PROBLEM 4: Isotropic Harmonic Oscillator

The Hamiltonian of a one-dimensional harmonic oscillator is

$$H = \frac{1}{2m} P_x^2 + \frac{1}{2} m\omega_0^2 X^2 .$$

The harmonic oscillator wave function is often written as

$$\psi_n(\xi) = A_n H_n(\xi) e^{-\frac{1}{2}\xi^2}, \qquad n = 0, 1, 2, \cdots$$

where A_n = normalization constant, $H_n(\xi)$ is a Hermite polynomial and

$$\xi = \alpha x, \quad \text{with} \quad \alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2} .$$

(~~a~~) What are the energy and the parity of the eigenstate associated with quantum number n? *(2 points)*

Let us now consider a 3-dimensional isotropic harmonic oscillator with the following Hamiltonian

$$\begin{aligned} H &= H_x + H_y + H_z \\ &= \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2) + \frac{1}{2} m\omega^2 (X^2 + Y^2 + Z^2) . \end{aligned}$$

The wave function is given by

$$\Psi_{n_x, n_y, n_z}(x, y, z) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z) .$$

(~~b~~) Find the energy, parity, and degeneracy of the lowest three distinct groups of energy levels. *(3 points)*

(~~c~~) What is the degeneracy of the energy levels with the same quantum number $n = n_x + n_y + n_z$? *(2 points)*

(d) The 3-dimensional harmonic oscillator can also be solved in spherical coordinates. Apply your knowledge of angular dependence for various states to find the angular momentum quantum number (ℓ) for the lowest two energy levels studied in part (b). *(3 points)*

(a)

We know

$$X = \sqrt{\frac{\hbar}{2m\omega_0}}\left(a + a^{+}\right)$$

$$P = -i\sqrt{\frac{m\hbar\omega_0}{2}}\left(a - a^{+}\right),$$

so

$$H = \frac{1}{2m}\left(-i\sqrt{\frac{m\hbar\omega_0}{2}}\left(a-a^{+}\right)\right)^2 + \frac{1}{2}m\omega_0^2\left(\sqrt{\frac{\hbar}{2m\omega_0}}\left(a+a^{+}\right)\right)^2$$

$$= -\frac{\hbar\omega_0}{4}\left(a-a^{+}\right)^2 + \frac{\hbar\omega_0}{4}\left(a+a^{+}\right)^2$$

$$= \frac{\hbar\omega_0}{4}\left(aa^{+}+a^{+}a\right) + \frac{\hbar\omega_0}{4}\left(aa^{+}+a^{+}a\right)$$

$$= \frac{\hbar\omega_0}{2}\left(aa^{+}+a^{+}a\right).$$

We know

$$[a, a^{+}] = aa^{+} - a^{+}a = 1$$

$$aa^{+} = 1 + a^{+}a,$$

so

$$H = \frac{\hbar\omega_0}{2}\left(2a^{+}a + 1\right)$$

$$H = \hbar\omega_0\left(a^{+}a + 1/2\right).$$

We know

$$H|n\rangle = E|n\rangle,$$

so

$$H|n\rangle = \hbar\omega_0\left(a^{+}a + \tfrac{1}{2}\right)|n\rangle$$

(a), cont'd...

$$H|n\rangle = \hbar\omega_0 \left(\sqrt{n}\, a^\dagger |n-1\rangle + \tfrac{1}{2}|n\rangle\right)$$
$$= \hbar\omega_0 \left(n|n\rangle + \tfrac{1}{2}|n\rangle\right)$$
$$= \hbar\omega_0 \left(n + \tfrac{1}{2}\right)|n\rangle .$$

So

$$\boxed{E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega_0}.$$

We know

$$H_n(-\xi) = (-1)^n H_n(\xi).$$

So since

$$\psi_n(\xi) = A_n H_n(\xi) e^{-\frac{1}{2}\xi^2},$$

we have

$$\psi_n(-\xi) = A_n H_n(-\xi)\, e^{-\frac{1}{2}(-\xi)^2}$$
$$= (-1)^n A_n H_n(\xi)\, e^{-\frac{1}{2}\xi^2}$$
$$= (-1)^n \psi_n(\xi).$$

Thus, the parity of the eigenstate ψ_n is

$$\boxed{\Pi \psi_n = (-1)^n \psi_n}$$

(b)

The total energy is given by

$$E_{n_x n_y n_z} = (n_x + n_y + n_z + 3/2)\hbar\omega_0$$

or

$$\boxed{E_N = (N + 3/2)\hbar\omega_0},$$

where $N = 0, 1, 2, \ldots$

The three lowest energy levels are

$$\boxed{\begin{aligned} E_0 &= \tfrac{3}{2}\hbar\omega_0 \\ E_1 &= \tfrac{5}{2}\hbar\omega_0 \\ E_2 &= \tfrac{7}{2}\hbar\omega_0 \end{aligned}}.$$

The degeneracy of the E_0 level is $\boxed{1}$.
The degeneracy of the E_1 level is $\boxed{3}$.
The degeneracy of the E_2 level is $\boxed{6}$.

We have

$$\psi_{n_x n_y n_z}(x, y, z) = \psi_{n_x}(x)\,\psi_{n_y}(y)\,\psi_{n_z}(z).$$

So we want to determine the parity.

(b), cont'd...

$$\psi_{n_x n_y n_z}(-x,-y,-z) = (-1)^{n_x}(-1)^{n_y}(-1)^{n_z}\psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z).$$

In the E_0 case, we have

$$\psi_{n_x n_y n_z}(-x,-y,-z) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z). \qquad (+)$$

In the E_1 case, we have

$$\psi_{n_x n_y n_z}(-x,-y,-z) = -\psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z). \qquad (-)$$

In the E_2 case, we have

$$\psi_{n_x n_y n_z}(-x,-y,-z) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z). \qquad (+)$$

(c)

Choose n_x. Then

$$n_y + n_z = n - n_x .$$

There are

$$n - n_x + 1$$

different pairs of n_y and n_z. Then

$$\sum_{n_x=0}^{n} (n - n_x + 1) = \sum_{0}^{n} (n+1) - \sum_{0}^{n} n_x$$

$$= (n+1)(n+1) - \frac{1}{2} n (n+1)$$

$$\boxed{g_n = \frac{(n+1)(n+2)}{2}}$$