F-2010

PROBLEM 1: Motion of a Particle in One Dimension

Consider a particle of mass m moving along the $+x$ direction in free space.

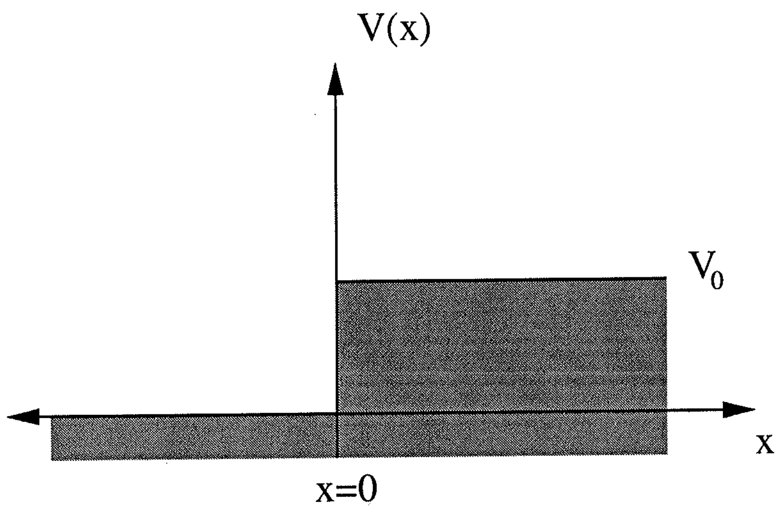
(a) [2 points] Suppose the particle is in a momentum eigenstate where the particles momentum is known precisely to be p_0. Write a wavefunction $\Psi(x,t)$ that describes such a state.

(b) [2 points] Suppose the particle is in a state where it is equally probable for the particle to have any momentum between $p_0 - \Delta p/2$ and $p_0 + \Delta p/2$ at time $t = 0$. Construct a wavefunction $\Psi(x,t)$ that describes such a state.

(c) [2 points] Suppose a beam of particles, each in the state described in part (a), encounters an abrupt step in potential energy at $x = 0$. The step height V_0 is less than the particles total energy E. Construct the wavefunction, $\Psi(x,t)$ with $-\infty \leq x \leq \infty$, that describes this situation.

(d) [2 points] Calculate the probability that a particle is reflected by the potential energy step described in part (c).

(e) [2 points] Consider the situation described in part (c), except with V_0 greater than E. Compare the probability of finding a particle at a distance x inside the barrier to the probability of finding a particle at $x = 0$.



F-2010

① (A) $P_0 = \hbar k_0$

$$\Rightarrow \boxed{\psi(x,t) = A e^{i(k_0 x - \omega t)}}$$

(B) $P_0 - \dfrac{\Delta P}{2} = \hbar k_1$ $\qquad$ $P_0 + \dfrac{\Delta P}{2} = \hbar k_2$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{k_1}^{k_2} \psi(x,0)\, e^{-ikx}\, dk$$

$$\psi(x,0) = A e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{k_1}^{k_2} A\, dk = \frac{A}{\sqrt{2\pi}} (k_2 - k_1)$$

$$= \frac{1}{\hbar}\left[\cancel{P_0} + \frac{\Delta P}{2} - \cancel{P_0} + \frac{\Delta P}{2}\right]$$

$$= \frac{\Delta P}{\hbar} \frac{A}{\sqrt{2\pi}}$$

$$\Rightarrow \psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \omega t)}\, dx$$

$$= \frac{A}{2\pi\hbar} \Delta P\, e^{-i\omega t} \underbrace{\int_{-\infty}^{\infty} e^{ikx}\, dx}_{2\pi \delta(k) = 2\pi \delta\left(\frac{P}{\hbar}\right)}$$

C $V_0 < E$ where $p_0 = \hbar k$

$$\rightarrow \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

$$\rightarrow \frac{d^2\psi}{dx^2} = \underbrace{\frac{-2m}{\hbar^2}(E-V)}_{k_2^2}\psi$$

$$= -k_2^2\psi$$

$$\rightarrow \psi(x,t) = \cancel{A}\, Be^{i(k_2 x - \omega t)} + Ce^{-i(k_2 x - \omega t)}$$

I think the particle has constant energy so $e^{i\omega t}$ stays the same

$$\boxed{\begin{aligned}\psi(x,t) = {} & Ae^{i(k_0 x - \omega t)} && x < 0 \\ & Be^{i(k_2 x - \omega t)} + Ce^{-i(k_2 x - \omega t)} && x > 0\end{aligned}}$$

D $\frac{|B|^2}{|A|^2}$

E ~~probability~~ ?

S-2007

PROBLEM 5

A particle of mass m is confined to a two-dimensional plane. The potential energy of the particle is

$$V(\rho) = \begin{cases} 0 & \rho < \rho_o \\ \infty & \rho \geq \rho_o, \end{cases}$$

where ρ is the radial coordinate of plane polar coordinate (ρ, φ). This potential confines the particle to the region of space $\rho \leq \rho_o$. The particle in this "circular square well" is the quantum analog of a marble on the head of a drum. The stationary-state Hamiltonian eigenfunctions of the particle are $\Psi_{n,m_\ell}(\rho, \varphi)$ with eigenenergies E.

a) [4pts] Write down a second-order differential equation for the radial function $R_{n,m_\ell}(\rho)$ in the bound-state Hamiltonian eigen functions

$$\psi_{n,m_\ell}(\rho, \varphi) = R_{n,m_\ell}(\rho)\Phi_{m_\ell}(\varphi),$$

where $\Phi_{m_\ell}(\varphi)$ is an eigenfunction of the orbital angular momentum operator $\hat{L} = -i\hbar\partial/\partial\varphi$. Write down and justify the boundary conditions that physically admissible solutions to your differential equation must satisfy, and write down the normalization integral for the radial functions.

b) [2pts] What, if anything, can you conclude from your differential equation about the degree of degeneracy of the bound-state energies E_{n,m_ℓ}. Justify your answer.

