F-2011

PROBLEM 3: Angular Momentum Operators

Consider a state space formed from the direct sum of the two subspaces: $\mathcal{E}$(j=0) spanned by $|j=0, m_y=0\rangle$ and $\mathcal{E}$(j=1) spanned by $|j=1, m_y=1\rangle, |j=1, m_y=0\rangle$, and $|j=1, m_y=-1\rangle$,: i.e.

$$\mathcal{E} = \mathcal{E}(\mathrm{j}=1) \oplus \mathcal{E}(\mathrm{j}=0)$$

where

$$J^2|j, m_y\rangle = j(j+1)\hbar^2|j, m_y\rangle$$
$$J_y|j, m_y\rangle = m_y\hbar|j, m_y\rangle$$

Let

$$|\Psi\rangle = \frac{1}{\sqrt{5}}|j=1, m_y=1\rangle + \frac{\sqrt{3}}{\sqrt{10}}|j=1, m_y=0\rangle - \frac{1}{\sqrt{2}}|j=0, m_y=0\rangle$$

(a) Consider the measurement of the two observables J^2 and J_y. Do these observables commute? Demonstrate explicitly the value of the commutator of J^2 and J_y. **(2 points]**

(b) Determine the probability of measuring J^2 and getting $2\hbar^2$, i.e. determine $P_{|\Psi\rangle}(2\hbar^2 \text{ for } \mathrm{J}^2)$. What is the resulting normalized state vector, $|\Psi'\rangle$ after this measurement? **(2 points]**

(c) If J_y is then measured after the measurement in part (b), what is the probability of obtaining $m_y = 0$, i.e. what is $P_{|\Psi'\rangle}(0 \text{ for } \mathrm{J_y})$? What is the resulting normalized state vector after this measurement? [2 points]

(d) What is the composite probability of measuring J^2 and getting $2\hbar^2$ and then measuring J_y and getting zero, i.e. what is $P_{|\Psi\rangle}(2\hbar^2 \text{ for } \mathrm{J}^2, 0 \text{ for } \mathrm{J_y})$? **(1 point)**

(e) Now starting with the original $|\Psi\rangle$ reverse the measurements, measuring J_y first and getting zero, and then measuring J^2 and getting $2\hbar^2$. Determine four quantities: 1) $P_{|\Psi\rangle}(0 \text{ for } \mathrm{J_y})$; 2) the resulting normalized state $|\Psi''\rangle$; 3) $P_{|\Psi''\rangle}(2\hbar^2 \text{ for } \mathrm{J}^2)$; and 4) the final normalized state after both measurements have been taken. [2 points]

(f) What is the new composite probability when the measurements are reversed, i.e. what is: $P_{|\Psi\rangle}(0 \text{ for } \mathrm{J_y}, 2\hbar^2 \text{ for } \mathrm{J}^2)$? Are your two composite probabilities the same or different? Discuss in detail. [1 point]

F-2011

3 A $\boxed{\text{yes}}$

$J^2 = J_x^2 + J_y^2 + J_z^2$ They commute because J_y helps to define J^2

Also, $[J^2, J_y] \, |j, m_y\rangle$ do this and will see they commute, ket stays the same

B $\frac{1}{5} + \frac{3}{10} + \frac{1}{2} = \frac{2}{10} + \frac{3}{10} + \frac{5}{10} = 1$

$2\hbar^2 = 1(1+1)\hbar^2 \quad , \quad j=1$

$P(j=1) = |\langle j=1, m_y=0|\psi\rangle|^2 + |\langle j=1, m_y=1|\psi\rangle|^2 + |\langle j=1, m_y=-1|\psi\rangle|^2$

$= \frac{3}{10} + \frac{1}{5} + 0$

$\rightarrow \boxed{P(j=1) = \frac{1}{2}}$

$$|\psi'\rangle = \sum_{m_y} |j=1, m_y\rangle\langle j=1, m_y|\psi\rangle$$

$$= \frac{1}{\sqrt{5}}|j=1, m_y=1\rangle + \frac{\sqrt{3}}{\sqrt{10}}|1,0\rangle$$

$$||\psi'\rangle| = \sqrt{\langle\psi'|\psi'\rangle} = \sqrt{\frac{1}{5} + \frac{3}{10}} = \sqrt{\frac{1}{2}}$$

$$\rightarrow \boxed{|\psi'\rangle = \sqrt{\frac{2}{5}}\,|j=1, m_y=1\rangle + \sqrt{\frac{3}{5}}\,|j=1, m_y=0\rangle}$$

(C) $P(m_y = \phi) = \boxed{\frac{3}{5}}$

$|\psi''\rangle = \sqrt{\frac{3}{5}}\,|j=1, m_y=\phi\rangle$

$|\psi''| = \sqrt{\langle\psi''|\psi''\rangle} = \sqrt{\frac{3}{5}}$

$\boxed{|\psi''\rangle = |j=1, m_y=0\rangle}$

(D) $\boxed{P(j=1, m_y=0) = \frac{3}{10}}$

(E) $P(m_y=\phi) = \frac{3}{10} + \frac{1}{2} = \frac{8}{10} = \boxed{\frac{4}{5}}$

