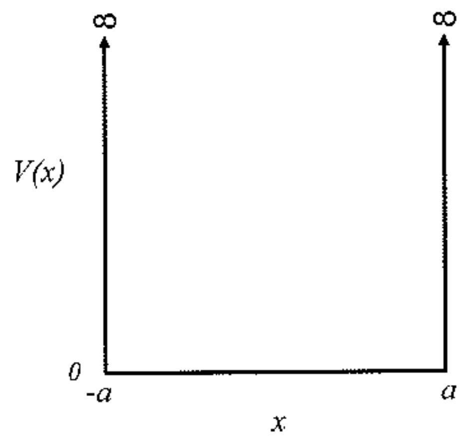
F2007

PROBLEM 2




Consider the one-dimensional infinite-well potential shown above.

[a] (4pts) Derive expressions for the energy eigenfunctions and energy eigenvalues for a particle in the one-dimensional infinite well shown above. Show your work.

[b] (4pts) Now suppose a perturbation of the form

$$\Delta V(x) = V_o a\,\delta(x)$$

is added with $V_o << \frac{\hbar^2 \pi^2}{m a^2}$. Here $\delta(x)$ is the Dirac-delta function. According to first order perturbation theory, what are the eigenenergies of each state?

[c] (2pts) According to first order perturbation theory, what is the wave function of the ground state? Write your answer in terms of $a$, $V_o$, fundamental constants, and the unperturbed wave functions $\phi_n(x)$. You do not have to normalize the wave function.

(a)

The Schrödinger equation tells us

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi,$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}.$$

The general solution is

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

Our boundary conditions require that

$$\psi(-a) = \psi(a) = 0.$$

So

$$\psi(a) = A\sin(ka) + B\cos(ka) = 0.$$

Let

$$ka = n\frac{\pi}{2}.$$

If $n = 1, 3, 5, \ldots$ then we must have $A = 0$.

If $n = 2, 4, 6, \ldots$ then we must have $B = 0$.

(a), cont'd...

So our unnormalized wavefunction is

$$\psi(x) = \begin{cases} b\cos\left(\frac{n\pi x}{2a}\right) & , n = 1,3,5,\ldots \\ A\sin\left(\frac{n\pi x}{2a}\right) & , n = 2,4,6,\ldots \end{cases}$$

Normalizing...

$$b^2 \int_{-a}^{a} \cos^2\left(\frac{n\pi x}{2a}\right) dx = 1$$

$$\frac{b^2}{2} \int_{-a}^{a} \left(1 + \cos\left(\frac{n\pi x}{a}\right)\right) dx = 1$$

$$\frac{b^2}{2}\left[x + \frac{a}{n\pi}\sin\left(\frac{n\pi x}{a}\right)\right]\Big|_{-a}^{a} = 1$$

$$\frac{b^2}{2}\left[2a\right] = 1$$

$$b = \sqrt{\frac{1}{a}}.$$

Similarly, we have $A = \sqrt{1/a}$, so our eigenstates are

$$\boxed{\psi^{(0)}(x) = \begin{cases} \sqrt{1/a}\cos\left(\frac{n\pi x}{2a}\right), & n \in \text{odds} \\ \sqrt{1/a}\sin\left(\frac{n\pi x}{2a}\right), & n \in \text{evens} \end{cases}}$$

We know

$$k = \frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{2a},$$

so

$$\boxed{E_n^{(0)} = \frac{n^2\pi^2\hbar^2}{8ma^2}}$$

(b)

To first order, we know

$$E^{(1)} = \langle n | V' | n \rangle$$
$$= \frac{1}{a} \int_{-a}^{a} \sin^2\left(\frac{n\pi x}{2a}\right) V_0 a \, \delta(x) \, dx$$
$$= V_0 \int_{-a}^{a} \sin^2\left(\frac{n\pi x}{2a}\right) \delta(x) \, dx$$

$$= V_0 \sin^2(0)$$
$$= 0$$

for $n \in$ evens. For $n \in$ odds,

$$E^{(1)} = \langle n | V' | n \rangle$$
$$= \frac{V_0 a}{a} \int_{-a}^{a} \cos^2\left(\frac{n\pi x}{2a}\right) \delta(x) dx$$

$$= V_0 \cos^2(0)$$
$$E^{(1)} = V_0 .$$

So we have

$$\boxed{E_n = \begin{cases} \dfrac{n^2\pi^2\hbar^2}{8ma^2} + V_0 & , n \in \text{odds} \\ \dfrac{n^2\pi^2\hbar^2}{8ma^2} & , n \in \text{evens} \end{cases}}$$

(c)

We have $n=1$, so

$$|n_1^{(1)}\rangle = \sum_{m\neq 1} \frac{\langle m|V'|1\rangle}{E_1^{(0)}-E_m^{(0)}} |m\rangle$$

$$= \sum_{m\neq 1} \frac{1}{E_1^{(0)}-E_m^{(0)}} \frac{1}{a}\int_{-a}^{a} \cos\left(\frac{m\pi x}{2a}\right) V_0 a\,\delta(x) \cos\left(\frac{\pi x}{2a}\right) dx \sqrt{\frac{1}{a}}\cos\left(\frac{m\pi x}{2a}\right)$$

$$= \sum_{m\neq 1} \frac{1}{E_1^{(0)}-E_m^{(0)}} V_0 \int_{-a}^{a} \cos\left(\frac{m\pi x}{2a}\right) \cos\left(\frac{\pi x}{2a}\right) \delta(x)\, dx \sqrt{\frac{1}{a}}\cos\left(\frac{m\pi x}{2a}\right)$$

$$= \sum_{m\neq 1} \frac{V_0}{E_1^{(0)}-E_m^{(0)}} \sqrt{\frac{1}{a}}\cos\left(\frac{m\pi x}{2a}\right)$$

$$= \sum_{m\neq 1} \frac{V_0}{\frac{\pi^2\hbar^2}{8m'a^2} - \frac{m^2\pi^2\hbar^2}{8m'a^2}} \sqrt{\frac{1}{a}}\cos\left(\frac{m\pi x}{2a}\right)$$

$$= \sum_{m\neq 1} \frac{8V_0 m'a^2}{\pi^2\hbar^2(1-m^2)} \sqrt{\frac{1}{a}}\cos\left(\frac{m\pi x}{2a}\right)$$

$$= \sum_{m\neq 1} \frac{8V_0 m'a^2}{\pi^2\hbar^2(1-m^2)} \phi_m(x)$$

Letting $m=n$, we thus have

$$|n_1\rangle = \sqrt{\frac{1}{a}}\cos\left(\frac{\pi x}{2a}\right) + \sum_{n\neq 1} \frac{8V_0 ma^2}{\pi^2\hbar^2(1-n^2)} \phi_n(x)$$

$$\boxed{|n_1\rangle = \phi_1(x) + \sum_{n\neq 1} \frac{8V_0 ma^2}{\pi^2\hbar^2(1-n^2)} \phi_n(x)}.$$