Angular momentum F2016

Problem 3: Angular momentum (10 pts)

One particle has spin j_1 and another particle has spin j_2.

(a) [1 point] What are the good quantum numbers for the two-particle system with $\vec{J} = \vec{J_1} + \vec{J_2}$ in the direct product basis? Write down the basis vectors labelled according to their eigenvalues.

(b) [1 points] Write down the basis vectors in the total j basis. What are the good quantum numbers in this case?

(c) [2 points] Write down the completeness relation for the direct product basis states.

(d) [2 points] Use the completeness relation to relate the total -j basis to the direct product basis. Identify the Clebsch-Gordon coefficient.

(e) [2 points] Write down the relation between total-j and direct product bases for $j_1 = 1/2$ and $j_2 = 1/2$. Recall

$$J_{\pm}|j, m >= \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1 >$$

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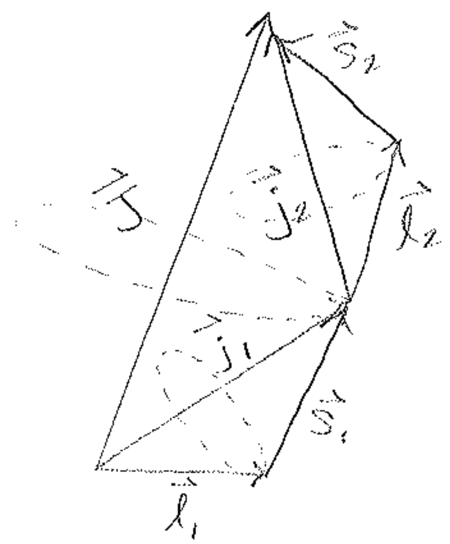
(f) [2 points] Suppose you have an interaction of the form $H_I = A\vec{J_1} \cdot \vec{J_2}$ where $\vec{J} = \vec{J_1} + \vec{J_2}$. Which basis vectors are best to use and why?

Problem 3 — Angular Momentum

Part (a)



In this case, we have jj-coupling, so we know

$$\langle H_{S\text{-}O} \rangle \gg \langle H_{res} \rangle ,$$

where $H_{S\text{-}O}$ is the spin-orbit coupling Hamiltonian and H_{res} is the Hamiltonian representing the residual electrostatic interaction. In this coupling scheme, H_{res} acts as a perturbation on $H_{S\text{-}O}$, and it is usually the case for heavier atoms.

The direct product basis is essentially a basis built up from each separate particle in our coupling scheme. For each individual particle, j_i $(i=1,2)$ is fixed, while $\hat{l}_i$ and $\hat{s}_i$ process about $\hat{j}_i$, so our direct product basis is

$$|j_1\, m_1\rangle \otimes |j_2\, m_2\rangle ,$$

where

$$-j_1 \le m_1 \le j_1$$

$$-j_2 \le m_2 \le j_2 .$$

Part (b)

In the total $\vec{J}$ basis, we have J fixed, so we know J and M_J are good quantum numbers. We write the basis vectors as

$$|J \; M_J \; j_1 \; j_2\rangle .$$

Part (c)

We can write our direct product basis state as

$$|j_1 \; m_1\rangle \otimes |j_2 \; m_2\rangle = |j_1 \, j_2 ; m_1 \, m_2\rangle ,$$

so our completeness relation is

$$I = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 \, j_2 ; m_1 \, m_2\rangle\langle j_1 \, j_2 ; m_1 \, m_2|$$

Part (d)

We have

$$|J\, M_J\, j_1 j_2\rangle = I\, |J\, M_J\, j_1 j_2\rangle$$

since I is just the identity matrix. Inserting completeness, we have

$$|J\, M_J\, j_1 j_2\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 j_2 ; m_1 m_2\rangle \underbrace{\langle j_1 j_2 ; m_1 m_2 | J\, M_J\, j_1 j_2\rangle}_{\text{This is our Clebsch-Gordan coefficient.}}$$

Part (e)

Let $j_1 = 1/2$ and $j_2 = 1/2$. For the sake of time, we let

$$|J\ M_j\ j_1\ j_2\rangle = |J\ M_J\rangle$$

since j_1 and j_2 are constant. We know

$$|j_1 - j_2| \leq J \leq |j_1 + j_2|$$

$$0 \leq J \leq 1,$$

so $J = 0, 1$. We also have

$$-J \leq M_J \leq J,$$

so our total basis vectors are

$\|J\ M_J\rangle$
$\|1\ 1\rangle$
$\|1\ 0\rangle$
$\|1\ -1\rangle$
$\|0\ 0\rangle$.

We want to write these in terms of our direct product basis vectors.