$|\psi''\rangle = \frac{\sqrt{3}}{\sqrt{10}}|j=1, m_y=0\rangle - \frac{1}{\sqrt{2}}|j=0, m_y=0\rangle$

$|\psi''\rangle = \sqrt{\frac{3}{10} + \frac{1}{2}} = \sqrt{\frac{4}{5}}$

$\sqrt{\frac{3}{10}}\sqrt{\frac{5}{4}}$

$\frac{3}{8}$

$\rightarrow \boxed{|\psi''\rangle = \sqrt{\frac{3}{8}}\,|j=1, m_y=0\rangle - \sqrt{\frac{5}{8}}\,|j=0, m_y=0\rangle}$

$\sqrt{\frac{5}{4}\frac{1}{2}} = \frac{5}{8}$

$P(j=1) = \boxed{\frac{3}{8}}$

$\boxed{|\psi'\rangle = |j=1, m_y=0\rangle}$

(F) $P(m_4 = 0, s = 1)$ (3)

$= \cancel{\frac{3}{10} \frac{4}{5}} = \cancel{\frac{3}{5} \cdot \frac{2}{5}} = \cancel{\frac{6}{25}}$

$$\frac{3}{8} \cdot \frac{4}{5} = \frac{3}{5} \frac{1}{2} = \boxed{\frac{3}{10}}$$

Probabilities are the same because they commute

S-2009

Problem 4: Angular momentum (10 points)

A $|jm\rangle = |1,0\rangle$ state scatters from a $|jm\rangle = |\frac{1}{2},\frac{1}{2}\rangle$ state via a $|jm\rangle = |\frac{1}{2},\frac{1}{2}\rangle$ resonance.

a) Relate the highest weight (highest possible m) states in the total j basis to the highest weight states in the direct product basis for this system of $\frac{1}{2}\otimes 1$. (1 pt)

b) Acting on the highest weight states with lowering operators, give an expansion of each total-j state in terms of direct product states and their Clebsch-Gordon co-efficients. (5 pts) *Hint:* $J_\pm|jm\rangle = \hbar[(j\mp m)(j\pm m+1)]^{1/2}|j,m\pm 1\rangle$

c) How often do the above-mentioned spin states scatter elastically, and how often do they scatter inelastically? (4 pts)

S-2009

4) $|jm\rangle = |1,0\rangle$ & $|jm\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle$

A) $|j_1 - j_2| \le j \le j_1 + j_2$

$m = m_1 + m_2$

$$|j\,m\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

$$= |j_1 j_2, m_1 m_2\rangle$$

$$\rightarrow \boxed{|\tfrac{3}{2},\tfrac{3}{2}\rangle = |1,\tfrac{1}{2},1,\tfrac{1}{2}\rangle}$$

B) $j = \frac{3}{2}$ or $\frac{1}{2}$ $\rightarrow$ $\frac{3}{2} - \frac{2}{2} = \frac{1}{2} - \frac{2}{2} = -\frac{1}{2} - \frac{2}{2} = -\frac{3}{2}$

$$\rightarrow J_-|\tfrac{3}{2},\tfrac{3}{2}\rangle = J_{-,1}|1\tfrac{1}{2}\,1\tfrac{1}{2}\rangle +$$

$$J_{-,2}|1\tfrac{1}{2}\,1\tfrac{1}{2}\rangle$$

$$\hbar\underbrace{\sqrt{(\tfrac{3}{2}+\tfrac{3}{2})(\tfrac{3}{2}-\tfrac{3}{2}+1)}}_{\frac{6}{2} = \sqrt{3}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle = \underbrace{\sqrt{(1+\underset{0}{1})(1-1+1)}}_{\sqrt{2}}\,|1,0,1,\tfrac{1}{2}\rangle$$

$$+ \underbrace{\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}}_{\sqrt{1}}\,|1\tfrac{1}{2},1,-\tfrac{1}{2}\rangle$$

$$\rightarrow |\tfrac{3}{2},\tfrac{1}{2}\rangle = \frac{\sqrt{2}}{\sqrt{3}}|1\,0\,1\tfrac{1}{2}\rangle + \frac{1}{\sqrt{3}}|1\,\tfrac{1}{2}\,1\,-\tfrac{1}{2}\rangle$$

B (Continued)

$$\boxed{\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|1\,0\,1\,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1\,\tfrac{1}{2}\,1\,-\tfrac{1}{2}\right\rangle}$$

C ?