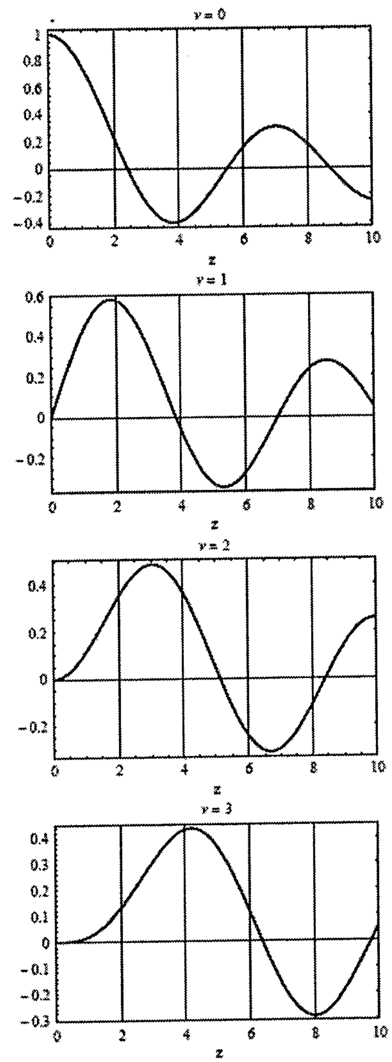
c) [2pts] Derive an equation for the bound-state energies E_{n,m_ℓ} in terms of the zeros $\varsigma_{n,\nu}$ of the cylindrical Bessel function of the first kind, $J_{\pm\nu}(z)$. (See the hint below.)

d) [2pts] Estimate the energies of the *lowest three* bound states of the cylindrical square well. Express your answers in terms of fundamental constants, the mass m, and the well radius ρ_o.

Hint: The cylindrical Bessel functions are solutions of Bessel's equation

$$\left[z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + (z^2 - \nu^2)\right] J_{\pm\nu}(z) = 0$$

The so-called cylindrical Bessel functions of the first kind, $J_{\pm\nu}(z)$, are regular at the origin and normalizable. These functions oscillate with increasing z and have an infinite number of *nodes*, i.e., values for which $z = \varsigma_{n,\nu} > 0$ at which $J_{\pm\nu}(z) = 0$; these nodes are indexed by $n = 1, 2, \ldots$ The figure shows the first four cylindrical Bessel functions.



First four cylindrical Bessel functions of the first kind (for use in problem 5.)

S-2007 ①

(5) $V(\rho) = \begin{cases} 0 & \rho < \rho_0 \\ \infty & \rho \geq \rho_0 \end{cases}$

$\psi_{n,m_\ell}(\rho, \phi)$, E

(A) $\psi_{n,m_\ell}(\rho,\phi) = R_{n,m_\ell}(\rho)\,\Phi_{m_\ell}(\phi)$

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}$$

$$\frac{p^2}{2m}\psi + V\psi = E\psi$$

$$p = -i\hbar\nabla \rightarrow p^2 = -\hbar^2\nabla^2$$

$$\rightarrow \frac{-\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}\right] = E\psi$$

$$\frac{-\hbar^2}{2m}\left[\Phi\frac{\partial^2 R}{\partial\rho^2} + \frac{\Phi}{\rho}\frac{\partial R}{\partial\rho} + \frac{R}{\rho^2}\underbrace{\frac{\partial^2\Phi}{\partial\phi^2}}_{-\frac{L^2\Phi}{\hbar^2} = m^2\Phi}\right] = E R\Phi$$

$$\rightarrow \boxed{\frac{-\hbar^2}{2m}\left[\frac{1}{R}\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho R}\frac{\partial R}{\partial\rho} - \frac{1}{\rho^2}m^2\right] = E}$$

$R(\rho \geq \rho_0) \rightarrow 0$

$R(\rho \geq \rho_0)|_{\rho_0} = R(\rho < \rho_0)|_{\rho_0}$

& derivatives

A continued

$$\boxed{1 = A^2 \int_0^{\rho_0} R^2\, r\, dr}$$

B will be degenerate (TWO-FOLD) m^2

C

$$\frac{1}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho R}\frac{\partial R}{\partial \rho} - \frac{m^2}{\rho^2} = \frac{-2mE}{\hbar^2}$$

$$\Rightarrow \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + R\left(\frac{-m^2}{\rho^2} + \frac{2mE}{\hbar^2}\right) = \phi$$

$$\Rightarrow \rho^2 \frac{\partial^2 R}{\partial \rho^2} + \rho\frac{\partial R}{\partial \rho} + \left(\rho^2 \underbrace{\left(\frac{2mE}{\hbar^2}\right)}_{k^2} - m^2\right) = \phi$$

$\cancel{z = k\rho}$ $\qquad z = k\rho$ $\qquad \frac{d}{d\rho} = \frac{dz}{d\rho}\frac{d}{dz} = k\frac{d}{dz}$

$\cancel{\frac{dz}{d\rho} = k \rightarrow \frac{d^2 z}{d\rho^2}}$

$\cancel{\frac{z^2}{k^2}\frac{\partial^2 R}{\partial z^2} + \frac{z}{k}\frac{\partial R}{\partial \rho} + z^2}$ $\qquad \frac{d^2}{d\rho^2} = \frac{d^2 z}{d\rho^2}\frac{d^2}{dz^2} = k^2\frac{d^2}{dz^2}$

$$\Rightarrow z^2 \frac{\partial^2 R}{\partial \rho^2} + z\frac{\partial R}{\partial \rho} + (z^2 - m^2)R = \phi$$

R or J is ϕ at $\rho = \rho_0$

$$\xi_{n,v} = k\rho_0 = \sqrt{\frac{2mE}{\hbar^2}}\,\rho_0$$

$$\Rightarrow \boxed{\frac{\xi^2}{\rho_0^2}\frac{\hbar^2}{2m} = E_{n,m,l}}$$

D $\xi = K\rho_0 \approx 2.5,\ 4,\ 5$ (where J is ϕ)

z

$$\boxed{\begin{aligned} E_{n,0} &= \frac{2.5\,\hbar^2}{2\rho_0^2 m} \\ E_{n,1} &= \frac{2\hbar^2}{\rho_0^2 m} \\ E_{n,3} &= \frac{5}{2}\frac{\hbar^2}{\rho_0^2 m} \end{aligned}}$$

F-2013

Problem 3: Barrier Scattering

Consider a particle of mass m in one dimension scattering off of a square barrier of width L:

$$\begin{aligned} V(x) &= 0, \quad x < 0 \\ V(x) &= V, \quad 0 < x < L, \quad V > 0 \\ V(x) &= 0, \quad x > L \end{aligned} \tag{1}$$

Assume the particle has an energy $E > V$ and is incoming from the left ($x < 0$).

