Time Evolution

F'2016

## Problem 1: Time dependent solutions to Schrodinger's Equation (10 pts)

Consider a particle of mass $m$ in an infinite square well.

$$V(x) = \begin{cases} 0, & -\frac{a}{2} \leq x \leq -\frac{a}{2} \\ \infty, & x < -\frac{a}{2} \text{ or } x > +\frac{a}{2} \end{cases}$$

The solutions to the time independent Schrodinger Equation are:
$H|\Psi_n\rangle = E_n|\Psi_n\rangle$ for n=1,2,3, ... where $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ and

$$\langle x|\Psi_n\rangle = \Psi_n(x) = \sqrt{\frac{2}{a}}\cos(\frac{n\pi x}{a})\ n = 1, 3, 5, ... \quad \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})\ n = 2, 4, 6, ...$$

Assume at $t_o$, the particle is in the state:

$$|\Psi(t_o = 0)\rangle = \sqrt{3/10}\,|\Psi_1\rangle - i\sqrt{7/10}\,|\Psi_3\rangle$$

Answer the following questions:

a) Using Dirac notation, write down the expression for the time evolution operator, $U(t, t_o = 0)$ in terms of energy eigenvalues and eigenstates. (1 pt)

b) Find $|\Psi(t)\rangle = U(t, t_o = 0)|\Psi(t_o = 0)\rangle$ (1 pt)

c) Does your $|\Psi(t)\rangle$ in part b) satisfy the time independent Schrodinger Equation? Demonstrate explicitly. (1 pt)

d) Does your $|\Psi(t)\rangle$ in part b) satisfy the time dependent Schrodinger Equation? Demonstrate explicitly. (1 pt)

e) Is the uncertainty in the energy $\Delta E > 0$, $< 0$ or $= 0$ for $|\Psi(t)\rangle$? Discuss. (1 pt)

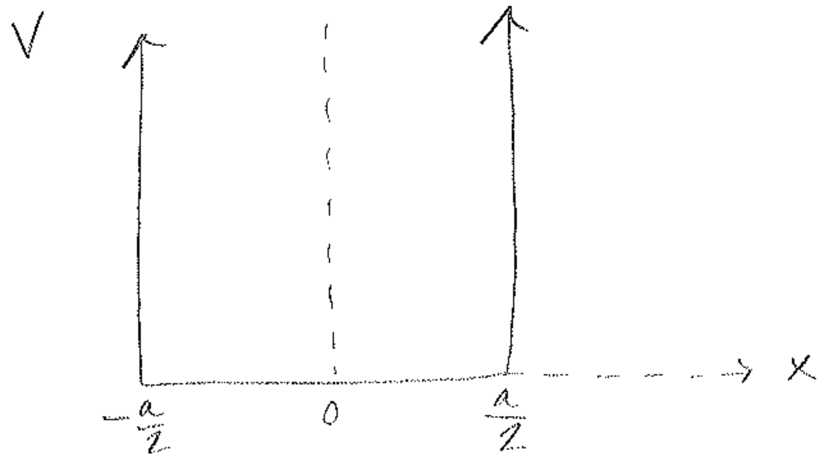
f) State whether the following properties are time dependent or time independent for a system in the state $|\Psi(t)\rangle$. (4 pts)

i) $\Delta E$
ii) $\langle x^2\rangle$
iii) $\langle p\rangle$
iv) $\langle P\rangle$, where $P$ is the parity operator

if observable commutes with $H$, $\langle\ \rangle$ does not change with time.

g) How do your answers to part f) change after the energy is measured at time t and the result is $E = \frac{9\pi^2\hbar^2}{2ma^2}$ ? (1 pt)

Problem 1 — Time dependent solutions to Schrödinger's Equation




$$\psi_{odd}(x) = \sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right) \qquad (n = 1,3,5,\dots)$$

$$\psi_{even}(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad (n = 2,4,6,\dots)$$

To derive these:

