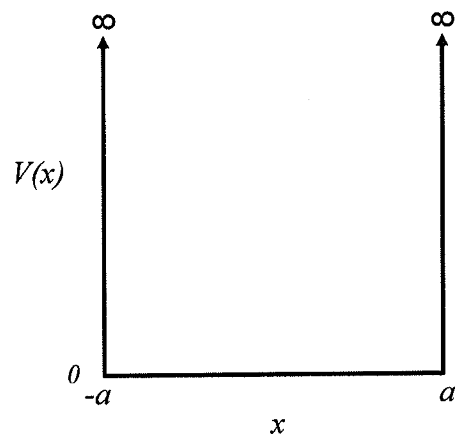
F-2007

PROBLEM 2



Consider the one-dimensional infinite-well potential shown above.

[a] (4pts) Derive expressions for the energy eigenfunctions and energy eigenvalues for a particle in the one-dimensional infinite well shown above. Show your work.

[b] (4pts) Now suppose a perturbation of the form

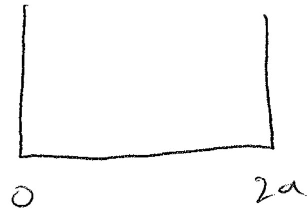
$$\Delta V(x) = V_o a\, \delta(x)$$

is added with $V_o << \frac{\hbar^2 \pi^2}{ma^2}$. Here $\delta(x)$ is the Dirac-delta function. According to first order perturbation theory, what are the eigenenergies of each state?

[c] (2pts) According to first order perturbation theory, what is the wave function of the ground state? Write your answer in terms of a, V_o, fundamental constants, and the unperturbed wave functions $\phi_n(x)$. You do not have to normalize the wave function.

F-2007

2 A Shift well



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad \text{Assume } E > \phi$$

$$\rightarrow \frac{d^2\psi}{dx^2} = \underbrace{-\frac{2mE}{\hbar^2}}_{k^2}\psi \rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi$$

$$\rightarrow \psi = A\sin kx + B\cos kx$$

ψ must be ϕ at $x = \phi$

$\hookrightarrow B \rightarrow \phi$

$$\rightarrow \psi = A\sin kx \quad \rightarrow \psi \text{ must be } \phi \text{ at } 2a$$

$$k(2a) = n\pi$$

$$\rightarrow k = \frac{n\pi}{2a}$$

$$\rightarrow \psi = A\sin\frac{n\pi}{2a}x$$

Normalize $\rightarrow$ $$1 = \int_0^{2a} A^2 \sin^2\frac{n\pi}{2a}x\,dx = A^2\left[\frac{x}{2} - \frac{\sin 2bx}{4b}\right]\Bigg|_0^{2a}$$

$$= A^2\left[\frac{x}{2} - \frac{\sin\frac{n\pi}{a}x}{\frac{4n\pi}{2a}}\right]\Bigg|_0^{2a}$$

$$= A^2 a = 1$$

$$\rightarrow A = \frac{1}{\sqrt{a}}$$

A Continued

$$\psi = \sqrt{\frac{1}{a}} \sin \frac{n\pi x}{2a}$$

$$\rightarrow \frac{n^2\pi^2}{4a^2} = \frac{2mE}{\hbar^2} \rightarrow \boxed{E_n = \frac{\hbar^2 n^2 \pi^2}{8ma^2}}$$

shift well back $0 \rightarrow -a$

$$\psi = \sqrt{\frac{1}{a}} \underbrace{\sin\left[\frac{n\pi}{2a}(x-a)\right]}$$

$$= \sin\frac{n\pi x}{2a}\cos\frac{n\pi a}{2a} + \cos\frac{n\pi x}{2a}\sin\frac{n\pi a}{2a}$$

$$\text{if } \boxed{\begin{array}{l} n = 1,3,5 \rightarrow \frac{1}{\sqrt{a}} \cos\frac{n\pi x}{2a} \\ n = 2,4,6 \rightarrow \frac{1}{\sqrt{a}} \sin\frac{n\pi x}{2a} \end{array}}$$

