Harmonic Oscillator

F2016

Problem 5: Expanding Harmonic Oscillator (10 pts)

Consider a particle of mass m confined in a 1D harmonic oscillator potential with frequency ω_0

$$H_a = \frac{P^2}{2m} + \frac{m}{2}\omega_0^2\, X^2 \tag{1}$$

The raising and lowering operators are useful for harmonic oscillator problems:

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{X}{\lambda} - i\frac{\lambda}{\hbar}P\right) \quad a = \frac{1}{\sqrt{2}}\left(\frac{X}{\lambda} + i\frac{\lambda}{\hbar}P\right) \tag{2}$$

where $\lambda = \sqrt{\frac{\hbar}{m\omega_0}}$ is the length scale for the harmonic oscillator:

(a) [2 pts] Use the raising and lowering operators to derive the ground state wavefuction, $\psi_0(x)$, and the first excited state wavefuction, $\psi_1(x)$, for the Hamiltonian H_a. Be sure to show your work.

(b) [1 pt] Consider a sudden change in the potential, modeled by a change in the original frequency of the oscillator by some multiplicative value f, to the new Hamiltonian:

$$H_b = \frac{P^2}{2m} + \frac{m}{2}\omega_1^2\, X^2, \quad \omega_1 = f\omega_0, \quad 0 < f < 1 \tag{3}$$

"Sudden" in this case means that one can ignore the time it takes to change the potential.

If $\phi_0(x)$ and $\phi_1(x)$ are the ground and first excited state wavefunctions of H_b, what are the functional forms for these wavefunctions? Explain your answer.

(c) [3 pts] The oscillator is in the ground state $\psi_0(x)$ when the potential suddenly changes. What is the expectation value of the energy of the oscillator after the potential changes? Show your work.

(d) [2 pts] If the oscillator is in the state $\psi_0(x)$ when the potential suddenly changes, what is the probability of the oscillator being in the ground state of H_b after the potential changes? Show your work.

(e) [1 pt] If the oscillator is in the state $\psi_0(x)$ when the potential suddenly changes, what is the probability of the oscillator being in the first excited state of H_b after the potential changes? Explain your answer.

(f) [1 pt] Finally, assume the oscillator is in the first excited state of H_a, $\psi_1(x)$, when the potential suddenly changes. What is the expectation value of the energy of the oscillator after the potential changes? Is the change in the expectation value of the energy, from H_a to H_b, for ψ_1 larger than, smaller than, or the same as ψ_0? Explain.

Remember that the Gaussian integrals have the form:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-ax^2}dx &= \sqrt{\frac{\pi}{a}}\\ \int_{-\infty}^{\infty} x^{2n}e^{-ax^n}dx &= \frac{1\cdot 3\cdot 5...(2n-1)}{2^n a^2}\sqrt{\frac{\pi}{a}}\end{aligned} \tag{4}$$

Problem 5 - Expanding Harmonic Oscillator

Part (a)

We know

$$a\psi_0 = 0,$$

where ψ_0 is the ground state wavefunction. Then

$$a\psi_0 = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} + i\frac{\lambda}{\hbar}p\right)\psi_0 = 0$$

$$\frac{x}{\lambda}\psi_0 + i\frac{\lambda}{\hbar}\left(-i\hbar\frac{d}{dx}\right)\psi_0 = 0$$

$$\frac{x}{\lambda}\psi_0 + \lambda\frac{d}{dx}\psi_0 = 0$$

$$\lambda\frac{d\psi_0}{dx} = -\frac{x}{\lambda}\psi_0$$

$$\frac{d\psi_0}{\psi_0} = -\frac{x}{\lambda^2}dx$$

Integrating,

$$\int\frac{1}{\psi_0}d\psi_0 = -\frac{1}{\lambda^2}\int x\,dx$$

$$\ln\psi_0 = -\frac{1}{2\lambda^2}x^2 + c_1$$

$$\psi_0 = e^{-\frac{1}{2\lambda^2}x^2 + c_1}$$

$$\psi_0 = Ce^{-\frac{1}{2\lambda^2}x^2}$$

We must normalize to determine the constant, C.

