S-2015

## Problem 5: Interaction Picture of Quantum Mechanics

The "Interaction Picture" of quantum mechanics is in some ways in-between the Schrödinger formulation and the Heisenberg formulation.

Consider a system with the Hamiltonian $H = H_0 + V(t)$ where $H_0$ is independent of time and $V(t)$ may or may not be time dependent. The Interaction Picture is defined by the transformation of the Schrödinger states:

$$\begin{aligned} |\psi\rangle_I &= {U_0}^{-1}|\psi\rangle_S \\ U_0 &= e^{-\frac{i}{\hbar}(t-t_0)H_0}. \end{aligned} \tag{1}$$

The subscripts $I$ and $S$ refer to the Interaction Picture and Schrödinger Picture respectively. $t_0$ is a time when the pictures coincide, and we will set $t_0 = 0$ for this problem.

(a) [1 pt] Show that $U_0$ is a unitary operator. Why is it important for the transformation between pictures be unitary?

(b) [3 pts] The transformation between $|\psi\rangle_S$ and $|\psi\rangle_I$ implies that there is also a transformation of the observables between the pictures. If $A_S$ and $A_I$ are operators for an observable in the Schrödinger and Interaction pictures respectively, derive the relation between $A_S$ and $A_I$. Show that this implies that $H_0$ is the same in the two pictures.

(c) [3 pts] Derive the differential equation that determines the time dependence of the Interaction Picture states, $|\psi(t)\rangle_I$. Be sure to show and explain your work. Explain why the Interaction Picture may be particularly useful when $V(t)$ is "small".

(d) [1 pt] Define the eigenstates of $H_0$ to be

$$H_0|\lambda\rangle_S = E_\lambda|\lambda\rangle_S \tag{2}$$

Show that if $V(t) = 0$, the Interaction Picture energy eigenstates $|\lambda\rangle_I$ are equal to $|\lambda(t=0)\rangle_S$ and independent of time.

(e) [2 pts] Consider a potential of the form

$$V(t) = 0, \quad t \leq 0 \quad V(t) \neq 0, \quad t > 0 \tag{3}$$

The system is in a state $|\psi_0\rangle_I$ for $t < 0$. For $t > 0$ the Interaction Picture state will depend on time. It can be expanded as:

$$|\psi(t)\rangle_I = \sum_\lambda c_\lambda(t)|\lambda(0)\rangle_I \tag{4}$$

In this expression, $c_\lambda(t)$ are time-dependent expansion coefficients for the state and $|\lambda(0)\rangle_I$ is the complete set of time-independent eigenstates of $H_0$ in the interaction picture.

Use the time dependence found in part (c) to derive a set of coupled equations relating $c_\lambda(t)$ and $\partial_t c_\lambda(t)$.

(5) $|\psi\rangle_I = U_0^{-1}|\psi\rangle_S$

$U_0 = e^{-\frac{i}{\hbar}H_0}$

(A) $U_0 U_0^{-1} = e^{-iH_0/\hbar} e^{iH_0/\hbar} = 1$, $\boxed{\text{unitary}}$

$\boxed{\text{Transformation should be unitary in order to preserve the length of the state vector.}}$

(B) $\cancel{{}_I\langle\psi|A_I|\psi\rangle_I = {}_S\langle\psi|e^{-itH_0/\hbar}A\,e^{itH_0/\hbar}|\psi\rangle_S}$

$\cancel{{}_S\langle\psi|A_S|\psi\rangle_S = {}_S\langle\psi|U_0 A_S U_0^{-1}|\psi\rangle_S = {}_I\langle\psi|A_I|\psi\rangle_I}$

$\cancel{U_0 A_S U_0^{-1} = A_I \to A_I}$

$\cancel{{}_S\langle\psi|A_S|\psi\rangle_S = {}_S\langle\psi|e^{-iH_0/\hbar}A_S e^{iH_0/\hbar}|\psi\rangle_S}$

$\cancel{= {}_I\langle\psi|A_I|\psi\rangle_I = {}_S\langle\psi|U_0^{-1}A_S|U_0|\psi\rangle_S}$

$\to {}_I\langle\psi|A_I|\psi\rangle_I = {}_S\langle\psi|U_0 A_I U^{-1}|\psi\rangle_S = {}_S\langle\psi|A_S|\psi\rangle_S$

$\to U_0 A_I U_0^{-1} = A_S$

$\to \boxed{A_I = U_0^{-1} A_S U_0 = e^{iH_0t/\hbar} A_S e^{-iH_0t/\hbar}}$

B Continued

$$\boxed{[A_S, H_0] = [A_I, H_0] = \emptyset}$$

C $|\psi\rangle_I = e^{iH_0 t/\hbar} |\psi_s\rangle$

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle_I = \frac{i\hbar i H_0}{\hbar} e^{iH_0 t/\hbar} |\psi_s\rangle + e^{iH_0 t/\hbar} \underbrace{i\hbar \frac{\partial}{\partial t} |\psi_s\rangle}_{(H_0 + V)|\psi_s\rangle}$$

$$= \cancel{-\frac{\hbar H_0}{\hbar} e^{iH_0 t/\hbar} + H_0 e^{iH_0 t/\hbar}} + e^{iH_0 t/\hbar} V |\psi_s\rangle$$

$$= e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar} e^{iH_0 t/\hbar} |\psi_s\rangle$$

$$\boxed{i\hbar \frac{\partial}{\partial t} |\psi\rangle_I = V_I |\psi\rangle_I}$$