S-2012

PROBLEM 4: Angular Momentum

The hydrogen atom including hyperfine splitting can be described by a Hamiltonian

$$\mathbf{H} = \frac{\mathbf{P}_p^2}{2m_p} + \frac{\mathbf{P}_e^2}{2m_e} - \frac{e^2}{r} + \mathbf{H}_{HF}$$

where $\mathbf{H}_{HF} = A\vec{S}_p \cdot \vec{S}_e$ describes the spin-spin or hyperfine interaction and the total spin angular momentum is given by $\vec{S} = \vec{S}_p + \vec{S}_e$. The subscripts ($p$ and e) refer to proton and electron, respectively

(a) Write down the form of the spin-spin direct product state vectors. What are the "good", *i.e.* diagonal operators for this set of state vectors? [2 points]

(b) Write down the form of the "total-s" state vectors. What are the "good", *i.e.* diagonal operators for this set of state vectors? [2 points]

(c) Choosing an appropriate set of state vectors, calculate the H_{HF} energy eigenvalues, and the energy splitting due to the hyperfine interaction. [5 points]

(d) If the photon wavelength (λ) is 21 cm from the hyperfine transition, evaluate the constant A in H_{HF}. *Hint:* $\hbar c = 1.97 \times 10^{-5}$ $eV{\cdot}cm$. [1 point]

S-2012

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(4) $$H = \frac{P_p^2}{2M_p} + \frac{P_e^2}{2m_e} - \frac{e^2}{r} + H_{HF}$$

$$H_{HF} = A\,\vec{S}_p \cdot \vec{S}_e$$

$$\vec{S} = \vec{S}_p + \vec{S}_e$$

(A) $\cancel{|S_{z,p}, S_{z,e}\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle \; |\tfrac{1}{2},-\tfrac{1}{2}\rangle \; |-\tfrac{1}{2},\tfrac{1}{2}\rangle \; |-\tfrac{1}{2},-\tfrac{1}{2}\rangle}$

~~or~~

$\cancel{|m_p, m_e\rangle}$

~~good operators would be~~

$\cancel{\vec{S}_+ = S_{p,+} + S_{e,+}}$

$\cancel{\vec{S}_- = S_{p,-} + S_{e,-}}$

Direct product $\rightarrow$ $|S_{z,p}\rangle \otimes |S_{z,e}\rangle$

$(2s+1)$ $\quad$ $(2s+1) = 4$

$\rightarrow$ $\boxed{\text{basis should be } |S_p, S_{p,z}, S_e, S_{e,z}\rangle}$

$[S^2, S_z] = \phi$

$+ [S_i^2, S_{j}] = \phi$

from Different spaces

$\boxed{\text{"good" operators are } S_p^2, S_e^2, S_{z,p}, S_{z,e}}$

B Direct sum

$$|S=1\rangle \oplus |S=\emptyset\rangle$$

Total S   Total S

Triplet   Singlet $\rightarrow$ 4

$$\boxed{|S_p, S_e, S, S_z\rangle}$$

$S_p + S_e$ $S_{z,p} + S_{z,e}$

$$\boxed{\text{good operators are } \quad S_p^2, S_e^2, S^2, S_z}$$

C ~~$H_{HF} = A S_p \cdot S_e = A[S_{px}S_{ex} + S_{py}S_{ey} + S_{pz}S_{ez}]$~~

~~$= A\left[\frac{1}{2}(S_{p+}S_{e-} + S_{p-}S_{e+}) + S_{pz}S_{ez}\right]$~~

~~choice is $\{S_p, S_{pz}, S_e, S_{e,z}\}$~~

~~$H_{HF}|m_p, m_e\rangle \rightarrow S_{p+}S_{e-}|m_p, m_e\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar$~~

~~$\sqrt{\left(\frac{1}{2} - m_p\right)\left(\frac{3}{2} + m_p\right)}\sqrt{\left(\frac{1}{2} + m_e\right)\left(\frac{3}{2} - m_e\right)}|m_p+1, m_e-1\rangle$~~

~~$S_e - S_p$~~

$$H_{HF} = A\vec{S_p} \cdot \vec{S_e} \rightarrow S = S_p + S_e \rightarrow S^2 = S_p^2 + S_e^2 + 2S_p \cdot S_e$$

$$\rightarrow S_p \cdot S_e = \frac{1}{2}\left[S^2 - S_p^2 - S_e^2\right]$$

(3)

good basis is $|S^2, S_z, S_p^2, S_e\rangle$

$\rightarrow S^2|S, S_z\, S_p\, S_e\rangle = S(S+1)\hbar^2|S, S_z, S_p, S_e\rangle$

$\rightarrow S_p^2|S, S_z\, S_p\, S_e\rangle = \frac{1}{2}\frac{3}{2}\hbar^2|S, S_z, S_p, S_e\rangle$ $\Big\rangle$ $\frac{6}{4} = \frac{3}{2}$

