F-2015

Problem 3: Vector Spaces and Dirac Notation

Consider a quantum system that can be described by three basis states, $|n\rangle$, $n = 1, 2, 3$, and an operator defined by its action on these three states:

$$\begin{aligned} A|1\rangle &= -i\alpha|3\rangle \\ A|2\rangle &= \alpha|2\rangle \\ A|3\rangle &= i\alpha|1\rangle \end{aligned} \tag{1}$$

where α is real.

(a) [2 pts] Write the operator A as a matrix using these basis states:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

(b) [1 pt] Show that A is Hermitian.

(c) [3 pts] Compute the eigenvalues and corresponding eigenvectors of A.

(d) [2 pts] In your result for part (c), you found one non-degenerate eigenstate, call it $|\gamma\rangle$, with eigenvalue γ. The other eigenstates are degenerate.

Define the projection operator $\mathcal{P}_\gamma = |\gamma\rangle\langle\gamma|$. Write the operator $\mathcal{P}_\gamma$ as a matrix using the basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$.

Check your results to show that this matrix form for the projection operator is correct.

(e) [2 pts] Consider the system in the state:

$$|\phi\rangle = \frac{2}{3}|1\rangle + \frac{2}{3}|2\rangle - \frac{i}{3}|3\rangle \tag{3}$$

Write down an expression for the probability that a measurement of A would result in the value γ in terms of the projection operator $\mathcal{P}_\gamma$. Solve for this probability.

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3 $A|1\rangle = -i\alpha|3\rangle$

$A|2\rangle = \alpha|2\rangle$

$A|3\rangle = i\alpha|1\rangle$

A $A \doteq \sum_{ij} |i\rangle\langle i|A|j\rangle\langle j|$

$$\boxed{\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 0 & i\alpha \\ 0 & \alpha & 0 \\ -i\alpha & 0 & 0 \end{pmatrix} \end{array}}$$

$$\begin{pmatrix} 0 & 0 & i\alpha \\ 0 & \alpha & 0 \\ -i\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \phi \\ \phi \end{pmatrix} =$$

B A^{+} = Row becomes Column $\rightarrow$

$$\begin{matrix} 0 & 0 & -i\alpha \\ 0 & \alpha & 0 \\ i\alpha & 0 & 0 \end{matrix}$$

$\rightarrow$ complex conjugate $\rightarrow$ $\boxed{\checkmark}$

(c) $$\begin{vmatrix} -\lambda & 0 & i\alpha \\ 0 & \alpha-\lambda & 0 \\ -i\alpha & 0 & -\lambda \end{vmatrix} = \emptyset$$

$$(\alpha+\lambda)$$

$$\rightarrow \quad -\lambda(-\lambda(\alpha-\lambda)) - \emptyset + i\alpha(0-(\alpha-\lambda)(-i\alpha)) = \emptyset$$

$$= -\lambda(-\lambda\alpha+\lambda^2) + i\alpha(i\alpha^2 - i\alpha\lambda) = \emptyset$$

$$\rightarrow \quad \lambda^2\alpha - \lambda^3 - \alpha^3 + \alpha^2\lambda = \emptyset$$

$$\rightarrow \quad (\lambda^2\alpha - \lambda^3) + (\alpha^2\lambda - \alpha^3) = \emptyset$$

$$\rightarrow \quad \lambda^2(\alpha-\lambda) + \alpha^2(\lambda-\alpha) = \emptyset$$

$$\rightarrow \quad \underbrace{(\lambda^2-\alpha^2)}\underbrace{(\alpha-\lambda)} = \emptyset$$

$$\boxed{\lambda = \pm\alpha \qquad \lambda = \alpha}$$

$\underline{\lambda = -\alpha}$ $$\rightarrow \begin{pmatrix} \alpha & 0 & i\alpha \\ 0 & 2\alpha & 0 \\ -i\alpha & 0 & \alpha \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} \quad \rightarrow B = \emptyset$$

$$A\alpha = -i\alpha C \rightarrow C = Ai$$

$$\rightarrow \begin{pmatrix} 0 & 0 & i\alpha \\ 0 & \alpha & 0 \\ -i\alpha & 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \begin{pmatrix} -\alpha \\ 0 \\ -i\alpha \end{pmatrix} \checkmark$$

$$\rightarrow \boxed{|\lambda = -\alpha\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}}$$