B $\langle n|V|n\rangle = E_0^{(1)} = \int_{-a}^{a} \frac{1}{a} \cos^2\frac{n\pi x}{2a} V_0 a\delta(x)\,dx$

$$= \frac{V_0 a}{a} = \boxed{V_0 = E_0^{(1)} \text{ if } n = \text{odd}}$$

$$= \int_{-a}^{a} \frac{1}{a} \sin^2\frac{n\pi x}{2a} V_0 a\,\delta(x)\,dx$$

$$= \boxed{0 = \text{if } n = \text{even}}$$

(c) $\psi_0^{(1)} = \sum_{k \neq n} \frac{\langle k|V|n\rangle}{E_n - E_k} |k\rangle$

$$\langle k|V|\phi\rangle = \frac{1}{a}\int_{-a}^{a} \cos\frac{\pi x}{2a} \sin\frac{n\pi x}{2a} \, a V_0 \delta(x) dx$$

$= \phi$ IF n IS even

$$= \frac{1}{a}\int_{-a}^{a} \cos\frac{\pi x}{2a} \cos\frac{n\pi x}{2a} \, a\delta(x) V_0 dx$$

$= V_0$

$$\boxed{\psi = \phi_1 + \sum_{\substack{k\neq 0 \\ \text{odd}}} \frac{V_0 \, 8ma^2}{\hbar^2\pi^2(1-n^2)} \phi_k}$$

$E_0 - E_k$

$= \frac{\hbar^2\pi^2}{8ma^2} - \frac{n^2\hbar^2\pi^2}{8ma}$

$= \frac{8ma^2}{\hbar^2\pi^2}(1-n^2)$

S-2010

PROBLEM 1: The Delta-Function Potential

Let us consider a single particle of mass m moving in one dimension with the Hamiltonian

$$H = T + V(x)\,,$$

where the kinetic energy is

$$T = \frac{P^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\,,$$

the potential energy is

$$V(x) = -V_0\,\delta(x)\,,$$

and $\delta(x)$ is the Dirac delta function.

(a) [2 points] Find an expression for the discontinuity of the derivative of the wave function at $x = 0$.

(b) [3 points] Find the ground state wave function.

(c) [2 points] Find the ground state energy.

(d) [3 points] Find the expectation value for the kinetic energy, $\langle T\rangle$.

S-2010

1. $H = T + V(x)$

$V = -V_0 \delta(x)$

A. $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$ (all $E < \emptyset$ (bound state)

$$\frac{d^2\psi}{dx^2} = \underbrace{\frac{2mE}{\hbar^2}}_{\kappa^2}\psi \rightarrow \psi = \underbrace{Ae^{\kappa x}}_{\text{blows up at } \infty} + \underbrace{Be^{-\kappa x}}_{\text{blows up at } -\infty}$$

$$\psi = \begin{cases} Ae^{\kappa x} & x < \emptyset \\ Be^{-\kappa x} & x > \emptyset \end{cases}$$

$\rightarrow$ at $x = 0$ $\rightarrow$ $\boxed{A = B}$

ψ =

$$\lim_{\epsilon \to \emptyset} -\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}dx - V_0\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)dx = \underbrace{\int_{-\epsilon}^{\epsilon}E\psi(x)}_{\emptyset}$$

$$= -\frac{\hbar^2}{2m}\Delta\frac{d\psi}{dx} = V_0\psi(\emptyset)$$

$$\rightarrow \boxed{\Delta\frac{d\psi}{dx}\bigg|_{x=\emptyset} = -\frac{2mV_0}{\hbar^2}\psi(\emptyset)}$$

B $\cancel{}$ $\left.\frac{d\psi}{dx}\right|_{x>0} = -kB$

$\left.\frac{d\psi}{dx}\right|_{x<0} = kA = kB$

$\rightarrow \Delta\frac{d\psi}{dx} = \cancel{+} \cancel{2}k\cancel{B} = \cancel{+}\frac{\cancel{2}mV_0}{\hbar^2}\cancel{B}$

