PROBLEM 3: Time-Dependent Perturbation Theory

Consider a non-relativistic particle of mass m and charge q with the potential energy:

$$V(x) = \frac{1}{2}\, k\, X^2$$

A homogeneous electric field $\mathcal{E}(t)$ directed along the x-axis is switched on at time $t = 0$. This causes a perturbation of the form

$$H' = -q\, X\, \mathcal{E}(t)$$

where $\mathcal{E}(t)$ has the form

$$\mathcal{E}(t) = \mathcal{E}_o e^{-t/\tau}$$

where $\mathcal{E}_o$ and τ are constants.

The particle is in the ground state at time $t \leq 0$. This problem will deal with calculating the probability that it will be found in an excited state as $t \to \infty$.

The probability that the particle makes a transition from an initial state i to a final state f is given by:

$$P_{fi}(t, t_o) = \frac{1}{\hbar^2} \left| \int_{t_o}^{t} dt' \, \langle \phi_f | H'(t') | \phi_i \rangle \, e^{i\omega_{fi} t'} \right|^2 .$$

where the particle originally is in state ϕ_i and finally in state ϕ_f.

(a) [2 points] In terms of known quantities, what is the value of ω_{fi} ?

(b) [2 points] How many excited states can the particle make a transition to?

(c) [6 points] Derive an expression for the probability that the particle will be found in any allowed excited state as $t \to \infty$.

S-2010

3

A $\omega_{fi} = \frac{E_f - E_i}{\hbar} = \frac{\hbar\omega(n+\frac{1}{2}) - \hbar\omega(\frac{1}{2})}{\hbar}$

$= \boxed{\omega n}$

B $H' = -qx\mathcal{E}(t)$ $\hbar = kg\frac{m^2}{s}$

$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)$

$\rightarrow \left[\cancel{\langle n|a|0\rangle}^{0} + \underbrace{\langle n|a^+|0\rangle}_{\underbrace{\langle n|1\rangle}_{\delta_{n,1}}}\right]\sqrt{\frac{\hbar}{2m\omega}}(-q\mathcal{E})$

$\boxed{1 \text{ state, } n=1}$

C $P_{fi} = \frac{1}{\hbar^2}\frac{\hbar}{2m\omega}q^2\left|\underbrace{\int_0^\infty e^{-t/\tau}e^{i\omega t}\,dt}_{\underbrace{\left[\frac{e^{-(t/\tau - i\omega t)}}{-\frac{1}{\tau}+i\omega}\right|_0^\infty}_{(\frac{1}{\tau}+i\omega)(\frac{1}{\tau}-i\omega) = \frac{1}{\tau^2}+\omega^2}}\right|^2$

$= 1 + \tau^2\omega^2$

$\rightarrow \boxed{P_{fi} = \frac{1}{2\hbar}\frac{q^2}{m\omega}\frac{1}{1+\tau^2\omega^2}}$

F-2011

PROBLEM 5: Stationary Perturbation Theory

Consider a particle of mass m confined in a 2D infinite square well:

$$V(x,y) = \begin{array}{ll} 0, & \text{for } 0 \leq x \leq L \text{ and } 0 \leq y \leq L, \\ \infty, & \text{otherwise,} \end{array}$$

with energy eigenfunctions

$$\psi_{n_x,n_y}(x,y) = \frac{2}{L}\sin\left(\frac{n_x\pi}{L}x\right)\sin\left(\frac{n_y\pi}{L}y\right).$$

(a) What are the energies and degeneracies of the first four energy levels (eigenenergies) of the particle? Explain your answer. [1 point]

Impurities in the well will shift these energy levels. Assume we can model the effect of an impurity through a local potential:

$$W(x,y) = -V_0\ L\ \delta(x-x_0)\delta(y-y_0)$$

where the point (x_0, y_0) is the position of the impurity.

(b) For the case where $x_0 = y_0 = L/2$, what are the energy shifts (including splitting of energy levels) to first order in V_0 for the first two energy levels of the particle? Show your work. [3 points]

Which of the energy eigenstates will not be changed by this impurity? Explain. (You should not have to do any calculations to answer this second question.)

(c) Again for $x_0 = y_0 = L/2$, what is the shift in the ground state energy that is second order in V_0? You should write your result in terms of sums, and approximate the result by summing over the largest terms. [3 points]

(d) For the case where $x_0 = L/3$ and $y_0 = L/4$, what are the energy shifts (including splitting of energy levels) to first order in V_0 for the first two energy levels of the particle? Show your work. [3 points]

F-2011

5 A $\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow k^2 = \frac{2mE}{\hbar^2}$

$\rightarrow k^2 = \frac{n^2\pi^2}{L^2}$

$\rightarrow E = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$\rightarrow E_{x,y} = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2+n_y^2)$

$\rightarrow \boxed{E_{11} = \frac{\pi^2\hbar^2}{mL^2}}$ $\boxed{E_{12} = E_{21} = \frac{5}{2}\frac{\pi^2\hbar^2}{mL^2}}$

$\rightarrow \boxed{E_{22} = \frac{4\pi^2\hbar^2}{mL^2}}$ $\boxed{E_{31} = E_{13} = \frac{5\pi^2\hbar^2}{mL^2}}$

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B $E_{11}^{(1)} = \int_0^L\int_0^L \psi W \psi \, dx\,dy = -\frac{4V_0}{L}\int_0^L\int_0^L \sin^2\left(\frac{\pi}{L}x\right)\sin^2\left(\frac{\pi}{L}y\right)\delta\left(x-\frac{L}{2}\right)\delta\left(y-\frac{L}{2}\right)dx\,dy$

$= \boxed{-\frac{4V_0}{L} = E_{11}^{(1)}}$

B Continued

$E_{12} = E_{21} \rightarrow W \doteq$

$$\begin{array}{ccc} & |12\rangle & |21\rangle \\ |12\rangle & W_{11} & W_{12} \\ |21\rangle & W_{21} & W_{22} \end{array}$$

$$\Rightarrow W_{11} = \frac{-4V_0}{L} \int_0^L \int_0^L \sin^2\left(\frac{\pi x}{L}\right) \sin^2\left(\frac{2\pi x}{L}\right) \delta\left(x - \frac{L}{2}\right) \delta\left(y - \frac{L}{2}\right)$$

These will all be zero

Since you'll always get

at least 1 $\sin 2\pi$)

$$\boxed{E_{12}^{(1)} = E_{21}^{(1)} = \phi}$$

C $\displaystyle\sum_{k \neq n} \frac{\langle n | V' | k \rangle^2}{E_n - E_k}$ you can do this

D you can do this

F-2010

PROBLEM 4: Stationary Perturbation Theory

Consider a non-relativistic particle of mass m moving in the three dimensional potential:

$$V(x) = \frac{1}{2}k(x^2 + y^2 + z^2).$$

(a) [1 point] What is the ground state energy and first excited state energy for this potential?

Now there is a perturbation applied so the potential becomes

$$V(x) = \frac{1}{2}k(x^2 + y^2 + z^2) + \lambda xy$$

where λ is a small parameter.

(b) [1 point] Calculate the ground state energy to first order in λ.

(c) [4 point] Calculate the ground state energy to second order in λ.

(d) [4 point] Calculate the first excited state energies to first order in λ.

F-2010

4 $V(x) = \frac{1}{2}k(x^2+y^2+z^2)$

A $E = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)$

$$\boxed{\begin{gathered} E_0 = \frac{3}{2}\hbar\omega \\ E_{100} = E_{010} = E_{001} = \hbar\omega\left(\frac{5}{2}\right) \end{gathered}}$$

B $V' = \lambda xy$

$E_0^{(1)} = \langle 0,0,0 | \lambda xy | 0,0,0 \rangle$

$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$

$xy = \frac{\hbar}{2m\omega}(a_x + a_x^\dagger)(a_y + a_y^\dagger)$

$= \frac{\hbar}{2m\omega}(a_x a_y + a_x a_y^\dagger + a_x^\dagger a_y + a_x^\dagger a_y^\dagger)$

$E_0^{(1)} = \frac{\hbar}{2m\omega}\Big[\cancel{\langle 0,0 | a_x a_y | 0,0 \rangle}^{\to\phi} + \cancel{\langle 0,0 | a_x a_y^\dagger | 0,0 \rangle}^{\to\phi} + \cancel{\langle 0,0 | a_x^\dagger a_y | 0,0 \rangle}^{\to\phi} + \cancel{\langle 0,0 | a_x^\dagger a_y^\dagger | 0,0 \rangle}^{\to\phi}\Big]$

$$\boxed{E_0^{(1)} = \phi}$$

c) $E_0^{(2)} = \sum_{k \neq n} \frac{\langle 0 | \lambda xy | n \rangle^2}{E_0 - E_n}$

$= \frac{\lambda^2 \hbar^2}{4m^2\omega^2 \hbar\omega} \left[\frac{\langle 0 | a_x a_y | n \rangle^2 + \langle 0 | a_x a_y^\dagger | n \rangle^2 + \langle 0 | a_x^\dagger a_y | n \rangle^2 + \langle 0 | a_x^\dagger a_y^\dagger | n \rangle^2}{-n_x - n_y} \right]$

$= \frac{\lambda^2 \hbar^2}{4m^2\omega^2 \hbar\omega} \left[\frac{\delta_{0,0,n_x-1,n_y-1} \sqrt{n_x}^2 \sqrt{n_y}^2}{-n_x - n_y} + \frac{\delta_{0,0,n_x-1,n_y+1} \sqrt{n_x}^2 \sqrt{n_y+1}^2}{-n_x - n_y} + \frac{\delta_{0,0,n_x+1,n_y-1} \sqrt{n_x+1}^2 \sqrt{n_y}^2}{-n_x - n_y} + \frac{\delta_{0,0,n_x+1,n_y+1} \sqrt{n_x+1}^2 \sqrt{n_y+1}^2}{-n_x - n_y} \right]$

$n_x = 1 = 0$

$= \frac{\lambda^2 \hbar}{4m^2\omega^3} \left[\frac{1}{-2} + 0 + 0 + 0 \right] = \boxed{\frac{-\lambda^2 \hbar}{8m^2\omega^3} = E_0^{(2)}}$

D $V' \doteq \begin{array}{c|cc} & |0\,1\rangle & |1\,0\rangle \\ |0\,1\rangle & V_{11} & V_{12} \\ |1\,0\rangle & V_{21} & V_{22} \end{array}$