(C) (Continued)

$$\lambda = \alpha, 1 \rightarrow \begin{pmatrix} -\alpha & 0 & i\alpha \\ 0 & 0 & 0 \\ -i\alpha & 0 & -\alpha \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$B = \text{anything} \rightarrow \quad B = 1$

$$\boxed{|\lambda = \alpha, 1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & 0 & i\alpha \\ 0 & \alpha & 0 \\ -i\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix} \checkmark$$

$$\boxed{|\lambda = \alpha, 2\rangle \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}}$$

(D) $P = |-\alpha\rangle\langle-\alpha|$

$$= \sum_{ij} |i\rangle \langle i|-\alpha\rangle\langle-\alpha|j\rangle\langle j|$$

$$\downarrow \qquad \downarrow$$

$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} (1\ 0\ 0)\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$

$0 = \frac{1}{\sqrt{2}} (0\ 1\ 0)\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \qquad \frac{1}{\sqrt{2}} (1\ 0\ -i)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$

$\frac{i}{\sqrt{2}} = \frac{1}{\sqrt{2}} (0\ 0\ 1)\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \qquad \frac{1}{\sqrt{2}} (1\ 0\ -i)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$

$\frac{1}{\sqrt{2}} (1\ 0\ -i)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{-i}{\sqrt{2}}$

(D) Continued

$$\boxed{P \doteq \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix}}$$

$$P|1\rangle = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{matrix} \frac{1}{2} \\ 0 \\ \cancel{\sqrt{2}}\ \frac{i}{2} \end{matrix}$$

✓

(e) $|\phi\rangle = \frac{2}{3}|1\rangle + \frac{2}{3}|2\rangle - \frac{i}{3}|3\rangle$

$$|\phi\rangle = \cancel{\sum} |-\alpha\rangle\langle-\alpha|\phi\rangle$$

$$= \frac{1}{\sqrt{2}}(1 \quad 0 \quad -i)\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{i}{3} \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\frac{2}{3} - \frac{1}{3}\right]$$

$$= \cancel{\sqrt{2}}\ \frac{1}{3\sqrt{2}}$$

$$\Rightarrow \boxed{\frac{1}{18}}$$

(5) ~~ex~~ as a check

$$|\psi\rangle = |-\alpha\rangle\langle-\alpha|\psi\rangle + |\alpha,1\rangle\langle\alpha,1|\psi\rangle + |\alpha,2\rangle\langle\alpha,2|\psi\rangle$$

$$\frac{1}{\sqrt{2}}|-\alpha\rangle + (0\ 1\ 0)\begin{pmatrix}\frac{2}{3}\\ \frac{2}{3}\\ -\frac{i}{3}\end{pmatrix}|\alpha,1\rangle + \frac{1}{\sqrt{2}}(1\ 0\ i)\begin{pmatrix}\frac{2}{3}\\ \frac{2}{3}\\ -\frac{i}{3}\end{pmatrix}|2\rangle$$

$$\frac{2}{3}|\alpha,1\rangle + \frac{1}{\sqrt{2}}$$

$$\cancel{\frac{1}{2} + \frac{4}{9} + \frac{1}{2} = \frac{9}{9} + \frac{4}{9} = \sqrt{\frac{13}{9}}}$$

$$\frac{1}{3\sqrt{2}} + \frac{2}{3} + \frac{1}{\sqrt{2}} = \frac{1}{18} + \frac{4}{9} + \frac{1}{2}$$

$$1 + 8 + 9$$

$$= \frac{18}{18}$$ ✓

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Problem 2: Quantum Operators

In this problem you will work with the ladder operators for angular momentum:

$$L_+ = L_x + iL_y, \quad L_- = L_x - iL_y \tag{1}$$

where

$$\begin{aligned} L^2 &= L_x^2 + L_y^2 + L_z^2 \\ L^2|\ell, m\rangle &= \ell(\ell+1)\hbar^2|\ell, m\rangle \\ L_z|\ell m\rangle &= m\hbar|\ell, m\rangle \end{aligned} \tag{2}$$

(a) [1 pt] Show that the eigenvalues of any Hermitian operator are real.

(b) [2 pt] Is the the operator L_+L_-, the product of the angular momentum ladder operators, Hermitian? Show your work to justify your answer.

(c) [4 pt] Determine the results of the operations: $\hat{L}_+|\ell, m\rangle$ and $\hat{L}_-|\ell, m\rangle$. Show all of your work and make sure you determine all constants correctly.

Hint: The commutation relation $[L_z, L_\pm]$ and the matrix elements $\langle \ell, m|L_\pm L_\mp|\ell, m\rangle$ might be useful.

