F2011

PROBLEM 3: Perturbation Theory

Consider a particle of mass m trapped inside a 1D parabolic potential

$$V(x) = \frac{1}{2} m\omega^2 x^2,$$

where ω sets the frequency of oscillation inside the potential.

a) If the particle is perturbed by a *static* potential

$$V_I = \alpha x,$$

with α small, compute energy correction of the energy levels in the lowest order where the result is non-zero. (3 Points)

b) What is the pertubed ket in the ground state? Compute the expectation value $\langle x \rangle$ in this state. Interpret the sign of $\langle x \rangle$. (3 Points)

c) Assume from now on that $\alpha = 0$. Imagine that the particle is charged and sits in the ground state at $t = -\infty$. Suppose an electric field is gradually tuned on, increases to a maximum at $t = 0$ and then slowly dies away,

$$V_I'(t) = -e|\mathbf{E}|x\, e^{-t^2/\tau^2},$$

where e is the electric charge, and $\mathbf{E}$ is the electric field . Write down the general expression for the amplitude of transition from a generic level i to level f. (Do not solve the integral yet) (2 Points).

d) Evaluate the probability of having the particle in the first excited state at $t = +\infty$. (2 Points).

Hint: $\int_{-\infty}^{\infty} dt\, e^{-t^2/\tau^2} e^{i\omega t} = \sqrt{\pi}\tau\, e^{-\omega^2\tau^2/4}$

(a)

The first-order correction to the energy is, in general,

$$E^{(1)} = \langle n | V_I | n \rangle,$$

where $|n\rangle$ is the unperturbed eigenstate and V_I is the perturbation to the potential. So we have

$$E^{(1)} = \langle n^{(0)} | \alpha x | n^{(0)} \rangle$$
$$= \alpha \langle n | x | n \rangle .$$

(superscript removed for the sake of simplicity)

In terms of raising and lowering operators, we know

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a + a^{+} \right),$$

so

$$E^{(1)} = \alpha \langle n | \sqrt{\frac{\hbar}{2m\omega}} \left(a + a^{+} \right) | n \rangle$$
$$= \alpha \sqrt{\frac{\hbar}{2m\omega}} \left(\langle n | a | n \rangle + \langle n | a^{+} | n \rangle \right)$$
$$= \alpha \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n} \langle n | n-1 \rangle + \sqrt{n+1} \langle n | n+1 \rangle \right)$$
$$= 0 .$$

The second-order correction to the energy is, in general,

$$E^{(2)} = \sum_{m \neq n} \frac{|\langle m | V_I | n \rangle|^2}{E_n^{(0)} - E_m^{(0)}},$$

where

$$E_n^{(0)} = \hbar\omega \left(n + 1/2 \right)$$

$$E_m^{(0)} = \hbar\omega \left(m + 1/2 \right).$$

(a), cont'd...

So

$$E^{(2)} = \sum_{m\neq n} \frac{|\langle m|\cancel{\alpha}\sqrt{\frac{\hbar}{2m\omega}}(a+a^{+})|n\rangle|^2}{\hbar\omega(n+1/2)-\hbar\omega(m+1/2)}$$

$$= \frac{\alpha^2\hbar}{2m\omega}\cdot\frac{1}{\hbar\omega}\sum_{m\neq n}\frac{|\langle m|a+a^{+}|n\rangle|^2}{n-m}$$

$$= \frac{\alpha^2}{2m\omega^2}\sum_{m\neq n}\frac{|\langle m|n-1\rangle\sqrt{n}+\langle m|n+1\rangle\sqrt{n+1}|^2}{n-m}$$