IF This is small, You can use $H_0$ basis For $H$

D

$$\cancel{(H_0 + V)|\psi\rangle_s = H_0 e^{iH_0 t/\hbar} |\psi\rangle_I + V e^{-iH_0 t/\hbar} |\psi\rangle_I}$$

$$\cancel{= E_\lambda}$$

D IF $V(t) = \phi$ $\rightarrow$ $i\hbar \frac{\partial}{\partial t} |\lambda\rangle_I = \phi$

in other words, $|\lambda\rangle_I = e^{iH_0 t/\hbar} \underset{t=\phi}{\uparrow} |\lambda\rangle_S$

$$\boxed{|\lambda\rangle_I = |\lambda\rangle_S}$$

E

$$|\psi(t)\rangle_I = \sum_\lambda c_\lambda(t) |\lambda(0)\rangle_I$$

$$\cancel{i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_I = \sum_\lambda c_\lambda(t) V |\lambda\rangle_I}$$

$$\cancel{i\hbar \frac{\partial}{\partial t} \langle\lambda|\psi\rangle_I =}$$

$$i\hbar \frac{\partial}{\partial t} \langle\lambda|\psi\rangle_I = \sum_\phi \langle\lambda|V_I|\phi\rangle\langle\phi|\psi\rangle_I$$

$$\rightarrow \boxed{i\hbar \frac{\partial}{\partial t} c_\lambda = \sum_\phi V_{\lambda\phi} c_\phi}$$

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**PROBLEM 5: Interaction Picture**

There is a 3rd 'picture' in quantum mechanics in addition to the Schrödinger and Heisenberg pictures that is often used. This picture is called the interaction picture. The interaction picture is related to the Schrödinger picture through the following unitary transformation for a Hamiltonian, $H = H_0 + V$.

$$\Psi_I(x,t) = \mathbf{U}_0^{-1}\Psi_S(x,t)$$

where

$$\mathbf{U}_0 = e^{-\frac{i}{\hbar}(t-t_0)H_0}\,.$$

The Hamiltonian $\mathbf{H}_0$ is assumed to be time independent, $V$ is considered to be small in comparison to $\mathbf{H}_0$, $I$ denotes interaction picture and $S$ denotes Schrödinger picture, $t_0$ is the time when the two pictures coincide (you can take this to be $t_0 = 0$) and $t$ is the time from when the two pictures coincide.

(a) Use this information to find the equation, analogous to the Schrödinger equation, that gives the time evolution for $\Psi_I$. To receive full credit justify all steps. (4 Points)

(b) How are operators in the interaction picture ($\mathbf{\Omega}_I$) and the Schrödinger picture ($\mathbf{\Omega}_S$) related? (2 Points)

(c) These 2 pictures are related to each other through a unitary transformation. In general, what is a unitary transformation and what are the important quantities that a unitary transformation preserves? (3 Points)

(d) Why do you think this is called the interaction picture? Why is it useful? To receive credit you must explain how the name relates to the dynamics. (1 Points)

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(5) (A) $\psi_I = e^{itH_0/\hbar}\psi_S$

$$i\hbar\frac{\partial\psi_I}{\partial t} = i\hbar\left(\frac{iH_0}{\hbar}\right)e^{itH_0/\hbar}\psi_S + e^{itH_0/\hbar}\underbrace{i\hbar\frac{\partial\psi_S}{\partial t}}_{(H_0+V)\psi_S}$$

$$= \cancel{-H_0 e^{itH_0/\hbar}\psi_S} + \cancel{e^{itH_0/\hbar}H_0\psi_S} + e^{itH_0/\hbar}V\psi_S$$

$$= e^{itH_0/\hbar}V\psi_S$$

$$= e^{itH_0/\hbar}Ve^{-itH_0/\hbar}\underbrace{e^{itH_0/\hbar}\psi_S}_{\psi_I}$$

$$\psi_I V_I \psi_I$$

$$= \psi_S e^{-itH_0/\hbar}V_I e^{itH_0/\hbar}\psi_S$$

$$= \psi_S V_S \psi_S$$

$$\to V_S = e^{-itH_0/\hbar}V_I e^{itH_0/\hbar}$$

$$\to V_I = e^{itH_0/\hbar}V_S e^{-itH_0/\hbar}$$

$$\boxed{i\hbar\frac{\partial\psi_I}{\partial t} = V_I\psi_I}$$

(B)

c) a unitary transformation is such that $U_0^{-1} U_0 = 1$

Thus it ~~preserves~~ preserves the inner product (physically observable quantities, probability etc. do not change).

The length & angle between ~~N~~ of state vectors remains constant.

D) interaction places time dependence on both operators & state vectors.

Allows for operators to act on state vectors at different times which is helpful if an operator itself evolves in time.