Schrodinger equation: $\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right]\psi = E\psi$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi$$

Let

$$k = \frac{\sqrt{2mE}}{\hbar},$$

then differential equation is

$$\frac{\partial^2\psi}{\partial x^2} = -k^2\psi$$

General solution is

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

Next apply boundary conditions. Must have

$$\psi(-\tfrac{a}{2}) = 0$$
$$\psi(\tfrac{a}{2}) = 0.$$

Consider the latter. Then

$$\psi(a/2) = A\sin\left(\frac{ka}{2}\right) + B\cos\left(\frac{ka}{2}\right) = 0$$

Let

$$\frac{ka}{2} = n\frac{\pi}{2}.$$

When $n = 1, 3, 5, \ldots$ we know

$$\cos\left(\frac{ka}{2}\right) = 0,$$

so we must have $A = 0$. Then our unnormalized odd solution is

$$\psi_{odd}(x) = B\cos(kx)$$
$$\psi_{odd}(x) = B\cos\left(\frac{n\pi x}{a}\right).$$

When $n = 2, 4, 6, \ldots$ we know

$$\sin\left(\frac{ka}{2}\right) = 0,$$

so we must have $B = 0$. Then our unnormalized even solution is

$$\psi_{even}(x) = A\sin(kx)$$
$$\psi_{even}(x) = A\sin\left(\frac{n\pi x}{a}\right).$$

Now we want to normalize both solutions. We must have

$$\int_{-a/2}^{a/2} |\psi(x)|^2 dx = 1.$$

Beginning with the odd solutions, we have

$$\int_{-a/2}^{a/2} \left|B\cos\left(\frac{n\pi x}{a}\right)\right|^2 dx = 1$$

$$B^2 \int_{-a/2}^{a/2} \cos^2\left(\frac{n\pi x}{a}\right) dx = 1$$

$$B^2 \int_{-a/2}^{a/2} \frac{1}{2}\left[1+\cos\left(\frac{2n\pi x}{a}\right)\right]dx = 1$$

$$\frac{B^2}{2} \int_{-a/2}^{a/2} \left[1+\cos\left(\frac{2n\pi x}{a}\right)\right]dx = 1$$

$$\frac{B^2}{2}\left[x + \frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]\Bigg|_{-\frac{a}{2}}^{a/2} = 1$$

$$\frac{B^2}{2}\left[a + \frac{a}{2n\pi}\sin(n\pi) - \frac{a}{2n\pi}\sin(-n\pi)\right] = 1$$

$$\frac{B^2}{2}\left[a + \frac{a}{n\pi}\sin(n\pi)\right] = 1$$

But $\sin(n\pi) = 0$ for all values of $n$, so we have

$$\frac{B^2 a}{2} = 1$$

$$B = \sqrt{\frac{2}{a}}$$

and so

$$\psi_{odd}(x) = \sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right) \qquad (n=1,3,5,\ldots)$$

Now for the even solutions...

$$\int_{-a/2}^{a/2} \left| A \sin\left(\frac{n\pi x}{a}\right)\right|^2 dx = 1$$

$$A^2 \int_{-a/2}^{a/2} \sin^2\left(\frac{n\pi x}{a}\right) dx = 1$$

$$A^2 \int_{-a/2}^{a/2} \frac{1}{2}\left[1 - \cos\left(\frac{2n\pi x}{a}\right)\right] dx = 1$$

$$\frac{A^2}{2} \int_{-a/2}^{a/2} \left[1 - \cos\left(\frac{2n\pi x}{a}\right)\right] dx = 1$$

$$\frac{A^2}{2}\left[x - \frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]\Big|_{-a/2}^{a/2} = 1$$

$$\frac{A^2}{2}\left[a - \frac{a}{n\pi}\sin(n\pi)\right] = 1$$

Again, $\sin(n\pi) = 0$ for all values of $n$, so we have

$$\frac{A^2 a}{2} = 1$$

$$A = \sqrt{\frac{2}{a}}$$

and so

$$\psi_{even}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad (n = 2, 4, 6, \ldots).$$