$\rightarrow S_e^2|S, S_z, S_p, S_z\rangle = \frac{3}{4}\hbar^2|S, S_z, S_p, S_e\rangle$

$$H_{HF} \doteq \begin{array}{cccc} |11\rangle & |10\rangle & |1-1\rangle & |00\rangle \end{array}$$

$$\begin{pmatrix} \frac{1}{4}A\hbar^2 & 0 & 0 & 0 \\ 0 & \frac{1}{4}A\hbar^2 & 0 & 0 \\ 0 & 0 & \frac{1}{4}A\hbar^2 & 0 \\ 0 & 0 & 0 & -\frac{3}{4}A\hbar^2 \end{pmatrix}$$

$S=1 \rightarrow S^2 \rightarrow 2\hbar^2$

$S=\emptyset \rightarrow S^2 \rightarrow 0$

$\frac{4}{2} - \frac{3}{2} =$

$$\boxed{\begin{array}{c} \text{eigenvalues} \\ \hline -\frac{1}{4}A\hbar^2 \\ -\frac{3}{4}A\hbar^2 \end{array}}$$

Splitting is $-\frac{13.6\,eV}{n^2} +$

(D) $E = \hbar\omega$

$= \frac{\hbar c}{\lambda}$

$\Rightarrow \frac{1}{4} + \frac{3}{4}A\hbar^2 = E = \frac{\hbar c}{\lambda}$

$\rightarrow A\hbar^2 = \frac{\hbar c}{\lambda} \rightarrow \boxed{A = \frac{\hbar c}{\hbar^2 \lambda}}$

S-2015

Problem 2: Angular Momentum States

Consider the electron in a hydrogen atom in the presence of a homogeneous magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. In this problem, ignore the electron spin and only consider the orbital angular momentum. The Hamiltonian of the system is

$$\mathcal{H} = \mathcal{H}_0 - \omega L_z, \tag{1}$$

where $\mathcal{H}_0$ is the Hamiltonian for the hydrogen atom, $\omega \equiv |e|B/2m_e c$, and L_z is the angular momentum operator along the z direction. The eigenstates $|n, \ell, m\rangle$ and eigenvalues $E_n^{(0)}$ of the unperturbed hydrogen atom are to be considered as known. Assume that initially (at $t = 0$) the system is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|2,1,-1\rangle - |2,1,1\rangle\right). \tag{2}$$

(a) [1 pt] Write down the time-dependent state for this atom, $|\psi(t)\rangle$, given the initial state and the full Hamiltonian.

(b) [2 pts] Calculate the probability of finding the atom at some later time $t > 0$ in the state

$$|2p_y\rangle = \frac{1}{\sqrt{2}}\left(|2,1,-1\rangle + |2,1,1\rangle\right). \tag{3}$$

When is the probability equal to 1?

(c) [3 pts] Define the state $|\mathbf{e}_\phi\rangle$ defined by

$$(\mathbf{e}_\phi \cdot \mathbf{L})\,|\mathbf{e}_\phi\rangle = \hbar|\mathbf{e}_\phi\rangle, \qquad \mathbf{L}^2|\mathbf{e}_\phi\rangle = 2\hbar^2|\mathbf{e}_\phi\rangle. \tag{4}$$

$\mathbf{e}_\phi$ is a unit vector in the $x-y$ plane, $\mathbf{e}_\phi = \cos(\phi)\mathbf{e_x} + \sin(\phi)\mathbf{e_y}$.

This state has quantum number $\ell = 1$ and angular momentum projection along the direction $\mathbf{e}_\phi$ equal to $+\hbar$. Solve for the state $|\mathbf{e}_\phi\rangle$ in the basis of states $|2,1,m\rangle$, with $m = \pm 1, 0$.

(d) [2 pts] Calculate the time-dependent probability of finding the system in the state $|\mathbf{e}_\phi\rangle$, if it starts in the state $|\psi(0)\rangle$ above, and show that this is a periodic function of time. Calculate the times when the probability is maximum and minimum.

(e) [2 pts] If the electron starts in the state $|\psi(0)\rangle$, calculate the expectation value of the magnetic dipole

$$\langle\vec{\mu}\rangle(t) = \frac{e}{2m_e c}\langle\mathbf{L}\rangle(t), \quad \mathbf{L} = L_x\mathbf{e_x} + L_y\mathbf{e_y} + L_z\mathbf{e_z} \tag{5}$$

as a function of time.

