S-2012

PROBLEM 1: Stationary States

For a quantum system with a time independent Hamiltonian ($\mathbf{H}$), the wave function ($\Psi(x,t)$) is a linear combination of stationary state solutions ($\Psi_n(x,t)$) to the Schrödinger equation:

$$\Psi_n(x,t) = u_n(x) \exp\left(-iE_n t/\hbar\right)$$

where $u_n(x)$ are eigenfunctions of the Hamiltonian

$$\mathbf{H}u_n(x) = E_n u_n(x)$$

and they form a complete orthonormal basis.

(a) Evaluate the uncertainty in the energy for a system in a stationary state with the wave function $\Psi(x,t) = \Psi_n(x,t)$. [Show all work.] (2 Points)

(b) Derive the time evolution operator $\mathbf{U}(t, t_0)$ in terms of the Hamiltonian ($\mathbf{H}$), and apply it to a stationary state $\Psi_n(x, t_0 = 0)$. Describe the change in the stationary state. (2 Points)

Now consider a particle that starts out in a normalized wave function

$$\Psi(x,0) = c_1 u_1(x) + c_2 u_2(x)$$

where the $u_n(x)$ are real eigenfunctions of the Hamiltonian and c_n are real.

(c) Determine an expression for the wave function $\Psi(x,t)$ at subsequent times. (2 Points)

(d) Evaluate the probability density and describe its motion in time. (3 Points)

(e) Determine the uncertainty in the energy ΔE with $\Delta t = \tau$ that is the period of oscillation in (d). (1 Points)

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(1) (A) $\psi_n(x,t) = U_n(x)\exp\left(-i\frac{E_n t}{\hbar}\right)$

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2}$$

$$\to \langle E^2\rangle = \int_{-\infty}^{\infty} \psi_n^* E^2 \psi \, dx$$

$$= \int_{-\infty}^{\infty} U_n(x)^2 \underset{\substack{\uparrow \\ \text{Constant}}}{E^2}\, dx = E^2 \underbrace{\int_{-\infty}^{\infty} U_n^2 dx}_{\text{normalized } = 1}$$

$$\to \langle E^2\rangle = E^2$$

$$\to \langle E\rangle = \int_{-\infty}^{\infty} \psi_n^* E \psi_n \, dx = E \underbrace{\int_{-\infty}^{\infty} U_n^2 dx}_{=1}$$

$$= E$$

$$\to \Delta E = \sqrt{E^2 - E^2} = \boxed{\emptyset = \Delta E}$$

B) $\psi_n(x, t_0 = \emptyset) = u_n(x)$

Time evolution operator →

$$U(t)\,\psi_n(x) = \psi_n(x,t)$$

→ $i\hbar \dfrac{\partial \psi}{\partial t} = H\psi$

→ $i\hbar \dfrac{\partial}{\partial t}\left[U_{(t)}\psi(x)\right] = H\,U(t)\,\psi(t)$

→ $\psi(x)$ does not depend on time

→ $i\hbar \dfrac{\partial}{\partial t} U(t) = H\,U(t)$

→ $\dfrac{\partial}{\partial t} U(t) = -i\dfrac{H}{\hbar} U(t)$

→ $\boxed{U(t) = e^{-iHt/\hbar}}$

→ $\boxed{U(t)\,u_n(x) = e^{-iE_n t/\hbar}\,u_n(x)}$

→ Standing wave that oscillates with angular frequency of $\dfrac{E_n}{\hbar}$ (ωt)

(c) $\psi(x,0) = C_1 U_1(x) + C_2 U_2(x)$

$$\boxed{\psi(x,t) = \cancel{\;} U(t)\,\psi(x,0) = C_1 e^{-iE_1 t/\hbar} U_1(x) + C_2 e^{-iE_2 t/\hbar} U_2(x)}$$

(D) probability density $= \langle \psi(x,t) | \psi(x,t) \rangle$ ~~dx~~

$$= \left[C_1 e^{iE_1 t/\hbar} U_1(x) + C_2 e^{iE_2 t/\hbar} U_2(x) \right]$$

$$\times \left[C_1 e^{-iE_1 t/\hbar} U_1(x) + C_2 e^{-iE_2 t/\hbar} U_2(x) \right]$$

$$= C_1^2 U_1^2(x) + C_1 C_2 e^{it/\hbar (E_1 - E_2)} U_1 U_2$$

$$+ C_2^2 U_2^2(x) + C_2 C_1 e^{it/\hbar (E_2 - E_1)} U_2 U_1$$

$$= \boxed{C_1^2 U_1^2(x) + C_2^2 U_2^2(x) + \cancel{\;} C_1 U_1 C_2 U_2 \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)}$$

The probability of finding a particle in $x + dx$ of space oscillates with time

if $E_1 = E_2$ $\quad |C_1 U_1 + C_2 U_2|^2 =$ probability density,

The probability density is stationary with time

E $\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle}$

This is the proper way to calculate $\langle E\rangle$ math should work

$\langle\psi|H|\psi\rangle = \langle\psi| i\hbar \frac{\partial}{\partial t}|\psi\rangle \overset{?}{=} \langle\psi|E|\psi\rangle$