$V_{11} = \langle 01 | \lambda xy | 01 \rangle = V_{22} = \emptyset$

$\langle 10 | 10 \rangle$

$V_{12} = \langle 10 | \lambda xy | 01 \rangle = \lambda \Big[\emptyset + \emptyset + \langle 10 | a_x^{\dagger} a_y | 01 \rangle$

$+ \langle 10 | a_x^{\dagger} a_y^{\dagger} | 01 \rangle \Big]$

$\langle 10 | 12 \rangle = \emptyset$

$= \dfrac{\lambda \hbar}{2m\omega}$

$\rightarrow V' \doteq \begin{pmatrix} -E & \frac{\lambda\hbar}{2m\omega} \\ \frac{\lambda\hbar}{2m\omega} & -E \end{pmatrix} = E^2 - \dfrac{\lambda^2 \hbar^2}{2^2 m^2 \omega^2} = \emptyset$

$\rightarrow \boxed{E_{\pm} = \pm \dfrac{\lambda \hbar}{2m\omega}}$

F-2008

Problem 2: Near Degenerate Perturbation (10 Points)

Consider a system with two energy levels that are very close to each other while all others are far away. In this system, the unperturbed Hamiltonian (H_0) has two eigenstates $|\psi_1^{(0)}\rangle$ and $|\psi_2^{(0)}\rangle$ with energy eigenvalues $E_1^{(0)}$ and $E_2^{(0)}$ that are very close to each other

$$|E_1^{(0)} - E_2^{(0)}| \simeq 0 \,. \tag{1}$$

We often choose a state of the form

$$|\psi\rangle = a|\psi_1^{(0)}\rangle + b|\psi_2^{(0)}\rangle \tag{2}$$

and try to diagonalize the complete Hamiltonian ($H = H_0 + H_1$) with

$$\begin{aligned} H|\psi\rangle &= E|\psi\rangle &(3)\\ H_0|\psi_i^{(0)}\rangle &= E_i^{(0)}|\psi_i^{(0)}\rangle &(4)\\ H_{ij} &= \langle\psi_i^{(0)}|H|\psi_j^{(0)}\rangle, i,j = 1,2 &(5) \end{aligned}$$

as well as

$$\tan\beta = \frac{2H_{12}}{H_{11} - H_{22}} \,. \tag{6}$$

(a) **(2 Points)** Solve the characteristic equation and find the energy eigenvalues E_1 and E_2.

(b) **(3 Points)** Show that the normalized states corresponding to the energy values E_1 and E_2 are

$$\begin{aligned} |\psi_1\rangle &= \cos(\beta/2)|\psi_1^{(0)}\rangle + \sin(\beta/2)|\psi_2^{(0)}\rangle &(7)\\ |\psi_2\rangle &= -\sin(\beta/2)|\psi_1^{(0)}\rangle + \cos(\beta/2)|\psi_2^{(0)}\rangle \,. &(8) \end{aligned}$$

In (c) and (d), consider the limit

$$|H_{11} - H_{22}| \gg |H_{12}| = |(H_1)_{12}| \,. \tag{9}$$

(c) **(3 Points)**

Find the energy eigenvalues E_1 and E_2 for the Hamiltonian H to the order of H_{12}^2 in terms of H_{11}, H_{22}, and H_{12} as well as in terms of $E_i^{(0)}$ and $|\psi_i^{(0)}\rangle, i = 1,2$.

(d) **(2 Points)** Find the eigenstates $|\psi_i\rangle, i = 1,2$.

F-2008

② $H = H_0 + H_1$

Ⓐ $$H_{ij} \doteq \begin{pmatrix} \langle \psi^\circ_i | H | \psi^\circ_i \rangle & \langle \psi^\circ_i | H | \psi^\circ_j \rangle \\ \langle \psi^\circ_j | H | \psi^\circ_i \rangle & \langle \psi^\circ_j | H | \psi^\circ_j \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{H_{ii}}\ H_{11} - E & H_{12} \\ H_{21} & H_{22} - E \end{pmatrix} \rightarrow (H_{11} - E)(H_{22} - E) - H_{21}H_{12} = \phi$$

$$\rightarrow H_{11}H_{22} + E^2 - H_{11}E - H_{22}E - H_{21}H_{12} = \phi$$

$$\rightarrow E^2 \cancel{\ldots} + E(-H_{11} - H_{22}) + (H_{11}H_{22} - H_{21}H_{12}) = \phi$$

$$(-H_{11} - H_{22})(-H_{11} - H_{22})$$

$$E = \frac{(H_{11} + H_{22}) \pm \sqrt{(-H_{11} - H_{22})^2 - 4(H_{11}H_{22} - H_{21}H_{12})}}{2}$$

$$\rightarrow \sqrt{} = H_{11}^2 + H_{22}^2 + 2H_{11}H_{22} - 4H_{11}H_{22} + 4H_{21}H_{12}$$

$$= H_{11}^2 + H_{22}^2 - 2H_{11}H_{22} + 4H_{21}H_{12}$$

$$= \sqrt{(H_{11} - H_{22})^2 + 4H_{21}H_{12}}$$

$$\rightarrow E = \frac{(H_{11} + H_{22})}{2} \pm \frac{(H_{11} - H_{22})}{2} \underbrace{\sqrt{1 + \frac{4H_{21}H_{12}}{(H_{11} - H_{22})^2}}}_{\sqrt{1 + \tan^2\beta}}$$

A Continued

$$\boxed{E_{1/2} = \frac{(H_{11}+H_{22})}{2} \pm \frac{(H_{11}-H_{22})}{2}\sec\beta}$$

$\sin 2\beta = 2\sin\beta\cos\beta$

$\sin\beta = 2\sin\frac{\beta}{2}\cos\frac{\beta}{2}$

B $$E_1 = \frac{(H_{11}+H_{22})}{2} \pm \frac{(H_{11}-H_{22})}{2}\sec\beta$$

$$\begin{pmatrix} H_{11} - \frac{(H_{11}+H_{22})}{2} - \frac{(H_{11}-H_{22})}{2}\sec\beta & H_{12} \\ H_{21} & H_{22} - \frac{(H_{11}+H_{22})}{2} - \frac{H_{11}-H_{22}}{2}\sec\beta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \phi$$

$$\rightarrow \left(H_{11} - \frac{H_{11}}{2} - \frac{H_{22}}{2} - \frac{H_{11}}{2}\sec\beta + \frac{H_{22}}{2}\sec\beta\right)a + H_{12}b = \phi$$

$$= \left[\frac{H_{11}}{2}(1-\sec\beta) + \frac{H_{22}}{2}\sec\beta\right]a + H_{12}b = \phi$$

$$\cancel{b = -\frac{H_{11}}{2H_{12}}(1-\sec\beta) - \frac{H_{22}}{2H_{12}}\sec\beta\, a}$$

$$\cancel{a\left(-\frac{H_{11}}{2H_{12}}(1-\sec\beta) - \frac{H_{22}}{2H_{12}}\sec\beta\, a\right)}$$

$$= \left[\frac{H_{11}}{2}\left(1 - \frac{1}{\cos\beta}\right) + \frac{H_{22}}{2}\frac{1}{\cos\beta}\right]a + \frac{H_{12}}{2}\frac{\sin\beta}{\cos\beta/2} = \phi$$

NOT working → Moving on

F-2014

PROBLEM 3: Perturbation Theory

Consider a particle of mass m trapped inside a 1D parabolic potential

$$V(x) = \frac{1}{2} m \omega^2 x^2,$$

where ω sets the frequency of oscillation inside the potential.

a) If the particle is perturbed by a *static* potential

$$V_I = \alpha x,$$

with α small, compute energy correction of the energy levels in the lowest order where the result is non-zero. (3 Points)

b) What is the pertubed ket in the ground state? Compute the expectation value $\langle x \rangle$ in this state. Interpret the sign of $\langle x \rangle$. (3 Points)

c) Assume from now on that $\alpha = 0$. Imagine that the particle is charged and sits in the ground state at $t = -\infty$. Suppose an electric field is gradually tuned on, increases to a maximum at $t = 0$ and then slowly dies away,

$$V_I'(t) = -e|\mathbf{E}| x \, \mathrm{e}^{-t^2/\tau^2},$$

where e is the electric charge, and $\mathbf{E}$ is the electric field . Write down the general expression for the amplitude of transition from a generic level i to level f. (Do not solve the integral yet) (2 Points).

d) Evaluate the probability of having the particle in the first excited state at $t = +\infty$. (2 Points).

Hint: $\int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{-t^2/\tau^2} \mathrm{e}^{i\omega t} = \sqrt{\pi} \tau \, \mathrm{e}^{-\omega^2 \tau^2/4}$

F-2014

3 $V(x) = \frac{1}{2} m\omega^2 x^2$

A $V_I = \alpha x$, $E_n^{(m)}$? $x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$

$$E_n^{(1)} = \sqrt{\frac{\hbar}{2m\omega}} \left[\langle n|a|n\rangle + \langle n|a^\dagger|n\rangle \right]$$

$$\rightarrow \boxed{E_n^{(1)} = \phi}$$

$$E_n^{(2)} = \sum_{n \neq k} \frac{(\langle n|V|k\rangle)^2}{E_n - E_k}$$

$$= \frac{\hbar\alpha^2}{2m\omega} \sum_{n\neq k} \frac{\left[\langle n|a|k\rangle + \langle n|a^\dagger|k\rangle\right]^2}{\hbar\omega(n+\frac{1}{2}) - \hbar\omega(k+\frac{1}{2})}$$

$$= \frac{\alpha^2}{2m\omega^2} \left[\frac{(\sqrt{k}\,\delta_{n,k-1})^2 + (\sqrt{k+1}\,\delta_{n,k+1})^2}{n-k} \right]$$

$n = k-1$

$n = k+1$

$$= \frac{\alpha^2}{2m\omega^2} \left[\frac{n+1}{n-(n+1)} + \frac{n}{n-(n-1)} \right] = \frac{\alpha^2}{2m\omega^2} \left[-n - 1 + n \right]$$

$$\boxed{E_n^{(2)} = -\frac{\alpha^2}{2m\omega^2}}$$

B $|n'\rangle^{(1)} = \sum_{n\neq k} \frac{\langle k|V|0\rangle}{E_0 - E_k} |k\rangle$