Hint: It will be useful to use:

$$\begin{aligned} J_\pm &\doteq J_x \pm iJ_y \\ J_\pm|j,m\rangle &= \hbar\sqrt{j(j+1) - m(m\pm 1)}\;|j, m\pm 1\rangle \end{aligned} \tag{6}$$

S-2015

①

② $\vec{B} = B\hat{z}$

$H = H_0 - \omega L_z$

$|n,l,m\rangle$ & $E_n^{(0)}$ are known

$\underline{t=0} \Rightarrow \quad |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|2,1,-1\rangle - |2,1,1\rangle)$

(A) $U|\psi(0)\rangle = \exp\left(\frac{-iHt}{\hbar}\right)\left[\frac{1}{\sqrt{2}}(|2,1,-1\rangle - |2,1,1\rangle)\right]$

$H|2,1,-1\rangle = E_2^{(0)} + \omega\hbar$

$H|2,1,1\rangle = E_2^{(0)} - \omega\hbar$

$$\rightarrow \boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i(E_2^{(0)}+\omega\hbar)t/\hbar}|2,1,-1\rangle - e^{-i(E_2^{(0)}-\omega\hbar)t/\hbar}|2,1,1\rangle\right]}$$

(B) $P = |\langle 2p_y|\psi(t)\rangle|^2 = \left|\frac{1}{2}e^{-iE_2^{(0)}t/\hbar}\underbrace{\left[e^{-i\omega t} - e^{i\omega t}\right]}_{2\sin\omega t}\right|^2$

$$\boxed{\sin^2\omega t = P}$$

$\sin = 1$ when $\boxed{\omega t = \frac{\pi}{2} = P = 1}$

(c) $\hat{e}_\phi = \cos\phi\, \hat{e}_x + \sin\phi\, \hat{e}_y$

$(\hat{e}_\phi \cdot \vec{L}) | \hat{e}_\phi \rangle = \hbar | \hat{e}_\phi \rangle \qquad \vec{L}^2 | \hat{e}_\phi \rangle = 2\hbar^2 | \hat{e}_\phi \rangle$

$\ell = \phi$

$\cos\phi\, \vec{L}_x\, \hat{e}_x | \hat{e}_\phi \rangle = \hbar | \hat{e}_\phi \rangle$ Solve For $|e_\phi\rangle$

$\sin\phi\, \vec{L}_y\, \hat{e}_y | \hat{e}_\phi \rangle = \hbar | \hat{e}_\phi \rangle$

<u>change of basis</u> $|b\rangle = U|a\rangle$

$\langle a|U|a\rangle = \langle a|b\rangle$

$$U \doteq \begin{array}{c} \\ \langle 2,1,1| \\ \langle 2,1,0| \\ \langle 2,1,-1| \end{array} \begin{array}{c} |\hat{e}_\phi\rangle \\ \left(\begin{array}{c} \\ \\ \\ \end{array} \right) \end{array}$$

$\hat{e}_\phi \cdot \vec{L} = \cos\phi\, \vec{L}_x + \sin\phi\, \vec{L}_y$

$\cancel{L^2 = L_x^2 + L_y^2 + L_z^2 = J_+ J_- + J_z^2 - \hbar J_z}$

$\vec{L}_x = \frac{L_+ + L_-}{2}$

$\vec{L}_y = -\frac{1}{2i}(J_+ - J_-)$

$J_+ - J_-$

$= \cancel{J_x} + iJ_y - \cancel{J_x} + iJ_y$

$-2iJ_y$

$\hat{e}_\phi \cdot \vec{L} = \left[\frac{\cos\phi}{2}(L_+ + L_-) + \frac{i\sin\phi}{2}(J_+ - J_-) \right] |e_\phi\rangle = \hbar |e_\phi\rangle$

$= \left[\frac{\cos\phi}{2} + \frac{i\sin\phi}{2} \right] J_+ + \left[\frac{\cos\phi}{2} - \frac{i\sin\phi}{2} \right] J_-$ $|e_\phi\rangle = \hbar |e_\phi\rangle$

$\left[\frac{e^{i\phi}}{2} J_+ + \frac{e^{-i\phi}}{2} J_- \right] |e_\phi\rangle = \hbar |e_\phi\rangle$

c) continued

$$L_+ \doteq \begin{array}{c|ccc} & |1\rangle & |0\rangle & |-1\rangle \\ \hline |1\rangle & 0 & \sqrt{2}\hbar & 0 \\ |0\rangle & 0 & 0 & \sqrt{2}\hbar \\ |-1\rangle & 0 & 0 & 0 \end{array}$$

$\hbar\sqrt{1(1+1)-0}$

$\hbar\sqrt{2+1(-1+1)}$

$$L_- \doteq \begin{array}{c|ccc} & |1\rangle & |0\rangle & |-1\rangle \\ \hline |1\rangle & 0 & 0 & 0 \\ |0\rangle & \sqrt{2}\hbar & 0 & 0 \\ |-1\rangle & 0 & \sqrt{2}\hbar & 0 \end{array}$$

