Problem 1: Spin $\frac{1}{2}$ particles (10 points)

Consider a system made up of spin 1/2 particles. If one measures the spin of the particles, one can only measure spin up or spin down. The general spin state of a spin 1/2 particle can be expressed as a two-element column matrix.

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

The spin matrices are:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) Can one simultaneously measure S_x, S_y and S_z? Explain your answer. (1 pt)

b) Can one simultaneously measure S^2 and S_z? Explain your answer. (1 pt)

c) Show S_z is Hermetian. (1 pt)

d) Calculate the normalized eigenvectors and eigenvalues of S_z. (2 pts)

Suppose a spin 1/2 particle is in the state

$$\chi = A\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

e) Normalize the state in order to determine A (1 pt)

f) If one measures S_z, what is the probability of getting -$\hbar/2$? (1 pt)

g) If one measures S_x, what is the probability of getting $+\hbar/2$? (2 pts)

h) What is the expectation value of S_y (1 pt)

(a)

No, one cannot simultaneously measure S_x, S_y, and S_z. We know that

$$[S_i, S_j] = i\hbar\, \varepsilon_{ijk} S_k, \quad (i,j,k = x,y,z)$$

which implies that $\{S_x, S_y, S_z\}$ do not form a complete set of commuting observables and thus, cannot be measured simultaneously.

(b)

Yes, we know that S^2 and S_z can simultaneously be measured because

$$[S^2, S_i] = 0 \qquad (i = x,y,z).$$

This means these observables are compatible.

(c)

A Hermitian operator is one that is equal to its adjoint, or complex transpose. We know

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so

$$S_z^{\dagger} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus,

$$S_z = S_z^{\dagger}$$

and S_z is Hermitian.

(d)

$$\begin{vmatrix} \frac{\hbar}{2}-\lambda & 0 \\ 0 & -\frac{\hbar}{2}-\lambda \end{vmatrix} = 0$$

$$\left(\frac{\hbar}{2}-\lambda\right)\left(-\frac{\hbar}{2}-\lambda\right) = 0$$

$$\boxed{\lambda = \pm\frac{\hbar}{2}} \quad \text{or} \quad \boxed{S_z^+ = \hbar/2 \,,\; S_z^- = -\hbar/2}.$$

$|S_z^+ = \hbar/2\rangle$

$$\begin{pmatrix} \frac{\hbar}{2}-\frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2}-\frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} 0c_1 + 0c_2 = 0 \\ 0c_1 - \hbar c_2 = 0 \end{array}$$

So c_1 can be anything and $c_2 = 0$. Then we have

$$\boxed{|S_z^+ = \hbar/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$|S_z^- = \hbar/2\rangle$

$$\begin{pmatrix} \frac{\hbar}{2}+\frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2}+\frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} \hbar c_1 + 0c_2 = 0 \\ 0c_1 + 0c_2 = 0 \end{array}$$

Similarly, we have $c_1 = 0$ and c_2 can be anything, so

$$\boxed{|S_z^- = \hbar/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

(e)

We have

$$\chi = A\begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$

To normalize, we want

$$\langle \chi | \chi \rangle = 1,$$

so

$$\langle \chi | \chi \rangle = A^2 (1-i, 2)\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

$$= A^2[(1-i)(1+i)+4]$$

$$= A^2(1+1+4)$$

$$= 6A^2 = 1$$

$$\boxed{A = \frac{1}{\sqrt{6}}}$$

and

$$\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$

(f)

We want to determine $P_{|\chi\rangle}(S_z = -\hbar/2)$. So

$$P_{|\chi\rangle}(S_z = -\hbar/2) = \left|\langle S_z = -\hbar/2 | \chi \rangle\right|^2$$
$$= \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}}(1+i) \\ 2/\sqrt{6} \end{pmatrix} \right|^2$$
$$= \frac{1}{6} \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right|^2$$
$$= \frac{1}{6} |2|^2$$
$$\boxed{P_{|\chi\rangle}\left(S_z = -\frac{\hbar}{2}\right) = \frac{2}{3}}$$

(g)

We need to find the eigenvector associated with $|S_x = \hbar/2\rangle$.
So

$$\begin{pmatrix} -\frac{\hbar}{2} & \hbar/2 \\ \hbar/2 & -\frac{\hbar}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -\frac{\hbar}{2} c_1 + \frac{\hbar}{2} c_2 = 0 \\ \frac{\hbar}{2} c_1 - \frac{\hbar}{2} c_2 = 0 \end{matrix}$$

Then

$$|S_x = \hbar/2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So

$$P_{|\chi\rangle}(S_x = \hbar/2) = \left| \frac{1}{\sqrt{6 \cdot 2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right|^2$$
$$= \frac{1}{12} \left| 1 + i + 2 \right|^2$$
$$= \frac{1}{12}(-i+3)(i+3) = \frac{1}{12}(1+9)$$
$$\boxed{P_{|\chi\rangle}(S_x = \hbar/2) = 5/6}$$

The expectation value is

$$\langle S_y \rangle = \langle \chi | S_y | \chi \rangle = \frac{\hbar}{12}(1-i,\ 2)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

$$= \frac{\hbar}{12}(1-i,\ 2)\begin{pmatrix} -2i \\ i-1 \end{pmatrix}$$

$$= \frac{\hbar}{12}(-2i-2+2i-2)$$

$$\boxed{\langle \chi | S_y | \chi \rangle = -\frac{\hbar}{3}}$$