$= \alpha\sqrt{\frac{\hbar}{2m\omega}} \left[\frac{\cancel{\langle k|a|0\rangle}^{\to\phi} + \langle k|a^\dagger|0\rangle}{E_0 - E_k} \right] |k\rangle$

$= \alpha\sqrt{\frac{\hbar}{2m\omega}} \left[\frac{\delta_{k,1}}{\hbar\omega(\frac{1}{2}) - \hbar\omega(k+\frac{1}{2})} \right]$

$= \frac{\alpha}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}} \left[\frac{\delta_{k,1}}{-k} \right] = -\alpha\sqrt{\frac{1}{2\hbar m\omega^3}} |1\rangle$

$$\boxed{|0\rangle' = \frac{-\alpha}{\sqrt{2\hbar m\omega^3}} |1\rangle}$$

$\to \langle 0|X|0\rangle = \left[\langle 0| + {}'\langle 0|\right] X \left[|0\rangle + |0\rangle'\right]$

$= {}'\langle 0|x|0\rangle + \cancel{{}'\langle 0|x|0\rangle'}^{\to\phi} + \cancel{\langle 0|x|0\rangle}^{\to\phi}$
$+ \langle 0|x|0\rangle'$

$\to {}'\langle 0|x|0\rangle = \frac{-\alpha}{\sqrt{2\hbar m\omega^3}} \sqrt{\frac{\hbar}{2m\omega}} \left[\cancel{\langle 1|a|0\rangle}^{\to\phi} + \underbrace{\langle 1|a^\dagger|0\rangle}_{1} \right]$

$= \frac{-\alpha}{2m\omega^2}$

$\to \langle 0|x|0\rangle' = \frac{-\alpha}{2m\omega^2}$

$$\to \boxed{\langle 0|x|0\rangle = -\frac{\alpha}{m\omega^2}}$$

C $V_I'(t) = -eExe^{-t^2/\tau^2}$

$$\boxed{C_n^{(1)} = \frac{-i}{\hbar}\int_{-\infty}^{\infty} \langle f|V|i\rangle e^{i\omega_{fi}t'}dt' \qquad \text{where } \omega_{fi} = \omega_f - \omega_i}$$

D $P = |C_n^{(1)}|^2$

$\rightarrow \langle 1|V|\phi\rangle = -eEe^{-t^2/\tau^2}\sqrt{\frac{\hbar}{2m\omega}}\left[\cancel{\langle 1|a|\phi\rangle} + \underbrace{\langle 1|a^\dagger|\phi\rangle}_{1}\right]$

$\rightarrow \omega_{fi} = \omega_{10} = \frac{1}{\hbar}\left(\hbar\omega\left(\frac{3}{2}\right) - \hbar\omega\left(\frac{1}{2}\right)\right) = \frac{\hbar\omega}{\hbar} = \omega$

$\rightarrow C_n^{(1)} = \frac{+ieE}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}\underbrace{\int_{-\infty}^{\infty} e^{-t'^2/\tau^2}e^{i\omega t'}dt'}_{\sqrt{\pi\tau}\,e^{-\omega^2\tau^2/4}}$

$\rightarrow P = \frac{e^2E^2}{\hbar^2}\frac{\hbar}{2m\omega}\pi\tau e^{-\omega^2\tau^2/2}$

$$\rightarrow \boxed{P = \frac{\pi\tau e^2E^2}{2\hbar m\omega}e^{-\omega^2\tau^2/2}}$$

S-2013

Problem 5: Perturbing a Square Well

Consider a particle of mass m in a 1D infinite square well of width a,

$$V(x) = 0, \quad 0 \leq x \leq a \qquad V(x) = \infty, \quad x < 0, \quad x > a. \tag{1}$$

(a) [2 pts] Derive the eigenfunctions and eigenenergies of the particle in this potential. Be sure to normalize the states.

(b) [2 pts] Show that if the well is perturbed by a potential $V'(x) = \alpha\, x$, the energy of all the unperturbed states shift by the same amount to first order in α. Find an expression for this energy shift. Give a physical explanation for why this perturbation results in an equal first-order energy shift for all states.

(c) [3 pts] Next, instead of the perturbing potential from part (b), the well is perturbed by a potential

$$V'(x) = V_0, \quad \frac{a}{2} - \delta \leq x \leq \frac{a}{2} + \delta \qquad V'(x) = 0, \quad x < \frac{a}{2} - \delta, \quad x > \frac{a}{2} + \delta \tag{2}$$

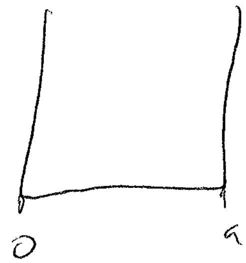
Compute the energy shift to first order in α for the unperturbed energy eigenstates $\psi_n(x)$. Explain the limit of this result as n, the unperturbed energy level, gets large.

(d) [2 pts.] What is the energy shift of the states $\psi_n(x)$ to first order in δ as $\delta \to 0$? (V_0 is constant.) Give a physical explanation of this result. Note: You should be able to answer this question even if you did not get a solution to part (c).

(e) [1 pt] What is the energy shift of the states $\psi_n(x)$ as $\delta \to \frac{a}{2}$? (V_0 is constant.) Give a physical explanation of this result. Note: You should again be able to answer this question even if you did not get a solution to part (c).

5 $V(x) = \emptyset$ for $0 \le x \le a$

$= \infty$ for $x < \emptyset,\ x > a$



A $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = -\underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi$

$= -k^2\psi$

$\rightarrow \psi = A\sin kx + B\cos kx$

$\hookrightarrow$ at $x=\emptyset$, $B\cos kx \neq \emptyset$

$\hookrightarrow B \rightarrow \emptyset$

$\rightarrow \psi = A\sin kx \rightarrow kx = n\pi \rightarrow k = \frac{n\pi}{a}$

$\rightarrow \psi = A\sin\frac{n\pi}{a}x \rightarrow 1 = A^2\int_0^a \sin^2\frac{n\pi}{a}x\,dx$

$= A^2\left[\frac{x}{2} - \frac{\sin\frac{2n\pi}{a}x}{4\left(\frac{n\pi}{a}\right)}\right]\Bigg|_0^a$

$= A^2\left[\frac{a}{2} - \emptyset - (0-0)\right]$

$\rightarrow A = \sqrt{\frac{2}{a}}$

$\rightarrow \boxed{\psi_n = \sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x} \rightarrow \frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{a^2}$

$\rightarrow \boxed{E = \frac{n^2\pi^2\hbar^2}{2ma^2}}$

B) $E_n^{(1)} = \langle n|V'|n\rangle$

$$= \frac{2\alpha}{a}\int_0^a \sin^2\frac{n\pi}{a}x \; x \, dx$$

$$= \frac{2\alpha}{a}\left[\frac{x^2}{4} - \frac{x\sin 2\frac{n\pi}{a}x}{4\frac{n\pi}{a}} - \frac{\cos 2\frac{n\pi}{a}x}{8\left(\frac{n\pi}{a}\right)^2}\right]\Bigg|_0^a$$

$$= \frac{2\alpha}{a}\left[\frac{a^2}{4} - \phi - \frac{1}{\cancel{8\frac{n^2\pi^2}{a^2}}} - \left(0 - 0 - \frac{1}{\cancel{8\frac{n^2\pi^2}{a^2}}}\right)\right]$$

$$= \boxed{\frac{\alpha a}{2} = E_n^{(1)}}$$

For all states,

Changing length of well, effect is seen by all states equally

C) $V'(x) = V_0$ for $\frac{a}{2} - \delta \le x \le \frac{a}{2} + \delta$

$$E_n^{(1)} = \underbrace{\int_0^{a/2-\delta}}_{\substack{\text{No } \rho \\ (1)}} + \underbrace{\int_{\frac{a}{2}-\delta}^{\frac{a}{2}+\delta}}_{\substack{\rho \\ (2)}} + \underbrace{\int_{\frac{a}{2}+\delta}^{a}}_{\substack{\text{No } \rho \\ (3)}}$$

← over all space

(c) Continued

(1) → $\frac{2}{a}\int_0^{a/2-\delta} \sin^2\frac{n\pi}{a}x\,dx = \frac{2}{a}\left[\frac{x}{2} - \frac{\sin 2ax}{4a}\right]\Bigg|_0^{a/2-\delta}$

$= \frac{2}{a}\left[\frac{a}{4} - \frac{\delta}{2} - \frac{1}{\frac{4n\pi}{a}} \sin\left[n\pi - \frac{2n\pi\delta}{a}\right]\right]$

$= \frac{1}{2} - \frac{\delta}{a} - \frac{1}{2n\pi}\left[\cancel{\sin n\pi}^{\,0}\cos\frac{2n\pi\delta}{a} - \cos n\pi \sin\frac{2n\pi\delta}{a}\right]$

$= \frac{1}{2} - \frac{\delta}{a} + \frac{\sin}{2n\pi}\frac{2n\pi\delta}{a}$

(3) → $\frac{2}{a}\int_{a/2+\delta}^{a}\sin^2\frac{n\pi x}{a}\,dx = \frac{2}{a}\left[\frac{x}{2}-\frac{\sin 2ax}{4a}\right]\Bigg|_{\frac{a}{2}+\delta}^{a}$

$= \frac{2}{a}\left[\frac{a}{2} - \cancel{\frac{\sin 2n\pi}{4a}}^{\,0}\right] - \frac{2}{a}\left[\frac{a}{4}+\frac{\delta}{2} - \frac{1}{\frac{4n\pi}{a}}\sin\left(n\pi + \frac{2n\pi\delta}{a}\right)\right]$

$= \underset{\frac{2}{2}}{1} - \frac{1}{2} - \frac{\delta}{2} + \frac{1}{2n\pi}\left[\cancel{\sin n\pi}^{\,0} + \cos n\pi \sin\frac{2n\pi}{a}\delta\right]$

$= \frac{1}{2} - \frac{\delta}{2} + \frac{1}{2n\pi}\sin\frac{2n\pi}{a}\delta$

(2) → $\frac{2}{a}V_0\int_{\frac{a}{2}-\delta}^{a/2\,\delta}\sin^2\frac{n\pi}{a}x\,dx = \frac{2}{a}V_0\left[\frac{x}{2}-\frac{\sin 2ax}{4a}\right]_{\frac{a}{2}-\delta}^{\frac{a}{2}+\delta}$