(d) [3 pt] Using the results from part (c), prove that $-\ell \le m \le +\ell$. Explain the physics of this result in terms of the operators L^2 and L_z.

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(1)

(2)

(A) $\cancel{\text{}}$ $\langle a_j | H | a_i \rangle = a_i \langle a_j | a_i \rangle$

$\langle a_j | H | a_i \rangle = a_j^* \langle a_j | a_i \rangle$

$\rightarrow (a_i - a_j^*) \underbrace{\langle a_j | a_i \rangle}_{1} = \phi$

$\rightarrow \boxed{a_i = a_i^* \quad \text{eigenvalues are real}}$

(B) $L_+ L_- = (L_x + iL_y)(L_x - iL_y)$

$= L_x^2 \underbrace{- iL_xL_y + iL_yL_x}_{\substack{i[L_y, L_x] \\ = -i\, i\hbar L_z \\ = \hbar L_z}} + L_y^2$

$= \underbrace{L_x^2 + L_y^2}_{L^2 - L_z^2} + \hbar L_z = \boxed{\underbrace{L^2 - L_z^2 + \hbar L_z}_{\text{All hermitian}} = L_+ L_-}$

(c) $[L_z, L_+] = L_z L_+ - L_+ L_z$

$\cancel{L_+|l, m_{max}\rangle = 0}$

$\cancel{\rightarrow L_z L_+ - L_+ L_z \rightarrow L_z L_+|l, m_{max}\rangle}$

$\cancel{= L_+ L_z |l, m_{max}\rangle}$

$\cancel{= 0}$

$$[L_z, L_+] = [L_z, L_x + iL_y] = [L_z, L_x] + i[L_z, L_y]$$

$$= i\hbar L_y - i i \hbar L_x$$

$$= i\hbar L_y + \hbar L_x$$

$$L_z L_+ - L_+ L_z = \hbar J_+$$

$$L_z [L_+|l,m\rangle] = \hbar [L_+|l,m\rangle] + L_+ L_z |l,m\rangle$$

$$= \underbrace{\hbar (m+1) [L_+|l,m\rangle]}$$

implies $L_+|l,m\rangle = C|l,m+1\rangle$

$$L_- L_+ = (L_x - iL_y)(L_x + iL_y) = L_x^2 + L_y^2 + \underbrace{iL_xL_y - iL_yL_x}_{ii\hbar L_z}$$

$$= L^2 - L_z^2 - \hbar L_z$$

$$\rightarrow \langle l,m|L_- L_+|l,m\rangle = \langle l,m|L^2 - L_z^2 - \hbar L_z|l,m\rangle$$

$$= C^2 \quad = l(l+1)\hbar^2 - \hbar^2 m^2 - \hbar^2 m$$

c continued

$l^2+l-m^2-m = l^2-m+l-m^2 = (l-m)(l+m+1)$

$l^2+\cancel{lm}+l-\cancel{ml}-m^2-m$

$\rightarrow \boxed{C_+ = \sqrt{(l-m)(l+m+1)}\,\hbar}$

$\rightarrow [L_z, L_-] = [L_z, L_x - iL_y] = [L_z, L_x] - i[L_z, L_y]$

$= i\hbar L_y + ii\hbar L_x$

$= i\hbar L_y - \hbar L_x$

$\rightarrow L_z L_- + L_- L_z = \hbar L_-$

$\rightarrow L_z[L_-|l,m\rangle] = \hbar[L_-|l,m\rangle] - L_- L_z|l,m\rangle$

$= \hbar(m-1)[L_-|l,m\rangle]$

$\hookrightarrow L_-|l,m\rangle = c|l,m-1\rangle$

$\rightarrow L_+L_- = L^2 - L_z^2 + \hbar L_z$

$\rightarrow \langle l,m|L_+L_-|l,m\rangle = \langle l,m|L^2 - L_z^2 + \hbar L_z|l,m\rangle$

