F2009

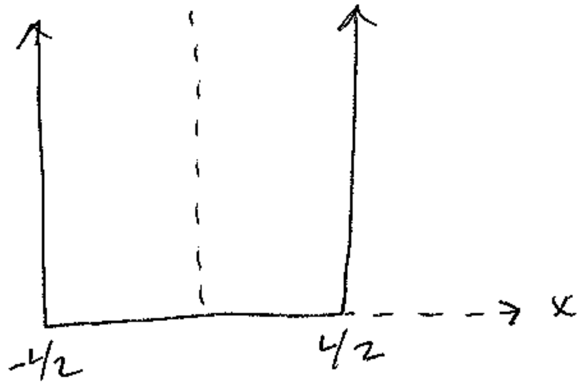
Problem 5: Quantum statistics (10 points)

1. Write down the energy eigenvalues and wave functions for a particle of mass m in an infinite square well, with $V = 0$ for $-L/2 < x < L/2$ and $V = \infty$ for $|x| > L/2$. (2 pts)

2. What is the ground state energy and wave-function if 2 identical non-interacting bosons are in the well? (4 pts)

3. What is the ground state energy and wave-function if 2 identical non-interacting spin-up fermions are in the well? (4 pts)

(a)



The Schrödinger equation, for an infinite square well, is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi,$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}.$$

The general solution is

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

Our boundary conditions require that

$$\psi(-L/2) = \psi(L/2) = 0.$$

So

$$\psi(-L/2) = A\sin\left(-\frac{kL}{2}\right) + B\cos\left(-\frac{kL}{2}\right)$$
$$= B\cos\left(\frac{kL}{2}\right) - A\sin\left(\frac{kL}{2}\right)$$
$$= 0$$

and

$$\psi(L/2) = A\sin\left(\frac{kL}{2}\right) + B\cos\left(\frac{kL}{2}\right)$$
$$= 0$$

(a), cont'd...

Let

$$\frac{kL}{2} = \frac{n\pi}{2}.$$

Consider $n \in$ evens $(n = 2, 4, 6, \ldots)$. Then

$$\psi(L/2) = A\sin\left(\frac{n\pi}{2}\right) + B\cos\left(\frac{n\pi}{2}\right)$$

$$= A(0) + B(1)$$

$$B = 0,$$

and our general solution for $n = 2, 4, 6, \ldots$ is

$$\psi_{even} = A\sin\left(\frac{n\pi x}{L}\right).$$

Normalizing...

$$A^2 \int_{-L/2}^{L/2} \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

$$A^2\left[\frac{1}{2}\int_{-L/2}^{L/2}\left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)dx\right] = 1$$

$$\frac{A^2}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]\Bigg|_{-L/2}^{L/2} = 1$$

$$\frac{A^2}{2}\left[L\right] = 1$$

$$A = \sqrt{\frac{2}{L}}$$

and

$$\boxed{\psi_{even}(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \quad n = 2, 4, 6, \ldots}$$

(a), cont'd...

Similarly, for $n \in$ odds $(n=1,3,5,\dots)$, we have

$$\boxed{\psi_{odd}(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x}{L}\right), \quad n=1,3,5,\dots}.$$

We let

$$k = \frac{\sqrt{2mE}}{\hbar},$$

but we also set

$$k = \frac{n\pi}{L}.$$

So our energy eigenvalues are

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{L}$$

$$2mE = \frac{n^2\pi^2\hbar^2}{L^2}$$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}}.$$

(b)

Since bosons do not obey the Pauli Exclusion Principle, both particles can be in the same state. The total energy is the sum of the single particle energies, so

$$E = \frac{(n_1^2 + n_2^2)\pi^2\hbar^2}{2mL^2}.$$

In the ground state, we have $n_1 = n_2 = 1$, so the ground state energy is

$$\boxed{E = \frac{\pi^2\hbar^2}{mL^2}}.$$

The wavefunction must be symmetric since these are bosons. In general, for indistinguishable, non-interacting bosons we have

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_2)\psi_{n_2}(x_1)\right].$$