Part (a)

In general, the time evolution operator is given by

$$U(t, t_0 = 0) = \sum_n |\psi_n\rangle\langle\psi_n| \, e^{-iE_n t/\hbar},$$

where the $|\psi_n\rangle$ are in the energy eigenbasis. For this system, we have

$$\boxed{U(t, t_0 = 0) = |\psi_1\rangle\langle\psi_1| \, e^{-iE_1 t/\hbar} + |\psi_2\rangle\langle\psi_2| \, e^{-iE_2 t/\hbar} + \ldots}$$

where for $n = 1, 3, 5, \ldots$ we have

$$\psi_n(x) = \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right)$$

and for $n = 2, 4, 6, \ldots$ we have

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right).$$

For any value of $n$, the energy is given by

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}.$$

Part (b)

We want to determine the time-dependent wave function. We know

$$|\Psi(t)\rangle = U(t, t_0 = 0)\,|\Psi(t_0 = 0)\rangle.$$

In our case, the initial wavefunction is

$$|\Psi(t_0=0)\rangle = \sqrt{\frac{3}{10}}\,|\Psi_1\rangle - i\sqrt{\frac{7}{10}}\,|\Psi_3\rangle.$$

The eigenstates are orthogonal, so the inner product $\langle\Psi_n|\Psi_m\rangle$ for $n \neq m$ is zero. Thus, we have

$$|\Psi(t)\rangle = \left(|\Psi_1\rangle\langle\Psi_1|\,e^{-iE_1t/\hbar} + |\Psi_2\rangle\langle\Psi_2|\,e^{-iE_2t/\hbar} + \ldots\right)\left(\sqrt{\frac{3}{10}}\,|\Psi_1\rangle - i\sqrt{\frac{7}{10}}\,|\Psi_3\rangle\right)$$

$$= |\Psi_1\rangle\langle\Psi_1|\,e^{-iE_1t/\hbar}\sqrt{\frac{3}{10}}\,|\Psi_1\rangle - |\Psi_3\rangle\langle\Psi_3|\,e^{-iE_3t/\hbar}\,i\sqrt{\frac{7}{10}}\,|\Psi_3\rangle$$

$$= \sqrt{\frac{3}{10}}\,|\Psi_1\rangle\langle\Psi_1|\Psi_1\rangle\,e^{-iE_1t/\hbar} - i\sqrt{\frac{7}{10}}\,|\Psi_3\rangle\langle\Psi_3|\Psi_3\rangle\,e^{-iE_3t/\hbar}$$

$$\boxed{|\Psi(t)\rangle = \sqrt{\frac{3}{10}}\,|\Psi_1\rangle\,e^{-iE_1t/\hbar} - i\sqrt{\frac{7}{10}}\,|\Psi_3\rangle\,e^{-iE_3t/\hbar}}$$

Part (c)

The time-independent Schrödinger Equation is given by

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right]\psi = E\psi.$$

Plugging in our expression for $|\Psi(t)\rangle$, we get

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + 0\right]\left(\sqrt{\frac{3}{10}}\,|\psi_1\rangle e^{-iE_1 t/\hbar} - i\sqrt{\frac{7}{10}}\,|\psi_3\rangle e^{-iE_3 t/\hbar}\right)$$

$$= -\frac{\hbar^2}{2m}\sqrt{\frac{3}{10}}\,\frac{\partial^2|\psi_1\rangle}{\partial x^2}e^{-iE_1 t/\hbar} + \frac{\hbar^2}{2m}\,i\sqrt{\frac{7}{10}}\,\frac{\partial^2|\psi_3\rangle}{\partial x^2}e^{-iE_3 t/\hbar}$$

