F2014

PROBLEM 2: Oscillator Model of Angular Momentum

Arbitrary angular momentum can be constructed from spin-1/2. The latter can be described in terms of the Pauli matrices

$$\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}.$$

The construction of a general angular momentum can be done by introducing two sets of independent harmonic oscillators, in terms of creation ($a_\zeta^\dagger$) and annihilation (a_ζ) operators,

$$[a_+, a_-] = 0, \quad [a_+^\dagger, a_-^\dagger] = 0, \quad [a_\zeta, a_{\zeta'}^\dagger] = \delta_{\zeta,\zeta'},$$

with $\zeta, \zeta' = \pm$ indexing oscillators of type $\pm$. Now define

$$\mathbf{J} = \frac{\hbar}{2} a^\dagger \boldsymbol{\sigma} a,$$

where a is a two component operator,

$$a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}.$$

a) Given the form of the Pauli matrices, give the explicit form for J_x, J_y, J_z in terms of $a_\zeta^\dagger$ and a_ζ operators (2 Points).

b) Show that $J_\pm = J_x \pm iJ_y$ have particularly simple forms in terms of a_ζ and $a_\zeta^\dagger$ operators (1 Point).

c) Compute the commutator $[J_x, J_y]$. How is this generalized for the other components? (2 Points)

d) Show that

$$J^2 = J_z^2 + J_+J_- + i[J_x, J_y],$$

and then write this in terms of the number operators for the two harmonic oscillators,

$$n_+ = a_+^\dagger a_+, \quad n_- = a_-^\dagger a_-.$$

Show that this implies that the eigenvalues of J^2 are $j(j+1)\hbar^2$, where j is an integer or an integer plus $\frac{1}{2}$ (Hint: apply the J^2 operator in the two harmonic oscillator state $|n_+, n_-\rangle$) (3 Points).

e) Using the properties of the harmonic oscillators, show that the state in which J^2 has the eigenvalue $j(j+1)\hbar$ and $J_z = m\hbar$ can be constructed from the state in which both n_+ and n_- have the value zero, $|0\rangle$, by

$$|jm\rangle = \frac{(a_+^\dagger)^{j+m}}{\sqrt{(j+m)!}} \frac{(a_-^\dagger)^{j-m}}{\sqrt{(j-m)!}} |0\rangle.$$

(2 Points)

(a)

In general, we have

$$\vec{J} = \frac{\hbar}{2} a^{\dagger} \vec{\sigma} a, \qquad (\vec{J} = a^{\dagger} \vec{S} a)$$

where

$$a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}.$$

We know

$$\vec{J} = J_x + J_y + J_z$$

and

$$\vec{\sigma} = \sigma_x + \sigma_y + \sigma_z,$$

so we must have

$$J_x = \frac{\hbar}{2} (a_+^{\dagger} \; a_-^{\dagger}) \sigma_x \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$= \frac{\hbar}{2} (a_+^{\dagger} \; a_-^{\dagger}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$= \frac{\hbar}{2} (a_+^{\dagger} \; a_-^{\dagger}) \begin{pmatrix} a_- \\ a_+ \end{pmatrix}$$

$$\boxed{J_x = \frac{\hbar}{2} \left(a_+^{\dagger} a_- + a_-^{\dagger} a_+ \right)}$$

(a), cont'd...

$$J_y = \frac{\hbar}{2}\left(a_+^\dagger\ a_-^\dagger\right)\sigma_y\begin{pmatrix}a_+\\a_-\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(a_+^\dagger\ a_-^\dagger\right)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a_+\\a_-\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(a_+^\dagger\ a_-^\dagger\right)\begin{pmatrix}-ia_-\\ ia_+\end{pmatrix}$$

$$\boxed{J_y = -\frac{i\hbar}{2}\left(a_+^\dagger a_- - a_-^\dagger a_+\right)}$$

$$J_z = \frac{\hbar}{2}\left(a_+^\dagger\ a_-^\dagger\right)\sigma_z\begin{pmatrix}a_+\\a_-\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(a_+^\dagger\ a_-^\dagger\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_+\\a_-\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(a_+^\dagger\ a_-^\dagger\right)\begin{pmatrix}a_+\\ -a_-\end{pmatrix}$$

$$\boxed{J_z = \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)}$$

(b)

Now we want to show that $J_\pm = J_x \pm iJ_y$ can be written in simple form.

