F2012

PROBLEM 1: Eigenvalue Equation and Time Evolution

The Hamiltonian for a certain three-level system is represented by the matrix

$$H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix},$$

where a, b, and c are real numbers and $a - c \neq \pm b$.

(a) Find the eigenvalues E_n and normalized eigenvectors $|E_n\rangle, n = 1, 2, 3$ of H. [4 points]

(b) If the system starts out in the state

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

what is $|\psi(t)\rangle$? [3 points]

(c) If the system starts out in the state

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

what is $|\psi(t)\rangle$? [3 points]

(a)

We want to find the eigenvalues of H.

$$\det(H - \lambda I) = \begin{vmatrix} a-\lambda & 0 & b \\ 0 & c-\lambda & 0 \\ b & 0 & a-\lambda \end{vmatrix} = 0$$

$$\left|\begin{array}{ccc|cc} a-\lambda & 0 & b & a-\lambda & 0 \\ 0 & c-\lambda & 0 & 0 & c-\lambda \\ b & 0 & a-\lambda & b & 0 \end{array}\right.$$

$$= (a-\lambda)(c-\lambda)(a-\lambda) - (b)(c-\lambda)(b) = 0$$

$$(a^2 - 2\lambda a + \lambda^2)(c-\lambda) - b^2(c-\lambda) = 0$$

or

$$(a-\lambda)^2(c-\lambda) = b^2(c-\lambda)$$

So we can have

$$c - \lambda = 0 \quad \text{or} \quad (a-\lambda)^2 = b^2.$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\lambda = c \qquad\qquad a - \lambda = \pm b$$

$$\lambda = a+b, a-b$$

So the energy eigenvalues are

$$\boxed{E_n = c,\ a+b,\ a-b}.$$

(a), cont'd...

$|E_n = c\rangle$

$$\begin{pmatrix} a-c & 0 & b \\ 0 & 0 & 0 \\ b & 0 & a-c \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(a-c)k_1 + bk_3 = 0 \qquad \Rightarrow \qquad k_1 = -\frac{b}{a-c}k_3$$

$$bk_1 + (a-c)k_3 = 0 \qquad k_1 = -\frac{(a-c)}{b}k_3$$

The only way these equations make sense is if $k_1 = k_3 = 0$. Then since k_2 can be anything, we choose $k_1 = 1$. So our eigenvector is

$$\boxed{|E_n = c\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

$|E_n = a+b\rangle$

$$\begin{pmatrix} a-(a+b) & 0 & b \\ 0 & c-(a+b) & 0 \\ b & 0 & a-(a+b) \end{pmatrix} \rightarrow \begin{pmatrix} -b & 0 & b \\ 0 & c-a-b & 0 \\ b & 0 & -b \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-bk_1 + bk_3 = 0 \qquad bk_1 = bk_3 \Rightarrow k_1 = k_3$$

$$(c-a-b)k_2 = 0 \Rightarrow \qquad k_2 = 0$$

$$bk_1 - bk_3 = 0$$

Since $k_1 = k_2$, they can be anything. Choose $k_1 = k_2 = 1$. Then

$$\boxed{|E_n = a+b\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

(a), cont'd...

$|E_n = a-b\rangle$

$$\begin{pmatrix} a-(a-b) & 0 & b \\ 0 & c-(a-b) & 0 \\ b & 0 & a-(a-b) \end{pmatrix} \rightarrow \begin{pmatrix} b & 0 & b \\ 0 & c-a+b & 0 \\ b & 0 & b \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} bk_1 + bk_3 &= 0 \\ (c-a+b)k_2 &= 0 \\ bk_1 + bk_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} k_2 &= 0 \\ bk_1 &= -bk_3 \end{aligned} \quad \Rightarrow \quad k_1 = -k_3$$

Choose $k_1 = 1$, then $k_3 = -1$. We must have $k_2 = 0$, so

$$\boxed{|E_n = a-b\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}.$$

(b)

We must first determine the time evolution operator. In general,

$$U(t,t_0) = \sum_n |n\rangle\langle n| e^{-iE_n t/\hbar}.$$

So if we let

$$|E_n=c\rangle = |E_1=c\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$$
$$|E_n=a+b\rangle = |E_2=a+b\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$$
$$|E_n=a-b\rangle = |E_3=a-b\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix},$$

we have

$$U(t,t_0) = |E_1=c\rangle\langle E_1=c| e^{-iE_1 t/\hbar} + |E_2=a+b\rangle\langle E_2=a+b| e^{-iE_2 t/\hbar} + |E_3=a-b\rangle\langle E_3=a-b| e^{-iE_3 t/\hbar}.$$

Since these are eigenvectors, they are all orthonormal. So since

$$|\psi(t)\rangle = U(t,t_0)|\psi(0)\rangle,$$

we have

$$|\psi(t)\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} e^{-iE_1 t/\hbar} + 0 + 0$$

$$= \begin{pmatrix}0\\1\\0\end{pmatrix} e^{-iE_1 t/\hbar}$$

or $$\boxed{|\psi(t)\rangle = |E_1=c\rangle e^{-iE_1 t/\hbar}}$$

This makes sense because we started in a stationary state.

where $E_1 = c$.

(c)

We have

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

or

$$|\psi(0)\rangle = \frac{\sqrt{2}}{2}|E_2 = a+b\rangle - \frac{\sqrt{2}}{2}|E_3 = a-b\rangle .$$

So

$$\boxed{|\psi(t)\rangle = \frac{\sqrt{2}}{2}|E_2 = a+b\rangle e^{-iE_2 t/\hbar} - \frac{\sqrt{2}}{2}|E_3 = a-b\rangle e^{-iE_3 t/\hbar}}$$

where

$$E_2 = a+b$$
$$E_3 = a-b .$$