Since

$$M_J = m_1 + m_2,$$

we can easily say that

$$|J=1\ M_J=1\rangle = |j_1=\tfrac{1}{2}\ j_2=\tfrac{1}{2};\ m_1=\tfrac{1}{2}\ m_2=\tfrac{1}{2}\rangle$$

and

$$|J=1\ M_J=-1\rangle = |j_1=\tfrac{1}{2}\ j_2=\tfrac{1}{2};\ m_1=-\tfrac{1}{2}\ m_2=-\tfrac{1}{2}\rangle.$$

To determine the relationship between $|J=1\ M_J=0\rangle$ and the direct product basis, we apply the lowering operator to our $|1\ 1\rangle$ state. So

$$J_-|1\ 1\rangle = (J_1^- + J_2^-)|\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle.$$

For the left-hand side, we get

$$J_-|1\ 1\rangle = \hbar\sqrt{((1)+(1))((1)-(1)+1)}\ |1\ 0\rangle$$
$$= \hbar\sqrt{2}\ |1\ 0\rangle.$$

For the right-hand side, we have

$$(J_1^- + J_2^-)|\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle = J_1^-|\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle + J_2^-|\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$
$$= \hbar\sqrt{(j_1+m_1)(j_1-m_1+1)}\ |\tfrac{1}{2}\ \tfrac{1}{2};-\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \hbar\sqrt{(j_2+m_2)(j_2-m_2+1)}\ |\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$
$$= \hbar\sqrt{((\tfrac{1}{2})+(\tfrac{1}{2}))((\tfrac{1}{2})-(\tfrac{1}{2})+1)}\ |\tfrac{1}{2}\ \tfrac{1}{2};-\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \hbar\sqrt{((\tfrac{1}{2})+(\tfrac{1}{2}))((\tfrac{1}{2})-(\tfrac{1}{2})+1)}\ |\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$
$$= \hbar\ |\tfrac{1}{2}\ \tfrac{1}{2};-\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \hbar\ |\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

Then we have

$$\hbar\sqrt{2}\,|1\ 0\rangle = \hbar\,|\tfrac{1}{2}\ \tfrac{1}{2};\ -\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \hbar\,|\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

or

$$|J=1\ M_J=0\rangle = \frac{1}{\sqrt{2}}\left[|j_1=\tfrac{1}{2}\ j_2=\tfrac{1}{2};m_1=-\tfrac{1}{2}\ m_2=\tfrac{1}{2}\rangle + |j_1=\tfrac{1}{2}\ j_2=\tfrac{1}{2};m_1=\tfrac{1}{2}\ m_2=-\tfrac{1}{2}\rangle\right]$$

The only other vector we have left to find a relation for is $|J=0\ M_J=0\rangle$. We know that all of these vectors must be orthonormal, so we should have

$$\langle J=0\ M_J=0|J=1\ M_J=0\rangle = 0.$$

Then

$$\langle 0\ 0|1\ 0\rangle = \left[\alpha\langle\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ -\tfrac{1}{2}| + \beta\langle\tfrac{1}{2}\ \tfrac{1}{2};-\tfrac{1}{2}\ \tfrac{1}{2}|\right]\left[\frac{1}{\sqrt{2}}|\tfrac{1}{2}\tfrac{1}{2};-\tfrac{1}{2}\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2}\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle\right]$$

$$= \frac{\alpha}{\sqrt{2}}\langle\tfrac{1}{2}\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}|\tfrac{1}{2}\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle + \frac{\beta}{\sqrt{2}}\langle\tfrac{1}{2}\tfrac{1}{2};-\tfrac{1}{2}\tfrac{1}{2}|\tfrac{1}{2}\tfrac{1}{2};-\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}} = 0$$

But $|0\ 0\rangle$ should be normalized, so that

$$\alpha^2 + \beta^2 = 1.$$

Then since

$$\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}} = 0$$

$$\beta = -\alpha,$$

we have

$$\alpha^2 + (-\alpha)^2 = 1$$
$$2\alpha^2 = 1$$
$$\alpha = \frac{1}{\sqrt{2}}.$$

Thus, we know

$$|J=0\ M_J=0\rangle = \frac{1}{\sqrt{2}}\left[\left|\tfrac{1}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2}\,\tfrac{1}{2};-\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle\right].$$

Then overall, our relation between the total J and direct product bases are:

$\lvert J\ M_J\rangle$	$\lvert j_1\, j_2; m_1\, m_2\rangle$
$\lvert 1\ 1\rangle$	$\left\lvert\tfrac{1}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle$
$\lvert 1\ 0\rangle$	$\frac{1}{\sqrt{2}}\left[\left\lvert\tfrac{1}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle + \left\lvert\tfrac{1}{2}\,\tfrac{1}{2};-\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle\right]$
$\lvert 1\ -1\rangle$	$\left\lvert\tfrac{1}{2}\,\tfrac{1}{2};-\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle$
$\lvert 0\ 0\rangle$	$\frac{1}{\sqrt{2}}\left[\left\lvert\tfrac{1}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle - \left\lvert\tfrac{1}{2}\,\tfrac{1}{2};-\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle\right]$.

Part (f)

We know that

$$\vec{J} = \vec{J}_1 + \vec{J}_2,$$

so

$$J^2 = (\vec{J}_1 + \vec{J}_2)^2$$
$$= J_1^2 + J_2^2 + 2\vec{J}_1 \cdot \vec{J}_2 .$$

Then we have

$$\vec{J}_1 \cdot \vec{J}_2 = \frac{1}{2}\left(J^2 - J_1^2 - J_2^2\right).$$

The operators J^2, J_1^2, and J_2^2 act on the $|J\ M\ j_1\ j_2\rangle$ vectors, so clearly, these are the best basis vectors to use.

$$J^2 |J\ M\rangle = J(J+1)\hbar^2 |J\ M\rangle$$
$$J_1^2 |J\ M\rangle = j_1(j_1+1)\hbar^2 |J\ M\rangle$$
$$J_2^2 |J\ M\rangle = j_2(j_2+1)\hbar^2 |J\ M\rangle$$