PROBLEM 6: Perturbation Theory

The unperturbed interaction Hamiltonian of an electron with a magnetic dipole moment $\vec{\mu}_s$ in a strong magnetic field $\vec{B}_0 = B_0\hat{z}$ is

$$H_0 = -\vec{\mu}_s \cdot \vec{B}_0$$

where

$$\vec{\mu}_s = -\frac{g\mu_B}{\hbar}\vec{S} = -\frac{g\mu_B}{2}\vec{\sigma}$$

and

$$\mu_B = \frac{e\hbar}{2mc} .$$

If the electron is in the state with $s_z = \hbar/2$, and we add a small magnetic field $\vec{B}_1 = B_1\hat{x}$ with $B_1 \ll B_0$, then we can consider the Hamiltonian as

$$\begin{aligned} H &= H_0 + H_1 \\ H_1 &= \frac{g}{2}(\mu_B)(\vec{B}_1 \cdot \vec{\sigma}) \end{aligned}$$

where H_1 is a perturbing potential and $g = 2$ for the electron.

(a) Find the first order change in the energy. *(2 point)*

(b) Find the second order change in the energy. *(3 point)*

(c) Find the first order correction to the state vector. *(2 point)*

(d) Calculate the exact energies for $H = H_0 + H_1$. Expand the larger energy in powers of B_1/B_0 with $B_1 \ll B_0$. Show that the term proportional to B_1^2 corresponds to the answer derived in (b). *(3 point)*

N.B. You must solve parts (a) - (c) by applying perturbation theory.

(a)

Our perturbing potential is

$$H_1 = \frac{g}{2}(\mu_B)(\vec{B}_1 \cdot \vec{\sigma}).$$

We know $g=2$, $\vec{B}_1 = B_1\hat{x}$, and $\vec{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$. Then

$$\vec{B}_1 \cdot \vec{\sigma} = B_1 \sigma_x,$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The electron is in the state with $s_z = \hbar/2$, so we know

$$|n\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then our first order correction is

$$\begin{aligned} E^{(1)} &= \langle n | H_1 | n \rangle \\ &= \langle n | \mu_B B_1 \sigma_x | n \rangle \\ &= (1 \quad 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_B B_1 \\ &= (0 \quad 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_B B_1 \end{aligned}$$

$$\boxed{E^{(1)} = 0}$$

(b)

The second order change in energy is given by

$$E^{(2)} = \sum_{m \neq n} \frac{|\langle m|H_1|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$= \sum_{m \neq n} \frac{|\langle m|\mu_B B_1 \sigma_x|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

Since $m \neq n$, the only other option is the spin up state, so

$$|m\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have

$$H_0 = -\mu_B \vec{\sigma} \cdot \vec{B}_0$$

$$= -\mu_B \sigma_z B_0$$

$$= -\mu_B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B_0$$

$$= \begin{pmatrix} -\mu_B B_0 & 0 \\ 0 & \mu_B B_0 \end{pmatrix},$$

so the energy eigenvalues are $\lambda = \pm \mu_B B_0$. Then

$$E^{(2)} = \frac{\left| \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_B B_1 \right|^2}{\mu_B B_0 - (-\mu_B B_0)}$$

$$= \frac{\left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_B B_1 \right|^2}{2\mu_B B_0}$$

$$= \frac{\mu_B^2 B_1^2}{2\mu_B B_0}, \quad \text{so} \quad \boxed{E^{(2)} = \frac{\mu_B B_1^2}{2B_0}}.$$

(c)

The first order correction is

$$|n^{(1)}\rangle = \sum_{m \neq n} \frac{\langle m | H_1 | n \rangle}{E_n^{(0)} - E_m^{(0)}} |m\rangle$$

$$= \frac{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_B b_1}{\mu_B b_0 - (-\mu_B b_0)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{\mu_B b_1}{2 \mu_B b_0} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{|n^{(1)}\rangle = \frac{b_1}{2 b_0} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

(d)

The total Hamiltonian is

$$H = -\mu_B \sigma_z b_0 + \mu_B b_1 \sigma_x$$

$$= \mu_B \left[b_1 \sigma_x - b_0 \sigma_z \right]$$

$$= \begin{pmatrix} -\mu_B b_0 & \mu_B b_1 \\ \mu_B b_1 & \mu_B b_0 \end{pmatrix}.$$

The eigenvalues are

$$\begin{vmatrix} -\mu_B b_0 - \lambda & \mu_B b_1 \\ \mu_B b_1 & \mu_B b_0 - \lambda \end{vmatrix} = 0$$

$$(-\mu_B b_0 - \lambda)(\mu_B b_0 - \lambda) - (\mu_B^2 b_1^2) = 0$$

$$\lambda = \pm \mu_B (b_0^2 + b_1^2)^{1/2}$$

These are the exact energies.

(d), cont'd...

Expanding the positive energy...

$$\mu_B B_0 \left(1 + B_1^2/B_0^2\right)^{1/2} = \mu_B B_0 \left[1 + \frac{1/2}{1!}\frac{B_1^2}{B_0^2} + \frac{1/2(-1/2)}{2!}\left(\frac{B_1^2}{B_0^2}\right)^2 + \cdots\right]$$

$$\approx \mu_B B_0 \left[1 + \frac{B_1^2}{2B_0^2} - \frac{B_1^4}{8B_0^4}\right]$$

$$\approx \mu_B B_0 + \frac{\mu_B B_1^2}{2B_0} - \frac{\mu_B B_1^4}{8B_0^3}$$

The first term corresponds to the unperturbed energy

$$E^{(0)} = \mu_B B_0 .$$

The second term corresponds to the second order energy correction, which agrees with our result in part (b).