Define the usual wavenumbers for this problem:

$$\frac{\hbar^2 k^2}{2m} = E, \quad \frac{\hbar^2 k'^2}{2m} = E - V \tag{2}$$

(a) [1 pt] Write down general expressions for the scattering wave function, the un-normalized eigenfunction of the scattering Hamiltonian, in the three regions, $x < 0$, $0 < x < L$, and $x > L$.

(b) [1 pt] Using the expressions from part (a), write down the boundary conditions on the scattering wave function. Explain the physics of each of these boundary conditions.

(c) [2 pt] Using your boundary conditions from part (b), show that

$$\frac{A}{E} = e^{ikL}\left(\cos k'L) - i\frac{k^2 + k'^2}{2kk'}\sin(k'L)\right) \tag{3}$$

where A is the amplitude of the incoming wave (from $x = -\infty$) and E is the amplitude of the outgoing wave (to $x = \infty$). Hint: We're not interested in the amplitude of the reflected wave.

(d) [3 pt] Solve for the transmission coefficient, T, for the barrier scattering. You may express this in terms of k, k', and L, but it will be useful for later parts of the question to write it in terms of E, V, L, and constants in the problem.

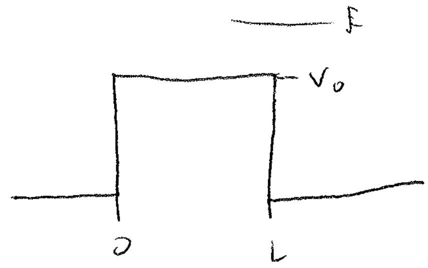
(e) [1 pt] What is the limit for the transmission coefficient T in the limit that $E \gg V$? Show your work and explain the physics of this result.

(f) [1 pt] There are energies where $T = 1$. What are these energies and the wavelength of the particle wave function? Give a physical argument of why the transmission coefficient is a maximum for these energies.

(g) [1 pt] What is the value for the transmission coefficient, T, in the limit that $E \to V$? Hint: To solve this you might define $\delta = E - V$.

F-2013

③



$E > V$

(A) $x < 0$ $\rightarrow$ $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$

$$\rightarrow \frac{d^2\psi}{dx^2} = -\underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi$$

$$= -k^2\psi$$

$$\rightarrow \boxed{\psi = Ae^{+ikx} + Be^{-ikx}}$$

$0 < x < L$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

$$\rightarrow \frac{d^2\psi}{dx^2} = -\underbrace{\frac{2m(E-V)}{\hbar^2}}_{k'^2}\psi$$

$$= -k'^2\psi$$

$$\rightarrow \boxed{\psi = C\sin k'x + D\cos k'x}$$

$L < x$ $\rightarrow$ $\boxed{\psi = Ee^{ikx}}$

(B) at $x=0$ $\rightarrow$ $A+B=D$

$ikA - ikB = Ck'$

at $x=L$

$C\sin k'L + D\cos k'L = Ee^{ikx}$

$k'C\cos k'L - k'D\sin k'L = ikEe^{ikL}$

$$\boxed{2ikA = ikD + Ck'}$$

$(k'\sin k'L)\, C\sin k'L + (k'\sin k'L)\, D\cos k'L = k'\sin k'L\, Ee^{ikx}$

$(\cos k'L)\, k'C\cos k'L - \cos k'L\, k'D\sin k'L = \cos k'L\, ikEe^{ikL}$

$\rightarrow k'C\sin^2 k'L + k'C\cos^2 k'L = Ee^{ikx}(k'\sin k'L + ik\cos k'L)$

$$\rightarrow \boxed{C = \frac{E}{k'}e^{ikx}(k'\sin k'L + ik\cos k'L)}$$

$(k'\cos k'L)\, C\sin k'L + k'\cos^2 k'L\, D = Ee^{ikx}k'\cos k'L$

$(-\sin k'L)(k'C\cos k'L) + k'D\sin^2 k'L = -\sin k'L\,(ik)Ee^{ikL}$

$\rightarrow D = \frac{Ee^{ikx}}{k'}(k'\cos k'L - ik\sin k'L)$

$\rightarrow 2ikA = i\frac{k}{k'}Ee^{ikx}(k'\cos k'L - ik\sin k'L) + Ee^{ikx}(k'\sin k'L + ik\cos k'L)$

$\rightarrow \frac{A}{E} = \frac{1}{2k'}e^{ikx}(k'\cos k'L - ik\sin k'L) - \frac{i}{2k}e^{ikx}(k'\sin k'L + ik\cos k'L)$

(3)

(B) Continued

$$\frac{A}{E} = \frac{e^{ikx}}{2}\left(\cos k'L - \frac{ik}{k'}\sin k'L\right) - \frac{e^{ikx}}{2}\left(\frac{k'}{k} i \sin k'L + \cos k'L\right)$$

$$= e^{ikx}\cos k'L - \frac{e^{ikx} i}{2}\underbrace{\left(\frac{k}{k'}\sin k'L + \frac{k'}{k}\sin k'L\right)}_{\sin k'L \underbrace{\left(\frac{k}{k'} + \frac{k'}{k}\right)}_{\frac{k^2+k'^2}{k'k}}}$$

$$\Rightarrow \boxed{\frac{A}{E} = e^{ikx}\left(\cos k'L - \frac{i(k^2+k'^2)}{2kk'}\sin k'L\right)}$$

(D) $T = \frac{|E|^2}{|A|^2}$ $\Rightarrow e^{ikL}\frac{(2kk'\cos k'L - i(k^2+k'^2)\sin k'L)}{2kk'} = \frac{A}{E}$

$$\Rightarrow \frac{E}{A} = e^{-ikL}\frac{(2kk')}{2kk'\cos k'L - i(k^2+k'^2)\sin k'L}$$

$$\Rightarrow \boxed{\frac{|E|^2}{|A|^2} = \frac{4k^2k'^2}{4k^2k'^2\cos^2 k'L + \sin^2 k'L\,(k^4+k'^4+2k^2k'^2)}}$$

E $T = \dfrac{1}{\cos^2 k'L + \dfrac{\sin^2 k'L}{4}\left(\dfrac{k^2}{k'^2} + \dfrac{k'^2}{k^2} + 2\right)}$

$$\frac{k^2}{k'^2} = \frac{E}{E-V}$$

$E \gg V$

$\rightarrow \dfrac{k^2}{k'^2} = \dfrac{k'^2}{k^2} = 1$

$$\frac{k'^2}{k^2} = \frac{E-V}{E}$$

$$\rightarrow \boxed{T = 1}$$

F $\sin^2 k'L = \emptyset \rightarrow k'L = n\pi$

$\dfrac{\sqrt{2m(E-V)}}{\hbar} L \rightarrow \boxed{\dfrac{n^2\pi^2\hbar^2}{2mL^2} + V = E}$

G $4k^2k'^2 = 4\left(\dfrac{4m^2 E(E-V)}{\hbar^4}\right) = \emptyset$ when $E \rightarrow V$

F-2015

Problem 1: Quantum Currents

For a 1D quantum mechanical system of particles with mass m, the current in a state $\Psi(x,t)$ can be defined as:

$$j(x,t) = \frac{1}{m} Re\left(\Psi^*(x,t) P \Psi(x,t)\right) \tag{1}$$

where P is the momentum operator and Re signifies the real part.

