F2011

PROBLEM 4: Spin Angular Momentum

A Stern-Gerlach experiment is set up with the axis of the inhomogeneous magnetic field in the $x-y$ plane, at an angle θ relative to the $x-$axis. Let us call this direction $\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}$. Then the spin operator in the $\hat{r}$ direction is $S_r = \cos\theta S_x + \sin\theta S_y$. Let us describe the common eigenvectors for S^2 and S_i as $|s, m_i\rangle$, e.g. $|s, m_x\rangle$ or $|s, m_z\rangle$.

(a) For a spin$-1/2$ particle, calculate the matrix corresponding to S_r. [1 point]

(b) Evaluate the eigenvalues of S_r. [1 point]

(c) Find the normalized eigenvectors of S_r. [2 points]

(d) Suppose a measurement of the spin of the particle in the $\hat{r}$ direction is made and it is determined that the spin is in the positive $\hat{r}$ direction, i.e. $S_r|\psi\rangle = (+\hbar/2)|\psi\rangle$. Now a second measurement is made to determine m_x (the component of the spin in the x direction). What is the probability that $m_x = -1/2$? [3 points]

(e) Suppose that the particle has spin in the positive $\hat{r}$ direction as in part (d). The z component of the spin is measured and it is discovered that $m_z = +1/2$. Now a third measurement is made to determine m_x. What is the probability that $m_x = -1/2$? [3 points]

(a)

We have

$$S_r = \cos\theta\, S_x + \sin\theta\, S_y,$$

where

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

So our matrix is

$$S_r = \frac{\hbar\cos\theta}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar\sin\theta}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{\hbar\cos\theta}{2} \\ \frac{\hbar\cos\theta}{2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{i\hbar\sin\theta}{2} \\ \frac{i\hbar\sin\theta}{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{\hbar}{2}(\cos\theta - i\sin\theta) \\ \frac{\hbar}{2}(\cos\theta + i\sin\theta) & 0 \end{pmatrix}$$

$$\boxed{S_r = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}}$$

(b)

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2}e^{-i\theta} \\ \frac{\hbar}{2}e^{i\theta} & -\lambda \end{vmatrix} = 0$$

$$(-\lambda)(-\lambda) - \left(\frac{\hbar}{2}e^{-i\theta}\right)\left(\frac{\hbar}{2}e^{i\theta}\right) = 0$$

$$\lambda^2 - \frac{\hbar^2}{4} = 0$$

$$\lambda = \pm\frac{\hbar}{2}$$

So our eigenvalues are $\boxed{S_r^+ = \frac{\hbar}{2}}$ and $\boxed{S_r^- = -\frac{\hbar}{2}}$.

(c)

<u>$|S_r^+ = \hbar/2\rangle$</u>

$$\begin{pmatrix} -\hbar/2 & \frac{\hbar}{2}e^{-i\theta} \\ \frac{\hbar}{2}e^{i\theta} & -\hbar/2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -\frac{\hbar}{2}c_1 + \frac{\hbar}{2}e^{-i\theta}c_2 = 0 \\ \frac{\hbar}{2}e^{i\theta}c_1 - \frac{\hbar}{2}c_2 = 0 \end{matrix}$$

So $c_1 = e^{-i\theta}c_2$. Then let $c_1 = 1$, and we must have $c_2 = e^{i\theta}$. So

$$|S_r^+ = \hbar/2\rangle = \frac{1}{\sqrt{|c_1|^2 + |c_2|^2}}\begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} = \frac{1}{\sqrt{1 + e^{i\theta}e^{-i\theta}}}\begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix}$$

$$\boxed{|S_r^+ = \hbar/2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix}}$$

(c), cont'd...