$$\int_{-\infty}^{\infty} |\psi_0(x)|^2 \, dx = 1$$

$$C^2 \int_{-\infty}^{\infty} e^{-\frac{1}{\lambda^2}x^2} \, dx = 1$$

$$C^2 \left(\sqrt{\frac{\pi}{(1/\lambda^2)}} \right) = 1$$

$$C^2 = \frac{1}{\sqrt{\pi \lambda^2}}$$

$$C = \left(\frac{1}{\pi \lambda^2} \right)^{1/4}$$

But $\lambda = \sqrt{\frac{\hbar}{m\omega_0}}$, so

$$C = \left(\frac{1}{\pi \left(\frac{\hbar}{m\omega_0} \right)} \right)^{1/4}$$

$$C = \left(\frac{m\omega_0}{\pi \hbar} \right)^{1/4}$$

and our ground state wavefunction is

$$\boxed{\psi_0(x) = \left(\frac{m\omega_0}{\pi \hbar} \right)^{1/4} e^{-\frac{m\omega_0}{2\hbar} x^2}}$$

Now we want to apply the raising operator to determine our first excited state. So

$$
\begin{aligned}
\psi_1 &= a^{+}\psi_0 \\
&= \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - i\frac{\lambda}{\hbar}p\right)\psi_0 \\
&= \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - \lambda\frac{d}{dx}\right)\psi_0 \\
&= \frac{x\psi_0}{\sqrt{2}\lambda} - \frac{\lambda}{\sqrt{2}}\frac{d\psi_0}{dx} \\
&= \left(\frac{m\omega_0}{2\hbar}\right)^{1/2} x \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}x^2} - \sqrt{\frac{\hbar}{2m\omega_0}}\,\frac{d\psi_0}{dx} \\
&= \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\left(\frac{m\omega_0}{2\hbar}\right)^{1/2} x e^{-\frac{m\omega_0}{2\hbar}x^2} - \left(\frac{\hbar}{2m\omega_0}\right)\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\frac{d}{dx}e^{-\frac{m\omega_0}{2\hbar}x^2}
\end{aligned}
$$

$$
\begin{aligned}
\psi_1 &= \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\left(\frac{m\omega_0}{2\hbar}\right)^{1/2} x e^{-\frac{m\omega_0}{2\hbar}x^2} - \left(\frac{\hbar}{2m\omega_0}\right)^{1/2}\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\left(-\frac{m\omega_0}{\cancel{2}\hbar}\right)\cancel{2}x\, e^{-\frac{m\omega_0}{2\hbar}x^2} \\
&= \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\left(\frac{m\omega_0}{2\hbar}\right)^{1/2} x e^{-\frac{m\omega_0}{2\hbar}x^2} + \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\left(\frac{m\omega_0}{2\hbar}\right)^{1/2} x e^{-\frac{m\omega_0}{2\hbar}x^2} \\
&= 2\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\left(\frac{m\omega_0}{2\hbar}\right)^{1/2} x e^{-\frac{m\omega_0}{2\hbar}x^2}
\end{aligned}
$$

$$
\boxed{\psi_1(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega_0}{\hbar}}\, x e^{-\frac{m\omega_0}{2\hbar}x^2}}
$$

Part (b)

Since ω_1 is just a constant and the Hamiltonian has the same form as before, we have

$$\phi_0(x) = \left(\frac{m\omega_1}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_1}{2\hbar}x^2}$$

$$\phi_1(x) = \left(\frac{m\omega_1}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega_1}{\hbar}}\, x\, e^{-\frac{m\omega_1}{2\hbar}x^2},$$

or in terms of ω_0, since

$$\omega_1 = f\omega_0 \qquad (0<f<1),$$

$$\phi_0(x) = \left(\frac{mf\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{mf\omega_0}{2\hbar}x^2}$$

$$\phi_1(x) = \left(\frac{mf\omega_0}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2mf\omega_0}{\hbar}}\, x\, e^{-\frac{mf\omega_0}{2\hbar}x^2}.$$

Part (c)