C Continued

$$\frac{2}{a}V_0\left[\frac{a}{4}+\frac{\delta}{2}-\frac{1}{4\frac{n\pi}{a}}\sin\left[\frac{2n\pi}{a}\frac{a}{2}+\frac{2n\pi}{a}\delta\right]\right]$$

$$-\frac{2}{a}V_0\left[\frac{a}{4}-\frac{\delta}{2}-\frac{1}{4\frac{n\pi}{a}}\sin\left[\frac{2n\pi}{a}\frac{a}{2}-\frac{2n\pi}{a}\delta\right]\right]$$

$$\rightarrow \sin n\pi \cancel{\cos\frac{2n\pi\delta}{a}} + \underbrace{\cos n\pi}_{1}\sin\frac{2n\pi\delta}{a}$$

$$\rightarrow \sin n\pi \cancel{\cos\frac{2n\pi\delta}{a}} - \cos n\pi \sin\frac{2n\pi\delta}{a}$$

$$= \cancel{\frac{V_0}{2}} + \frac{V_0\delta}{a} - \frac{1}{2n\pi}\sin\frac{2n\pi\delta}{a} - \cancel{\frac{V_0}{2}} + \frac{V_0\delta}{a} - \frac{1}{2n\pi}\sin\frac{2n\pi\delta}{a}$$

$$= \frac{2V_0\delta}{a} - \cancel{\frac{1}{n\pi}\sin\frac{2n\pi\delta}{a}}$$

$$\rightarrow 1+2+3 = \frac{1}{2} - \frac{\delta}{2} + \cancel{\frac{\sin}{2n\pi}\frac{2n\pi\delta}{a}} + \frac{1}{2} - \frac{\delta}{2} + \cancel{\frac{1}{2n\pi}\sin\frac{2n\pi\delta}{a}}$$

$$= 1 - \delta + \frac{2V_0\delta}{a}$$

This isn't right, its how you do it but things should cancel as they are

D Delta ~~A~~ spike. $\to V'(x) = V_0\,\delta(x - \frac{a}{2})$

$$E_n^{(1)} = \frac{2}{a} V_0 \int_0^a \sin^2 \frac{n\pi x}{a}\, \delta(x - \tfrac{a}{2})\, dx$$

$$= \frac{2}{a} V_0 \sin^2 \frac{n\pi}{2} \to \boxed{\begin{matrix} \frac{2}{a} V_0 & \text{if } n = \text{odd} \\ 0 & \text{if } n = \text{even} \end{matrix} = E_n^{(1)}}$$

E $E_n^{(1)} = \frac{2}{a} V_0 \int_0^a \sin^2 \frac{n\pi x}{a} = \boxed{V_0}$

S-2015

Problem 6: Perturbations in a 2D well

Consider a spinless particle of mass m and charge q confined to a hard-walled square well (in two dimensions) with sides of length L. The potential can be written:

$$\begin{aligned} V(x,y) &= 0, \quad -\frac{L}{2} \leq x \leq \frac{L}{2}, \quad -\frac{L}{2} \leq y \leq \frac{L}{2} \\ V(x,y) &= \infty \quad otherwise \end{aligned}$$

(a) [2 pts] Write down the eigenenergies, eigenstates, and degeneracies of the first three energy levels for this well. You do not have to solve for these explicitly, but you must explain and justify how you obtained these results.

(b) [2 pts] Consider applying a constant electric field in the x-direction to this system,

$$\vec{E} = E_0 \hat{e}_x \tag{1}$$

Assuming that E_0 is small, determine the first order shift in the energies for the ground state and first excited states. Be sure to show your work.

(c) [3 pts] The second-order, in E_0, energy shift of the ground state can be written in terms of a sum. Write down an expression for this sum using the general form for the eigentstates you determined in part (a). Calculate an approximate value for this energy shift by solving for the largest term in the sum. Your answer should be in terms of the parameters given in the problem, and fundamental constants.

(d) [1 pt] Considering the sum you wrote down in part (c), what is the next largest term that will contribute a non-zero value to the sum? Explain your answer, but you do not need to compute this term.

(e) [2 pts] Finally, instead of an electric field, consider the effect of a localized perturbation:

$$V(x,y) = V_0 L^2 \delta(x - x_0)\delta(y - y_0) \tag{2}$$

where (x_0, y_0) is some point in the well. Write down an expression for the first order energy shift for the ground state, showing how the energy shift depends on the position of the perturbation (x_0, y_0).

Determine a position for the perturbation where the ground state energy changes, but the first excited state does not.

Determine a position for the perturbation that splits the degeneracy of the first excited state.

S-2015

6 $s=0$ $m,1$

$V(x,y) = \phi, \quad -\frac{L}{2} \le x \le \frac{L}{2}, \quad \frac{L}{2} \le y \le \frac{L}{2}$

$= \infty$, otherwise

A For 1D, asymmetric, $\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = k^2\psi$

$\uparrow$

$\frac{2mE}{\hbar^2}$

$\rightarrow$ Solution of the form $\psi = A\sin kx + \cancel{B\cos kx}$ $\rightarrow \phi$

doesn't vanish at boundary, so $B = \phi$

$\rightarrow kx = n\pi \rightarrow k = \frac{n\pi}{L} \rightarrow \sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{L} \rightarrow \frac{n^2\pi^2\hbar^2}{2mL^2} = E$

$\rightarrow$ Shift well $\rightarrow$ $n_{odd} \rightarrow \psi_{odd} = \sqrt{\frac{2}{L}} \cos\frac{n\pi x}{L}$

$\rightarrow n_{even} \rightarrow \psi_{even} = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$

n_x, n_y	ψ_{n_x,n_y}	E_{n_x,n_y}	Deg
1 1	$\frac{2}{L}\cos\frac{\pi x}{L}\cos\frac{\pi y}{L}$	$\frac{\pi^2\hbar^2}{mL^2}$	1
1 2	$\frac{2}{L}\cos\frac{\pi x}{L}\sin\frac{2\pi y}{L}$	$\frac{3}{2}\frac{\pi^2\hbar^2}{mL^2}$	2
2 1	$\frac{2}{L}\sin\frac{2\pi x}{L}\cos\frac{\pi y}{L}$		
2 2	$\frac{2}{L}\sin\frac{2\pi x}{L}\sin\frac{2\pi y}{L}$	$\frac{2\pi^2\hbar^2}{mL^2}$	1

(2)

(B) $\vec{E} = F_0 \hat{e}_x \rightarrow$ ~~[illegible]~~ $V = -\int F \cdot dl \rightarrow \vec{F} = e\vec{E}\hat{e}_x$

$E_n^{(1)} = \langle n|V|n\rangle \qquad \hookrightarrow V = -eEx$

$$E_{11}^{(1)} = \langle 11|V|11\rangle = \sum_x \underbrace{\langle 11|x\rangle}_{\psi_{1n}} \langle x|V|x\rangle \underbrace{\langle x|11\rangle}_{\psi_{1n}}$$

$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi_{1n} \underbrace{\langle x|V|x\rangle}_{\underbrace{-eE\langle x|x|x\rangle}_{-eEx\underbrace{\langle x|x\rangle}_{1}}} \psi_{1n}\, dx$$

You ~~[illegible]~~
Should Integrate
over y Too,
but its $\emptyset$
So doent
Matter here

$$= \frac{2}{L}\cos\frac{\pi y}{L} \underbrace{\int_{-\frac{L}{2}}^{\frac{L}{2}} x\cos^2\frac{\pi x}{2}\, dx}$$

$$\left. \frac{x^2}{4} + \frac{x\sin\frac{2\pi}{L}x}{\frac{4\pi}{L}} + \frac{\cos\frac{2\pi x}{L}}{8\left(\frac{\pi}{L}\right)^2} \right|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$\downarrow \qquad \downarrow \qquad \downarrow$

$=\emptyset \qquad \frac{2\pi}{L}\frac{L}{2} = \emptyset \qquad \frac{2\pi x}{L}\frac{L}{2} = \pi$

~~[illegible]~~

$=\emptyset$

$$\rightarrow \boxed{E_{11}^{(1)} = \emptyset}$$

B Continued

degenerate

$$V \doteq \begin{array}{c|cc} & |12\rangle & |21\rangle \\ \hline |12\rangle & \langle 12|V|12\rangle & \langle 12|V|21\rangle \\ |21\rangle & \langle 21|V|12\rangle & \langle 21|V|21\rangle \end{array}$$

ADDition Formulas ☆☆

$\rightarrow \langle 12|V|12\rangle = \langle 21|V|21\rangle = \phi$

$$\rightarrow \langle 12|V|21\rangle = -\frac{2^2}{L^2} eE \int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} x \cos\frac{\pi x}{L} \sin\frac{2\pi x}{L} \cos\frac{\pi y}{L} \sin\frac{2\pi y}{L}\, dx\, dy$$

$$\rightarrow \int x \cos\frac{\pi x}{L} \sin\frac{2\pi x}{L}\, dx \rightarrow x\left[-\frac{\cos(2\pi-\pi)x}{2(2\pi-\pi)} - \frac{\cos(3\pi)x}{2(3\pi)}\right]\Bigg|_{-L/2}^{L/2}$$

$$+ \int \frac{\cos \pi x}{2\pi}\, dx + \int \frac{\cos 3\pi x}{6\pi}$$

apparently This is ϕ

$$\boxed{E_{12}^{(1)} = E_{21}^{(1)} = \phi}$$

c $E_0^{(2)} = \sum_{k\neq n} \frac{\langle n|V|k\rangle^2}{E_n - E_k}$

$= -Eq \sum_{n_x n_y} \frac{\langle 11|x|n_x n_y\rangle^2}{\frac{\pi^2 \hbar^2}{mL^2} - \frac{\pi^2\hbar^2}{mL^2}(n_x^2+n_y^2)}$

$= \frac{mL^2 Eq}{\pi^2\hbar^2} \sum_{n_x n_y} \frac{\langle 11|x|n_x n_y\rangle^2}{n_x^2+n_y^2-1}$

$\rightarrow \langle 1|x|n_x\rangle\langle 1|n_y\rangle = \langle 11|x|n_x n_y\rangle$

Since $\psi = \psi_x \psi_y$

$= \langle 1|x|n_x\rangle \delta_{n_y,1}$

$\hookrightarrow \int x \cos\frac{\pi x}{L} \psi$

$\rightarrow \langle 1|x|2\rangle = \int x \cos\frac{\pi x}{L} \sin\frac{2\pi x}{L}\, dx$

$= \frac{1}{2}\left[\int x \sin\frac{\pi x}{L}\, dx + \int x \sin\frac{3\pi x}{L}\, dx\right]$

$\propto (1-0)+(1-0)$ $\qquad = \frac{1}{2}\left[\frac{\sin\frac{\pi x}{L}}{(\pi/L)^2} - \frac{x\cos\frac{\pi x}{L}}{\pi/L}\right]\Bigg|_{-L/2}^{L/2}$

$\rightarrow \langle 1|x|3\rangle$ should be $\emptyset$

S-2012

PROBLEM 6: Stationary Perturbation Theory

Let us consider the Hamiltonian $\mathbf{H}$ for a harmonic oscillator with a charged particle in a constant electric field (E):

$$\begin{aligned} \mathbf{H} &= \mathbf{H}_0 + \mathbf{H}_1 \\ \mathbf{H}_0 &= \frac{\mathbf{P}^2}{2m} + \frac{1}{2}k\mathbf{X}^2 \quad \text{and} \\ \mathbf{H}_1 &= \lambda \mathbf{X} \end{aligned}$$

where $\lambda = qE$ and q is the electric charge.