The cross-terms will always go to zero, so we have

$$E^{(2)} = \frac{\alpha^2}{2m\omega^2}\sum_{m\neq n}\frac{n\langle m|n-1\rangle^2+(n+1)\langle m|n+1\rangle^2}{n-m}$$

$$= \frac{\alpha^2}{2m\omega^2}\sum_{m\neq n}\frac{n\,\delta_{m,n-1}+(n+1)\,\delta_{m,n+1}}{n-m}$$

$$= \frac{\alpha^2}{2m\omega^2}\left[\sum_{m\neq n}\frac{n\,\delta_{m,n-1}}{n-m}+\sum_{m\neq n}\frac{(n+1)\,\delta_{m,n+1}}{n-m}\right]$$

$$= \frac{\alpha^2}{2m\omega^2}\left[\frac{n}{n-(n-1)}+\frac{n+1}{n-(n+1)}\right]$$

$$= \frac{\alpha^2}{2m\omega^2}\left(\frac{n}{n-n+1}+\frac{n+1}{n-n-1}\right)$$

$$= \frac{\alpha^2}{2m\omega^2}\left(\frac{n}{1}+\frac{n+1}{-1}\right)$$

$$= \frac{\alpha^2}{2m\omega^2}\left(n-(n+1)\right)$$

$$\boxed{E^{(2)} = -\frac{\alpha^2}{2m\omega^2}}.$$

(b)

Now we want to determine the perturbed ket in the ground state. In general, we know the correction to the ground state ket is given by

$$|n^{(1)}\rangle = \sum_{m\neq n} \frac{\langle m^{(0)}|V_{\pm}|n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}} |m^{(0)}\rangle$$

$$= \alpha\sqrt{\frac{\hbar}{2m\omega}} \cdot \frac{1}{\hbar\omega} \sum_{m\neq n} \frac{\langle m|a+a^{+}|n\rangle}{n-m} |m\rangle$$

$$= \frac{\alpha}{\sqrt{2m\hbar\omega^3}} \left[\sum_{m\neq n} \frac{\sqrt{n}\langle m|n-1\rangle}{n-m}|m\rangle + \sum_{m\neq n} \frac{\sqrt{n+1}\langle m|n+1\rangle}{n-m}|m\rangle \right]$$

$$= \frac{\alpha}{\sqrt{2m\hbar\omega^3}} \left(\frac{\sqrt{n}}{n-(n-1)}|n-1\rangle + \frac{\sqrt{n+1}}{n-(n+1)}|n+1\rangle \right)$$

$$= \frac{\alpha}{\sqrt{2m\hbar\omega^3}} \left(\sqrt{n}\,|n-1\rangle - \sqrt{n+1}\,|n+1\rangle \right)$$

but we are in the ground state, so $n=0$ and our lowest possible state is $|0\rangle$, giving us

$$|n^{(1)}\rangle = -\frac{\alpha}{\sqrt{2m\hbar\omega^3}} |1\rangle,$$

where $|1\rangle$ is the unperturbed first excited state. Thus, our perturbed ground state ket is

$$\boxed{|0^{(1)}\rangle = |0\rangle - \frac{\alpha}{\sqrt{2m\hbar\omega^3}} |1\rangle}.$$

The expectation value is given by

$$\langle x\rangle = \langle 0^{(1)}|x|0^{(1)}\rangle$$

$$= \left(\langle 0| - \frac{\alpha}{\sqrt{2m\hbar\omega^3}}\langle 1| \right) x \left(|0\rangle - \frac{\alpha}{\sqrt{2m\hbar\omega^3}}|1\rangle \right)$$

$$= \langle 0|x|0\rangle - \frac{\alpha}{\sqrt{2m\hbar\omega^3}}\langle 1|x|0\rangle - \frac{\alpha}{\sqrt{2m\hbar\omega^3}}\langle 0|x|1\rangle + \frac{\alpha^2}{2m\hbar\omega^3}\langle 1|x|1\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left[\langle 0|a+a^{+}|0\rangle - \frac{\alpha}{\sqrt{2m\hbar\omega^3}}\langle 1|a+a^{+}|0\rangle - \frac{\alpha}{\sqrt{2m\hbar\omega^3}}\langle 0|a+a^{+}|1\rangle + \frac{\alpha^2}{2m\hbar\omega^3}\langle 1|a+a^{+}|1\rangle \right]$$

(b), cont'd...

