2014

Problem 6: Perturbation Theory

An isotropic Harmonic oscillator in two dimensions has the Hamiltonian

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2),$$

where x and y are position operators in Cartesian coordinates x and y.

a) What is the energy of the *three* lowest energy levels and their respective degeneracies? (2 Points)

b) Consider a perturbative potential of the form:

$$V(x, y) = Am\omega^2 xy.$$

Compute the energy correction of the lowest level in the lowest order in perturbation theory where the result is non-zero. (3 Points)

c) Compute the energy splitting of the first excited energy level (which is degenerate), due to the perturbation. Compute the split ket states in terms of the original unperturbed kets. (3 Points) Griffiths, 6.7.2

? d) Suppose that there are three indistinguishable spin 1/2 particles in the system. Compute the total energy of the ground state in first order in perturbation theory. (2 Points)

(a)

The general form of the energy is

$$E = (n_x + n_y + 1)\hbar\omega .$$

Then the three lowest energies and their degeneracies are :

Energy	Degeneracy	States
$E = \hbar\omega$	1	ψ_{00}
$E = 2\hbar\omega$	2	ψ_{10}, ψ_{01}
$E = 3\hbar\omega$	3	$\psi_{11}, \psi_{20}, \psi_{02}$

(b)

Now we want to compute the energy correction of the lowest level in the lowest order where the result is non-zero.
To first order, we know

$$\begin{aligned}
E^{(1)} &= \langle n^{(0)} | V' | n^{(0)} \rangle \\
&= \langle n_x n_y | Am\omega^2 xy | n_x n_y \rangle \\
&= Am\omega^2 \langle n_x n_y | xy | n_x n_y \rangle \\
&= Am\omega^2 \langle n_x n_y | \sqrt{\tfrac{\hbar}{2m\omega}}(a_x + a_x^\dagger)\sqrt{\tfrac{\hbar}{2m\omega}}(a_y + a_y^\dagger) | n_x n_y \rangle \\
&= \frac{A\omega\hbar}{2}\left[\langle n_x n_y | a_x a_y + a_x a_y^\dagger + a_x^\dagger a_y + a_x^\dagger a_y^\dagger | n_x n_y \rangle\right] \\
&= 0
\end{aligned}$$

(b), cont'd ...

To second order, we know

$$E^{(2)} = \sum_{m \neq n} \frac{|\langle m^{(0)}|V'|n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}.$$

Since we are in the lowest level, we have $|n^{(0)}\rangle = |n_x n_y\rangle = |00\rangle$.
Then

$$E^{(2)} = \sum_{m \neq n} \frac{|\langle m_x m_y | A m\omega^2 xy | 00\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$= A^2 m^2 \omega^4 \sum_{m \neq n} \frac{|\langle m_x m_y | \sqrt{\frac{\hbar}{2m\omega}}(a_x + a_x^+)\sqrt{\frac{\hbar}{2m\omega}}(a_y + a_y^+)|00\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$= \frac{A^2\omega^2\hbar^2}{4} \sum_{m \neq n} \frac{|\langle m_x m_y | a_x a_y + a_x a_y^+ + a_x^+ a_y + a_x^+ a_y^+ |00\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$= \frac{A^2\omega^2\hbar^2}{4} \sum_{m \neq n} |\langle m_x m_y | a_x a_y | 00\rangle + \langle m_x m_y | a_x a_y^+ | 00\rangle + \langle m_x m_y | a_x^+ a_y | 00\rangle + \langle m_x m_y | a_x^+ a_y^+ | 00\rangle|^2 / (E_n^{(0)} - E_m^{(0)})$$

$$= \frac{A^2\omega^2\hbar^2}{4} \sum_{m \neq n} \frac{|0 + 0 + 0 + \langle m_x m_y | 11\rangle|^2}{(n_x + n_y + 1)\hbar\omega - (m_x + m_y + 1)\hbar\omega}$$

$$= \frac{A^2\omega\hbar}{4} \sum_{m \neq n} \frac{|\langle m_x m_y | 11\rangle|^2}{-m_x - m_y}$$

$$= \frac{A^2\omega\hbar}{4} \sum_{m \neq n} \frac{|\delta_{m_x,1}\delta_{m_y,1}|^2}{-m_x - m_y}$$

$$E^{(2)} = -\frac{A^2\omega\hbar}{8}$$

So our corrected energy is

$$E_{n_x n_y} = \boxed{E_{00} = \hbar\omega - \frac{A^2}{8}\hbar\omega}.$$

(c)

