PROBLEM 3: Angular Momentum Operators

The eigenvector of L^2 and L_z is usually expressed as $|\ell, \ell_z\rangle = |\ell, m\rangle$. This is the ℓ_z basis with

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

for $\ell = 1$.

(a) Apply the raising and lowering operators and determine L_y with

$$(L_y)_{mn} = \langle \ell = 1, \ell_z = m | L_y | \ell = 1, \ell_z = n \rangle$$

in the form of a 3×3 matrix. *(3 points)*

(b) Find the eigenvalues and normalized eigenvectors of L_y. *(3 points)*

(c) If a particle is in the state with $\ell_z = -1$, and L_y is measured, what are the possible outcomes and their probabilities? *(3 points)*

(d) Take the state in which $\ell_z = 1$. In this state what is the uncertainty $\Delta L_y = \langle (L_y - \langle L_y \rangle)^2 \rangle^{1/2}$? *(1 points)*

(a)

We know

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

for $l=1$. We want to find L_y, where a given matrix element of L_y is

$$(L_y)_{mn} = \langle l=1, l_z=m | L_y | l=1, l_z=n \rangle .$$

The raising and lowering operators are, respectively,

$$L_+ = L_x + iL_y$$
$$L_- = L_x - iL_y .$$

So

$$iL_y = L_+ - L_x$$

and

$$iL_y = L_x - L_- ,$$

So

$$2iL_y = (L_+ - L_x) + (L_x - L_-)$$
$$2iL_y = L_+ - L_-$$
$$L_y = \frac{L_+ - L_-}{2i} .$$

Substituting this in for L_y in the matrix element definition,

$$(L_y)_{mn} = \langle 1, m | \frac{L_+ - L_-}{2i} | 1, n \rangle$$

(a), cont'd...

$$(L_y)_{mn} = -\frac{i}{2}\left[\langle 1,m|L_+|1,n\rangle - \langle 1,m|L_-|1,n\rangle\right]$$

In general, we know

$$L_+|\ell\, m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m+1)}\;|\ell, m+1\rangle$$
$$L_-|\ell\, m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m-1)}\;|\ell, m-1\rangle.$$

So

$$\begin{aligned}\langle 1,m|L_+|1,n\rangle &= \langle 1,m|\hbar\sqrt{(1)((1)+1)-n(n+1)}\;|1,n+1\rangle\\ &= \hbar\sqrt{2-n(n+1)}\;\langle 1,m|1,n+1\rangle\\ &= \hbar\sqrt{2-n(n+1)}\;\delta_{m,n+1}\end{aligned}$$

and

$$\begin{aligned}\langle 1,m|L_-|1,n\rangle &= \langle 1,m|\hbar\sqrt{(1)((1)+1)-n(n-1)}\;|1,n-1\rangle\\ &= \hbar\sqrt{2-n(n-1)}\;\langle 1,m|1,n-1\rangle\\ &= \hbar\sqrt{2-n(n-1)}\;\delta_{m,n-1}\end{aligned}$$

We know

$$L_z = \hbar \begin{array}{cc} \begin{array}{ccc} m=1 & m=0 & m=-1 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \begin{array}{l} n=1 \\ n=0 \\ n=-1 \end{array} \end{array}$$

(a), cont'd...

Starting with $\langle 1,m|L_+|1,n\rangle$ ...

<u>$m=1$</u>

$n=1$: $\langle 1,1|L_+|1,1\rangle = \hbar\sqrt{2-(1)((1)+1)}\ \delta_{1,2}$
$= 0$

$n=0$: $\langle 1,1|L_+|1,0\rangle = \hbar\sqrt{2-(0)((0)+1)}\ \delta_{1,1}$
$= \hbar\sqrt{2}$

$n=-1$: $\langle 1,1|L_+|1,-1\rangle = \hbar\sqrt{2-(-1)((-1)+1)}\ \delta_{1,0}$
$= 0$

<u>$m=0$</u>

$n=1$: $\langle 1,0|L_+|1,1\rangle = \hbar\sqrt{2-(1)((1)+1)}\ \delta_{0,2}$
$= 0$

$n=0$: $\langle 1,0|L_+|1,0\rangle = \hbar\sqrt{2}\ \delta_{0,1}$
$= 0$

$n=-1$: $\langle 1,0|L_+|1,-1\rangle = \hbar\sqrt{2}\ \delta_{0,0}$
$= \hbar\sqrt{2}$

<u>$m=-1$</u>

$n=1$: $\langle 1,-1|L_+|1,1\rangle = 0$

$n=0$: $\langle 1,-1|L_+|1,0\rangle = \hbar\sqrt{2}\ \delta_{-1,1}$
$= 0$

$n=-1$: $\langle 1,-1|L_+|1,-1\rangle = \hbar\sqrt{2}\ \delta_{-1,0}$
$= 0$

(a), cont'd...

