Problem 3: 2-d potential (10 points)

A particle of mass m is confined by two impenetrable parallel walls at $x = \pm a$ to move on a two-dimensional strip defined by

$$-a < x < a$$
$$-\infty < y < \infty$$

The wave function for this system can be expressed as the product of two functions: one that depends only on the spatial co-ordinates (x and y), and one that depends only on time t.

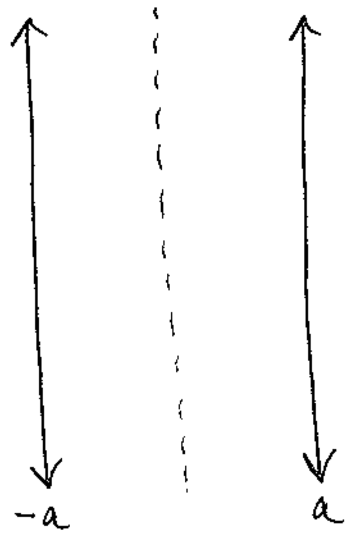
a) Use the separation of variables technique to find the time dependent function. (2 points)

b) The part of the wave function that depends only on spatial co-ordinates can be expressed as the product of two functions: one that depends only on x and one that depends only on y. Use the separation of variables technique to find these two functions. (3 points)

c) What is the minimum energy of the particle that measurement can yield? (2 points)

d) Suppose that two additional walls are inserted at $y = \pm a$. Can a measurement of the particle's energy yield the value $3\pi^2\hbar^2/8ma^2$ Explain your answer. (3 points)

(a)



We know that

$$\Psi_{tot} = \psi(x,y)\,\varphi(t).$$

The time-dependent Schrödinger equation tells us

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi.$$

Since the particle will behave as a free particle in this region, we set $V=0$ so that

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi.$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi.$$

Assuming that $\psi(x,y)$ has no time-dependence and $\varphi(t)$ has no spatial dependence, we can plug in Ψ_{tot} and use separation of variables to determine $\varphi(t)$. So

$$i\hbar\,\psi(x,y)\,\frac{\partial \varphi(t)}{\partial t} = -\frac{\hbar^2}{2m}\varphi(t)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y)$$

(a), cont'd...

Dividing by $\psi(x,y)$ and $\psi(t)$, we have

$$i\hbar \frac{1}{\psi(t)} \frac{d\psi(t)}{dt} = -\frac{\hbar^2}{2m} \frac{1}{\psi(x,y)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y). \qquad (1)$$

Since the two sides are independent of one another, we must have

$$i\hbar \frac{1}{\psi(t)} \frac{d\psi(t)}{dt} = k,$$

where k is a constant. Then

$$i\hbar \frac{d\psi(t)}{\psi(t)} = k\,dt$$

and integrating, we have

$$i\hbar \int \frac{d\psi(t)}{\psi(t)} = \int k\,dt$$

$$i\hbar \ln(\psi(t)) = kt + \phi,$$

where ϕ is a constant. Simplifying...

$$\ln(\psi(t)) = -\frac{i}{\hbar}(kt + \phi)$$

$$\boxed{\psi(t) = e^{-\frac{i}{\hbar}(kt+\phi)}}.$$

(b)

Now we consider the right-hand side of Eqn. (1).
We are given that

$$\psi(x,y) = \psi(x)\psi(y),$$

so

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)\psi(y)}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x)\psi(y) = k,$$

where again k is a constant. Then

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)\psi(y)}\left[\psi(y)\frac{d^2\psi(x)}{dx^2}+\psi(x)\frac{d^2\psi(y)}{dy^2}\right] = k$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2}+\frac{1}{\psi(y)}\frac{d^2\psi(y)}{dy^2}\right] = k$$

$$\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2}+\frac{1}{\psi(y)}\frac{d^2\psi(y)}{dy^2} = -\frac{2mk}{\hbar^2}$$

Then we must have

$$\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} = -c_1$$

and

$$\frac{1}{\psi(y)}\frac{d^2\psi(y)}{dy^2} = -c_2,$$

where c_1 and c_2 are constants. Then

$$\frac{d^2\psi(x)}{dx^2} = -c_1\psi(x)$$

$$\frac{d^2\psi(y)}{dy^2} = -c_2\psi(y)$$

(b), cont'd...

The general solutions are then

$$\psi(x) = A\sin(c_1 x) + B\cos(c_1 x)$$
$$\psi(y) = De^{-ic_2 y} + Fe^{ic_2 y}$$

We must have

$$\psi(-a) = \psi(a) = 0$$

for $\psi(x)$. So

$$\psi(a) = A\sin(c_1 a) + B\cos(c_1 a) = 0$$
$$\psi(-a) = A\sin(-c_1 a) + B\cos(-c_1 a) = 0.$$

If

$$c_1 a = n\frac{\pi}{2},$$

where $n = 1, 3, 5, \ldots$ then we have $A = 0$. If $n = 2, 4, 6, \ldots$ then we have $B = 0$. So

$$\psi_{odd}(x) = B\cos\left(\frac{n\pi x}{2a}\right) \quad (n \in \text{odds})$$
$$\psi_{even}(x) = A\sin\left(\frac{n\pi x}{2a}\right) \quad (n \in \text{evens}).$$

Normalizing gives us

$$\boxed{\begin{aligned} \psi_{odd}(x) &= \sqrt{\frac{1}{a}}\cos\left(\frac{n\pi x}{2a}\right) \quad (n \in \text{odds}) \\ \psi_{even}(x) &= \sqrt{\frac{1}{a}}\sin\left(\frac{n\pi x}{2a}\right) \quad (n \in \text{evens}) \end{aligned}}$$

Since we can't normalize $\psi(y)$, we simply have

$$\boxed{\psi(y) = De^{-iky} + Fe^{iky}}$$

where k is a constant.

(c)

The minimum energy the particle can have is

$$E = \hbar\omega_y + \frac{\pi^2\hbar^2}{8ma^2},$$

where ω_y is the frequency of the particle on the y-axis and the second term is the ground state energy with respect to x.

(d)

The total energy, in general, would be

$$E = \frac{(n_x^2 + n_y^2)\pi^2\hbar^2}{8ma^2}.$$

Therefore, it would <u>not</u> be possible to measure

$$E = \frac{3\pi^2\hbar^2}{8ma^2}$$

since $n_x, n_y \in \mathbb{Z}$.