We want to determine the energy correction for the two-fold degenerate first excited state. The W matrix is, in general,

$$W = \begin{pmatrix} W_{aa} & W_{ab} \\ W_{ba} & W_{bb} \end{pmatrix}.$$

We also know

$$W_{ij} = \langle \psi_i^{(0)} | H' | \psi_j^{(0)} \rangle,$$

so letting $\psi_a = \psi_{10}$ and $\psi_b = \psi_{01}$, our degenerate eigenstates, we have:

$$W_{aa} = \langle \psi_a | V | \psi_a \rangle$$

$$= \langle \psi_{10} | V | \psi_{10} \rangle = \langle 1\,0 | V | 1\,0 \rangle$$

$$= \langle 1\,0 | Am\omega^2 xy | 1\,0 \rangle$$

$$= Am\omega^2 \cdot \frac{\hbar}{2m\omega} \langle 1\,0 | (a_x + a_x^+)(a_y + a_y^+) | 1\,0 \rangle$$

$$= Am\omega^2 \cdot \frac{\hbar}{2m\omega} \langle 1\,0 | a_x a_y + a_x a_y^+ + a_x^+ a_y + a_x^+ a_y^+ | 1\,0 \rangle$$

① $W_{aa} = 0$

$$W_{bb} = \langle \psi_b | V | \psi_b \rangle$$

$$= \langle \psi_{01} | V | \psi_{01} \rangle$$

$$= \langle 0\,1 | Am\omega^2 xy | 0\,1 \rangle$$

② $W_{bb} = 0$

(c), cont'd...

$$
\begin{aligned}
W_{ab} &= \langle \psi_a | V | \psi_b \rangle \\
&= \langle \psi_{10} | V | \psi_{01} \rangle \\
&= \langle 1\,0 | V | 0\,1 \rangle \\
&= \frac{A\omega\hbar}{2} \langle 1\,0 | a_x a_y + a_x a_y^\dagger + a_x^\dagger a_y + a_x^\dagger a_y^\dagger | 0\,1 \rangle \\
&= \frac{A\hbar\omega}{2} \left[0 + 0 + \langle 1\,0 | a_x^\dagger a_y | 0\,1 \rangle + \langle 1\,0 | a_x^\dagger a_y^\dagger | 0\,1 \rangle \right] \\
&= \frac{A\hbar\omega}{2} \left(\langle 1\,0 | 1\,0 \rangle + \sqrt{2} \langle 1\,0 | 1\,2 \rangle \right)
\end{aligned}
$$

(3) $$W_{ab} = \frac{A\hbar\omega}{2}$$

$$
\begin{aligned}
W_{ba} &= \langle \psi_b | V | \psi_a \rangle \\
&= \langle \psi_{01} | V | \psi_{10} \rangle \\
&= \frac{A\hbar\omega}{2} \langle 0\,1 | a_x a_y + a_x a_y^\dagger + a_x^\dagger a_y + a_x^\dagger a_y^\dagger | 1\,0 \rangle \\
&= \frac{A\hbar\omega}{2} \left[0 + \langle 0\,1 | 0\,1 \rangle + 0 + \sqrt{2} \langle 0\,1 | 2\,1 \rangle \right]
\end{aligned}
$$

(4) $$W_{ba} = \frac{A}{2}\hbar\omega$$

Then our matrix is

$$
W = \begin{pmatrix} 0 & \frac{A}{2}\hbar\omega \\ \frac{A}{2}\hbar\omega & 0 \end{pmatrix}.
$$

Now we must determine the eigenvalues.

(c), cont'd...

$$\det(W - \lambda I) = \begin{vmatrix} -\lambda & \frac{A}{2}\hbar\omega \\ \frac{A}{2}\hbar\omega & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \left(\frac{A}{2}\hbar\omega\right)^2 = 0$$

$$\lambda^2 = \left(\frac{A}{2}\hbar\omega\right)^2$$

$$\lambda = \pm \frac{A}{2}\hbar\omega$$

Then our corrected first excited energy is

$$\boxed{E_{10} = E_{01} = 2\hbar\omega \pm \frac{A}{2}\hbar\omega}.$$

We know

$$(W - \lambda I)\psi = 0,$$

where ψ is our eigenvector. So

$$\begin{pmatrix} -\frac{A}{2}\hbar\omega & \frac{A}{2}\hbar\omega \\ \frac{A}{2}\hbar\omega & -\frac{A}{2}\hbar\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 - c_1 = 0$$

$$c_1 - c_2 = 0$$

So $c_1 = c_2$. Let

$$|\lambda = +\tfrac{A}{2}\hbar\omega\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Normalizing...

$$|\lambda = +\tfrac{A}{2}\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(c), cont'd...

We also have

$$\begin{pmatrix} \frac{A}{2}\hbar\omega & \frac{A}{2}\hbar\omega \\ \frac{A}{2}\hbar\omega & \frac{A}{2}\hbar\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_1 + c_2 = 0,$$

so $c_2 = -c_1$. Let

$$|\lambda = -\tfrac{A}{2}\hbar\omega\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Normalizing...

$$|\lambda = -\tfrac{A}{2}\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In general, we know

$$\psi^\circ = c_1\psi_a + c_2\psi_b.$$

So

$$\psi^\circ = \begin{cases} (\psi_a + \psi_b)/\sqrt{2} \\ (\psi_a - \psi_b)/\sqrt{2} \end{cases}$$

$$\boxed{\psi^\circ = \begin{cases} \frac{1}{\sqrt{2}}(\psi_{10} + \psi_{01}) \\ \frac{1}{\sqrt{2}}(\psi_{10} - \psi_{01}) \end{cases}}.$$

(d) THIS PART NOT DONE CORRECTLY...