(a) [2 pts] Consider a 1D step-potential

$$\begin{aligned} V(x) &= 0,\ x < 0, \\ V(x) &= V_0,\ x > 0 \end{aligned} \tag{2}$$

where $V_0 > 0$, and the 1D scattering eignestates for the Hamiltonian for particles incident from $x < 0$

$$\begin{aligned} \Psi_E(x) &= \psi_I(x) + \psi_R(x),\ x < 0, \\ \Psi_E(x) &= \psi_T(x),\ x > 0, \\ H\Psi_E &= E\Psi_E \end{aligned} \tag{3}$$

where ψ_I, ψ_R, and ψ_T represent the incoming, reflected, and transmitted waves respectively.

Write down the functional form for $\Psi_E(x)$, and solve for the amplitudes of ψ_T and ψ_R in terms of the amplitude of ψ_I for $E > V_0$.

(b) [2 pts] What is the ratio of the transmitted to incoming currents,

$$\frac{j_T}{j_I}, \tag{4}$$

as a function of the energy E, for $E > V_0$? Check your result for $E >> V_0$ and $E \to V_0$.

(c) [1 pt] What is J_T for $E < V_0$? Show your work.

(d) [2 pts] Next, consider a 1D Hamiltonian, H, that has a series of bound, non-degenerate, real eigenfunctions $\psi_n(x)$: $H\psi_n(x) = E_n\psi_n(x)$. Show that the current for these states,

$$j_n(x,t) = \frac{1}{m} Re\left(\Psi_n^*(x,t) P \Psi_n(x,t)\right) = 0 \tag{5}$$

(e) [3 pts] Now consider a bound state of H from part (c) given, at $t = 0$, by

$$\Psi(x, t=0) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right) \tag{6}$$

where $\psi_1(x)$ and $\psi_2(x)$ are the ground state and first excited state of H.

Show that the current for this state will not be zero, and derive the time-dependence of the current.

F-2015

1. $V(x) = \emptyset \quad x < \emptyset$

$V(x) = V_0 \quad x > \emptyset$

A. $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad E > V_0$

$\underline{x < 0} \rightarrow \quad \frac{d^2\psi}{dx^2} = -l^2 E\psi \qquad \text{where } l^2 = \frac{2mE}{\hbar^2}$

$\rightarrow \psi = A \sin lx + B\cos lx$

or

$$\boxed{\psi_E = \cancel{Ae^{-ilx}}\ \underbrace{Ae^{-ilx}}_{\psi_R} + \underbrace{Be^{ilx}}_{\psi_I}}$$

$\underline{x > 0} \rightarrow \quad \frac{d^2\psi}{dx^2} = -k^2\psi$

$\psi_E = Ce^{ikx} + De^{-ikx}$

$D = \emptyset$ since Trave is to the right

$$\rightarrow \boxed{\psi_E = \underbrace{Ce^{ikx}}_{\psi_T}}$$

(A) Continued Solve for C & A in terms of B

$$\rightarrow \quad A + B = C$$

&

$$-i\ell A + i\ell B = ikC$$

$$\Rightarrow \quad -\ell A + \ell B = kC$$

C $\rightarrow$ $\ell A + \ell B = \ell C$

$$-\ell A + \ell B = kC$$

$$2\ell B = (\ell + k)C$$

$$\rightarrow \quad \boxed{C = \frac{2\ell}{\ell + k} B}$$

$$\cancel{A\ell - \ell B = -A\ell}$$

$$\cancel{-\ell A + \ell B = kC}$$

A $\rightarrow$ $\cancel{-Ak - Bk = -kC}$

$$\cancel{-\ell A + \ell B = kC}$$

$$kA + kB = -\ell A + \ell B$$

$$\rightarrow (k + \ell) A = (\ell - k) B$$

$$\rightarrow \quad \boxed{A = \left(\frac{\ell - k}{k + \ell}\right) B}$$

B

$\cancel{\frac{C}{B}}$

$$\cancel{\frac{C}{B} = \frac{2k}{l+k} = \frac{2\sqrt{\frac{2mE}{\hbar^2}}}{\sqrt{\frac{2mE}{\hbar^2}} + \sqrt{\frac{2m(E-V_0)}{\hbar^2}}}}$$

$$\cancel{= \frac{2\sqrt{E}}{\sqrt{E} + \sqrt{E-V_0}}}$$

$$J_T = \frac{1}{m} \mathrm{RE}\left(\psi_T^* p \psi_T\right)$$

$$\rightarrow \quad -i\hbar \frac{\partial}{\partial x} C e^{ikx} = -i\hbar\, ik C e^{ikx}$$
$$= \hbar k C e^{ikx}$$

$$\rightarrow J_T = \frac{1}{m} \hbar k C^2$$

$$\rightarrow J_F \rightarrow \quad -i\hbar B\, il\, e^{ilx} = \hbar l B e^{ilx}$$

$$J_F = \frac{1}{m} \hbar l B^2$$

$$\rightarrow \quad \frac{J_T}{J_F} = \frac{kC^2}{lB^2} \qquad \rightarrow \quad \frac{C^2}{B^2} = \frac{4l^2}{(l+k)^2}$$

B Continued

$$\frac{J_T}{J_F} = \frac{2kl^{\cancel{2}}}{\cancel{l}(l+k)^2} = \frac{4kl}{l^2+k^2+2lk}$$

$$= \frac{4\sqrt{\frac{2mE}{\hbar^2}}\sqrt{\frac{2m(E-V_0)}{\hbar^2}}}{\frac{2mE}{\hbar^2} + \frac{2m(E-V_0)}{\hbar^2} + 2\sqrt{\frac{2mE}{\hbar^2}}\sqrt{\frac{(E-V_0)2m}{\hbar^2}}}$$

$$\boxed{\frac{J_T}{J_I} = \frac{4\sqrt{E(E-V_0)}}{E + (E-V_0) + 2\sqrt{E(E-V_0)}}}$$