$$= \frac{\hbar^2}{2m}\sqrt{\frac{3}{10}}\sqrt{\frac{2}{a}}\,\frac{\pi^2}{a^2}\cos\left(\frac{\pi x}{a}\right)e^{-iE_1 t/\hbar} - \frac{\hbar^2}{2m}\,i\sqrt{\frac{7}{10}}\sqrt{\frac{2}{a}}\,\frac{9\pi^2}{a^2}\cos\left(\frac{3\pi x}{a}\right)\cdot e^{-iE_3 t/\hbar}$$

$$= \frac{\pi^2\hbar^2}{2ma^2}\sqrt{\frac{3}{10}}\sqrt{\frac{2}{a}}\cos\left(\frac{\pi x}{a}\right)e^{-iE_1 t/\hbar} - \frac{9\pi^2\hbar^2}{2ma^2}\,i\sqrt{\frac{7}{10}}\sqrt{\frac{2}{a}}\cos\left(\frac{3\pi x}{a}\right)e^{-iE_3 t/\hbar}$$

But

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2} \quad \text{and} \quad E_3 = \frac{9\pi^2\hbar^2}{2ma^2},$$

So

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]|\Psi(t)\rangle = E_1\sqrt{\frac{3}{10}}\,|\psi_1\rangle e^{-iE_1 t/\hbar} - E_3\, i\sqrt{\frac{7}{10}}\,|\psi_3\rangle e^{-iE_3 t/\hbar}$$

Thus, our wave-function from part (b) does not satisfy the time-independent Schrödinger Equation. This was to be expected since $|\Psi(t)\rangle$ is a non-stationary state.

$\boxed{\text{Part (d)}}$

The time-dependent Schrödinger Equation is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi,$$

or

$$i\hbar \frac{\partial |\Psi(t)\rangle}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right] |\Psi(t)\rangle$$

$$i\hbar \frac{\partial |\Psi(t)\rangle}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 |\Psi(t)\rangle}{\partial x^2}$$

for our system. Starting with the left-hand side, we have

$$i\hbar \frac{\partial}{\partial t}\left[\sqrt{\frac{3}{10}}\,|\psi_1\rangle e^{-iE_1 t/\hbar} - i\sqrt{\frac{7}{10}}\,|\psi_3\rangle e^{-iE_3 t/\hbar}\right]$$

$$= i\hbar\left[-\frac{iE_1}{\hbar}\sqrt{\frac{3}{10}}\,|\psi_1\rangle e^{-iE_1 t/\hbar} - \frac{E_3}{\hbar}\sqrt{\frac{7}{10}}\,|\psi_3\rangle e^{-iE_3 t/\hbar}\right]$$

$$= E_1\sqrt{\frac{3}{10}}\,|\psi_1\rangle e^{-iE_1 t/\hbar} - E_3\, i\sqrt{\frac{7}{10}}\,|\psi_3\rangle e^{-iE_3 t/\hbar}.$$

Looking at the right-hand side now, we have

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\sqrt{\frac{3}{10}}\sqrt{\frac{2}{a}}\cos\left(\frac{\pi x}{a}\right)e^{-iE_1 t/\hbar} - i\sqrt{\frac{7}{10}}\sqrt{\frac{2}{a}}\cos\left(\frac{3\pi x}{a}\right)e^{-iE_3 t/\hbar}\right]$$

$$= \frac{\hbar^2}{2m}\left[\sqrt{\frac{3}{10}}\sqrt{\frac{2}{a}}\frac{\pi^2}{a^2}\cos\left(\frac{\pi x}{a}\right)e^{-iE_1 t/\hbar} - i\sqrt{\frac{7}{10}}\sqrt{\frac{2}{a}}\frac{9\pi^2}{a^2}\cos\left(\frac{3\pi x}{a}\right)e^{-iE_3 t/\hbar}\right]$$

$$= \frac{\pi^2\hbar^2}{2ma^2}\sqrt{\frac{3}{10}}\,|\psi_1\rangle e^{-iE_1 t/\hbar} - \frac{9\pi^2\hbar^2}{2ma^2}\, i\sqrt{\frac{7}{10}}\,|\psi_3\rangle e^{-iE_3 t/\hbar}$$

$$= E_1\sqrt{\frac{3}{10}}\,|\psi_1\rangle e^{-iE_1 t/\hbar} - E_3\, i\sqrt{\frac{7}{10}}\,|\psi_3\rangle e^{-iE_3 t/\hbar},$$

which agrees with the left side. Thus, the time-dependent equation is satisfied. ☺