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[0 - \frac{\kappa}{\sqrt{2m\hbar\omega^3}}\left(0 + \langle 1|1\rangle\right) - \frac{\kappa}{\sqrt{2m\hbar\omega^3}}\left(\langle 0|0\rangle + 0\right) + 0\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(-\frac{2\kappa}{\sqrt{2m\hbar\omega^3}}\right)$$

$$= -\frac{2\kappa}{2m\omega^2}$$

$$\boxed{\langle x \rangle = -\frac{\kappa}{m\omega^2}}$$

In the unperturbed ground state, $\langle x^{(0)} \rangle = 0$. This tells us that by perturbing the ground state, the expectation value has been shifted more to the negative side of the potential well. This implies that the well must be deeper on this side.

(c)

Now we consider the time-dependent perturbation

$$V_I'(t) = -e|\vec{E}|\, x\, e^{-t^2/\tau^2}.$$

For a transition from $|a\rangle$ to $|b\rangle$, we know the transition amplitude is, to first order,

$$\boxed{c_b^{(1)} = -\frac{i}{\hbar}\int_{t_0}^{t} H'_{ba}(t)\, e^{i\omega_0 t'}\, dt'} \quad (t_0 = -\infty, t = \infty)$$

where $H'_{ba} = \langle b|H'|a\rangle$ and $\omega_0 = \frac{E_b^{(0)} - E_a^{(0)}}{\hbar}$. Then

$$H'_{1,0} = V'_{1,0} = \langle 1|V'|0\rangle$$

So

$$c_1^{(1)} = -\frac{i}{\hbar}\int_{-\infty}^{\infty}\langle 1|V'|0\rangle\, e^{i\left[\frac{E_1^{(0)} - E_0^{(0)}}{\hbar}\right]t'}\, dt'.$$

We have

$$\begin{aligned}
\langle 1|V'|0\rangle &= \langle 1|-e|\vec{E}|\, x\, e^{-t^2/\tau^2}|0\rangle \\
&= -e|\vec{E}|\, e^{-t^2/\tau^2}\langle 1|x|0\rangle \\
&= -e|\vec{E}|\, e^{-t^2/\tau^2}\langle 1|\sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)|0\rangle \\
&= -e|\vec{E}|\, e^{-t^2/\tau^2}\sqrt{\frac{\hbar}{2m\omega}}\left[0 + \langle 1|1\rangle\right] \\
&= -e\sqrt{\frac{\hbar}{2m\omega}}\, |\vec{E}|\, e^{-t^2/\tau^2}
\end{aligned}$$

(c), cont'd...

and so

$$c_1^{(1)} = \frac{ie}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}\, E \int_{-\infty}^{\infty} e^{i\left[\frac{E_1^{(0)}-E_0^{(0)}}{\hbar}\right]t'} e^{-t'^2/\tau^2}\, dt'.$$

We are told

$$\int_{-\infty}^{\infty} e^{-t^2/\tau^2} e^{i\omega t}\, dt = \sqrt{\pi}\tau\, e^{-\omega^2\tau^2/4},$$

so

$$c_1^{(1)} = \frac{ieE}{\sqrt{2m\hbar\omega}}\left(\sqrt{\pi}\tau\, e^{-\left(\frac{E_1^{(0)}-E_0^{(0)}}{\hbar}\right)^2\tau^2/4}\right)$$

$$\boxed{c_1^{(1)} = ieE\sqrt{\frac{\pi\tau}{2m\hbar\omega}}\, e^{-\left(\frac{E_1^{(0)}-E_0^{(0)}}{\hbar}\right)^2\tau^2/4}}$$

(d) Then the probability of having the particle in the first excited state at $t=+\infty$ is

$$\boxed{|c_1^{(1)}|^2 = \frac{e^2|\vec{E}|^2\pi\tau}{2m\hbar\omega}\, e^{-\left(\frac{E_1^{(0)}-E_0^{(0)}}{\hbar}\right)^2\tau^2/2}}$$