$\underline{E \gg V_0} \rightarrow \frac{4E}{E+E+2E} = \boxed{1 \text{ Total Transmission}}$

$\rightarrow E \rightarrow V_0 \rightarrow \boxed{\emptyset}$

C $\dfrac{d^2\psi}{dx^2} = -\dfrac{\overbrace{2m(E-V_0)}^{-}}{\hbar^2}\psi$

$= \underbrace{\dfrac{2m(V_0-E)}{\hbar^2}}_{K^2}\psi$

$\rightarrow \psi_T = \underbrace{\cancel{A}}_{=\phi}e^{\kappa x} + \cancel{B}e^{-\kappa x}$

$= \cancel{B}\,C\,e^{-\kappa x}$

$-i\hbar\dfrac{d}{dx}Ce^{-\kappa x} = i\hbar\kappa Ce^{-\kappa x}$

$\rightarrow Re\left| i\hbar c^2 e^{-2\kappa x}\right|$

$= \phi \rightarrow \boxed{J_T = 0}$

D Bound means E is Negative, Assuming positive V (or $E < V$)

$\rightarrow \dfrac{d^2\psi}{dx^2} = -\dfrac{\overbrace{2m(E-V)}^{\text{overall } - \#}}{\hbar^2}\psi$

$= \underbrace{- -}_{+} \dfrac{\overbrace{2m(V-E)}^{\text{overall } + \#}\psi}{\hbar^2}$

will always have an i in current ~~Ans~~

E $\psi(x,0) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right)$

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left(e^{-iE_1 t/\hbar}\psi_1(x) + e^{-iE_2 t/\hbar}\psi_2(x)\right)$$

$$\rightarrow -i\hbar\frac{\partial}{\partial x}\left[\left(\frac{1}{\sqrt{2}}\, e^{-iE_1 t/\hbar}\psi_1(x) + e^{-iE_2 t/\hbar}\psi_2(x)\right)\right]$$

$$= \frac{-i\hbar}{\sqrt{2}}\left[e^{-iE_1 t/\hbar}\frac{d\psi_1(x)}{dx} + e^{-iE_2 t/\hbar}\frac{d\psi_2}{dx}\right]$$

$$\rightarrow \frac{1}{m}\left[\left(e^{iE_1 t/\hbar}\psi_1 + e^{iE_2 t/\hbar}\psi_2\right)\left(e^{-iE_1 t/\hbar}\frac{d\psi_1(x)}{dx} + e^{-iE_2 t/\hbar}\frac{d\psi_2}{dx}\right)\right]$$

$+$

$\frac{-i\hbar}{2}$

$$= \frac{-i\hbar}{2m}\left[\psi_1\frac{d\psi_1}{dx} + e^{it/\hbar(E_1-E_2)}\psi_1\frac{d\psi_2}{dx} + \right.$$

Dont want to

do it any more but

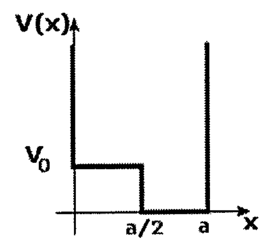
its easy

S-2015

Problem 3: Double Step Potential

Consider a single particle of mass m in a one dimensional well of width a and a potential, $V(x)$, given by:

$$V(x) = \begin{cases} \infty, & x < 0 \\ V_0, & 0 < x < \frac{a}{2} \\ 0, & \frac{a}{2} < x < a \\ \infty, & x > a \end{cases} \tag{1}$$



In this question, you will consider the special cases where this potential well has a bound state at the energy $E = V_0$. There are only certain values of V_0 and a where this will happen.

In this problem, use the constant

$$k = \sqrt{\frac{2mV_0}{\hbar^2}} \tag{2}$$

(a) [2 pts] For the energy $E = V_0$ in this potential, determine the general eigenfunction solutions to the time-independent Schrödinger equation in all regions of x. Show your work.

(b) [3 pts] Apply boundary conditions to determine relationships between the constants you introduced in writing the wave functions in part (a).

(c) [2 pts] From your results above, derive a transcendental equation that gives the values of V_0 where there is an energy eigenstate with $E = V_0$, for a fixed well width a. This equation will have the form $z = f(z)$ with $z = k\frac{a}{2}$. Plot this function and determine a relationship between the first energy V_0 that satisfies this equation and the bound state energies of a square well of width a.

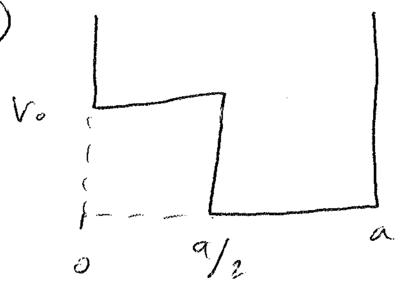
(d) [2 pts] Qualitatively sketch the wave function that corresponds to the smallest value of V_0 that satisfies the transcendental equation from part (c), for a fixed value of a.

(e) [1 pt] Finally, consider the case where the width of the well is fixed but the potential step, V_0, can be changed. There are an infinite number of possible values of V_0 where the well contains an energy eigenstate with $E = V_0$. Describe, qualitatively, the changes in the wavefunctions of these eigenstates as V_0 gets larger.

S - 2015

1

3



$E = V_0$

A $\quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$

$x < 0$ & $x > a$

$\rightarrow \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \infty\psi = E\psi$

↳ wave function must be

$0 \rightarrow \boxed{\psi_1 = 0}$

$0 < x < a/2 \quad \rightarrow \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$

$\rightarrow \frac{d^2\psi}{dx^2} = 0 \rightarrow \boxed{\psi = Ax + B}$

$a/2 < x < a \quad \rightarrow \quad \frac{d^2\psi}{dx^2} = -k^2\psi \rightarrow \boxed{\psi = \overset{C}{\cancel{A}} \sin kx + \overset{D}{\cancel{B}} \cos kx}$

(question asks for general solutions)

B) at $x = \emptyset \rightarrow \psi = B$

$\quad$ but $\psi = \emptyset$ at $x = \emptyset \longrightarrow B = \emptyset$

at $\underline{x = a} \rightarrow \psi = \emptyset \rightarrow \emptyset = C \sin ka + D \cos ka$

$$\hookrightarrow \boxed{D = -C \tan ka}$$

$\underline{\text{at } x = a/2} \rightarrow \boxed{\frac{Aa}{2} = C \sin k\frac{a}{2} + D \cos k\frac{a}{2}}$

