S2008

Problem 1: The Infinite Square Well: (10 Points)

A single particle is in a one dimensional infinitely deep potential well of width L where $V(x)$ is given by:

$$V(x) = \begin{cases} 0, & \text{if } 0 \le x \le L \\ \infty, & \text{otherwise} \end{cases}$$

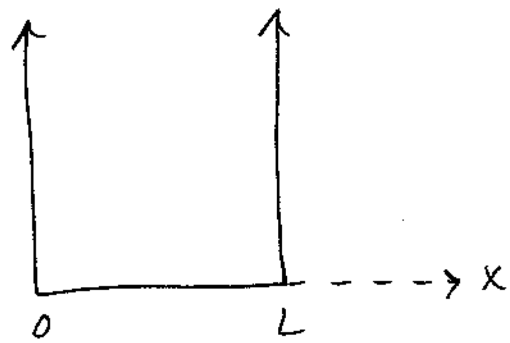
1. Find the allowed energies (E_n) and the normalized eigenfunctions ($\Psi(x)$) to Schrodinger's Equation for this potential. Show all your work. **(2 Points)**

2. Sketch the wave functions for the first three stationary states for this potential. **(2 Points)**

3. Now, if four spin-1/2 identical particles of mass m are placed in this potential, calculate the three lowest values for the total energy of the system of particles. **(3 Points)**

4. Determine the degeneracy for each of the three energy states found in part 3. **(3 Points)**

Part (i)



In general, Schrödinger's equation tells us

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right]\psi = E\psi .$$

For this potential, we can say

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi ,$$

where

$$k = \frac{\sqrt{2mE}}{\hbar} .$$

The general solution for this equation is

$$\psi(x) = A\sin(kx) + B\cos(kx) .$$

Our boundary conditions require that

$$\psi(0) = \psi(L) = 0 .$$

Part (i), cont'd ...

Then

$$\psi(0) = A\sin(0) + B\cos(0) = 0$$
$$B\cos(0) = 0$$
$$B = 0$$

and

$$\psi(L) = A\sin(kL) = 0,$$

which requires

$$kL = n\pi \quad \text{for } n = 1, 2, 3, \ldots$$
$$k = \frac{n\pi}{L}.$$

So our unnormalized wave function is

$$\psi(x) = A\sin\left(\frac{n\pi x}{L}\right).$$

The normalization condition requires

$$\int_0^L |\psi(x)|^2 dx = 1.$$

So

$$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$
$$A^2 \int_0^L \frac{1 - \cos\left(\frac{2n\pi x}{L}\right)}{2} dx = 1$$
$$A^2 \int_0^L \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) dx = 2$$

Part (i), cont'd...

$$A^2\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]\Big|_0^L = 2$$

$$A^2\left[L - \frac{L}{2n\pi}\sin(2n\pi)\right] = 2$$

But $\sin(2n\pi) = 0 \ \forall n$. So

$$A^2 L = 2$$

$$A = \sqrt{\frac{2}{L}}$$

and

$$\boxed{\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)}.$$

We said

$$k = \frac{\sqrt{2mE}}{\hbar},$$

but we also know

$$k = \frac{n\pi}{L}.$$

Then our eigenenergies are

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{L}$$

$$2mE = \frac{n^2\pi^2\hbar^2}{L^2}$$

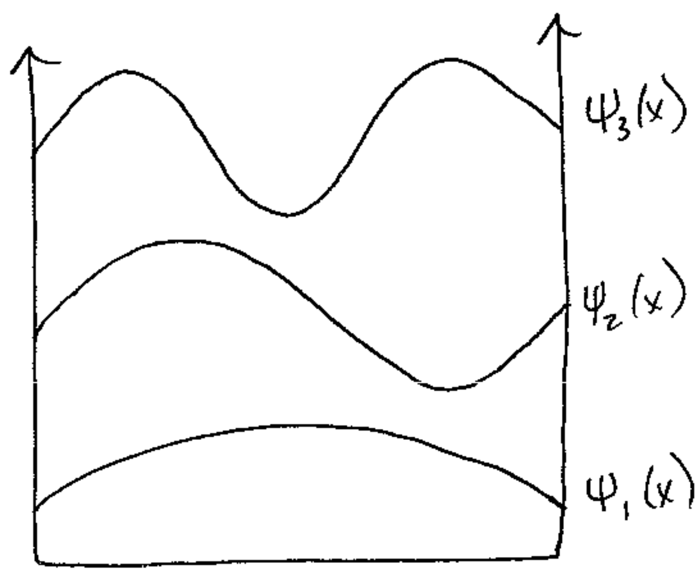
$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}}.$$

Part (ii)

The first three stationary states correspond to $n=1$, $n=2$, and $n=3$.
So

$$\psi_1(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$$
$$\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$
$$\psi_3(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right)$$

and our wavefunctions look like



Note: vertical scale is irrelevant.

Part (iii)

The energy is a sum of the single particle energies,

$$E = E_1 + E_2 + E_3 + E_4 .$$

So

$$E = \frac{(n_1^2 + n_2^2 + n_3^2 + n_4^2)\pi^2\hbar^2}{2mL^2} .$$

Since these particles are fermions, they cannot occupy the same state. So the lowest total energy is given by

$$E_{1234} = \frac{((1)^2 + (2)^2 + (3)^2 + (4)^2)\pi^2\hbar^2}{2mL^2}$$

$$\boxed{E_{1234} = \frac{15\pi^2\hbar^2}{mL^2}}$$

The second lowest total energy is then

$$E_{1235} = \frac{((1)^2 + (2)^2 + (3)^2 + (5)^2)\pi^2\hbar^2}{2mL^2}$$

$$\boxed{E_{1235} = \frac{39\pi^2\hbar^2}{2mL^2}}$$

and the third lowest total energy is

$$E_{1245} = \frac{((1)^2 + (2)^2 + (4)^2 + (5)^2)\pi^2\hbar^2}{2mL^2}$$

$$\boxed{E_{1245} = \frac{23\pi^2\hbar^2}{mL^2}}$$

Part (iv)

The degeneracies are the same for the three lowest energies. There are 16 different ways to order our n's, so the degeneracy is $\boxed{16}$.