$\rightarrow k = \frac{mV_0}{\hbar^2}$

$\rightarrow \psi = Be^{-\frac{mV_0}{\hbar^2}|x|}$

even Function because of absolute value

$\rightarrow$ Find B $\rightarrow$ $1 = B^2\int_{-\infty}^{\infty} e^{-\frac{2mV_0}{\hbar^2}|x|}dx$

$= 2B^2\int_0^{\infty} e^{-\frac{2mV_0}{\hbar^2}x}dx$

$= -\frac{2\hbar^2}{2mV_0}(0-1) = \frac{\hbar^2}{mV_0}B^2 \stackrel{?}{=} 1$

$\rightarrow B = \frac{\sqrt{mV_0}}{\hbar}$

$\rightarrow \boxed{\psi = \frac{\sqrt{mV_0}}{\hbar}e^{-\frac{mV_0}{\hbar^2}\cancel{\$}|x|}}$

C $\cancel{\frac{2mE}{\hbar^2} = \frac{mV_0}{\hbar^2} \to E = -\frac{mV_0}{2}}$

dimension's

$$\frac{2mE}{\hbar^2} = \frac{m^2 V_0^2}{\hbar^4} \to \boxed{E = -\frac{mV_0^2}{2\hbar^2}}$$

D $$\langle T \rangle = \int_{-\infty}^{\infty} -\frac{\hbar^2}{2m} \frac{mV_0}{\hbar^2} e^{-\frac{mV_0}{\hbar^2}|x|} \frac{d^2}{dx^2} e^{-\frac{mV_0}{\hbar^2}|x|} dx$$

or $\langle T \rangle = \langle E \rangle - \langle V \rangle$

$$\to \langle V \rangle = \int_{-\infty}^{\infty} -V_0 \delta(x) \frac{mV_0}{\hbar^2} e^{-\frac{mV_0}{\hbar^2}|x|} dx$$

$$= -\frac{V_0^2 m}{\hbar^2}$$

$\langle E \rangle$ is a constant

$$\to \langle T \rangle = -\frac{mV_0^2}{2\hbar^2} + \frac{2V_0^2 m}{2\hbar^2}$$

$$= \frac{mV_0^2}{2\hbar^2}$$

(D) Try other way

$$\langle T \rangle = -\frac{V_0}{2} \int_{-\infty}^{\infty} \frac{m^2 V_0^2}{\hbar^{\cancel{3}4}} e^{\cancel{Aaaa} -\frac{2mV_0}{\hbar^2}|x|} \, dx$$

↑ even Function

$$= \frac{-V_0^{\cancel{2}3} m^2}{\hbar^{\cancel{2}4}} \int_0^{\infty} e^{-\frac{2mV_0}{\hbar^2}|x|}$$

$$= \frac{+V_0^3 m^2}{\hbar^4} \frac{\hbar^2}{2mV_0} = \boxed{\frac{V_0^2 m}{2\hbar^2}}$$

S-2013

Problem 1: Bound States and Scattering for a Delta-Function Well

Consider a delta-fuction for a 1-D system,

$$V(x) = -g\;\delta(x) \tag{1}$$

where $g > 0$. We will consider the states of a particle of mass m interacting with this potential for both $E < 0$ and $E > 0$.

This potential has a single bound state $E_b < 0$.

(a) [1 pt] Explain why the bound state wavefunction for the particle will have the form $\Psi(x) = ce^{-|x|/\lambda}$. (You don't need to solve for anything to answer this question.)

(b) [2 pts] Derive the boundary conditions for $\Psi(x)$ and $\partial_x\Psi(x)$ at $x = 0$.

(c) [1 pt] Using the boundary conditions at $x = 0$, determine the value of λ.

(d) [1 pts] What is the energy of the bound state, E_b? What is the normalization constant c?

(e) [2 pts] What is the uncertainty in position, Δx for the particle in this bound state?