$\quad$ & $\boxed{A = kC \cos k\frac{a}{2} - kD \sin k\frac{a}{2}}$

$$\rightarrow \boxed{\frac{Aa}{2} = C \sin \frac{ka}{2} - C \cos \frac{ka}{2} \tan ka}$$

$$\rightarrow \boxed{A = kC \cos \frac{ka}{2} + kC \sin \frac{ka}{2} \tan ka}$$

C) $\rightarrow k\frac{a}{2} \cancel{C} \cos \frac{ka}{2} + k\frac{a}{2} \cancel{C} \sin \frac{ka}{2} \tan ka = \cancel{C} \sin \frac{ka}{2} - \cancel{C} \cos \frac{ka}{2} \tan ka$

$$\rightarrow \frac{ka}{2} = \frac{\sin \frac{ka}{2} - \cos \frac{ka}{2} \tan ka}{\cos \frac{ka}{2} + \sin \frac{ka}{2} \tan ka} \qquad \frac{\sin ka}{\cos ka}$$

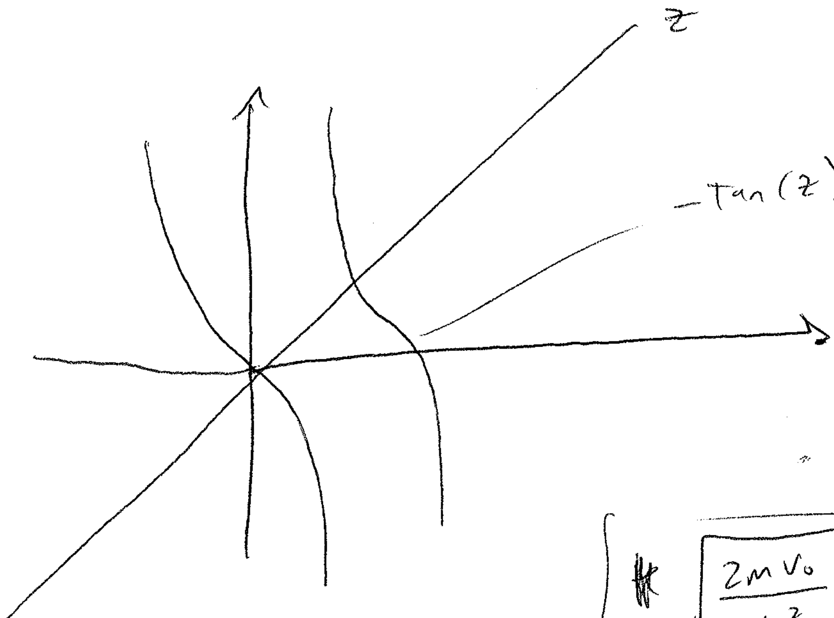
$$= \frac{\cos ka \sin \frac{ka}{2} - \cos \frac{ka}{2} \sin ka}{\cos ka \cos \frac{ka}{2} + \sin \frac{ka}{2} \sin ka}$$

(c) Continued

$$\to \frac{ka}{2} = \frac{\sin\left(\frac{ka}{2} - ka\right)}{\cos\left(\frac{ka}{2} - ka\right)} = \tan\left(-\frac{ka}{2}\right)$$

$$= -\tan\frac{ka}{2}$$

$$\to \boxed{z = -\tan z}$$

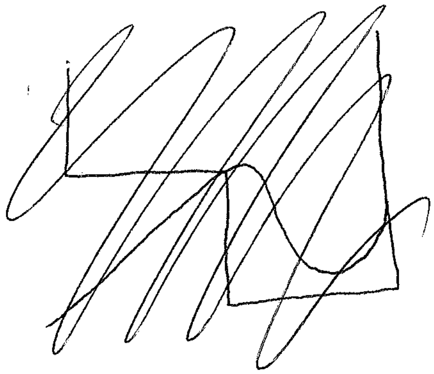


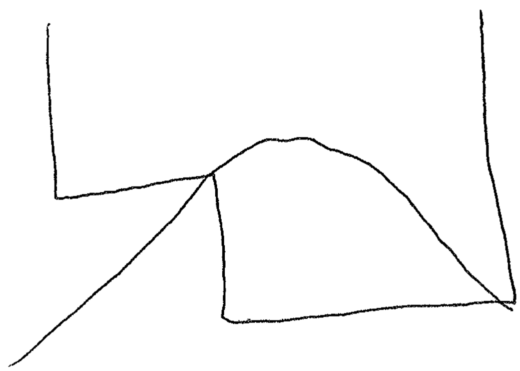
$$\boxed{\sqrt{\frac{2mV_0}{\hbar^2}}\,\frac{a}{2} = -\tan\left(\sqrt{\frac{2mV_0}{\hbar^2}}\,\frac{a}{2}\right)}$$

What is this question asking?

There are places
where these intersect
but how the Fuck
do I know
where?
besides 0

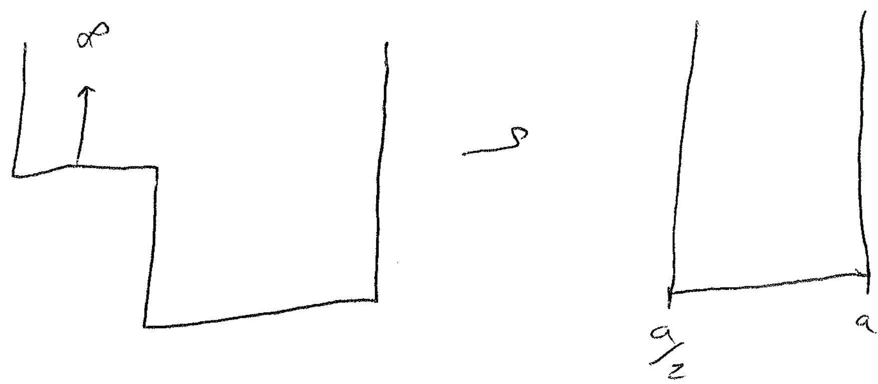
D





E

V_0



Infinite square well with

$V_0 = \infty \quad x < a/2$

$V_0 = 0 \quad a/2 < x < a$

$V_0 = \infty \quad a > a$

Wave function will become $\psi = A\sin ka + B\cos ka$

Problem 1: Step Potential (10 points)