Part (c)

In general, we know

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}.$$

Since $|\Psi(t)\rangle$ is not a stationary state, just a linear combination of stationary states, we know that $\Delta E \neq 0$.

Now we want to determine $\langle E^2 \rangle$ and $\langle E \rangle$. Starting with the latter...

$$\langle E \rangle = \langle \Psi(t) | i\hbar \frac{\partial}{\partial t} | \Psi(t) \rangle$$

$$= i\hbar \left[ \left( \sqrt{\tfrac{3}{10}} \langle \psi_1 | e^{\frac{iE_1 t}{\hbar}} + i\sqrt{\tfrac{7}{10}} \langle \psi_3 | e^{\frac{iE_3 t}{\hbar}} \right) \left( -i \frac{E_1}{\hbar} \sqrt{\tfrac{3}{10}} |\psi_1\rangle e^{\frac{-iE_1 t}{\hbar}} - \frac{E_3}{\hbar} \sqrt{\tfrac{7}{10}} |\psi_3\rangle e^{\frac{-iE_3 t}{\hbar}} \right) \right]$$

$$= i\hbar \left[ \frac{-i}{\hbar} \frac{3E_1}{10} \underbrace{\langle \psi_1 | \psi_1 \rangle}_{1} - \frac{i}{\hbar} \frac{7E_3}{10} \underbrace{\langle \psi_3 | \psi_3 \rangle}_{1} + \frac{\sqrt{21}}{10} \frac{E_1}{\hbar} \underbrace{\langle \psi_3 | \psi_1 \rangle}_{0} e^{\frac{-i(E_1 - E_3)t}{\hbar}} - \frac{\sqrt{21}}{10} \frac{E_3}{\hbar} \underbrace{\langle \psi_1 | \psi_3 \rangle}_{0} e^{\frac{i(E_1 - E_3)t}{\hbar}} \right]$$

$$\langle E \rangle = \frac{3E_1}{10} + \frac{7E_3}{10}$$

Now for $\langle E^2 \rangle$, we have...

$$\langle E^2 \rangle = \langle \Psi(t) | -\hbar^2 \frac{\partial^2}{\partial t^2} | \Psi(t) \rangle$$

$$= -\hbar^2 \left[ \left( \sqrt{\tfrac{3}{10}} \langle \psi_1 | e^{\frac{iE_1 t}{\hbar}} + i\sqrt{\tfrac{7}{10}} \langle \psi_3 | e^{\frac{iE_3 t}{\hbar}} \right) \left( -\frac{E_1^2}{\hbar^2} \sqrt{\tfrac{3}{10}} |\psi_1\rangle e^{\frac{-iE_1 t}{\hbar}} + \frac{i}{\hbar^2} E_3^2 \sqrt{\tfrac{7}{10}} |\psi_3\rangle e^{\frac{-iE_3 t}{\hbar}} \right) \right]$$

$$= -\hbar^2 \left[ -\frac{3E_1^2}{10\hbar^2} \underbrace{\langle \psi_1 | \psi_1 \rangle}_{1} - \frac{7E_3^2}{10\hbar^2} \underbrace{\langle \psi_3 | \psi_3 \rangle}_{1} + \frac{\sqrt{21}}{10} \frac{iE_3^2}{\hbar^2} \underbrace{\langle \psi_1 | \psi_3 \rangle}_{0} e^{\frac{i(E_1 - E_3)t}{\hbar}} - \frac{\sqrt{21}}{10} \frac{iE_1^2}{\hbar^2} \underbrace{\langle \psi_3 | \psi_1 \rangle}_{0} e^{\frac{-i(E_1 - E_3)t}{\hbar}} \right]$$