The non-perturbed Hamiltonian has the following eigenvalue equation

$$\mathbf{H}_0|n\rangle = E_n^{(0)}|n^{(0)}\rangle\,, \quad E_n^{(0)} = \hbar\omega(n + \frac{1}{2}) \quad \text{and} \quad \omega = \sqrt{k/m}\,.$$

(a) Apply perturbation theory and determine the first order energy $E_n^{(1)}$. [2 Points]

(b) Apply perturbation theory and evaluate the second order energy $E_n^{(2)}$. [3 Points]

(c) Solve this problem exactly and find the energy E_n. [3 Points]

(d) Determine the eigenvector to the first order $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle$. [2 Points]

S-2012

(6) $H = H_0 + H_1$

$H_0 = \frac{p^2}{2m} + \frac{1}{2} k x^2$

$H_1 = \lambda x$

$H_0 |n\rangle = E_n^{(0)} |n^{(0)}\rangle$

$E_n^{(0)} = \hbar\omega(n + \frac{1}{2})$

$\omega = \sqrt{\frac{k}{m}}$

(A) $E_n^{(1)} = \langle n | H_1 | n \rangle = \langle n | \lambda x | n \rangle$

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^+)$$

$$E_n^{(1)} = \lambda \sqrt{\frac{\hbar}{2m\omega}} \Big[\underbrace{\langle n | a | n \rangle}_{\sqrt{n}\, \delta_{n,n-1} = \phi} + \underbrace{\langle n | a^+ | n \rangle}_{\sqrt{n+1}\, \delta_{n,n+1} = \phi} \Big]$$

$$\rightarrow \boxed{E_n^{(1)} = \phi}$$

(B) $$E_n^{(2)} = \sum_{k \neq n} \frac{\langle n | H_1 | k \rangle^2}{E_n - E_k}$$ ~~$\frac{x^2 \hbar}{2m\omega} \sum_{k\neq n} \langle n|a|k\rangle + \langle n|a|k\rangle$~~

$$= \frac{\lambda^2 \hbar}{2m\omega} \sum_{k \neq n} \frac{|\langle n | a | k \rangle + \langle n | a^+ | k \rangle|^2}{\hbar\omega [n + \frac{1}{2} - k - \frac{1}{2}]}$$

$$= \frac{\lambda}{2m\omega^2} \sum_{k \neq n} \frac{|\sqrt{k}\, \delta_{n,k-1} + \sqrt{k+1}\, \delta_{n,k+1}|^2}{n - k}$$

Cross terms don't survive & sum can go away since δ functions

$$= \frac{\lambda}{2m\omega^2} \left[\frac{k\, \delta_{n,k-1}}{n-k} + \frac{(k+1)\, \delta_{n,k+1}}{n-k} \right]$$

(B) Continued

$n = k-1 \rightarrow \dfrac{n+1}{n-(n+1)} = -\dfrac{n+1}{1}$

$n = k+1 \rightarrow \dfrac{n}{n-(n-1)} = \dfrac{n}{1}$

$\rightarrow E_n^{(2)} = \dfrac{\lambda}{2m\omega^2}\left[n-(n+1)\right]$

$$\boxed{E_n^{(2)} = -\dfrac{\lambda}{2m\omega^2}}$$

(C)

$\left(\dfrac{p^2}{2m} + \dfrac{1}{2}kx^2 + \lambda x\right)\psi = E\psi$

$p = -i\hbar\dfrac{d}{dx} \rightarrow p^2 = -\hbar^2\dfrac{d^2}{dx^2}$

$\rightarrow \left[\dfrac{-\hbar^2}{2m}\dfrac{d^2}{dx^2} + \dfrac{1}{2}kx^2 + \lambda x\right]\psi = E\psi$

$(y-k)(y-k)$

y^2+k^2-2

$\dfrac{kg}{s^2} = k$

$\dfrac{\lambda}{k}$ has units of m

$kg\dfrac{m^2}{s^2} \rightarrow \lambda = kg\dfrac{m}{s^2}$

$x = y - \dfrac{\lambda}{k}$

$\rightarrow -\dfrac{\hbar^2}{2m}\dfrac{d^2}{dy^2} + \dfrac{1}{2}k\left(y-\dfrac{\lambda}{k}\right)^2 + \lambda\left(y-\dfrac{\lambda}{k}\right)$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}+\frac{k}{2}\left[y^2+\frac{\lambda^2}{k^2}-2\frac{y\lambda}{k}\right]+\lambda y-\frac{\lambda^2}{k}$$

$$=-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}+\frac{ky^2}{2}+\frac{\lambda^2}{2k}-\cancel{y\lambda+\lambda y}-\frac{2\lambda^2}{2k}$$

$$=\left[-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}+\frac{ky^2}{2}-\frac{1}{2}\frac{\lambda^2}{k}\right]\psi=E\psi$$

$$\rightarrow\left[-\frac{\hbar^2}{2m}\frac{d^2}{dy}+\frac{ky^2}{2}\right]\psi=\left(E+\frac{1}{2}\frac{\lambda^2}{k}\right)\psi$$

$$\hbar\omega(n+\tfrac{1}{2})=E_n+\frac{1}{2}\frac{\lambda^2}{k}$$

$$\rightarrow\boxed{E_n=\hbar\omega(n+\tfrac{1}{2})-\frac{1}{2}\frac{\lambda^2}{k}}$$

D $$|n^{(1)}\rangle=\sum_{n\neq k}\frac{\langle n|H_1|k\rangle}{E_n-E_k}|k\rangle$$

$\frac{1}{2}-\frac{2}{2}$

$$=\lambda\sqrt{\frac{\hbar}{2m\omega}}\frac{1}{\hbar\omega}\sum_{k\neq n}\left[\frac{\langle n|a|k\rangle}{n-k}+\frac{\langle n|a^+|k\rangle}{n-k}\right]|k\rangle$$

$\frac{1}{2}+\frac{2}{2}$

$$=\lambda\sqrt{\frac{1}{2\hbar\omega^3}}\left[\frac{\sqrt{k}\,\delta_{n,k-1}}{n-k}+\frac{\sqrt{k+1}\,\delta_{n,k+1}}{n-k}\right]|k\rangle$$

$n=k-1$

$$=\lambda\sqrt{\frac{1}{2\hbar\omega^3}}\left[\frac{\sqrt{n+1}}{n-(n+1)}+\frac{\sqrt{n}}{n-(n-1)}\right]|k\rangle$$

4

D Continued

$$\frac{\lambda}{\sqrt{2\hbar\omega^3}}\left[\underbrace{\sqrt{n+1}\,|n+1\rangle}_{-1} + \sqrt{n}\,|n-1\rangle\right] = |n'\rangle$$

F-2012

PROBLEM 5: Time Dependent Perturbation Theory

A particle of charge q, undergoing simple harmonic motion along the x-axis (1-D), is acted on by a time-dependent homogeneous electric field,

$$\vec{E}(t) = E_0 e^{-t^2/\tau^2}\hat{x}$$

where E_0 and τ are constants.

(a) What is the new interaction term in the Hamiltonian for the simple harmonic motion due to the specified electric field? [1 Point]

(b) If the oscillator is in its ground state at $t = -\infty$, find the probability that it will be in an excited state at $t = \infty$. Assume the interaction can be treated as a time-dependent perturbation. [3 Points]

(c) Consider the same charged particle linear harmonic oscillator as in (a). Assuming that dE/dt is small, and that at $t = -\infty$ the oscillator is in the ground state, use the adiabatic approximation to obtain the probability that the oscillator will be found in an excited state as $t \to \infty$. Compare your result with the one you obtained in (b). [3 Points]

(d) Again consider the charged particle harmonic oscillator but with a slightly different perturbation. For $t < 0$

$$H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2 .$$

For $t > 0$

$$H(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}k(x-a)^2 - ka^2$$

with

$$a = \frac{qE_0}{m\omega^2},$$

where $\omega = \sqrt{k/m}$. Show that in the weak coupling limit for $t > 0$ that the only *eigenstate of* H_0 which will be excited with any sizable probability is the first excited state, $\psi_1(x)$, and that the corresponding transition probability is

$$P_{10}(t) = \frac{2q^2E_0^2}{m\hbar\omega^3}\sin^2(\omega t/2).$$

Assume the perturbation is turned on suddenly (fast). [3 Points]

5) H.O.

$$\vec{E}(t) = E_0 e^{-t^2/\tau^2}\hat{x} \rightarrow \vec{F} = qE_0 e^{-t^2/\tau^2}\hat{x}$$

A) $$\vec{F} = -\frac{\partial V}{\partial x}\hat{x} \rightarrow \boxed{V = -qE_0 x e^{-t^2/\tau^2}}$$