Consider the potential $V(x)$

$$V(x) = \begin{cases} 0, & x \leq 0 \\ -\text{V}, & x > 0 \end{cases}$$

A particle of mass m and kinetic energy E approaches the step from $x < 0$.

a) Write the solution to Schrodinger's equation for $x < 0$. (1 pt)

b) Write the solution to Schrodinger's equation for $x > 0$. (1 pt)

c) Sketch the wave function for $x < 0$ as well as $x > 0$. Making sure to describe how the amplitude and frequency of the wave function changes. (1 pt)

d) What is the probability that particle will reflect back if $E = V/8$? (2 pts)

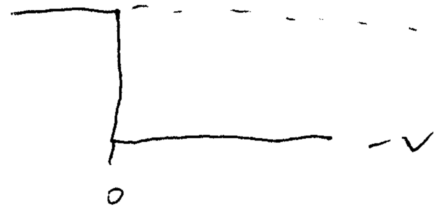
e) What is the probability that the particle will be transmitted if $E = V/8$. (2 pts) (Determine the transmission probability directly by using the flow of probability current and do not simply use $T = 1 - R$)

f) Show that $T + R = 1$. What does this mean physically? (1 pt)

g) If instead the particle approached the step from $x > 0$, how do your answers to parts a), b), d) and e) change? (2 pts)

F-2001

$$V(x) = \begin{cases} 0 & x \le 0 \\ -V & x > 0 \end{cases}$$



(A) $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\rightarrow \frac{d^2\psi}{dx^2} = \underbrace{\frac{-2mE}{\hbar^2}}_{-k^2}\psi$$

$$\rightarrow \boxed{\psi = Ae^{-ikx} + Be^{ikx}}$$

(B) $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V\psi = E\psi$$

$$\rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E+V)\psi \rightarrow \frac{d^2\psi}{dx^2} = \underbrace{\frac{-2m(E+V)}{\hbar^2}}_{-K^2}\psi$$

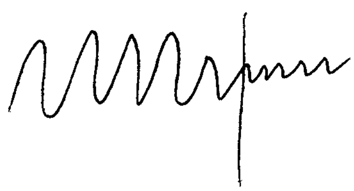
$$\rightarrow \psi = \cancel{A}\,C e^{iKx} + \cancel{B}\,D e^{-iKx}$$

$D = \phi$ since particle comes from left

$$\rightarrow \boxed{\psi = Ce^{iKx}}$$

(C) Come Back (2)

Incident $A >$ Transmitted



↑ Frequency

Larger

$K = \frac{kg^2 \frac{m^2}{s^2}}{kg^2 \frac{m^4}{s^2}} = \frac{1}{m^2}$

rp

$r \cdot mv$

$m \; kg \frac{m}{s}$

$V = \frac{m}{s}$

$c = \frac{\omega}{k}$

work

(D) $R = \frac{|B|^2}{|A|^2} \rightarrow A + B = C$

$-ikA + ikB = iKC$

$\rightarrow -kA + kB = KC$

$\rightarrow A + B = \frac{k}{K}(-A+B)$

in your case it's actually $\frac{A^2}{B^2}$

$\rightarrow B\left(\frac{k}{K} - 1\right) = A\left(\frac{k}{K} + 1\right)$

$\rightarrow \frac{B^2}{A^2} = \frac{\left(\frac{k}{K}+1\right)^2}{\left(\frac{k}{K}-1\right)^2} = \frac{\left(\frac{1}{3}+\frac{3}{3}\right)^2}{\left(\frac{1}{3}-\frac{3}{3}\right)^2}$

$\frac{4^2}{2^2} = \frac{k}{4}$

$K = \sqrt{\frac{2mV}{\hbar^2 p}}$

$K = \frac{\sqrt{2m\left(\frac{V}{p}+V\right)}}{\sqrt{\hbar^2}} = \frac{2m\left(\frac{9}{p}V\right)}{\hbar^2}$

$\rightarrow$ ~~$\frac{1}{9} + 1 = \frac{10}{9} = \frac{10}{8} = 4$~~ $= 4$

$\frac{k}{K} = \frac{V/4}{9V/4} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

$\rightarrow$ ~~$R = 4$~~ $\boxed{1/4 = R}$

~~$\frac{25}{16} = R$~~

E $T = \dfrac{|J_{\text{transmitted}}|}{|J_{\text{incident}}|}$

$$J_{\text{transmitted}} = \frac{i\hbar}{2m}\left[\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right]$$

$$= \frac{i\hbar}{2m}\left[Ce^{-iKx}\, iKCe^{iKx} + Ce^{iKx}\, Ce\, iKe^{iKx}\right]$$

$$= -\frac{\hbar}{2m}\left[C^2K + C^2K\right] = -\frac{\hbar C^2 K}{m}$$

$$J_{\text{incident}} = \frac{i\hbar}{2m}\left[Be^{-ikx}\, k\, iBe^{ikx} + Be^{ikx}\, ik\, Be^{-ikx}\right]$$

$$= -\frac{\hbar}{2m}\left[B^2k + B^2k\right] = -\frac{\hbar B^2 k}{m}$$

$$T = \frac{C^2 K}{B^2 k}$$

$$\rightarrow \quad -ikA + ikB = iKC$$

$$A + B = C \rightarrow A = C - B$$

$$\rightarrow \quad -ik(C-B) + ikB = iKC$$

$$= -ikC + ikB + ikB = iKC$$

$$= 2kB = (K + k)C \rightarrow \frac{C^2}{B^2} = \frac{2k^2}{(K+k)^2}$$

$$= \frac{4k^2}{K^2 + k^2 + 2Kk}$$

$$\rightarrow T = \frac{K}{\cancel{k}}\,\frac{4k^{2}}{K^2 + k^2 + 2kK}$$

$$= \sqrt{\frac{9mV}{4\hbar^2}}\;\frac{4\left(\sqrt{\frac{2mV}{\hbar^2 8}}\right)}{\frac{9mV}{4\hbar^2} + \frac{mV}{\hbar^2 4} + 2\sqrt{\frac{4m^2V^2\,9}{\hbar^4\,64}}}$$