$\langle\psi|HH|\psi\rangle = \langle\psi| -\hbar^2 \frac{\partial^2}{\partial t^2}|\psi\rangle = \langle E^2\rangle$

$\rightarrow \langle E^2\rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi^* \frac{\partial^2}{\partial t^2}\psi \, dx$

$\rightarrow \frac{\partial^2}{\partial t^2}\psi = \left(\left(-\frac{iE_1}{\hbar}\right)\left(-\frac{iE_1}{\hbar}\right)\right) e^{-iE_1 t/\hbar}\psi_1(x) + \left(-\frac{iE_2}{\hbar}\right)\left(-\frac{iE_2}{\hbar}\right) e^{-iE_2 t/\hbar}\psi_2(x)$

$= -\frac{E_1^2}{\hbar^2}\psi_1 - \frac{E_2^2}{\hbar^2} u_2$

You'll get cross terms that are orthogonal $\int u_1 u_2 = 0$

$= E_1^2 + E_2^2$

$\langle E\rangle = i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial}{\partial t}\psi \, dx$

$\frac{d}{dt}\psi = -\frac{iE_1}{\hbar} + -\frac{iE_2}{\hbar}$

$E_1 + E_2$

$\Rightarrow \Delta E = \sqrt{E_1^2 + E_2^2 - E_1 - E_2}$

$= E_1(E_1 - 1) + E_2(E_2 - 1)$

$-E_1 - E_2 = -2E_2E_1$

$(E_2 - E_1)^2$

$= E_2^2 + E_1^2 - 2E_2E_1$

F-2014

PROBLEM 1: Stationary and Non-Statonary States

Consider a quantum system whose particles are in the following state:

$$\Psi(x,t) = \frac{1}{\sqrt{8}}\psi_1(x)\mathrm{e}^{-iE_1t/\hbar} - i\sqrt{\frac{3}{8}}\psi_3(x)\mathrm{e}^{-iE_3t/\hbar} + \frac{1}{\sqrt{2}}\psi_5(x)\mathrm{e}^{-iE_5t/\hbar}, \tag{1}$$

where $\psi_n(x)$, $n = 1, 2, 3\ldots$ are stationary states of the Hamiltonian governing the system,

$$H\psi_n(x) = E_n\psi(x).$$

Answer the following questions:

a) Do you expect $\langle x\rangle$, $\langle x^2\rangle$ and $\langle E\rangle$ to be time dependent or time independent? Discuss briefly, but do not calculate. (2 Points)

b) Is the uncertainty ΔE positive, negative or zero? Is ΔE time dependent or time independent? Again, discuss briefly but do not calculate. (2 Points)

c) Is $\Psi(t)$ above a solution of the time dependent Schroedinger equation? Demonstrate. (2 Points)

d) If the stationary states $\psi_1(x)$, $\psi_3(x)$ and $\psi_5(x)$ are eigenstates of the harmonic oscillator, will any of your answers to part a) change? Justify. (2 Points)

e) Now assume the particles are in the state

$$\Psi(x,t) = \psi_3(x)\mathrm{e}^{-iE_3t/\hbar}.$$

Answer parts a) and b) for this state. (2 Points)

F-2014

1 A $\frac{d\langle A\rangle}{dt} = \langle \frac{1}{i\hbar}[H, A]\rangle$

if in an eigenbasis of the Hamiltonian.

$[x, H] \neq \emptyset$

$[x^2, H] \neq 0$

$[E, H]$ doesn't make sense but $[H, H] = \emptyset$

No time dependence for E

but time dependence for x & x^2

B $\Delta E > \emptyset$ Proportional to $E_2 - E_1$

has to be real

C $i\hbar \frac{\partial \psi}{\partial t} = H\psi$

$\rightarrow i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{\sqrt{8}} \psi_1 \left(\frac{-iE_1}{\hbar}\right) e^{-iE_1 t/\hbar} - i\sqrt{\frac{3}{8}} \psi_3 \left(\frac{-iE_3 t}{\hbar}\right) e^{-iE_3 t/\hbar}$

$+ \frac{1}{\sqrt{2}} \psi_5 e^{-iE_5 t/\hbar} \left(\frac{-iE_5}{\hbar}\right) + i\hbar$

$= E_1 \ldots$ $\boxed{\text{yes, it is.}}$

D $\langle x \rangle = \alpha \langle a^\dagger \rangle + \langle a \rangle$

$$= \psi_1 \cancel{\psi_2}^{0} + \psi_3 \cancel{\psi_4}^{0} + \cancel{\psi_5} \psi_6 + \cdots$$

Nothing will change

$\langle x^2 \rangle$ Just do the Calculation (don't worry about constants)

E $\langle X \rangle$ will be zero

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi^2 \, dx$$

Time independent

$\langle E \rangle$ Time Independent

$\Delta E = \emptyset$

definite energy