We want to determine

$$\langle H \rangle = \langle \Psi_0 | H | \Psi_0 \rangle.$$

So

$$\langle H \rangle = \int_{-\infty}^{\infty} \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}x^2} \left(\frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 X^2\right) \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}x^2} dx$$

$$= \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} \left[\frac{-\hbar^2}{2m}\int_{-\infty}^{\infty} e^{-\frac{m\omega_0}{2\hbar}x^2} \frac{d^2}{dx^2} e^{-\frac{m\omega_0}{2\hbar}x^2} dx + \frac{1}{2}m\omega_1^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega_0}{\hbar}x^2} dx\right]$$

We have

$$\frac{d}{dx} e^{-\frac{m\omega_0}{2\hbar}x^2} = -\frac{m\omega_0}{\hbar} x e^{-\frac{m\omega_0}{2\hbar}x^2}$$

$$\frac{d^2}{dx^2} e^{-\frac{m\omega_0}{2\hbar}x^2} = \left(\frac{m\omega_0}{\hbar}\right)^2 x^2 e^{-\frac{m\omega_0}{2\hbar}x^2} - \frac{m\omega_0}{\hbar} e^{-\frac{m\omega_0}{2\hbar}x^2}$$

Then our first integral becomes

$$\frac{-\hbar^2}{2m}\int_{-\infty}^{\infty} e^{-\frac{m\omega_0}{2\hbar}x^2} \left[\left(\frac{m\omega_0}{\hbar}\right)^2 x^2 e^{-\frac{m\omega_0}{2\hbar}x^2} - \frac{m\omega_0}{\hbar} e^{-\frac{m\omega_0}{2\hbar}x^2}\right] dx$$

$$= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \left[\left(\frac{m\omega_0}{\hbar}\right)^2 x^2 e^{-\frac{m\omega_0}{\hbar}x^2} - \left(\frac{m\omega_0}{\hbar}\right) e^{-\frac{m\omega_0}{\hbar}x^2}\right] dx$$

$$= -\frac{\hbar^2}{2m}\left(\frac{m\omega_0}{\hbar}\right)^2 \frac{1}{2}\sqrt{\frac{\pi}{(m\omega_0/\hbar)^3}} + \frac{\hbar^2}{2m}\left(\frac{m\omega_0}{\hbar}\right)\sqrt{\frac{\pi}{(m\omega_0/\hbar)}}$$

$$= -\frac{m\omega_0^2}{4}\left(\frac{\pi\hbar^3}{m^3\omega_0^3}\right)^{1/2} + \frac{\hbar\omega_0}{2}\left(\frac{\pi\hbar}{m\omega_0}\right)^{1/2}$$

$$= -\frac{m\omega_0^2}{4}\cdot\frac{\hbar}{m\omega_0}\left(\frac{\pi\hbar}{m\omega_0}\right)^{1/2} + \frac{\hbar\omega_0}{2}\left(\frac{\pi\hbar}{m\omega_0}\right)^{1/2}$$

$$= \left(\frac{\pi\hbar}{m\omega_0}\right)^{1/2}\left(\frac{\hbar\omega_0}{2} - \frac{\hbar\omega_0}{4}\right)$$

$$= \left(\frac{\pi\hbar}{m\omega_0}\right)^{1/2}\left(\frac{\hbar\omega_0}{4}\right)$$

Our second integral is

$$\frac{1}{2}m\omega_1^2\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega_0}{\hbar}x^2}\,dx = \frac{1}{2}m\omega_1^2\cdot\frac{1}{2}\sqrt{\frac{\pi}{(m\omega_0/\hbar)^3}}$$

$$= \frac{1}{4}m\omega_1^2\sqrt{\frac{\pi\hbar^3}{m^3\omega_0^3}}$$

$$= \frac{1}{4}\frac{\omega_1^2\hbar}{\omega_0}\sqrt{\frac{\pi\hbar}{m\omega_0}}$$