(f) [2 pts] Next consider a scattering state for this particle with energy $E > 0$

$$\begin{aligned} \Psi(x) &= e^{ikx} + ae^{-ikx}, \quad x < 0 \\ &= be^{ikx}, \quad x > 0 \end{aligned} \tag{2}$$

For this state, $E = \frac{\hbar^2 k^2}{2m}$

Using the boundary conditions you found in part (b), determine a and b, and the transmission and reflection coefficients for this scattering state.

1 $V(x) = -g\delta(x)$

A $E_b < \emptyset$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = \frac{-2mE}{\hbar^2}\psi$$

$\underline{E < \emptyset}$

$$\hookrightarrow \frac{d^2\psi}{dx^2} = \underbrace{\frac{2mE}{\hbar^2}}_{\kappa^2}\psi$$

$$\rightarrow \psi(x) = Ae^{\kappa x} + Be^{-\kappa x}$$

$$= Ce^{-\kappa|x|}$$

Both Terms blow up at either end

B $\psi(x) = \begin{cases} Ae^{\kappa x} & x < \emptyset \\ Be^{-\kappa x} & x > 0 \end{cases} \rightarrow$ $\boxed{A = B \quad \text{First boundary Condition}}$

<u>Discontinuity in derivative at $x=0$</u>

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} - g\delta(x)\psi = E\psi$$

B Continued

$$\int_{-\epsilon}^{\epsilon} -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\,dx - \int_{-\epsilon}^{\epsilon} g\,\delta(x)\,\psi\,dx = \int_{-\epsilon}^{\epsilon} E\psi\,dx$$

$$\underbrace{}_{-\frac{\hbar^2}{2m}\Delta\left(\frac{d\psi}{dx}\right)} \qquad g\psi(\phi) \qquad E\psi(\epsilon+\epsilon)$$

$\lim \epsilon \to \phi$

$= \phi$

$$\to \boxed{\Delta\left(\frac{d\psi}{dx}\right) = \frac{-2m}{\hbar^2} g\,\psi(\phi)}$$ Second boundary Condition

$$\to \left.\frac{d\psi}{dx}\right|_{x>0} = -Bk\,e^{-kx}$$

$$\to \left.\frac{d\psi}{dx}\right|_{x<\phi} = Ak\,e^{kx}$$

$A = B$

C

$$\to -2Bk\,e^{-kx} = \frac{-2m}{\hbar^2} g\,\psi(\phi)$$

$$\to k = \frac{1}{\lambda} \to -\frac{2B}{\lambda} e^{-\frac{x}{\lambda}} = \frac{-2m}{\hbar^2} g\,\psi(\phi)$$

$x = \phi$

$$\to -\frac{2B}{\lambda} = \frac{-2m}{\hbar^2} g B$$

$$\to \frac{1}{\lambda} = \frac{mg}{\hbar^2} \to \boxed{\lambda = \frac{\hbar^2}{mg}}$$

D $\cancel{\frac{1}{\lambda} = \frac{mg}{\hbar^2} = \frac{2mE}{\hbar^2}}$

$$\frac{1}{\lambda} = \sqrt{\frac{2mE}{\hbar^2}} \rightarrow \frac{1}{\lambda^2} = \frac{2mE}{\hbar^2}$$

$$\rightarrow E = \frac{\hbar^2}{2m\lambda^2}$$

$$= \frac{\hbar^2 m^2 g^2}{2m\hbar^4} = \boxed{-\frac{mg^2}{2\hbar^2} = E}$$

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2\int_0^{\infty} |\psi(x)|^2 dx$$

$$= 2B^2 \int_0^{\infty} e^{-2kx} dx$$

$$= 2B^2 \left[\frac{e^{-2kx}}{-2kx} \Big|_0^{\infty} \right]$$

$$= \frac{2B^2}{2k} \left(-(0-1) \right) \Big]$$

$$= \frac{2B^2}{2k} = B^2\lambda = 1$$

$$\rightarrow B^2 = \frac{1}{\lambda} = \frac{mg}{\hbar^2}$$

$$\rightarrow \boxed{B = \frac{\sqrt{mg}}{\hbar}}$$

E $\Delta x = \langle \Delta x^2 \rangle = \sqrt{\langle x^2 \rangle - \langle x \rangle}$