Thus, our matrix for L_+ is

$$L_+ = \hbar \begin{array}{cccc} m=1 & m=0 & m=-1 & \\ \end{array}$$
$$\begin{array}{c} \\ \end{array}\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{array}{l} n=1 \\ n=0 \\ n=-1 \end{array}$$

For $\langle 1,m|L_-|1,n\rangle \ldots$

$\underline{m=1}$

$n=1$: $\langle 1,1|L_-|1,1\rangle = \hbar\sqrt{2-(1)((1)-1)}\;\delta_{1,0} = 0$

$n=0$: $\langle 1,1|L_-|1,0\rangle = \hbar\sqrt{2-(0)((0)-1)}\;\delta_{1,-1} = 0$

$n=-1$: $\langle 1,1|L_-|1,-1\rangle = \hbar\sqrt{2-(-1)((-1)-1)}\;\delta_{1,-2} = 0$

$\underline{m=0}$

$n=1$: $\langle 1,0|L_-|1,1\rangle = \hbar\sqrt{2-(1)((1)-1)}\;\delta_{0,0} = \hbar\sqrt{2}$

$n=0$: $\langle 1,0|L_-|1,0\rangle = \hbar\sqrt{2-(0)((0)-1)}\;\delta_{0,-1} = 0$

$n=-1$: $\langle 1,0|L_-|1,-1\rangle = \hbar\sqrt{2-(-1)((-1)-1)}\;\delta_{0,-2} = 0$

$\underline{m=-1}$

$n=1$: $\langle 1,-1|L_-|1,1\rangle = \hbar\sqrt{2}\;\delta_{-1,0} = 0$

$n=0$: $\langle 1,-1|L_-|1,0\rangle = \hbar\sqrt{2}\;\delta_{-1,-1} = \hbar\sqrt{2}$

$n=-1$: $\langle 1,-1|L_-|1,-1\rangle = 0\;\delta_{-1,-2} = 0$

(a), cont'd...

Thus, our matrix for L_- is

$$L_- = \hbar \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{ccc} m=1 & m=0 & m=-1 \end{array}$$

$$L_- = \hbar \begin{array}{c c} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} n=1 \\ n=0 \\ n=-1 \end{array} \end{array}$$

Thus, our overall matrix for L_y is

$$L_y = -\frac{i}{2}\left[L_+ - L_-\right]$$

$$= -\frac{i\hbar}{2}\left[\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= -\frac{i\hbar}{2}\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\boxed{L_y = \frac{i\hbar\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}$$

(b)

The eigenvalues are...

$$\begin{vmatrix} -\lambda & \frac{i\hbar}{\sqrt{2}} & 0 \\ -\frac{i\hbar}{\sqrt{2}} & -\lambda & \frac{i\hbar}{\sqrt{2}} \\ 0 & -\frac{i\hbar}{\sqrt{2}} & -\lambda \end{vmatrix} \begin{matrix} -\lambda & \frac{i\hbar}{\sqrt{2}} \\ -\frac{i\hbar}{\sqrt{2}} & -\lambda \\ 0 & -\frac{i\hbar}{\sqrt{2}} \end{matrix} = 0$$

$$(-\lambda)(-\lambda)(-\lambda) - \left(-\frac{i\hbar}{\sqrt{2}}\right)\left(\frac{i\hbar}{\sqrt{2}}\right)(-\lambda) - (-\lambda)\left(-\frac{i\hbar}{\sqrt{2}}\right)\left(\frac{i\hbar}{\sqrt{2}}\right) = 0$$

$$-\lambda^3 + \frac{\hbar^2}{2}\lambda + \frac{\hbar^2}{2}\lambda = 0$$

$$\lambda^3 - \hbar^2\lambda = 0$$

$$\lambda(\lambda^2 - \hbar^2) = 0$$

So

$$\lambda = 0,\ \lambda = \pm\hbar$$

or

$$\boxed{l_y = 0, \pm\hbar}$$

$\underline{|l_y = 0\rangle}$

$$\begin{pmatrix} 0 & \frac{i\hbar}{\sqrt{2}} & 0 \\ -\frac{i\hbar}{\sqrt{2}} & 0 & \frac{i\hbar}{\sqrt{2}} \\ 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} 0c_1 + \tfrac{i\hbar}{\sqrt{2}}c_2 + 0c_3 &= 0 \\ -\tfrac{i\hbar}{\sqrt{2}}c_1 + 0c_2 + \tfrac{i\hbar}{\sqrt{2}}c_3 &= 0 \\ 0c_1 - \tfrac{i\hbar}{\sqrt{2}}c_2 + 0c_3 &= 0 \end{aligned}$$

We must have $c_2 = 0$. We know $c_3 = c_1$, so let $c_1 = c_3 = 1$. Then

$$\boxed{|l_y = 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

(b), cont'd...