$$J_+ = J_x + iJ_y$$

$$= \frac{\hbar}{2}\left(a_+^\dagger a_- + a_-^\dagger a_+\right) + i\cdot\frac{-i\hbar}{2}\left(a_+^\dagger a_- - a_-^\dagger a_+\right)$$

$$= \frac{\hbar}{2}\left[a_+^\dagger a_- + \cancel{a_-^\dagger a_+} + a_+^\dagger a_- - \cancel{a_-^\dagger a_+}\right]$$

$$\boxed{J_+ = \hbar\left(a_+^\dagger a_-\right)}$$

and

$$J_- = J_x - iJ_y$$

$$= \frac{\hbar}{2}\left(a_+^\dagger a_- + a_-^\dagger a_+\right) - i\cdot\frac{-i\hbar}{2}\left(a_+^\dagger a_- - a_-^\dagger a_+\right)$$

$$= \frac{\hbar}{2}\left(\cancel{a_+^\dagger a_-} + a_-^\dagger a_+ - \cancel{a_+^\dagger a_-} + a_-^\dagger a_+\right)$$

$$\boxed{J_- = \hbar\left(a_-^\dagger a_+\right)}$$

(c)

Now we want to compute $[J_x, J_y]$. We have

$$\begin{aligned}
[J_x, J_y] &= J_x J_y - J_y J_x \\
&= \left(\frac{\hbar}{2}\right)^2 \left[(a_+^\dagger a_- + a_-^\dagger a_+)(i a_-^\dagger a_+ - i a_+^\dagger a_-) \right] \\
&\quad - \left(\frac{\hbar}{2}\right)^2 \left[(i a_-^\dagger a_+ - i a_+^\dagger a_-)(a_+^\dagger a_- + a_-^\dagger a_+) \right] \\
&= \frac{i\hbar^2}{4} \left[a_+^\dagger a_- a_-^\dagger a_+ - a_+^\dagger a_- a_+^\dagger a_- + \cancel{a_-^\dagger a_+ a_-^\dagger a_+} - a_-^\dagger a_+ a_+^\dagger a_- \right] \\
&\quad - \frac{i\hbar^2}{4} \left[a_-^\dagger a_+ a_+^\dagger a_- + \cancel{a_-^\dagger a_+ a_-^\dagger a_+} - a_+^\dagger a_- a_+^\dagger a_- - a_+^\dagger a_- a_-^\dagger a_+ \right]
\end{aligned}$$

We know $[a_+, a_+^\dagger] = 1$ and $[a_-, a_-^\dagger] = 1$. We also know $[a_+, a_-^\dagger] = 0$ and $[a_-, a_+^\dagger] = 0$. Lastly, we can say $[a_+^\dagger, a_-^\dagger] = 0$ and $[a_+, a_-] = 0$. This implies

$$a_+ a_+^\dagger - a_+^\dagger a_+ = 1 \qquad (1)$$

$$a_- a_-^\dagger - a_-^\dagger a_- = 1 \qquad (2)$$

$$a_+ a_-^\dagger = a_-^\dagger a_+ \qquad (3)$$

$$a_- a_+^\dagger = a_+^\dagger a_- \qquad (4)$$

$$a_+^\dagger a_-^\dagger = a_-^\dagger a_+^\dagger \qquad (5)$$

$$a_+ a_- = a_- a_+ \qquad (6)$$

Simplifying our original expression,

$$\begin{aligned}
[J_x, J_y] &= \frac{i\hbar^2}{4} \left[2 a_+^\dagger a_- a_-^\dagger a_+ + 0 - 2 a_-^\dagger a_+ a_+^\dagger a_- \right] \\
&= \frac{i\hbar^2}{2} \left(a_+^\dagger a_- a_-^\dagger a_+ - a_-^\dagger a_+ a_+^\dagger a_- \right)
\end{aligned}$$

(c), cont'd...

Using (1), (2), (5), and (6), we have

$$[J_x, J_y] = \frac{i\hbar^2}{2}\left[a_+^\dagger a_+ + a_-^\dagger a_+^\dagger a_- a_+ - a_-^\dagger a_- - a_-^\dagger a_+^\dagger a_- a_+\right]$$

$$= \frac{i\hbar^2}{2}\left[a_+^\dagger a_+ + a_-^\dagger a_+^\dagger a_- a_+ - a_-^\dagger a_- - a_-^\dagger a_+^\dagger a_- a_+\right]$$

$$= \frac{i\hbar^2}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$$

$$\boxed{[J_x, J_y] = i\hbar J_z}$$

The generalized expression is

$$[J_i, J_j] = i\hbar\, \varepsilon_{ijk} J_k,$$

where

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{even permutations} \\ -1, & \text{odd permutations} \\ 0, & \text{otherwise} \end{cases}$$