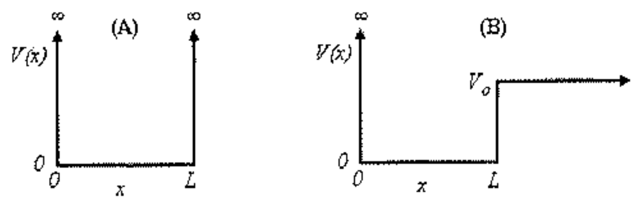
S2007

PROBLEM 2

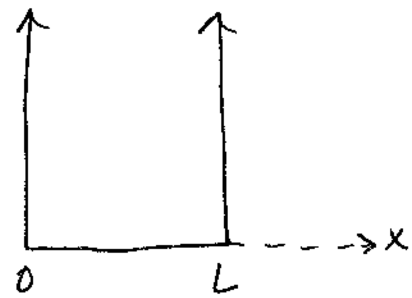


a) [2 pts] Calculate the energy eigenvalues for a particle of mass m in the one-dimensional infinite well shown in Figure A.

b) [4 pts] For the time-independent Schrödinger Equation corresponding to potential (B), find a transcendental equation in E giving the eigenenergies in terms of V_o, L, m, and $\hbar$

c) [4 pts] For the time-independent Schrödinger Equation corresponding to potential (B), what is the smallest value of V_o that gives one bound state? What is the smallest value of V_o that gives two bound states?

(a)



The Schrödinger equation tells us

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$
$$\frac{d^2\psi}{dx^2} = -k^2\psi,$$

where

$$k = \frac{\sqrt{2mE}}{\hbar},$$

for this potential. The general solution is

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

Boundary conditions require that

$$\psi(0) = \psi(L) = 0,$$

so

$$\psi(0) = A\sin(0) + B\cos(0) = 0$$

implies $B=0$. Then

$$\psi(L) = A\sin(kL) = 0$$

implies

$$kL = n\pi$$
$$k = \frac{n\pi}{L}. \quad (n = 1,2,3,\ldots)$$

(a), cont'd...

But we said

$$k = \frac{\sqrt{2mE}}{\hbar},$$

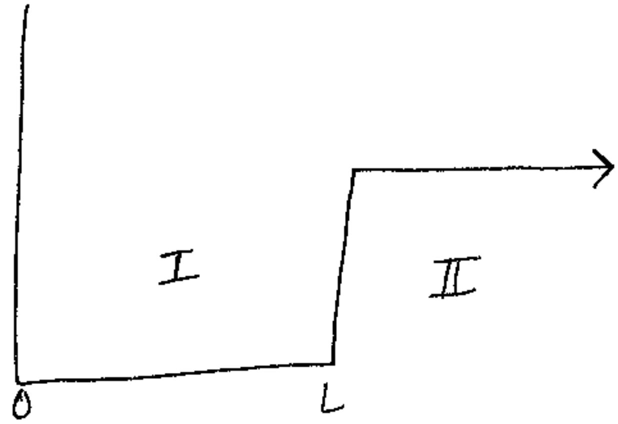
so our energy eigenvalues are

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{L}$$

$$2mE = \frac{n^2\pi^2\hbar^2}{L^2}$$

$$\boxed{E = \frac{n^2\pi^2\hbar^2}{2mL^2}}$$

(b)



In region I, we have

$$\psi_I(x) = A\sin(kx) + B\cos(kx)$$

and in region II, we have

$$\psi_{II}(x) = De^{\lambda x} + Fe^{-\lambda x}.$$

Our boundary conditions in region I still require that we have $B = 0$, so

$$\psi_I = A\sin(kx) = \sqrt{\tfrac{2}{L}}\sin(kx)$$

As $x \to \infty$, we must have $D = 0$ because the solution blows up. So

$$\psi_{II}(x) = Fe^{-\lambda x}.$$

Since the Schrödinger equation tells us

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (V_0 - E)\psi = 0$$

$$\frac{d^2\psi}{dx^2} = \frac{2m(V_0-E)}{\hbar^2}\psi,$$

we know

$$\lambda = \frac{\sqrt{2m(V_0-E)}}{\hbar}.$$

(b), cont'd...

Continuity requires that

$$\psi_I(L) = \psi_{II}(L)$$

and

$$\frac{d\psi_I}{dx}\Big|_L = \frac{d\psi_{II}}{dx}\Big|_L .$$

So

$$\sqrt{\frac{2}{L}}\sin(kL) = Fe^{-\lambda L}$$
$$F = \sqrt{\frac{2}{L}}\sin(kL)e^{\lambda L}$$

and

$$k\sqrt{\frac{2}{L}}\cos(kL) = -\lambda F e^{-\lambda L}$$
$$k\sqrt{\frac{2}{L}}\cos(kL) = -\lambda\sqrt{\frac{2}{L}}\sin(kL)$$
$$-\frac{k}{\lambda} = \tan(kL).$$

But

$$k = \frac{\sqrt{2mE}}{\hbar}$$

and

$$\lambda = \frac{\sqrt{2m(V_0-E)}}{\hbar},$$

so

$$\sqrt{\frac{2mE}{2m(V_0-E)}} = -\tan\left(\frac{\sqrt{2mE}}{\hbar}L\right)$$
$$\sqrt{\frac{V_0-E}{E}} = -\cot\left(\frac{\sqrt{2mE}}{\hbar}L\right)$$

(b), cont'd...

We could have left our transcendental equation in the form

$$k \cot(kL) = -\lambda$$

$$\frac{\sqrt{2mE}}{\hbar} \cot\left(\frac{\sqrt{2mE}}{\hbar} L\right) = -\frac{\sqrt{2m(V_0 - E)}}{\hbar},$$

as well.