F2009



# Problem 1: Step Potential (10 points)

Consider the potential $V(x)$

$$V(x) = \begin{cases} 0, & x \leq 0 \\ -\text{V}, & x > 0 \end{cases}$$

A particle of mass $m$ and kinetic energy $E$ approaches the step from $x < 0$ .

a) Write the solution to Schrodinger's equation for $x < 0$. (1 pt)

b) Write the solution to Schrodinger's equation for $x > 0$. (1 pt)

c) Sketch the wave function for $x < 0$ as well as $x > 0$. Making sure to describe how the amplitude and frequency of the wave function changes. (1 pt)

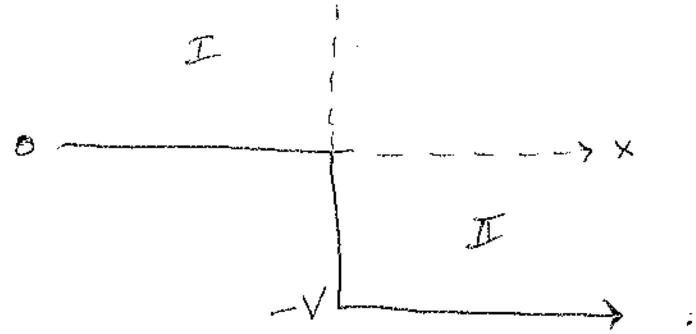
d) What is the probability that particle will reflect back if $E = V/8$? (2 pts)

e) What is the probability that the particle will be transmitted if $E = V/8$. (2 pts) (Determine the transmission probability directly by using the flow of probability current and do not simply use $T = 1 - R$)

f) Show that $T + R = 1$. What does this mean physically? (1 pt)

g) If instead the particle approached the step from $x > 0$, how do your answers to parts a), b), d) and e) change? (2 pts)

(a)




For $x<0$, the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_I}{dx^2} = E\psi_I$$

$$\frac{d^2\psi_I}{dx^2} = -k^2\psi_I,$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}.$$

The general solution is

$$\boxed{\psi_I(x) = Ae^{ikx} + Be^{-ikx}}$$

(b)

For $x>0$, the Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + (-V)\right]\psi_{II} = E\psi_{II}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} = (E+V)\psi_{II}$$

$$\frac{d^2\psi_{II}}{dx^2} = -l^2\psi_{II},$$

where

$$l = \frac{\sqrt{2m(E+V)}}{\hbar}.$$

Then the general solution is

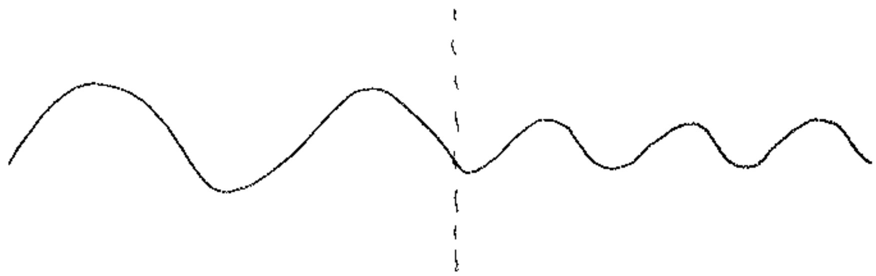
$$\psi_{II}(x) = Ce^{ilx} + De^{-ilx}.$$

(b), cont'd...

Assuming particles only come from the left, we let $D=0$, so

$$\boxed{\psi_{II}(x) = Ce^{ilx}}.$$

(c)



$x=0$

The amplitude decreases, but the frequency increases.

(d)

By our boundary conditions, we must have

$$\psi_I(0) = \psi_{II}(0)$$

and

$$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0}.$$

So

$$A+B = C$$

and

$$ik(A-B) = ilC$$

$$k(A-B) = lC.$$

(d), cont'd...

Then

$$A + B = \frac{k}{l}(A - B)$$

$$A + B = \frac{k}{l}A - \frac{k}{l}B$$

$$A(1 - k/l) = \left(-1 - \frac{k}{l}\right)B$$

$$\frac{B}{A} = \frac{1 - k/l}{-(1 + k/l)}$$

$$\frac{B}{A} = \frac{k/l - 1}{k/l + 1}.$$

The reflection coefficient is given by

$$R = \left|\frac{B}{A}\right|^2.$$

We let $E = V/8$. Then

$$k = \frac{\sqrt{2mE}}{\hbar} = \sqrt{\frac{mV}{4\hbar^2}}$$

$$l = \frac{\sqrt{2m(E+V)}}{\hbar} = \sqrt{\frac{9mV}{4\hbar^2}},$$

so

$$\frac{k}{l} = 1/3.$$

Then

$$\frac{B}{A} = \frac{1/3 - 1}{1/3 + 1} = -\frac{1}{2}$$

and the probability the particle will reflect back is $\boxed{R = 1/4}$.

(e)

We must have

$$\frac{\hbar k}{m}\left(|A|^2 - |B|^2\right) = \frac{\hbar \ell}{m}|C|^2$$

$$|A|^2 - |B|^2 = \frac{\ell}{k}|C|^2,$$

so

$$\frac{|B|^2}{|A|^2} + \frac{\ell}{k}\frac{|C|^2}{|A|^2} = 1.$$

Then our transmission probability is

$$T = \frac{\ell}{k}\frac{|C|^2}{|A|^2}.$$

We have

$$B = C - A$$

and

$$B = A - \frac{\ell}{k}C,$$

so

$$C - A = A - \frac{\ell}{k}C$$

$$C\left(1 + \ell/k\right) = 2A$$

$$\frac{C}{A} = \frac{2}{1+\ell/k}.$$

(e), cont'd...

We know

$$\frac{k}{l} = 1/3$$

from part (d), so

$$\frac{l}{k} = 3$$

and

$$\frac{C}{A} = \frac{2}{1+3} = \frac{1}{2}.$$

Thus, our transmission probability is

$$T = \frac{l}{k}\left|\frac{C}{A}\right|^2$$

$$= 3\left|\frac{1}{2}\right|$$

$$\boxed{T = \frac{3}{4}}.$$

(f) We have

$$T + R = 3/4 + 1/4$$

$$= 1,$$

as expected. This implies that the particle can only be transmitted or reflected.

(g)

For $x<0$, we would have

$$\psi_{x<0}(x) = Ce^{-ikx}$$

since we only have particles coming from the right.

For $x>0$, we would have

$$\psi_{x>0}(x) = Ae^{ilx} + Be^{-ilx}.$$

The reflection coefficient in this case would be

$$R = \left|\frac{A}{B}\right|^2.$$

We know

$$A+B = C$$

and

$$-ikC = ilA - ilB$$

$$C = -\frac{l}{k}(A-B),$$

so

$$A+B = -\frac{l}{k}A + \frac{l}{k}B$$

$$A(1+l/k) = B(-1+l/k)$$

$$\frac{A}{B} = \frac{l/k - 1}{1 + l/k}$$

(g), cont'd...



We know

$$\frac{l}{k} = 3,$$

so

$$\frac{A}{B} = \frac{3-1}{1+3}$$

$$\frac{A}{B} = \frac{1}{2}$$

and the reflection coefficient/probability is

$$R = 1/4.$$

The transmission probability will also be the same as before.