PROBLEM 6: Hyperfine Splitting

The hyperfine splitting in hydrogen comes from a spin-spin interaction between the electron and the proton. The total Hamiltonian can be written as

$$H = \frac{P_p^2}{2m_p} + \frac{P_e^2}{2m_e} - \frac{e^2}{r} + H_{HF}$$

where $H_{HF} = A\vec{S}_e \cdot \vec{S}_p$, and A is a real constant.

(a) [1 points] What are the spin quantum numbers s and m_s of the electron?

(b) [1 points] What are the spin quantum numbers s and m_s of the proton?

(c) [1 points] What are the spin quantum numbers s and m_s of the combined electron-proton system?

(d) [5 points] Diagonalize H_{HF} in the total $\vec{S} = \vec{S}_e + \vec{S}_p$ basis and compute the energy eigenvalues.

(e) [2 points] Write an expression for the energy of a photon that would be emitted from a hyperfine transition in terms of A, $\hbar$, and any other relevant constants.

(a)

For the electron,

$$S = 1/2$$
$$m_s = \pm 1/2 .$$

(b)

For the proton,

$$S = 1/2$$
$$m_s = \pm 1/2 .$$

(c)

For the combined system,

$$S = 0, 1$$
$$m_s = 0, \pm 1 .$$

(d)

We know

$$H_{HF} = A\vec{S}_e \cdot \vec{S}_p .$$

Since S_e and S_p are the spin matrices for two fermions, quantized with respect to the z-axis, we have

$$H_{HF} = A\left(S_x^2 + S_y^2 + S_z^2\right)$$

$$= A\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$$

$$= A\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$$

$$= A\begin{pmatrix}3&0\\0&3\end{pmatrix}$$

We want to diagonalize this in the $\vec{S} = \vec{S}_e + \vec{S}_p$ basis. So

$$S = \frac{2\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \frac{2\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \frac{2\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= \begin{pmatrix}0&2\\2&0\end{pmatrix} + \begin{pmatrix}0&-2i\\2i&0\end{pmatrix} + \begin{pmatrix}2&0\\0&-2\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}2&2-2i\\2+2i&-2\end{pmatrix}$$

We want to determine $S^{+}H_{HF}S$. We know

$$S^{+} = \frac{\hbar}{2}\begin{pmatrix}2, & 2-2i\\2+2i, & -2\end{pmatrix}.$$

(d), cont'd...

Then

$$H_{HF}^{S} = \frac{\hbar^2}{4}\begin{pmatrix} 2 & 2-2i \\ 2+2i & -2 \end{pmatrix}\begin{pmatrix} 3A & 0 \\ 0 & 3A \end{pmatrix}\begin{pmatrix} 2 & 2-2i \\ 2+2i & -2 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 2 & 2-2i \\ 2+2i & -2 \end{pmatrix}\begin{pmatrix} 6A & 6A-6Ai \\ 6A+6Ai & -6A \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 12A + (2-2i)(6A+6Ai) & (12A-12Ai) - (12A-12Ai) \\ (12A+12Ai) - (12A+12Ai) & (2+2i)(6A-6Ai) + 12A \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 12A+12A+12A & 0 \\ 0 & 12A+12A+12A \end{pmatrix}$$

$$\boxed{H_{HF}^{S} = \frac{\hbar^2}{4}\begin{pmatrix} 36A & 0 \\ 0 & 36A \end{pmatrix}}$$

The eigenvalues are...

$$\begin{vmatrix} 36A-E & 0 \\ 0 & 36A-E \end{vmatrix} = 0$$

$$(36A-E)(36A-E) = 0$$

$$E = \frac{\pm 36A\hbar^2}{4}$$

$$\boxed{E = \pm 9A\hbar^2}$$