$c^2 = l(l+1)\hbar^2 - \hbar^2 m^2 + \hbar^2 m$

$\rightarrow l^2+l-m^2+m = l^2+m+l-m^2$

$= (l+m)(l-m+1)$

$l^2-\cancel{lm}+l+\cancel{ml}-m^2+m$

$\boxed{C_- = \sqrt{(l+m)(l-m+1)}\,\hbar}$

D

$$\langle l,m | L_+L_- + L_-L_+ | l,m \rangle$$

$L^2 - L_z^2 + \hbar L_z$ $\quad$ $L^2 - L_z^2 - \hbar L_z$

$$= 2\underbrace{\langle l,m | L^2 - L_z^2 | l,m \rangle}_{l(l+1) - m^2 \geq \emptyset} \geq \emptyset$$

inner product

$$l(l+1) \geq m^2$$

There is a max value of m

$$J_+ | l, m_{max} \rangle = \emptyset$$

$$J_- J_+ | l, m_{max} \rangle = \emptyset = L^2 - L_z^2 + \hbar L_z = \emptyset$$

$$\Rightarrow l(l+1) - m^2 + m = \emptyset$$

$$\Rightarrow l(l+1) = m_{max}(m_{max} \mp 1)$$

Min value as well

$$J_+ J_- | l, m_{min} \rangle = \emptyset = L^2 - L_z^2 - \hbar L_z = \emptyset$$

$$l(l+1) - m_{min}^2 - m_{min}$$

$$l(l+1) = m_{min}(m_{min} + 1)$$

$$\rightarrow m_{min}(m_{min} - 1) = m_{max}(m_{max} + 1)$$

$$\hookrightarrow m_{min} = -m_{max}$$

$$-m_{min}(\cancel{m_{max}}+1) = m_{max}(\cancel{m_{max}} + 1)$$

D Continued

Therefore $-m_{max} \le m \le m_{max}$

$m_{max} = \cancel{m_{max}}\; m_{min} + n\hbar = -m_{max} + \cancel{\ }\, n\hbar$

$\rightarrow m_{max} = \frac{n\hbar}{2}$

$\rightarrow \ell(\ell+1) = m_{max}(m_{max}+1)$

$\ell = m_{max}$

$\& \ \ell = \cancel{\ }\, m_{min}$

S-2008

Problem 4: Measurement of Hermitian Observables: (10 Points)

Consider a system with three Hermetian observables that are represented in a three-dimensional Hilbert space using the orthonormal basis $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$

with

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The system at time t=0 is in the state:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{6}}|e_1\rangle - \frac{1}{\sqrt{6}}|e_2\rangle + \sqrt{\frac{2}{3}}|e_3\rangle$$

1. Find the eigenvalues and normalized eigenvectors of B and C. **(1 Point)**
2. Find the probability of measuring B at time $t = 0$ with the eigenvalue $b = 1$, and then immediately measuring C and finding the eigenvalue $c = 1$, i.e. find $P_{|\Psi(0)\rangle}(b = 1, c = 1)$. **(2 Points)**
3. Now find the probability if these measurements are performed in reverse order at $t = 0$, i.e. find $P_{|\Psi(0)\rangle}(c = 1, b = 1)$. **(2 Points)**
4. Are the probabilities obtained in part 1. and part 2. the same or different? Explain in detail. **(2 Points)**
5. Use the Generalized Uncertainty Principle to determine a lower bound on $\Delta B \Delta C$ for the system in the initial state $|\Psi(0)\rangle$. Discuss your results. **(2 Points)**
6. Discuss in detail, the conditions that would result in obtaining a lower bound of zero when using the Generalized Uncertainty Principle. Would you expect to get zero for a particular pair of the observables, A, B, and C in this problem? Or for other conditions? **(1 Point)**

S-2008

(4) (A) For B

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2i \\ 0 & -2i & 1-\lambda \end{vmatrix} = \emptyset = (1-\lambda)\left[(1-\lambda)^2 - 4\right]$$

Max powers of $\lambda = 3$, 3 eigenvalues

$$\boxed{\lambda = 1, -1, 3 = \text{eigenvalues}}$$

$\lambda = 1$ $\rightarrow$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \emptyset \rightarrow \left.\begin{matrix} \emptyset = \emptyset \\ 2ic = \emptyset \\ -2ib = \emptyset \end{matrix}\right\} \begin{matrix} c, b = \emptyset \\ a = 1 \end{matrix}$$

$$\rightarrow \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} = \boxed{|a = 1\rangle}$$

$\lambda = -1$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2i \\ 0 & -2i & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \emptyset \rightarrow \begin{matrix} 2a = \emptyset \\ 2b + 2ic = \emptyset \\ -2ib + 2c = \emptyset \rightarrow c = ib \end{matrix}$$

$$|a = -1\rangle \rightarrow \begin{pmatrix} 0 \\ b \\ ib \end{pmatrix} \rightarrow \begin{pmatrix} 0 & b & -ib \end{pmatrix}\begin{pmatrix} 0 \\ b \\ ib \end{pmatrix} = \cancel{b} + b^2 + b^2 = 1$$

$$\rightarrow b = \frac{1}{\sqrt{2}}$$

$\langle a=-1 | a=-1 \rangle = 1$

$$\rightarrow \boxed{|a = -1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}}$$