Since these are identical fermions, we know they cannot be in the same state. Then we know the lowest total energy will correspond to the particles being in states ψ_{00}, ψ_{01}, and ψ_{02} (or variations due to degeneracy).

We know

$$E_{00} = \hbar\omega - \frac{A^2}{8}\hbar\omega$$

and

$$E_{01} = 2\hbar\omega \pm \frac{A}{2}\hbar\omega.$$

Now we need to determine the correction for E_{02}. The energy is three-fold degenerate, so

$$W = \begin{pmatrix} W_{aa} & W_{ab} & W_{ac} \\ W_{ba} & W_{bb} & W_{bc} \\ W_{ca} & W_{cb} & W_{cc} \end{pmatrix}.$$

Let $\psi_a = \psi_{11}$, $\psi_b = \psi_{20}$, and $\psi_c = \psi_{02}$. Then

$$W_{aa} = \langle \psi_{11} | V | \psi_{11} \rangle$$
$$= 0$$

$$W_{bb} = \langle \psi_{20} | V | \psi_{20} \rangle$$
$$= 0$$

$$W_{cc} = \langle \psi_{02} | V | \psi_{02} \rangle$$
$$= 0$$

(d), cont'd...

$$W_{ab} = \langle \psi_{11} | V | \psi_{20} \rangle$$
$$= \langle 11 | Am\omega^2 xy | 20 \rangle$$
$$= \frac{A\hbar\omega}{2} \langle 11 | a_x a_y + a_x a_y^\dagger + a_x^\dagger a_y + a_x^\dagger a_y^\dagger | 20 \rangle$$
$$= \frac{A\hbar\omega}{2} \left(0 + \sqrt{2} \langle 11 | 11 \rangle + 0 + \sqrt{3} \langle 11 | 31 \rangle \right)$$
$$= \frac{A\hbar\omega}{\sqrt{2}}$$

$$W_{ac} = \langle \psi_{11} | V | \psi_{02} \rangle$$
$$= \langle 11 | Am\omega^2 xy | 0\,2 \rangle$$
$$= \frac{A\hbar\omega}{\sqrt{2}}$$

$$W_{ba} = \langle \psi_{20} | V | \psi_{11} \rangle$$
$$= \frac{A\hbar\omega}{2} \langle 20 | a_x a_y + a_x a_y^\dagger + a_x^\dagger a_y + a_x^\dagger a_y^\dagger | 11 \rangle$$
$$= \frac{A\hbar\omega}{\sqrt{2}}$$

$$W_{bc} = \langle \psi_{20} | V | \psi_{02} \rangle$$
$$= \frac{A\hbar\omega}{2} \langle 20 | a_x a_y + a_x a_y^\dagger + a_x^\dagger a_y + a_x^\dagger a_y^\dagger | 0\,2 \rangle$$
$$= 0$$

$$W_{ca} = \langle \psi_{02} | V | \psi_{11} \rangle$$
$$= \frac{A\hbar\omega}{\sqrt{2}}$$

$$W_{cb} = 0$$

(d), cont'd...

Then

$$W = \begin{pmatrix} 0 & \frac{A\hbar\omega}{\sqrt{2}} & \frac{A\hbar\omega}{\sqrt{2}} \\ \frac{A\hbar\omega}{\sqrt{2}} & 0 & 0 \\ \frac{A\hbar\omega}{\sqrt{2}} & 0 & 0 \end{pmatrix}.$$

and

$$\begin{vmatrix} -\lambda & \frac{A\hbar\omega}{\sqrt{2}} & \frac{A\hbar\omega}{\sqrt{2}} \\ \frac{A\hbar\omega}{\sqrt{2}} & -\lambda & 0 \\ \frac{A\hbar\omega}{\sqrt{2}} & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 - \left(\frac{A\hbar\omega}{\sqrt{2}}\right)^2(-\lambda) - \left(\frac{A\hbar\omega}{\sqrt{2}}\right)^2(-\lambda) = 0$$

$$-\lambda^3 + \frac{A^2\hbar^2\omega^2}{2}\lambda + \frac{A^2\hbar^2\omega^2}{2}\lambda = 0$$

$$-\lambda^3 + A^2\hbar^2\omega^2\lambda = 0$$

$$\lambda^2 = A^2\hbar^2\omega^2$$

$$\lambda = \pm A\hbar\omega.$$

So

$$E_{02} = E_{20} = E_{11} = 3\hbar\omega \pm A\hbar\omega.$$

Our total energy is then

$$\boxed{E = 6\hbar\omega - \frac{A^2}{8}\hbar\omega \pm \frac{A}{2}\hbar\omega \pm A\hbar\omega}.$$