B) $$C_n = \frac{-i}{\hbar}\int_{t_0}^{t'} e^{-i\omega_{ni}t} V_{ni}(t')\,dt'$$

$$\rightarrow \langle n|x|i\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\cancel{\langle n|a|i\rangle}^{\phi} + \langle n|a^{+}|i\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\sqrt{i+1}\,\delta_{n,i+1} \rightarrow i = \phi$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\delta_{n,1} \rightarrow n = 1$$

$$C_1 = \frac{-i}{\hbar}\int_{-\infty}^{\infty} e^{-i\omega_{01}t}\sqrt{\frac{\hbar}{2m\omega}}\left(-qE_0 e^{-t^2/\tau^2}\right)dt'$$

$$\rightarrow \omega_{01} = \hbar\omega_1 - \hbar\omega_0 = \frac{1}{\hbar}(\hbar\omega_1 - \hbar\omega_0) = \frac{1}{\hbar}\left(\frac{3}{2}\hbar\omega - \frac{1}{2}\hbar\omega\right)$$

$$= \omega$$

$$C_1 = +\frac{i}{\hbar}qE_0\sqrt{\frac{\hbar}{2m\omega}}\underbrace{\int_{-\infty}^{\infty} e^{-i\omega t}e^{-t^2/\tau^2}dt'}_{\sqrt{\pi\tau^2}\,e^{(-\omega^2/4/\tau^2)}}$$

$$= \sqrt{\pi\tau^2}\,e^{-\omega^2\tau^2/4}$$

B Continued

$$\boxed{P = c_1^2 = \cancel{\ldots} = \frac{q^2 E_0^2 \pi \tau^2}{2\hbar m\omega} e^{-\omega^2\tau^2/2}}$$

C Adiabatic Approximation = gradual change in time.

$$c_n^{(1)} = \frac{-i}{\hbar}\int_{-\infty}^{\infty} V_{ni}\, e^{i\omega_{ni}t'}\, dt'$$

$$\rightarrow \quad e^{i\omega_{ni}t'} = \frac{1}{i\omega_{ni}} \frac{\partial e^{i\omega_{ni}t}}{\partial t}$$

$$c_n^{(1)} = \frac{-1}{\omega_{ni}} \int_{-\infty}^{\infty} V_{ni} \frac{\partial e^{i\omega_{ni}t}}{\partial t}\, dt$$

$$= \left[-\frac{1}{\omega_{ni}} V_{ni}\, e^{i\omega_{ni}t} \right]\Bigg|_{-\infty}^{\infty} + \frac{1}{\omega_{ni}} \int_{-\infty}^{\infty} e^{i\omega_{ni}t} \frac{\partial V_{ni}}{\partial t}\, dt$$

↳ Switched on at $-\infty$ and off at ∞
→ Vanishes at boundaries

$$\rightarrow \boxed{c_n^{(1)} = \frac{1}{\omega_{ni}} \frac{\partial V_{ni}}{\partial t} \int_{-\infty}^{\infty} e^{i\omega_{ni}t}\, dt}$$

↑ So small, it's a constant

adiabatic approximation

C Continued

$$C_n = \frac{1}{\omega_{ni}}\frac{\partial V_{ni}}{\partial t}\left[\frac{1}{i\omega_{ni}} e^{i\omega_{ni}t}\Big|_{-\infty}^{\infty}\right]$$

$\hookrightarrow = \phi$, has to vanish at boundaries.

$$\Rightarrow \boxed{C_n = \phi}$$

~~D Sudden change (?)~~

$$\cancel{C_n^{(1)} = \frac{1}{\omega_{ni}} e^{i\omega_{ni}t}\int_0^t \frac{\partial V_{ni}}{\partial t}\,dt}$$

~~This is non constant~~

For Sudden, I think it's, just normal perturbation theory.

$$H(t) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}k(x-a)^2 - ka^2$$

$$\frac{kx^2}{2} + \frac{a^2k}{2} - xak - ka^2$$

$$= \frac{kx^2}{2}\underbrace{\left[-\frac{ka^2}{2} - xak\right]}_{V'}$$

$$\Rightarrow \langle n|V|i\rangle = -\left[\cancel{\langle n|\tfrac{ka^2}{2}|i\rangle}^{\,\phi} + \langle n|xak|i\rangle\right]$$

$$= -ak\sqrt{\frac{\hbar}{2m\omega}}\left[\phi + \sqrt{i+1}\,\delta_{n,i+1}\right] \quad n=1$$

1) Continued

$$\langle n|V|i\rangle = -ak\sqrt{\frac{\hbar}{2m\omega}}$$

$$\omega_{ni} = \omega$$

$$\cancel{\Rightarrow c_n^{(1)} = \frac{1}{\omega}\left(-ak\sqrt{\frac{\hbar}{2m\omega}}\right)(\cos\omega t + i\sin\omega t)}$$

$$\cancel{\Rightarrow P = \frac{a^2k^2}{\hbar\omega^2}\frac{\hbar}{2m\omega}(\cos\omega t + i\sin\omega t)(\cos\omega t - i\sin\omega t)}$$

$$\cancel{\cos^2\omega t}$$

$$c_n^{(1)} = -\frac{i}{\hbar}\left(-ak\sqrt{\frac{\hbar}{2m\omega}}\right)\underbrace{\int_0^t e^{i\omega t'}\,dt'}_{\frac{e^{i\omega t}}{i\omega}\Big|_0^t = \frac{e^{i\omega t}-1}{i\omega}}$$

$$\cancel{e^{i\omega t} = \cos\omega t + i\sin\omega t}$$

$$P = \frac{1}{\hbar^2\omega^2}a^2k^2\frac{\hbar}{2m\omega}\underbrace{(e^{i\omega t}-1)(e^{-i\omega t}-1)}_{1-e^{i\omega t}-e^{-i\omega t}+1 \;=\; 2-(\underbrace{e^{i\omega t}+e^{-i\omega t}}_{\frac{\cos\omega t}{2}})}$$

$$= \underbrace{2 - \frac{\cos\omega t}{2}}_{\text{This is related to } \sin^2 A}$$

There seems to be a minor mistake somewhere but this is definitely how you do it

F-2009

Problem 3: Artificial Atoms (10 points)

Modern techniques in nanotechnology research can create artificial atoms, man-made structures that confine electrons like real atoms but with properties that can be engineered. In this problem, consider a 2D atom (electrons tightly bound in the z-direction) with a parabolic potential in the x- and y-directions. The Hamiltonian is:

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2\right). \tag{1}$$

Note: In solving this problem, you might want to use the standard operators:

$$a_x = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} + i\frac{\lambda}{\hbar}p_x\right), \quad a_y = \frac{1}{\sqrt{2}}\left(\frac{y}{\lambda} + i\frac{\lambda}{\hbar}p_y\right) \tag{2}$$

and their Hermitian conjugates, where $\lambda = \sqrt{\frac{\hbar}{m\omega}}$.

a) What are the eigenenergies of this atom? What are the degeneracies of these energy levels? If the separation between adjacent levels is 20 meV (0.02 eV), approximately how large are the low-energy electron states in the atom (the radius)? (2 pts)

b) If the atom is put in a constant electric field, the Hamiltonian H_0 is perturbed by a potential:

$$H_1 = -eE_1 x \tag{3}$$

where E_1 is a constant (the electric field). Prove that to first order in the field, the energy levels of the atom do not change. (2 pts)

c) Next the atom is placed in a more complex field to study its properties. The new potential is:

$$H_2 = \frac{C_2}{\lambda^2}xy \tag{4}$$

To first order in C_2, what are the new eigenenergies of what were the first three energy levels of H_0? Show your work. (4 pts)

d) If a different perturbing potential:

$$H_3 = \frac{C_3}{\lambda^2}x^2 \tag{5}$$

is applied (rather than H_2), how would your answers to part (c) change? No computations should be necessary to answer this question. (2 pts)

F-2009

3 $H_0 = \frac{p^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2)$

$$a_x = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} + i\frac{\lambda}{\hbar}p_x\right), \quad a_y = \frac{1}{\sqrt{2}}\left(\frac{y}{\lambda} + i\frac{\lambda}{\hbar}p_y\right)$$

$$\lambda = \sqrt{\frac{\hbar}{m\omega}}$$

A Harmonic oscillator

$$\rightarrow \boxed{E_{xy} = \hbar\omega(n_x+n_y+1)}$$

$$n_x + n_y = n$$

$$\cancel{\hbar\omega(n_x+n_y+1)}$$

n	n_x	n_y	d
0	0	0	1
1	0 1	1 0	2
2	0 2 1	2 0 1	3
3	3 0 2 1	0 3 1 2	4

$$\boxed{\text{Degeneracy} = n+1}$$

$$\Delta E = \hbar\omega(n_2 - n_1 + 1 - 1) = \hbar\omega\,\Delta n$$

$\boxed{?}$

B $H_1 = -eE_1 x$

$$x = a^\dagger + a = \frac{1}{\sqrt{2}}\left(\frac{2x}{\lambda}\right) = \frac{\sqrt{2}\,x}{\lambda}$$

$$\frac{2}{2} - \frac{1}{2} = \frac{1}{2}$$

$$\rightarrow \frac{\lambda}{\sqrt{2}}(a^\dagger + a) = x$$

$$\boxed{\begin{aligned} E^{(1)}_{n} &= \langle n_x n_y | \frac{\lambda}{\sqrt{2}}(a^\dagger + a) | n_x n_y \rangle \\ &= \frac{\lambda}{\sqrt{2}}\Big[\underbrace{\langle n_x n_y | a^\dagger | n_x n_y \rangle}_{\sqrt{n_x+1}\,\underbrace{\langle n_x n_y | n_x + 1, n_y \rangle}_{\phi}} + \underbrace{\langle n_x n_y | a | n_x n_y \rangle}_{\sqrt{n_x}\,\underbrace{\langle n_x n_y | n_x - 1\, n_y \rangle}_{\phi}} \Big] \\ &= \phi \end{aligned}}$$

C $xy = \frac{2}{\lambda^2}(a_x^\dagger + a_x)(a_y^\dagger + a_y)$

$$= \frac{2}{\lambda^2}\left[a_x^\dagger a_y^\dagger + a_x a_y + a_x^\dagger a_y + a_x a_y^\dagger \right]$$

$$\rightarrow E^{(1)}_{xy} = \frac{2}{\lambda^4}\Big[\langle n_x n_y | a_x^\dagger a_y^\dagger | n_x n_y \rangle + \langle n_x n_y | a_x a_y | n_x n_y \rangle$$

$$+ \langle n_x n_y | a_x^\dagger a_y | n_x n_y \rangle + \langle n_x n_y | a_x a_y^\dagger | n_x n_y \rangle \Big]$$