$\hbar\sqrt{2-1(1-1)}$

$$\to \hat{e}_\phi \cdot \vec{L} = \begin{pmatrix} -\hbar & \frac{\hbar}{\sqrt{2}}e^{i\phi} & 0 \\ \frac{\hbar}{\sqrt{2}}e^{-i\phi} & -\hbar & \frac{\hbar}{\sqrt{2}}e^{i\phi} \\ 0 & \frac{\hbar}{\sqrt{2}}e^{-i\phi} & -\hbar \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \emptyset$$

$$\Rightarrow -A + \frac{1}{\sqrt{2}}e^{i\phi}B = \emptyset \to A = \frac{1}{\sqrt{2}}e^{i\phi}B$$

$$\frac{1}{\sqrt{2}}e^{-i\phi}A - B + \frac{1}{\sqrt{2}}e^{i\phi}C = \emptyset$$

$$\frac{1}{2}B - B + \frac{1}{\sqrt{2}}e^{i\phi}C = \emptyset$$

$$B\left(-\frac{1}{2}\right) + \frac{1}{\sqrt{2}}e^{i\phi}C = \emptyset$$

$$B = \sqrt{2}e^{i\phi}C = B$$

$$C = \frac{e^{-i\phi}}{\sqrt{2}}B$$

$$\to |\hat{e}_\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{i\phi}B \\ B \\ \frac{e^{-i\phi}}{\sqrt{2}}B \end{pmatrix}$$

C) Continued

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$$\langle \hat{e}_\phi | \hat{e}_\phi \rangle = \left(\frac{1}{\sqrt{2}} e^{-i\phi} B \quad B \quad \frac{e^{i\phi}}{\sqrt{2}} B \right) \begin{pmatrix} \frac{1}{\sqrt{2}} e^{i\phi} B \\ B \\ \frac{1}{\sqrt{2}} e^{-i\phi} B \end{pmatrix} = 1$$

This is True since I Calculated $L_\pm$ in $|2,1,\pm m\rangle$ basis

$$\frac{1}{2} B^2 + B^2 + \frac{1}{2} B^2 = 1 \rightarrow 2B^2 = 1$$

$$B = \frac{1}{\sqrt{2}}$$

$$\rightarrow \boxed{|\hat{e}_\phi\rangle = \frac{i}{2} e^{i\phi} |2,1,1\rangle + \frac{1}{\sqrt{2}} |2,1,0\rangle + \frac{1}{2} e^{-i\phi} |2,1,-1\rangle}$$

D) $$|\langle e_\phi | \psi(t)\rangle|^2 = \left[\frac{1}{2\sqrt{2}} e^{-iE_2^{(\phi)} t/\hbar} \left(-e^{i\omega t} e^{-i\phi} + e^{-i\omega t} e^{i\phi} \right) \right]^2$$

$$\frac{1}{8} \left(e^{i\omega t} e^{-i\phi} - e^{-i\omega t} e^{i\phi} \right) \left(e^{-i\omega t} e^{+i\phi} - e^{i\omega t} e^{-i\phi} \right)$$

$$\frac{1}{8} \left(1 - 2 e^{2i\omega t} e^{-2i\phi} \right)$$

$$\boxed{\frac{1}{8} - \frac{1}{4} e^{2i(\omega t - \phi)} = P}$$

E) Just plug & Chug $\rightarrow$ $\boxed{\langle \mu \rangle(t) = \frac{e}{2m_e c} \langle L_x \rangle + \langle L_y \rangle + \langle L_z \rangle}$

F-2014

PROBLEM 2: Oscillator Model of Angular Momentum

Arbitrary angular momentum can be constructed from spin-1/2. The latter can be described in terms of the Pauli matrices

$$\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}.$$

The construction of a general angular momentum can be done by introducing two sets of independent harmonic oscillators, in terms of creation ($a_\zeta^\dagger$) and annihilation (a_ζ) operators,

$$[a_+, a_-] = 0, \quad [a_+^\dagger, a_-^\dagger] = 0, \quad [a_\zeta, a_{\zeta'}^\dagger] = \delta_{\zeta,\zeta'},$$

with $\zeta, \zeta' = \pm$ indexing oscillators of type $\pm$. Now define

$$\mathbf{J} = \frac{\hbar}{2} a^\dagger \boldsymbol{\sigma} a,$$

where a is a two component operator,

$$a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}.$$

a) Given the form of the Pauli matrices, give the explicit form for J_x, J_y, J_z in terms of $a_\zeta^\dagger$ and a_ζ operators (2 Points).

b) Show that $J_\pm = J_x \pm iJ_y$ have particularly simple forms in terms of a_ζ and $a_\zeta^\dagger$ operators (1 Point).

c) Compute the commutator $[J_x, J_y]$. How is this generalized for the other components? (2 Points)

d) Show that

$$J^2 = J_z^2 + J_+J_- + i[J_x, J_y],$$

and then write this in terms of the number operators for the two harmonic oscillators,

$$n_+ = a_+^\dagger a_+, \quad n_- = a_-^\dagger a_-.$$

Show that this implies that the eigenvalues of J^2 are $j(j+1)\hbar^2$, where j is an integer or an integer plus $\frac{1}{2}$ (Hint: apply the J^2 operator in the two harmonic oscillator state $|n_+, n_-\rangle$) (3 Points).