$$= \frac{3}{10} E_1^2 + \frac{7}{10} E_3^2$$

Then

$$\langle E\rangle^2 = \left(\frac{3}{10}E_1 + \frac{7}{10}E_3\right)^2$$

$$= \frac{9}{100}E_1^2 + \frac{49}{100}E_3^2 + \frac{42}{100}E_1E_3$$

and so

$$\Delta E^2 = \langle E^2\rangle - \langle E\rangle^2$$

$$= \left(\frac{3}{10}E_1^2 + \frac{7}{10}E_3^2\right) - \left(\frac{9}{100}E_1^2 + \frac{49}{100}E_3^2 + \frac{42}{100}E_1E_3\right)$$

$$= \frac{30}{100}E_1^2 + \frac{70}{100}E_3^2 - \frac{9}{100}E_1^2 - \frac{49}{100}E_3^2 - \frac{42}{100}E_1E_3$$

$$= \frac{21}{100}E_1^2 + \frac{21}{100}E_3^2 - \frac{42}{100}E_1E_3$$

$$= \frac{21}{100}\left(E_3^2 - 2E_1E_3 + E_1^2\right)$$

$$= \frac{21}{100}\left(E_3 - E_1\right)^2$$

and so

$$\Delta E = \frac{\sqrt{21}}{10}\left(E_3 - E_1\right).$$

Since $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$, then our uncertainty in the energy is positive.

Part (f)

(i)

As we determined in part (e), $\Delta E$ has no time-dependence.

(ii)

$$\langle X^2\rangle = \langle\Psi(t)|x^2|\Psi(t)\rangle$$
$$= \left(\sqrt{\tfrac{3}{10}}\langle\psi_1|e^{\frac{iE_1t}{\hbar}} + i\sqrt{\tfrac{7}{10}}\langle\psi_3|e^{\frac{iE_3t}{\hbar}}\right) x^2 \left(\sqrt{\tfrac{3}{10}}|\psi_1\rangle e^{\frac{-iE_1t}{\hbar}} - i\sqrt{\tfrac{7}{10}}|\psi_3\rangle e^{\frac{-iE_3t}{\hbar}}\right)$$
$$= \frac{3}{10}\underbrace{\langle\psi_1|x^2|\psi_1\rangle}_{\text{even}} + \frac{7}{10}\underbrace{\langle\psi_3|x^2|\psi_3\rangle}_{\text{even}} - i\frac{\sqrt{21}}{10}\underbrace{\langle\psi_1|x^2|\psi_3\rangle}_{\text{even}} e^{\frac{i(E_1-E_3)t}{\hbar}} + i\frac{\sqrt{21}}{10}\underbrace{\langle\psi_3|x^2|\psi_1\rangle}_{\text{even}} e^{\frac{-i(E_1-E_3)t}{\hbar}}$$

This will be $\boxed{\text{time-dependent}}$ since all of our integrals are even and our time dependence will not cancel out.

(iii)

