangle \\
&= qE \sqrt{\frac{\hbar}{2m\omega}} \langle 1|a+a^\dagger|0\rangle \\
&= qE \sqrt{\frac{\hbar}{2m\omega}} \left[\langle 1|a|0\rangle + \langle 1|a^\dagger|0\rangle\right] \\
&= qE \sqrt{\frac{\hbar}{2m\omega}}
\end{aligned}$$

(e), cont'd...

$$c_1(t) = \frac{-i}{\hbar}\int_{t_0}^{t} qE\sqrt{\frac{\hbar}{2m\omega}}\, e^{i(E_1-E_0)t/\hbar}\,dt$$

$$= \frac{-iqE}{\sqrt{2m\hbar\omega}}\int_{t_0}^{t} e^{i(E_1-E_0)t/\hbar}\,dt$$

$$= \frac{-iqE}{\sqrt{2m\hbar\omega}}\int_{0}^{t} e^{i(E_1-E_0)t/\hbar}\,dt.$$

We know

$$E_n^{(0)} = (n+\tfrac{1}{2})\hbar\omega,$$

so

$$E_0 = \tfrac{1}{2}\hbar\omega, \quad E_1 = \tfrac{3}{2}\hbar\omega$$

and

$$c_1(t) = \frac{-iqE}{\sqrt{2m\hbar\omega}}\int_0^t e^{i\hbar\omega t/\hbar}\,dt$$

$$= \frac{-iqE}{\sqrt{2m\hbar\omega}}\int_0^t e^{i\omega t}\,dt$$

$$= \frac{-iqE}{\sqrt{2m\hbar\omega}}\left(-\frac{i}{\omega}e^{i\omega t}\right)\Big|_0^t$$

$$= -\frac{qE}{\sqrt{2m\hbar\omega^3}}\left(e^{i\omega t}-1\right).$$

Then the probability the oscillator is in state $|1\rangle$ at time t is

$$P_{|1\rangle}(t) = \left|-\frac{qE}{\sqrt{2m\hbar\omega^3}}\left(e^{i\omega t}-1\right)\right|^2$$

$$= \left|-\frac{qE}{\sqrt{2m\hbar\omega^3}}\right|^2\cdot\left(e^{i\omega t}-1\right)\left(e^{-i\omega t}-1\right)$$

$$\boxed{P_{|1\rangle}(t) = \frac{q^2E^2}{2m\hbar\omega^3}\left(1-2\cos(\omega t)\right)}.$$