e) Using the properties of the harmonic oscillators, show that the state in which J^2 has the eigenvalue $j(j+1)\hbar$ and $J_z = m\hbar$ can be constructed from the state in which both n_+ and n_- have the value zero, $|0\rangle$, by

$$|jm\rangle = \frac{(a_+^\dagger)^{j+m}}{\sqrt{(j+m)!}} \frac{(a_-^\dagger)^{j-m}}{\sqrt{(j-m)!}} |0\rangle.$$

(2 Points)

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2 $\tilde{S} = \frac{\hbar}{2}\tilde{\sigma}$

$$[a_+, a_-] = \phi, \quad [a_+^\dagger, a_-^\dagger] = \phi, \quad [a_\xi, a_{\xi'}^\dagger] = \delta_{\xi,\xi'}$$

$$\rightarrow \tilde{J} = \frac{\hbar}{2} a^\dagger \tilde{\sigma} a, \qquad a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

assume elements ~~are~~ may be imaginary

A $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\tilde{J}_x = \frac{\hbar}{2} \overbrace{(a_+^\dagger \;\; a_-^\dagger)}^{a^\dagger} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$= \frac{\hbar}{2} (a_+^\dagger \;\; a_-^\dagger) \begin{pmatrix} a_- \\ a_+ \end{pmatrix} = \boxed{\frac{\hbar}{2}[a_+^\dagger a_- + a_-^\dagger a_+] = \tilde{J}_x}$$

$$a_+^\dagger a_- - a_- a_+^\dagger = \phi$$

$$= \frac{\hbar}{2}[a_- a_+^\dagger + a_-^\dagger a_+]$$

IF $a_\pm$ are real then $a_\pm^\dagger = a_\pm$

$$\tilde{J}_y = \frac{\hbar}{2} (a_+^\dagger \;\; a_-^\dagger) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$= \frac{\hbar}{2} (a_+^\dagger \;\; a_-^\dagger) \begin{pmatrix} -ia_- \\ ia_+ \end{pmatrix} = \frac{\hbar}{2}[-i a_+^\dagger a_- + i a_-^\dagger a_+]$$

$$= \boxed{\frac{i\hbar}{2}[a_-^\dagger a_+ - a_+^\dagger a_-] = \tilde{J}_y}$$

$$\tilde{J}_z = \frac{\hbar}{2} (a_+^\dagger \;\; a_-^\dagger) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \frac{\hbar}{2} (a_+^\dagger \;\; a_-^\dagger) \begin{pmatrix} a_+ \\ -a_- \end{pmatrix}$$

$$= \boxed{\frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) = \tilde{J}_z}$$

(B) $\tilde{J}_+ = \hat{J}_x + i\tilde{J}_y$

$$= \frac{\hbar}{2}\left[a_+^\dagger a_- + \cancel{a_-^\dagger a_+ - a_-^\dagger a_+} + a_+^\dagger a_-\right]$$

$$\boxed{\tilde{J}_+ = \hbar a_+^\dagger a_-}$$

$\hat{J}_- = \hat{J}_x - i\tilde{J}_y$

$$= \frac{\hbar}{2}\left[\cancel{a_+^\dagger a_-} + a_-^\dagger a_+ + a_-^\dagger a_+ - \cancel{a_+^\dagger a_-}\right]$$

$$\boxed{\tilde{J}_- = \hbar a_-^\dagger a_+}$$

(C) $[\tilde{J}_x, \hat{J}_y] = \underbrace{i\hbar \hat{J}_z}$

This should be the answer

$$\rightarrow \frac{\hbar^2}{2^2}\left[a_+^\dagger a_- + a_-^\dagger a_+, i(a_-^\dagger a_+ - a_+^\dagger a_-)\right]$$

$$= \frac{\hbar}{2^2}\Big([a_+^\dagger a_-, i a_-^\dagger a_+] - [a_+^\dagger a_-, i a_+^\dagger a_-] + [a_-^\dagger a_+, i a_-^\dagger a_+] - [a_-^\dagger a_+, i a_+^\dagger a_-]\Big)$$

$$a_+^\dagger \underbrace{[a_-, a_-^\dagger]}_{1} i a_+ + a_- \underbrace{[a_+^\dagger, a_+]}_{-1} a_-^\dagger i = i[a_+^\dagger a_+ - a_- a_-^\dagger]$$

$$a_+^\dagger \underbrace{[a_-, a_+^\dagger]}_{\phi} a_- i + a_- \underbrace{[a_+^\dagger, a_-]}_{\phi} a_+^\dagger i$$

$$a_-^\dagger \underbrace{[a_+, a_-^\dagger]}_{\phi} i a_+ + a_+ \underbrace{[a_-^\dagger, a_+]}_{\phi} a_-^\dagger i$$

$$a_-^\dagger \underbrace{[a_+, a_+^\dagger]}_{1} a_- i + a_+ \underbrace{[a_-^\dagger, a_-]}_{-1} a_+^\dagger i = i(a_-^\dagger a_- - a_+ a_+^\dagger)$$

