F2014

PROBLEM 5: Addition of angular momenta

Consider an electron. We know its orbital angular momentum $\ell = 1$ and the z component $m = 1/2$ of its total angular momentum j.

a) What are the possible values of j? (2 Points).

b) Write down the kets $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$ in terms of products of spin and orbital angular momentum states (3 Points)

c) Calculate the expectation value of the spin operator $\mathbf{S}$ in the state $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$. Consider all possible values of j. (3 Points).

d) The magnetic dipole moment of the electron is

$$\boldsymbol{\mu} = \frac{e}{2m_e c}(\mathbf{L} + 2\mathbf{S}),$$

with $\mathbf{L}$ the orbital angular momentum operator, e the electron charge, m_e the mass and c the speed of light. Calculate the expectation value of $\boldsymbol{\mu}$ in the states $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$. (2 Points)

Raising and lowering angular momentum operators:

$$J_\pm|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

(a)

We know $l=1$ and $m_l = 1/2$. Since this is an electron, we know $S = 1/2$. We must have

$$|l-s| \leq j \leq |l+s|$$

$$|1-1/2| \leq j \leq |1+1/2|$$

$$1/2 \leq j \leq 3/2$$

So we can have $\boxed{j = 1/2, 3/2}$.

(b)

We want to write $|l=1, 1/2; j, m=1/2\rangle$ in terms of products of spin and orbital angular momentum states. We start in the state $|L=1, S=1/2; J=3/2, M=3/2\rangle$, where $M = m_l + m_s$. We then have

$$|1\ 1/2; 3/2\ 3/2\rangle = |l=1, m_l=1\rangle \otimes |S=1/2, m_s=1/2\rangle$$

Applying J_- to the left-hand side, we get

$$J_-|1\ 1/2; 3/2\ 3/2\rangle = \hbar\sqrt{(j+m)(j-m+1)}\,|1\ 1/2; 3/2\ 1/2\rangle$$

$$= \hbar\sqrt{((3/2)+(3/2))((3/2)-(3/2)+1)}\,|1\ 1/2; 3/2\ 1/2\rangle$$

$$= \sqrt{3}\hbar\,|1\ 1/2; 3/2\ 1/2\rangle$$

Applying $(J_-^L + J_-^S)$ to the right-hand side, we get

$$(J_-^L + J_-^S)\,|1\ 1\rangle \otimes |1/2\ 1/2\rangle$$

$$= (J_-^L|1\ 1\rangle)|1/2\ 1/2\rangle + |1\ 1\rangle(J_-^S|1/2\ 1/2\rangle)$$

(b), cont'd...

$$= \hbar\sqrt{(l+m_l)(l-m_l+1)}\,|1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \hbar\sqrt{(s+m_s)(s-m_s+1)}\,|1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$= \sqrt{2}\,\hbar\,|1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \hbar\,|1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

Then overall, we have

$$\sqrt{3}\,\hbar\,|1\ \tfrac{1}{2};\ \tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{2}\,\hbar\,|1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \hbar\,|1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$\boxed{|1\ \tfrac{1}{2};\ \tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle}.$$

Now we start in the state $|1\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$. We can use orthogonality to determine the product for this state. We must have

$$\langle 1\ \tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\,|\,1\ \tfrac{1}{2};\ \tfrac{3}{2}\ \tfrac{1}{2}\rangle = 0,$$

so letting

$$|1\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle = A\,|1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + B\,|1\ 1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle,$$

where A & B are constants, we have

$$A\sqrt{\tfrac{2}{3}} + B\sqrt{\tfrac{1}{3}} = 0,$$

or

$$B = -A\sqrt{2}.$$

(b), cont'd...

We must also normalize this state, which implies

$$A^2 + B^2 = 1,$$

or

$$A^2 + \left(-A\sqrt{2}\right)^2 = 1$$
$$A^2 + 2A^2 = 1$$
$$3A^2 = 1$$
$$A = \sqrt{\frac{1}{3}}.$$

Then

$$B = -\sqrt{\frac{1}{3}} \cdot \sqrt{2}$$
$$= -\sqrt{\frac{2}{3}}$$

and so

$$\boxed{|1\ \tfrac{1}{2}\,;\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle = \sqrt{\frac{1}{3}}\,|1\ 0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|1\ 1\rangle|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle}$$

(c)

Now we want to calculate the expectation value of the spin operator in the state $|1\ \tfrac{1}{2};\ j\ \tfrac{1}{2}\rangle$. In other words, we want to find

$$\langle S(j=3/2)\rangle = \langle 1\ \tfrac{1}{2};\ \tfrac{3}{2}\ \tfrac{1}{2}|S|1\ \tfrac{1}{2};\ \tfrac{3}{2}\ \tfrac{1}{2}\rangle$$

and

$$\langle S(j=1/2)\rangle = \langle 1\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}|S|1\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle.$$