$$\langle p\rangle = \langle\Psi(t)|-i\hbar\tfrac{\partial}{\partial x}|\Psi(t)\rangle$$
$$= -i\hbar\left(\sqrt{\tfrac{3}{10}}\langle\psi_1|e^{\frac{iE_1t}{\hbar}} + i\sqrt{\tfrac{7}{10}}\langle\psi_3|e^{\frac{iE_3t}{\hbar}}\right)\left(-\frac{\pi}{a}\sqrt{\tfrac{3}{10}}|\psi_{1,\sin}\rangle e^{\frac{-iE_1t}{\hbar}} + \frac{3\pi}{a}i\sqrt{\tfrac{7}{10}}|\psi_{3,\sin}\rangle e^{\frac{-iE_3t}{\hbar}}\right)$$
$$= -i\hbar\left(-\frac{\pi}{a}\frac{3}{10}\underbrace{\langle\psi_1|\psi_{1,\sin}\rangle}_{\text{odd}} - \frac{3\pi}{a}\frac{7}{10}\underbrace{\langle\psi_3|\psi_{3,\sin}\rangle}_{\text{odd}} - i\frac{\pi}{a}\frac{\sqrt{21}}{10}\underbrace{\langle\psi_3|\psi_{1,\sin}\rangle}_{\text{odd}} e^{\frac{-i(E_1-E_3)t}{\hbar}} + \frac{3\pi}{a}i\frac{\sqrt{21}}{10}\underbrace{\langle\psi_1|\psi_{3,\sin}\rangle}_{\text{odd}} e^{\frac{i(E_1-E_3)t}{\hbar}}\right)$$

All of these integrals are odd, so they will go to zero. Thus, $\langle p\rangle = 0$ and therefore, $\langle p\rangle$ is $\boxed{\text{time-independent}}$.

$\boxed{iv}$

We want to determine $\langle \pi \rangle$, where $\pi |\Psi(x)\rangle\rangle = |\Psi(-x)\rangle\rangle$.

$$\langle \pi \rangle = \langle \Psi(x,t) | \pi | \Psi(x;t)\rangle\rangle$$

$$= \langle \Psi(x,t) | \Psi(-x;t)\rangle\rangle$$

$$= \langle \Psi(x,t) | \Psi(x;t)\rangle\rangle$$

$$= \left( \sqrt{\frac{3}{10}} \langle \psi_1 | e^{\frac{iE_1 t}{\hbar}} + i\sqrt{\frac{7}{10}} \langle \psi_3 | e^{\frac{iE_3 t}{\hbar}} \right) \left( \sqrt{\frac{3}{10}} |\psi_1\rangle e^{\frac{-iE_1 t}{\hbar}} - i\sqrt{\frac{7}{10}} |\psi_3\rangle e^{\frac{-iE_3 t}{\hbar}} \right)$$

$$= \frac{3}{10} \underbrace{\langle \psi_1 | \psi_1 \rangle}_{1} + \frac{7}{10} \underbrace{\langle \psi_3 | \psi_3 \rangle}_{1} - i\frac{\sqrt{21}}{10} \underbrace{\langle \psi_1 | \psi_3 \rangle}_{0} e^{\frac{i(E_1 - E_3)t}{\hbar}} + i\frac{\sqrt{21}}{10} \underbrace{\langle \psi_3 | \psi_1 \rangle}_{0} e^{\frac{-i(E_1 - E_3)t}{\hbar}}$$

$$\langle \pi \rangle = 1.$$

This is clearly $\boxed{\text{time-independent}}$.

Part (g)

Our wave-function is now the stationary state

$$|\Psi'(t)\rangle = |\Psi_3\rangle e^{-iE_3 t/\hbar}.$$

(i)

$\Delta E$ will still be time-independent, and will in fact be zero.

(ii)

$$\langle \Psi'(t) | x^2 | \Psi'(t) \rangle = \langle \Psi_3 | x^2 | \Psi_3 \rangle$$

Time dependence cancels out, so $\langle x^2 \rangle$ is time-independent.

(iii)

$\langle p \rangle = 0$, so $\langle p \rangle$ is time-independent.

$[H, p] = 0$ for a free particle.

(iv)

The state is still even, so I expect

$$\langle \pi \rangle = 1.$$

$$\langle \Psi'(t) | \pi | \Psi'(t) \rangle = \langle \Psi'(x,t) | \Psi'(-x,t) \rangle$$
$$= \langle \Psi'(x,t) | \Psi'(x,t) \rangle$$
$$= 1.$$

Thus, $\langle \pi \rangle$ is time-independent.