1 (Continued)

<u>$n = 0$</u>

$$E^{(1)}_{xy} = \frac{2C_2}{\lambda^4}\Big[\langle 00|a_x^\dagger a_y^\dagger|00\rangle + \langle 00|a_x a_y|00\rangle$$

$$+ \langle 00|a_x^\dagger a_y|00\rangle + \langle 00|a_x a_y|00\rangle$$

$$= 0$$

<u>$n = 1$</u>

$|10\rangle$ & $|01\rangle$ <u>Degenerate perturbation</u>

$$V \doteq \begin{array}{c|cc} & |10\rangle & |01\rangle \\ \hline |10\rangle & -E & 2 \\ |01\rangle & 2 & -E \end{array} \quad \frac{2C_2}{\lambda^2}$$

$$E^2 - 4 = 0 \rightarrow E = \pm \frac{4C_2}{\lambda^2}$$

(c) Continued

$$\boxed{E_1 = 2\hbar\omega \pm \frac{4C_2}{\lambda^2}}$$

$\underline{n=2}$

$|02\rangle \quad |20\rangle \quad |11\rangle$

$$V \doteq \begin{array}{c|ccc} & |02\rangle & |20\rangle & |11\rangle \\ \hline |02\rangle & 0 & \cancel{} & \\ |20\rangle & \cancel{} & 0 & \\ |11\rangle & & & 0 \end{array} \quad \frac{2C^2}{\lambda^2}$$

~~$\langle 20|02\rangle$~~ $\quad \langle 02|20\rangle$

0 0 0 0

$\langle 02|11\rangle$

~~0 1 1 2 $\quad$ 1 2 1 0~~

Not Finishing, tedious

but doable

F-2015

Problem 5: Simple Harmonic Oscillator with External Perturbations

Consider a one-dimensional simple harmonic oscillator of mass m with a natural angular frequency ω. If there is no external perturbation, the Hamiltonian for this system is

$$H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2, \quad H_0|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle \tag{1}$$

(a) [2 pts] Consider the case where there is an external potential on the oscillator of the form $V_1(x) = \gamma_1 x$. Calculate the exact eigenenergies of $H_0 + V_1$.

Describe the difference between the new eigenstates of this total Hamiltonian and the eigenstates of H_0.

(Hint: The new Hamiltonian can be transformed back into a harmonic oscillator of frequency ω plus an extra term).

(b) [4 pts] Using perturbation theory to the first non-zero order, calculate the perturbed eigenenergies of $H_0 + V_1$. How do these compare with the exact solutions from (a)?

(c) [1 pts] Now consider the case where there is an external potential on the oscillator of the form $V_2(x) = \gamma_2 x^2$. Calculate the exact eigenenergies of $H_0 + V_2$.

Describe the new eigenstates of this total Hamiltonian, comparing them with the eigenstates of H_0.

(d) [3 pts] Using perturbation theory to the first non-zero order, calculate the perturbed eigenenergies of $H_0 + V_2$. How do these compare with the exact solutions from (c)?

F-2015

5) $H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2}$

A) $\gamma_1 x = V_1$

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2} + \gamma_1 x$$

$\to H|n\rangle = E|n\rangle$

Change of Variables $\to$ Shift x by a dimensionless constant

$$x = y - \frac{\gamma_1}{m\omega^2}$$

★ remember how to do this $\quad \frac{d^2}{dx^2} \equiv \frac{d^2}{dy^2}$

$\to$ would find

$$\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy} \quad \to \quad \text{Then } \frac{d^2}{dx} = \frac{d}{dx}\frac{d}{dx}$$

$\to$ Simple here

$$\to H = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m}{2}\omega^2\left[y - \frac{\gamma_1}{m\omega^2}\right]^2 + \gamma\left[y - \frac{\gamma_1}{m\omega^2}\right]$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m}{2}\omega^2\left[y^2 + \frac{\gamma_1^2}{m^2\omega^4} - \frac{2y\gamma_1}{m\omega^2}\right] + y\gamma - \frac{\gamma^2}{m\omega^2}$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m}{2}\omega^2 y^2 + \frac{\gamma_1^2}{2m\omega^2} \cancel{- y\gamma_1} \cancel{+ \gamma y} - \frac{\gamma^2}{m\omega^2}$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m}{2}\omega^2 y^2 - \frac{\gamma_1^2}{2m\omega^2}$$

A Continued

$H\psi_n = E_n \psi_n$

$$\rightarrow \left[-\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m}{2}\omega^2 y^2\right]\psi_n = \boxed{\underbrace{\left[E_n + \frac{\gamma_1^2}{2m\omega^2}\right]}_{E_n'}}\psi_n$$

where $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$

Shifted along x-axis, Centered at $x = \frac{\gamma}{m\omega^2}$

B First order $\rightarrow$ $\langle n|V'|n\rangle = \emptyset$ Since $V' \propto x$

Second order $\rightarrow$

$$E_n^{(2)} = \sum_{n\neq m}\frac{|\langle n|V'|m\rangle|^2}{E_n^\circ - E_m^\circ}$$

$$= \sum_{n\neq m}\frac{\gamma^2(\langle n|x|m\rangle)^2}{E_n^\circ - E_m^\circ}$$

$\rightarrow x \rightarrow a + a^\dagger$

$$= \frac{2}{\sqrt{2}}\beta x$$

$$= \sqrt{2}\,\beta x \rightarrow \frac{1}{\sqrt{2}\,\beta}(a+a^\dagger) = x$$

$$\rightarrow \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger) = x$$

B (continued) 5

$$E_n^{(2)} = \frac{\gamma^2 (\hbar)}{2m\omega} \sum_{n \neq m} \frac{(\langle n|a|m\rangle + \langle n|a^\dagger|m\rangle)^2}{E_n - E_m}$$

$$= \frac{\hbar \gamma^2}{2m\omega} \sum_{n\neq m} \frac{1}{E_n - E_m}\left[\sqrt{m}\,\delta_{n,m-1} + \sqrt{m+1}\,\delta_{n,m+1}\right]^2$$

$n = m-1 \qquad n = m+1$

$$\rightarrow \sqrt{n+1} + \sqrt{n}$$

$$\rightarrow \frac{n+1}{E_n - E_{n+1}} + \frac{n}{E_n - E_{n-1}}$$

$$\cancel{E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)} \qquad \cancel{E_{n-1} = \hbar\omega\left(n-\tfrac{1}{2}\right)}$$

$$\cancel{E_{n+1} = \hbar\omega\left(n+\tfrac{3}{2}\right)} \qquad \cancel{= \hbar\omega}$$

$$\cancel{E_n - E_{n+1} = \hbar\omega - 1}$$

$$\cancel{E_n^{(2)} = \frac{\gamma^2 \hbar}{2m\omega}\left(\frac{(n+1)\hbar\omega}{(\hbar\omega - 1)\hbar\omega} + \frac{n(\hbar\omega - 1)}{\hbar\omega(\hbar\omega-1)}\right) = \frac{\gamma^2\hbar}{2m\omega}}$$

$$\rightarrow \frac{n+1}{\hbar\omega(n+\frac{1}{2}) - \hbar\omega(m+\frac{1}{2})} + \frac{n}{\hbar\omega(n+\frac{1}{2}) - \hbar\omega(m+\frac{1}{2})}$$

$$\rightarrow \frac{n+1}{\hbar\omega(n-m)} + \frac{n}{\hbar\omega(n-m)}$$

B Continued

$$\Rightarrow \frac{\gamma^2}{2m\omega^2}\sum\left[\frac{m}{n-m} + \frac{m+1}{n-m}\right]$$

$m-1=n \qquad n=m+1$

$$\Rightarrow \frac{\gamma^2}{2m\omega^2}\sum \frac{n+1}{n-n-1} + \frac{n}{n-n+1}$$

$$\Rightarrow \frac{\gamma^2}{2m\omega^2}\sum \frac{n+1}{-1} + \frac{n}{1}$$

$$\Rightarrow \frac{-\gamma^2}{2m\omega^2}\sum \frac{n+1-n}{1}$$

$$\Rightarrow \boxed{\frac{-\gamma^2}{2m\omega^2} = E_n^{(2)}}$$

Matches A

(c) $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\left[\frac{m}{2}\omega^2+\gamma_2\right]x^2 = H$

~~VM~~ ~~VM~~ $kg\frac{m^2}{s^2}$ $\rightarrow$ $\gamma = \frac{kg}{s^2}$

$$\rightarrow \omega'^2 = \cancel{m}\omega^2 + \frac{2\gamma_2}{m}$$

$$\rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega'^2x^2 = H$$

$$\rightarrow E = \hbar\omega'(n+\tfrac{1}{2})$$

$$= \hbar \cancel{\omega}(n+\tfrac{1}{2})\sqrt{\omega^2+\frac{2\gamma_2}{m}}$$

$$= \hbar\omega(n+\tfrac{1}{2})\left(1+\frac{2\gamma}{m\omega^2}\right)^{1/2}$$

assume this term small

$$= \hbar\omega(n+\tfrac{1}{2}) + \frac{\hbar\gamma}{m\omega}(n+\tfrac{1}{2})$$

$$\boxed{E = \left(\hbar\omega + \underbrace{\frac{\hbar\gamma}{m\omega}}_{\text{correction}}\right)(n+\tfrac{1}{2})}$$

D $x^2 = \frac{\hbar}{2m\omega}[a a + a^\dagger a^\dagger + a a^\dagger + a^\dagger a]$

$\langle n|x^2|m\rangle = \frac{\hbar\gamma}{2m\omega}[\delta_{n+1,m-1}\sqrt{n+1}\sqrt{m} + \delta_{m+1,n-1}\sqrt{n}\sqrt{m+1} + \phi + \phi]$

$= \frac{\hbar\gamma}{2m\omega\,\hbar\omega}\left[\frac{(n+1)(m)}{n-m} + \frac{(n)(m+1)}{n-m}\right]$

$n+1 = m-1$ $\quad$ $m+1 = n-1$

$2+n = m$ $\quad$ $m = n-2$

$= \frac{\hbar\gamma}{2m\omega\hbar\omega}\left[\frac{(n+1)(n+2)}{n-2-n} + \frac{(n)(n-2+1)}{n-n+2}\right]$

$= \frac{\gamma}{2m\omega^2}\left[\frac{n^2+2+3n}{-2} + \frac{n^2-2n+n}{2}\right]$

$= \frac{\gamma}{2m\omega^2}\left[\frac{\cancel{n^2}+2+3n - \cancel{n^2}+2n-n}{2}\right]$

$= \frac{\gamma}{2m\omega^2}\left[\frac{2+3n+2n-n}{2}\right] = \frac{\gamma}{2m\omega^2}(1+2n) = E$

Made Mistake

This actually interesting,

Notice the First order Correction

is Not Zero,

You should have done Calculus

which is much easier, but none the less,

The answer is Almost identical

5-2008

Problem 5: Perturbation Theory: (10 Points)

A single particle is in a one dimensional infinite well of length L. The potential $V(x)$ is given by:

$$V(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq L \\ \infty, & \text{otherwise} \end{cases}$$

Suppose the potential energy inside the well is changed to

$$V(x) = \epsilon \sin \frac{\pi x}{L}$$

when $0 \leq x \leq L$.