In the ground state,

$$\psi_{n_1}(x_1) = \psi_1(x_1) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x_1}{L}\right)$$
$$\psi_{n_2}(x_2) = \psi_1(x_2) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x_2}{L}\right)$$
$$\psi_{n_1}(x_2) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x_2}{L}\right)$$
$$\psi_{n_2}(x_1) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x_1}{L}\right).$$

So

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\frac{2}{L}\cos\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{\pi x_2}{L}\right) + \frac{2}{L}\cos\left(\frac{\pi x_2}{L}\right)\cos\left(\frac{\pi x_1}{L}\right)\right]$$
$$= \frac{4}{L\sqrt{2}}\cos\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{\pi x_2}{L}\right)$$

(b), cont'd...

We must normalize this function. We know

$$\int_{-L/2}^{L/2} |\psi(x_1,x_2)|^2 dx_1 dx_2 = 1,$$

so

$$\frac{A^2 8}{L^2}\int_{-L/2}^{L/2}\cos^2\left(\frac{\pi x_1}{L}\right)dx_1 \int_{-L/2}^{L/2}\cos^2\left(\frac{\pi x_2}{L}\right)dx_2 = 1$$

$$\frac{A^2 8}{L^2}\int_{-L/2}^{L/2}\frac{1}{2}\left(1+\cos\left(\frac{2\pi x_1}{L}\right)\right)dx_1 \int_{-L/2}^{L/2}\frac{1}{2}\left(1+\cos\left(\frac{2\pi x_2}{L}\right)\right)dx_2 = 1$$

$$\frac{2A^2}{L^2}\int_{-L/2}^{L/2}\left(1+\cos\left(\frac{2\pi x_1}{L}\right)\right)dx_1 \int_{-L/2}^{L/2}\left(1+\cos\left(\frac{2\pi x_2}{L}\right)\right)dx_2 = 1$$

$$\frac{2A^2}{L^2}(L)(L) = 1$$

$$A = \frac{1}{\sqrt{2}}$$

and our normalized wavefunction is

$$\psi(x_1,x_2) = \frac{4}{L\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cos\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{\pi x_2}{L}\right)$$

$$\boxed{\psi(x_1,x_2) = \frac{2}{L}\cos\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{\pi x_2}{L}\right)}$$

NOTE: The spin part of the wavefunction must also be symmetric, so the spin portion can be any one of the three triplet spin states. Then our total wavefunction, including spin, is just

$$\Psi = \psi(x_1,x_2)\cdot\psi^{spin}_{sym}.$$

(c)

The wavefunction for fermions must be antisymmetric. Since we are considering two spin-up fermions, we know the spin part of the wavefunction is symmetric. This means the spatial part of the wavefunction is antisymmetric, and is, in general, given by

$$\psi(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)\right].$$

Since fermions must obey the Pauli Exclusion Principle, the particles cannot be in the same state (or the wavefunction would be equal to zero). Then the ground state energy of our system is

$$E = \frac{(n_1^2+n_2^2)\pi^2\hbar^2}{2mL^2}$$

$$= \frac{((1)^2+(2)^2)\pi^2\hbar^2}{2mL^2}$$

$$\boxed{E = \frac{5\pi^2\hbar^2}{2mL^2}}.$$

Our ground state wavefunction is then

$$\psi(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\frac{2}{L}\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \frac{2}{L}\cos\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)\right]$$

$$= \frac{2}{L\sqrt{2}}\left[\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \cos\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{2\pi x_1}{L}\right)\right],$$

which is already normalized. So

$$\boxed{\psi(x_1,x_2) = \frac{\sqrt{2}}{L}\left[\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \cos\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{2\pi x_1}{L}\right)\right]}.$$

NOTE: The total wavefunction would be $\Psi = \psi(x_1,x_2)\cdot\psi^{spin}_{sym}$, where ψ^{spin}_{sym} is one of the three symmetric spin states. In particular, $\psi^{spin}_{sym} = |\uparrow_1\uparrow_2\rangle$.