D $\frac{\hbar}{2}\, 2\, i\, \underbrace{\frac{\hbar}{2}\left(-a_-^\dagger a_- + a_+ a_+^\dagger\right)}_{\tilde{J}_z}$

$$\boxed{\left[\tilde{J}_x, \tilde{J}_y\right] = i\hbar \tilde{J}_z}$$

$$\boxed{\left[\tilde{J}_i, \tilde{J}_j\right] = i\hbar J_k \epsilon_{ijk}}$$

E $\tilde{J}^2 = \tilde{J}_x^2 + \tilde{J}_y^2 + \tilde{J}_z^2$

$$= \frac{\hbar^2}{4}\Big[a_+^\dagger a_- a_+^\dagger a_- + \overbrace{\left(a_+^\dagger a_- a_-^\dagger a_+ + a_-^\dagger a_+ a_+^\dagger a_-\right)}^{2}$$

$$+ \cancel{a_-^\dagger a_+ a_-^\dagger a_+} - \cancel{a_-^\dagger a_+ a_-^\dagger a_+} + a_-^\dagger a_+ a_+^\dagger a_-$$

$$+ a_+^\dagger a_- a_-^\dagger a_+ - a_+^\dagger a_- a_+^\dagger a_-$$

$$2$$

$$+ a_+^\dagger a_+ a_+^\dagger a_+ - a_+^\dagger a_+ a_-^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_+$$

$$+ a_-^\dagger a_- a_-^\dagger a_-\Big]$$

$$= \frac{\hbar^2}{4}\Big[a_+^\dagger a_- a_+^\dagger a_- + 2a_+^\dagger a_- a_-^\dagger a_+ + 2a_-^\dagger a_+ a_+^\dagger a_-$$

$$-2a_+^\dagger a_+ a_-^\dagger a_- - a_+^\dagger a_- a_+^\dagger a_- + a_+^\dagger a_+ a_+^\dagger a_+$$

$$+ a_-^\dagger a_- a_-^\dagger a_-$$

$\rightarrow J_z^2$

$$= J_z^2 + \frac{\hbar^2}{4}\Big[\underbrace{2a_+^\dagger a_- a_-^\dagger a_+ + 2a_-^\dagger a_+ a_+^\dagger a_-}_{4J_+J_-} + \underbrace{a_+^\dagger a_- a_+^\dagger a_- - a_+^\dagger a_- a_+^\dagger a_-}_{i[J_x, J_y]}\Big]$$

$\rightarrow J_+ J_- = \hbar^2 a_+^\dagger a_- a_-^\dagger a_+$

$$\boxed{\tilde{J}^2 = J_z^2 + J_+J_- + i[J_x, J_y]}$$

P Continued

$$\tilde{J}_z = \frac{\hbar}{2}[n_+ - n_-]$$

$$\tilde{J}_z^2 = \frac{\hbar^2}{4}[n_+n_+ - n_+n_- - n_-n_+ + n_-n_-]|n_+,n_-\rangle =$$

$$J_+J_- = \hbar^2[n_+n_-]$$

$$i[J_x,J_y] = -\hbar J_z = -\frac{\hbar^2}{2}[n_+ - n_-]$$

$$\to \tilde{J}_z^2 = \frac{\hbar^2}{4}[n_+^2 + n_-^2 - 2n_+n_-]$$

$$J^2 = \underbrace{\frac{\hbar^2}{4}n_+^2 + \frac{\hbar^2}{4}n_-^2 - \frac{\hbar^2}{2}n_+n_- + \hbar^2 n_+n_-}_{\frac{\hbar^2}{4}n_+^2 + \frac{\hbar^2}{4}n_-^2 + \frac{\hbar^2}{2}n_+n_-} + \underbrace{\frac{\hbar^2}{2}n_+ + \frac{\hbar^2}{2}n_-}_{\frac{\hbar^2}{2}N}$$

~~Mistake in math somewhere~~

~~n_+1~~

$$\underbrace{\frac{\hbar^2}{4}n_+^2 + \frac{\hbar^2}{4}n_-^2 + \frac{\hbar^2}{2}n_+n_-}_{\frac{\hbar^2}{4}(\underbrace{n_+ + n_-}_{N})^2 = \frac{\hbar^2}{4}N^2}$$

$$\boxed{\Rightarrow J^2 = \frac{\hbar^2N^2}{4} + \frac{\hbar^2}{2}N = \underbrace{\hbar^2\frac{N}{2}\left(\frac{N}{2}+1\right)}_{\hbar^2 J(J+1)}}$$

where N can be half integer

e Do This