$|l_y = \hbar\rangle$

$$\begin{pmatrix} -\hbar & \frac{i\hbar}{\sqrt{2}} & 0 \\ -\frac{i\hbar}{\sqrt{2}} & -\hbar & \frac{i\hbar}{\sqrt{2}} \\ 0 & -\frac{i\hbar}{\sqrt{2}} & -\hbar \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} -\hbar c_1 + \frac{i\hbar}{\sqrt{2}} c_2 + 0 c_3 = 0 \\ -\frac{i\hbar}{\sqrt{2}} c_1 - \hbar c_2 + \frac{i\hbar}{\sqrt{2}} c_3 = 0 \\ 0 c_1 - \frac{i\hbar}{\sqrt{2}} c_2 - \hbar c_3 = 0 \end{array}$$

So $c_2 = -i\sqrt{2}\, c_1$ and $c_3 = \frac{-i}{\sqrt{2}} c_2$, and $c_1 - i\sqrt{2}\, c_2 - c_3 = 0$:

$$\begin{array}{r} i\sqrt{2}\, c_1 + c_2 = 0 \\ \frac{i}{\sqrt{2}} c_2 + c_3 = 0 \\ c_1 - i\sqrt{2}\, c_2 - c_3 = 0 \end{array} \Rightarrow \begin{array}{l} c_3 = \frac{-i}{\sqrt{2}} c_2 = \\ -\frac{i}{\sqrt{2}}\left(-i\sqrt{2}\, c_1\right) \\ c_3 = -c_1 \end{array}$$

$$|l_y = \hbar\rangle = \frac{1}{\sqrt{|c_1|^2 + |-i\sqrt{2}\, c_1|^2 + |-c_1|^2}} \begin{pmatrix} c_1 \\ -i\sqrt{2}\, c_1 \\ -c_1 \end{pmatrix}$$

Let $c_1 = 1$, then

$$\boxed{|l_y = \hbar\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix}}$$

(b), cont'd...

$|l_y = -\hbar\rangle$

$$\begin{pmatrix} \hbar & \frac{i\hbar}{\sqrt{2}} & 0 \\ -\frac{i\hbar}{\sqrt{2}} & \hbar & \frac{i\hbar}{\sqrt{2}} \\ 0 & -\frac{i\hbar}{\sqrt{2}} & \hbar \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} \hbar c_1 + \frac{i\hbar}{\sqrt{2}} c_2 &= 0 \\ -\frac{i\hbar}{\sqrt{2}} c_1 + \hbar c_2 + \frac{i\hbar}{\sqrt{2}} c_3 &= 0 \\ -\frac{i\hbar}{\sqrt{2}} c_2 + \hbar c_3 &= 0 \end{aligned}$$

So

$$c_2 = i\sqrt{2}\, c_1$$

and

$$c_3 = \frac{i}{\sqrt{2}} c_2 = \frac{i}{\sqrt{2}}\left(i\sqrt{2}\, c_1\right) = -c_1 ,$$

so if $c_1 = 1$...

$$\boxed{|l_y = -\hbar\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}}$$

(c)

We want to determine

$$P_{|l_z=-\hbar\rangle}(l_y = 0, \hbar, \text{and} -\hbar).$$

So

$$P_{|l_z=\hbar\rangle}(l_y=0) = |\langle l_y=0 | l_z=-\hbar\rangle|.$$

But we need $|l_z=-\hbar\rangle$. So

$$\begin{pmatrix} 2\hbar & 0 & 0 \\ 0 & \hbar & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 1 \end{matrix}$$

and

$$|l_z=-\hbar\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$P_{|l_z=-\hbar\rangle}(l_y=0) = \left|\frac{1}{\sqrt{2}}(1\ 0\ 1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{2}(1) = \frac{1}{2}$$

$$P_{|l_z=\hbar\rangle}(l_y=\hbar) = \left|\frac{1}{2}(1, i\sqrt{2}, -1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{4}(1) = \frac{1}{4}$$

$$P_{|l_z=\hbar\rangle}(l_y=-\hbar) = \left|\frac{1}{2}(1, -i\sqrt{2}, -1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{4}(1) = \frac{1}{4}$$

So we have a 50% chance of measuring $|l_y=0\rangle$, and a 25% chance of measuring $|l_y=\hbar\rangle$ and $|l_y=-\hbar\rangle$, separately.

(d)

We know

$$|l_z = \hbar\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We want to determine

$$\Delta L_y = \left(\langle (L_y - \langle L_y \rangle)^2 \rangle\right)^{1/2}$$
$$= \sqrt{\langle L_y^2 \rangle - \langle L_y \rangle^2}.$$

So

$$\langle L_y \rangle = \langle l_z = \hbar | L_y | l_z = \hbar \rangle$$
$$= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{i\hbar\sqrt{2}}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= \frac{i\hbar\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$
$$= \frac{i\hbar\sqrt{2}}{2} (0)$$
$$= 0.$$

We have

$$L_y^2 = -\frac{\hbar^2}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$
$$= -\frac{\hbar^2}{2} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

(d), cont'd...

Then

$$\langle L_y^2 \rangle = \langle l_z = \hbar | L_y^2 | l_z = \hbar \rangle$$

$$= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \left(\frac{-\hbar^2}{2}\right) \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{-\hbar^2}{2} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{-\hbar^2}{2} (-1)$$

$$= \frac{\hbar^2}{2}$$

Then

$$\Delta L_y = \sqrt{\frac{\hbar^2}{2} - (0)^2}$$

$$\boxed{\Delta L_y = \frac{\hbar}{\sqrt{2}}}$$