E Continued

$$\frac{\frac{3}{2t_1}\sqrt{\frac{mv}{m}}\;\frac{4}{2t_1}\sqrt{mv}}{\frac{10}{4}\frac{mv}{t_1^2}+\frac{4\times 3}{8}\frac{mv}{t_1^2}} = \frac{\frac{12}{4t_1^2}\cancel{mv}}{\frac{10}{4}\frac{\cancel{mv}}{\cancel{t_1^2}}+\frac{12}{8}\frac{\cancel{mv}}{\cancel{t_1^2}}}$$

$$= \frac{\frac{12}{4}}{\frac{10}{4}+\frac{6}{4}}$$

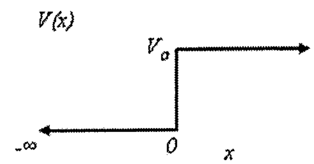
$$= \frac{12}{10+6} = \frac{12}{16} = \boxed{\frac{3}{4} = T}$$

F $\boxed{\frac{3}{4}+\frac{1}{4}=1}$ Conservation of # of Particles

G Basically redo problem, should be straight forward

S-2007

PROBLEM 1



Consider the step potential shown in the figure.

a) [1 pts] Consider a particle traveling from $x = -\infty$ to the right with an energy E. The appropriate wavefunction for this particle is given by

$$\phi = \begin{cases} e^{ik_L x} + Ae^{-ik_L x} & \text{for } x < 0 \\ Be^{ik_R x} & \text{for } x > 0 \end{cases}$$

Give expressions for k_L and k_R and define any undefined parameters/constants given in your expression.

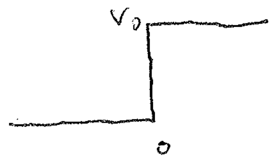
b) [3 pts] For the case that $E > V_o$, use appropriate boundary conditions to find the coefficients A and B.

c) [2 pts] For the case that $E > V_o$, find the probability that the particle will be reflected.

d) [2 pts] For the case that $E > V_o$, the probability that the particle will be transmitted is given by $T = 1 - R$. Determine and explain the physical meaning of the ratio $|B|^2/T$.

e) [2 pts] What is the probability for reflection when $E < V_o$?

1



(A) $\phi = \begin{cases} e^{ik_L x} + A e^{-ik_L x} & \text{For } x<0 \\ B e^{ik_R x} & x>0 \end{cases}$

$\rightarrow$ $\underline{x<0}$ $\rightarrow$ $-\dfrac{\hbar^2}{2m}\dfrac{d^2\phi}{dx^2} = E\phi$

$\rightarrow \dfrac{d^2\phi}{dx^2} = -\dfrac{2mE}{\hbar^2}\phi$

$$\boxed{k_L = \sqrt{\frac{2mE}{\hbar^2}}}$$

$\rightarrow$ $\underline{x>0}$ $\rightarrow$ $-\dfrac{\hbar^2}{2m}\dfrac{d^2\phi}{dx^2} + V_0 = E\phi$

$\rightarrow \dfrac{d^2\phi}{dx^2} = -\dfrac{2m(E-V_0)}{\hbar^2}\phi$

$\rightarrow \boxed{k_R = \sqrt{\dfrac{2m(E-V_0)}{\hbar^2}}}$

(B) $\phi(x=0)$ has to be continuous

$\left.\dfrac{d\phi}{dx}\right|_{x=0}$ has to be continuous

$\rightarrow 1 + A = B$

$\rightarrow ik_L - ik_L A = ik_R B \rightarrow k_L(1-A) = k_R B \rightarrow \dfrac{k_L}{k_R}(1-A) = B$

$\rightarrow 1 + A = \dfrac{k_L}{k_R}(1-A) \rightarrow 1 + A = \dfrac{k_L}{k_R} - \dfrac{k_L}{k_R}A \rightarrow A\left(\dfrac{k_R}{k_R} + \dfrac{k_L}{k_R}\right) = \dfrac{k_L}{k_R} - \dfrac{k_R}{k_R}$

$\rightarrow \boxed{A = \dfrac{k_L - k_R}{k_R + k_L}} \rightarrow B = \dfrac{k_R + k_L + k_L - k_R}{k_R + k_L} \rightarrow \boxed{B = \dfrac{2k_L}{k_R + k_L}}$

(C) $R = \frac{|A|^2}{\cancel{\quad}|^2} = \boxed{\frac{(k_L - k_R)^2}{(k_R + k_L)^2} = R}$ ②

(D) $T = 1 - R \rightarrow \frac{(k_R + k_L)^2 - (k_L - k_R)^2}{(k_R + k_L)^2} = T$

$\rightarrow \frac{(k_R + k_L)(k_R + k_L) - (k_L - k_R)(k_L - k_R)}{(k_R + k_L)^2} = T$

$\rightarrow (k_R + k_L)(k_R + k_L) =$

$k_R^2 + k_L^2 + 2k_R k_L$

$\rightarrow (k_R - k_L)(k_R - k_L) =$

$k_R^2 + k_L^2 - 2k_R k_L$

$\hookrightarrow \frac{4k_R k_L}{(k_R + k_L)^2} = T$

$\rightarrow B^2 = \frac{4k_L}{(k_R + k_L)^2} = Tk_R$

$\rightarrow \frac{B^2}{T} = k_R$

B won't change, as T gets smaller, k_R gets bigger

$\rightarrow k_R \propto E - V_0$

E) $E < V_0$

$$\to -\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} = (E-V_0)\phi$$

$$= -\underbrace{(V_0-E)}_{\text{positive } \#}\phi$$

$$\frac{d^2\phi}{dx^2} = \frac{2m(V_0-E)}{\hbar^2}\phi \to k_R = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$\to \phi, \underline{x<0} \to Be^{-k_R x}$$

has to NOT blow up at ∞

$$\to 1+A = B$$

$$\to ik_L - Aik_L = -k_R B \to -\frac{ik_L(1-A)}{k_R} = B$$

$$\to 1+A = -\frac{ik_L}{k_R}(1-A) \to 1+A = -\frac{ik_L}{k_R} + \frac{ik_L}{k_R}A$$

$$\to \left(\frac{k_R}{k_R} - \frac{ik_L}{k_R}\right)A = -\left(\frac{ik_L}{k_R} + \frac{k_R}{k_R}\right)$$

$$\to A = \frac{-ik_L - k_R}{k_R - ik_L}$$

$$\to R = |A|^2 = \frac{(-ik_L - k_R)(ik_L - k_R)}{(k_R - ik_L)(k_R + ik_L)}$$

$$\to R = \frac{k_L^2 + ik_Lk_R - ik_Rk_L + k_R^2}{k_R^2 + ik_Rk_L - ik_Rk_L + k_L^2} = \boxed{1 = R}$$