$$\rightarrow \langle x^2 \rangle = 2\int_0^\infty x^2 \frac{mg}{\hbar^2} e^{-2x/\lambda} \, dx$$

$$= \frac{2mg}{\hbar^2} \int_0^\infty x^2 e^{-2x/\lambda} \, dx$$

$$= \frac{2mg}{\hbar^2} \left[\frac{\Gamma(2+1)}{\left(\frac{2}{\lambda}\right)^3} \right]$$

$$= \frac{4mg}{8\hbar^2} \lambda^3 = \frac{\hbar^6}{m^3 g^3} \frac{mg}{2\hbar^2} = \frac{1}{2} \frac{\hbar^4}{m^2 g^2}$$

$$\rightarrow \langle x \rangle = \frac{mg}{\hbar^2} \int_{-\infty}^{\infty} x \, e^{2|x|/\lambda} \, dx$$

$e^{2|x|/\lambda}$

is an even function

Since absolute value $f(-x) = f(x)$

x is odd

$$\rightarrow \langle x \rangle = \emptyset$$

E Continued

$$\boxed{\Delta x = \frac{1}{\sqrt{2}} \frac{\hbar^2}{mg}}$$

F $1 + a = b$

$$\left.\frac{d\psi}{dx}\right|_{x>0} = ikbe^{ikx} \rightarrow ikb$$

$$\left.\frac{d\psi}{dx}\right|_{x<0} = ike^{ikx} - ikae^{-ikx} \rightarrow ik - ika$$

$$\rightarrow \Delta = ikb - ik + ika = \frac{-2m}{\hbar^2} g(1+a)$$

$$= ik(1+a) - ik + ika = \frac{-2m}{\hbar^2} g(1+a)$$

$$= 2ika = \underbrace{-\frac{2m}{\hbar^2} g(1+a)}_{\frac{-2mg}{\hbar^2} - \frac{2m}{\hbar^2} ga}$$

$$\rightarrow a\left(2ik + \frac{2mg}{\hbar^2}\right) = \frac{-2mg}{\hbar^2}$$

$$\rightarrow \boxed{a = \frac{-mg}{\hbar^2}\left(\frac{1}{ik + mg/\hbar^2}\right) = \frac{-mg}{ik\hbar^2 + mg}}$$

$$\cancel{R = \frac{m^2 g^2}{(ik\hbar^2 + mg)^2}}$$

$$a^2 = R = \frac{m^2 g^2}{(-ik\hbar^2 + mg)(ik\hbar^2 + mg)} = \boxed{\frac{m^2 g^2}{k^2 \hbar^4 + m^2 g^2} = R}$$

F Continued

$$\cancel{T = 1 - \frac{m^2g^2}{(Lkt^2+mg)^2} = \frac{(Lkt^2+mg)^2 - m^2g^2}{(Lkt^2+mg)^2}}$$

$$\cancel{= -k^2t^4 + m^2g^2 + 2Lkt^2mg - m^2g^2}$$

$$\cancel{\boxed{= \frac{2Lkt^2mg - k^2t^2}{(Lkt^2+mg)^2} = T}}$$

$$b = 1 + a \rightarrow \frac{Lkt^2 + mg - mg}{Lkt^2 + mg}$$

$$\rightarrow \boxed{b = \frac{Lkt^2}{Lkt^2 + mg}}$$

Probably better way to get T

$$\text{is } b^2 = \left(\frac{-Lkt^2}{-Lkt^2+mg}\right)\left(\frac{Lkt^2}{Lkt^2+mg}\right)$$

$$= \frac{k^2t^4}{k^2t^2 - Lkt^2mg + Lkt^2mg + m^2g^2}$$

$$= \boxed{\frac{k^2t^2}{k^2t^2 + m^2g^2} = T}$$

$$\boxed{R + T = 1 \checkmark}$$