A Continued

$\underline{\lambda = 3}$ $\rightarrow$ $\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 2i \\ 0 & -2i & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \emptyset \rightarrow$

$-2a = \emptyset$

$-2b + 2ic = \emptyset$

$-2ib - 2c = \emptyset$

$\hookrightarrow c = -ib$

$$\rightarrow \boxed{|a = 3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}}$$

For C

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = \emptyset \rightarrow -\lambda(-\lambda(1-\lambda)) = \emptyset$$

$$\rightarrow \lambda^2(1-\lambda) = \emptyset$$

$$\boxed{\lambda = 1, -1, 1 = \text{eigenvalues}}$$

$\underline{\lambda = 1}$ $\rightarrow$ $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \emptyset \rightarrow$

$-a + b = \emptyset$

$-b = \emptyset$

$0 = \emptyset$

$\hookrightarrow a = b$

$$|c=1\rangle = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix}$$

$$\langle c=1|c=1\rangle = (a\ a\ 0)\begin{pmatrix} a \\ a \\ 0 \end{pmatrix} = a^2 + a^2 = 1$$

$$\rightarrow a = \frac{1}{\sqrt{2}}$$

$$\rightarrow \boxed{|c=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

A Continued

$\lambda = 1'$ $\rightarrow$ $-a+b=\emptyset$

$-b=\emptyset$

$\emptyset=\emptyset \leftarrow c=1$

$$\rightarrow \boxed{|c=1'\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$\lambda = -1$ $\rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \emptyset \rightarrow$ $a+b=\emptyset$, $b=\emptyset$, $2c=\emptyset$ $\rightarrow a=-b$

$$|c=-1\rangle = \begin{pmatrix} a \\ -a \\ \emptyset \end{pmatrix}$$

$$\rightarrow \langle c=-1|c=1\rangle = (a \;\; -a \;\; \emptyset)\begin{pmatrix} a \\ -a \\ 0 \end{pmatrix} = a^2+a^2=1$$

$\rightarrow a = \frac{1}{\sqrt{2}}$

$$\rightarrow \boxed{|c=-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ \emptyset \end{pmatrix}}$$

B $|b=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |e_1\rangle$

$$\rightarrow P(b=1) = |\langle b=1|\psi(\emptyset)\rangle|^2 = \frac{1}{6}$$

$$|c=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle)$$

$$c=1' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |e_3\rangle$$

$$\rightarrow P_{lab}(c=1 \text{ or } c=1') = |\langle c=1|b=1\rangle|^2 + |\langle c=1'|b=1\rangle|^2$$

$$= \frac{1}{2}$$

(B) Continued

$$\boxed{P(b=1 \text{ or } C=1) = P(b=1)\, P_{\text{in } b}(C=1 \text{ or } C=1') = \frac{1}{12}}$$

(C) $P(C=1 \text{ or } C=1') = |\langle C=1 | \psi(\phi) \rangle|^2 + |\langle C=1' | \psi(0) \rangle|^2$

$$= \left| \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} \right|^2 + \frac{2}{3}$$

$$P_{C=1}(b=1) = |\langle b=1 | C=1 \text{ or } C=1' \rangle|^2$$

$$= \frac{1}{2}$$

$$\rightarrow \boxed{P(C=1 \text{ and } b=1) = \frac{1}{3}}$$

(D) $BC = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2i \\ -2i & 0 & 1 \end{pmatrix}$ $CB = \begin{pmatrix} 0 & 1 & 2i \\ 1 & 0 & 0 \\ 0 & -2i & 1 \end{pmatrix}$

$\boxed{[B,C] \neq \phi \text{ So probabilities would } \underline{\text{not}} \text{ be the Same}}$

(E) $\Delta B \Delta C \geq \frac{1}{2}[B,C]$ → apply This on $|\psi(\phi)\rangle$

$\langle B \rangle = \langle \psi(\phi) | B | \psi(\phi) \rangle$ → or, IT Could be $\Delta B = \sqrt{\langle B^2 \rangle - \langle B \rangle^2}$ ☆

☆ → I Think its This ☆ ☆ ☆

(F) would be ϕ if The 2 operators Commute