$|S_r^- = -\hbar/2\rangle$

$$\begin{pmatrix} \hbar/2 & \frac{\hbar}{2}e^{-i\theta} \\ \frac{\hbar}{2}e^{i\theta} & \hbar/2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \frac{\hbar}{2}c_1 + \frac{\hbar}{2}e^{-i\theta}c_2 = 0 \\ \frac{\hbar}{2}e^{i\theta}c_1 + \frac{\hbar}{2}c_2 = 0 \end{matrix}$$

$$c_2 = -e^{i\theta}c_1$$

Let $c_1 = 1$, then $c_2 = -e^{i\theta}$. So

$$|S_r^- = -\hbar/2\rangle = \frac{1}{\sqrt{1+e^{i\theta}e^{-i\theta}}}\begin{pmatrix} 1 \\ -e^{i\theta} \end{pmatrix}$$

$$\boxed{|S_r^- = -\hbar/2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{i\theta} \end{pmatrix}}$$

(d)

If we measure $S_r^+ = \hbar/2$, we must be in the state $|S_r^+ = \hbar/2\rangle$. Now we want to determine the probability of measuring $|S_x = -\hbar/2\rangle$. In other words, we want to find

$$P_{|S_r^+\rangle}(S_x = -\hbar/2) = \left|\langle S_x = -\hbar/2 \,|\, S_r^+ = \hbar/2\rangle\right|^2.$$

We must determine the eigenvectors of S_x. So

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm\hbar/2$$

(d), cont'd...

$|S_x = -\hbar/2\rangle$

$$\begin{pmatrix} \hbar/2 & \hbar/2 \\ \hbar/2 & \hbar/2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \hbar/2\, c_1 + \hbar/2\, c_2 = 0 \\ c_1 = -c_2 \end{matrix}$$

Let $c_1 = 1$. Then $c_2 = -1$ and

$$|S_x = -\hbar/2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So our probability is

$$P_{|S_r^+\rangle}(S_x = -\hbar/2) = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} \right|^2$$

$$= \frac{1}{4} \left| 1 - e^{i\theta} \right|^2$$

$$= \frac{1}{4} (1 - e^{-i\theta})(1 - e^{i\theta})$$

$$= \frac{1}{4} (1 - e^{i\theta} - e^{-i\theta} + 1)$$

$$= \frac{1}{4} (2 - (e^{i\theta} + e^{-i\theta}))$$

$$= \frac{1}{4} (2 - 2\cos\theta)$$

$$\boxed{P_{|S_r^+\rangle}(S_x = -\hbar/2) = \frac{1}{2}(1 - \cos\theta)}$$

or $$\boxed{P_{|S_r^+\rangle}(S_x = -\hbar/2) = \sin^2(\theta/2)}.$$

(e)

The S_z eigenvector is...

$$\begin{pmatrix} \hbar/2 - \hbar/2 & 0 \\ 0 & -\frac{\hbar}{2} - \frac{\hbar}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} 0c_1 + 0c_2 = 0 \\ 0c_1 - \hbar c_2 = 0 \end{array}$$

So $c_2 = 0$ and c_1 can be anything. Let $c_1 = 1$, so

$$|S_z^+ = \hbar/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then our probability of measuring $|S_x^- = -\hbar/2\rangle$ is

$$P_{|S_z^+ = \hbar/2\rangle}(S_x^- = -\hbar/2) \cdot P_{|S_r^+\rangle}(S_z^+ = \hbar/2).$$

So

$$P_{|S_r^+\rangle}(S_z^+ = \hbar/2) = \left| \langle S_z^+ = \hbar/2 | S_r^+ = \hbar/2 \rangle \right|^2$$

$$= \left| (1 \ 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} \right|^2$$

$$= \frac{1}{2} |1|^2$$

$$= \frac{1}{2}$$

and

$$P_{|S_z^+ = \hbar/2\rangle}(S_x^- = -\hbar/2) = \left| \langle S_x^- = -\hbar/2 | S_z^+ = \hbar/2 \rangle \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} (1 \ -1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2$$

$$= \frac{1}{2} |1|^2$$

$$= \frac{1}{2}$$

Thus, $\boxed{P_{tot} = \frac{1}{4}}$.