So

$$\langle H\rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2}\left[\left(\frac{\pi\hbar}{m\omega_0}\right)^{1/2}\frac{\hbar\omega_0}{4} + \frac{\hbar\omega_1^2}{4\omega_0}\left(\frac{\pi\hbar}{m\omega_0}\right)^{1/2}\right]$$

$$= \frac{\hbar\omega_0}{4} + \frac{\hbar\omega_1^2}{4\omega_0}$$

$$= \frac{\hbar\omega_1}{4f} + \frac{\hbar\omega_1 f}{4} = \hbar\omega_1\left(\frac{1}{4f} + \frac{f}{4}\right)$$

Now we want to relate this to the energy in the new potential, which should be given, in general, by

$$E_b = \hbar\omega_1(n+1/2)$$

Writing in terms of ω_0, we have

$$E_b = \hbar(f\omega_0)\left(n+\frac{1}{2}\right)$$
$$= \hbar\omega_0\left(nf+\frac{f}{2}\right)$$

This implies

$$nf+\frac{f}{2} = \frac{1}{2}$$
$$n+\frac{1}{2} = \frac{1}{2f}$$
$$n = \frac{1}{2f}-\frac{1}{2}$$
$$n = \frac{1}{2}\left(\frac{1}{f}-1\right).$$

The only way for n to be an integer, as we require, is if $1/f$ is odd. The smaller f is, the larger our expectation value of the energy will be.

So, our expectation value of the energy is

$$\langle E\rangle = \frac{\hbar\omega_0}{2},$$

which corresponds to the

$$n = \frac{1}{2}\left(\frac{1}{f}-1\right)$$

energy state in the new potential.

Part (d)

The ground state can be written as

$$\psi_0(x) = \sum_n c_n \phi_n .$$

So the probability that the oscillator is in the ground state of H_0 is

$$|c_0|^2 = \left| \int_{-\infty}^{\infty} \psi_0^*(x)\, \phi_0(x)\, dx \right|^2$$

$$= \left| \int_{-\infty}^{\infty} \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}x^2} \left(\frac{mf\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{mf\omega_0}{2\hbar}x^2}\, dx \right|^2$$

$$= \left| \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} f^{1/4} \int_{-\infty}^{\infty} e^{-\frac{m\omega_0}{2\hbar}(1+f)x^2}\, dx \right|^2$$

$$= \left| \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} f^{1/4} \left(\frac{\pi}{\frac{m\omega_0}{2\hbar}(1+f)}\right)^{1/2} \right|^2$$

$$= \left| \left(\frac{\cancel{m\omega_0}}{\cancel{\pi\hbar}}\right)^{1/2} f^{1/4} \left(\frac{2\cancel{\pi\hbar}}{\cancel{m\omega_0}(1+f)}\right)^{1/2} \right|^2$$

$$= \left| f^{1/4} \left(\frac{2}{1+f}\right)^{1/2} \right|^2$$

$$\boxed{|c_0|^2 = \frac{2\sqrt{f}}{1+f}}$$

Part (e)

The probability is

$$|c_1|^2 = \left| \int_{-\infty}^{\infty} \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}x^2} \left(\frac{m\omega_0 f}{\pi\hbar}\right)^{1/4} \left(\frac{2m\omega_0 f}{\hbar}\right)^{1/2} x\, e^{-\frac{m\omega_0 f}{2\hbar}x^2} dx \right|^2$$

$$= \left| \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} \left(\frac{2m\omega_0 f}{\hbar}\right)^{1/2} f^{1/4} \int_{-\infty}^{\infty} x\, e^{-\frac{m\omega_0}{2\hbar}(1+f)x^2} dx \right|^2$$

$$= \left| \left(\frac{2m^2\omega_0^2 f}{\pi\hbar^2}\right)^{1/2} f^{1/4} \int_{-\infty}^{\infty} x\, e^{-\frac{m\omega_0}{2\hbar}(1+f)x^2} dx \right|^2$$

$$= \left| \left(\frac{2m^2\omega_0^2 f}{\pi\hbar^2}\right)^{1/2} f^{1/4} \cdot 0 \right|^2$$

$$\boxed{|c_1|^2 = 0}$$

This tells us that if the particle is in the ground state before the potential changes, there is no chance of it being in the first excited state after the potential changes.

This makes sense because the initial state had even parity, so only the even-parity states of the new oscillator can be excited.