Note you may use your results from Problem 1 for this problem.

1. Calculate the energy shifts for the perturbed well to first order in ϵ. **(2 Points)**
2. Which energy level is shifted the most to first order in ϵ? **(1 Point)**
3. Calculate the second order (in ϵ) correction to the ground state energy **(2 Points)**
4. Calculate the corrections to the ground state wavefunction to first order in ϵ. **(2 Points)**
5. Suppose that ϵ is larger than the energy of the first excited state. Carefully sketch the wavefunction versus x for the ground state and for the first excited state. How many nodes, maxima, and minima does the wavefunction have in each state. **(2 Points)**
6. Suppose the wavefunction is a linear combination of the ground state and the first excited state from part 5. Describe how the maximum of the probability density depends on time. **(1 Point)**

~~QM~~ 5-2008

(5) $V(x) = \begin{cases} 0 & 0 \le x \le L \\ \infty & \text{~~elsewhere~~ else} \end{cases}$

$$\psi = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$V(x)' \to \epsilon \sin \frac{\pi x}{L}$$

(A) $E_n^{(1)} = \langle n^{(0)} | V' | n^{(0)} \rangle$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle n^{(0)} | x \rangle \underbrace{\langle x | V' | x' \rangle}_{V' \langle x | x' \rangle = V' \delta(x - x')} \langle x' | n'^{(0)} \rangle \, dx\, dx'$$

This took 2 hours & you're halfway done

$$= \frac{2\epsilon}{L} \int_{-\infty}^{\infty} \sin^2 \frac{n\pi x}{L} \sin \frac{\pi x}{L}\, dx$$

$$= \frac{2\epsilon}{L} \left[\int_{-\infty}^{\infty} \frac{1}{2} \sin \frac{\pi x}{L} - \frac{1}{2} \cos \frac{2n\pi x}{L} \sin \frac{\pi x}{2}\, dx \right]$$

$$= \frac{\epsilon}{L} \left[\int_0^L \sin \frac{\pi x}{L}\, dx - \int_0^L \cos \frac{2n\pi x}{L} \sin \frac{\pi x}{L}\, dx \right]$$

$$= \frac{\epsilon}{L} \left(\frac{-\cos \frac{\pi}{L} x}{\frac{\pi}{L}} \Bigg|_0^L \right) - \frac{\epsilon}{L} \left[\frac{-\cos\left(\frac{\pi x}{L} - \frac{2n\pi x}{L}\right)}{2\left(\frac{\pi}{L} - \frac{2n\pi}{L}\right)} + \frac{-\cos\left(\frac{\pi x + 2n\pi x}{L}\right)}{2\left(\frac{\pi}{L} + \frac{2n\pi}{L}\right)} \right.$$

$$= \frac{\epsilon}{L} \left[\frac{-(-1-1)}{\pi/L} \right] + \frac{\epsilon}{2L} \left[\underbrace{\frac{\cos(\pi - 2n\pi)}{\left(\frac{\pi}{L} - \frac{2n\pi}{L}\right)}}_{A} + \underbrace{\frac{\cos(\pi + 2n\pi)}{\left(\frac{\pi}{L} + \frac{2n\pi}{L}\right)}}_{B} - \underbrace{\left(\frac{1}{\frac{\pi}{L} - \frac{2n\pi}{L}} + \frac{1}{\frac{\pi}{L} + \frac{2n\pi}{L}} \right)}_{C} \right] \Bigg|_0^L$$

A Continued

$$= \frac{\epsilon}{L}\frac{L}{\pi}(1+1) + \frac{\epsilon}{2L}\left[\frac{\cos(\pi-2n\pi)}{A} + \frac{\cos(\pi+2n\pi)}{B} - C\right]$$

$$= \frac{2\epsilon L}{L\pi} + \frac{\epsilon}{2L}\left[\frac{\cos\pi\cos(2n\pi) + \sin\pi\,\cancel{\sin 2n\pi}^{\,0}}{A} + \frac{\cos\pi\cos 2n\pi - \sin\pi\,\cancel{\sin 2n\pi}^{\,0}}{B} - C\right]$$

$$= \frac{2\epsilon}{\pi} + \frac{\epsilon}{2L}\left[-\frac{2B}{AB} - \frac{2A}{AB}\right] = \frac{2\epsilon}{\pi} - \frac{\epsilon}{2L}\left[\frac{\frac{2\pi}{L} + \cancel{\frac{4n\pi}{L}} + \frac{2\pi}{L} - \cancel{\frac{4n\pi}{L}}}{\frac{\pi^2 - 4n^2\pi^2}{L^2}}\right]$$

$$AB = \left(\frac{\pi}{L} - \frac{2n\pi}{L}\right)\left(\frac{\pi}{L} + \frac{2n\pi}{L}\right)$$

$$\frac{\pi^2}{L^2} - \frac{2^2 n^2\pi^2}{L^2}$$

$$= \frac{\pi^2}{L^2} - \frac{4n^2\pi^2}{L^2}$$

$$= \frac{\pi^2 - 4n^2\pi^2}{L^2}$$

Made a mistke somewhere actually +

$$\frac{2\epsilon}{\pi} + \frac{\epsilon L}{2L}\left[\frac{4\cancel{\pi}}{\pi^2 - 4n^2\pi^2}\right]$$

$$\boxed{E_0^{(1)} = \frac{2\epsilon}{\pi} + \frac{\epsilon}{2\pi}\left[\frac{4}{1 - 4n^2}\right]}$$

B $E_0^{(1)} = \frac{2\epsilon}{\pi} - \frac{\epsilon}{2\pi}\left[\frac{4}{1-4n^2}\right]$ if $n=1$

$$\Rightarrow \frac{2\epsilon}{\pi} + \frac{\epsilon}{2\pi}\frac{4}{3}$$ as $n \uparrow$ $\downarrow$

$\boxed{\text{Max shift when } n=1}$

(c) $E_\phi^{(2)} = \sum_{m\neq n} \frac{(\langle m|V'|n\rangle)^2}{E_n^0 - E_m^0}$ (3)

$$= \frac{1}{E_1-E_2}\left[\int_0^L \epsilon \frac{2}{L}\sin^2\frac{\pi x}{L}\,\sin\frac{2\pi x}{L}\,dx\right]^2$$

$$= \underbrace{\frac{1}{E_1-E_2}\,\frac{4\epsilon^2}{L^2}}_{A}\left[\int_0^L \sin^2\frac{\pi x}{L}\,\sin\frac{2\pi x}{L}\,dx\right]^2$$

$kg\frac{m^2}{s}$

$$= \frac{A}{2^2}\left[\int_0^L \sin\frac{2\pi x}{L}\,dx - \int_0^L \cos\frac{\pi x}{L}\,\sin\frac{2\pi x}{L}\,dx\right]^2 \qquad \frac{2}{L}$$

$$= \frac{A}{4}\left[-\frac{\cos\frac{2\pi x}{L}}{\frac{2\pi}{L}}\Bigg|_0^L + \left(\frac{\cos\left(\frac{2\pi}{L}-\frac{\pi}{L}\right)x}{2\left(\frac{2\pi}{L}-\frac{\pi}{L}\right)} + \frac{\cos\left(\frac{2\pi}{L}+\frac{\pi}{L}\right)x}{2\left(\frac{2\pi}{L}+\frac{\pi}{L}\right)}\right)\Bigg|_0^L\right]$$

$\frac{mL^2}{t^2}$

$\frac{kg\,m^2 s^2}{kg^2 m^4}$

$\frac{s^2}{kg\,m^2}$

$$= \frac{A}{4}\left[-\frac{(\cancel{0}-1)}{\frac{2\pi}{L}} + \left(\frac{\cos\pi}{A} + \frac{\cos 3\pi}{B} - \frac{1}{A} - \frac{1}{B}\right)\right]^2$$

$$= \frac{A}{4}\left[\frac{L}{2\pi} - \left(\frac{-1}{A}\ \frac{-1}{B} + \frac{1}{A}\ \frac{-1}{B}\right)\right]^2$$

$$= \frac{A}{4}\left[\frac{L}{2\pi} - \left(\frac{2}{A} + \frac{2}{B}\right)\right]^2 \qquad A = \frac{2\pi}{L} \qquad \frac{36}{3} \quad 114$$

$$B = \frac{6\pi}{L}$$

$$= \frac{A}{4}\left[\frac{3L}{6\pi} - \frac{6L}{6\pi} - \frac{2L}{6\pi}\right]^2$$

$$= \frac{A}{4}\left[\frac{-5L}{6\pi}\right]^2 = \frac{1}{E_1-E_2}\,\frac{\cancel{4}\epsilon^2}{\cancel{4}\cancel{L^2}}\,\frac{5^2\cancel{L^2}}{6^2\pi^2} = \frac{5^2\epsilon^2}{6^2\pi^2}\,\frac{1}{E_1-E_2}$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \rightarrow E_1 - E_2 = \frac{\pi^2\hbar^2}{2mL^2}(1-4) = \frac{-3\pi^2\hbar^2}{2mL^2}$$

$$\text{so } \boxed{E_\phi^{(2)} = -\frac{50\epsilon^2 mL^2}{114\hbar^2}} \qquad \boxed{\text{Is This right?}}$$

S-2014

Problem 6: Perturbation Theory

An isotropic Harmonic oscillator in two dimensions has the Hamiltonian

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2),$$

where x and y are position operators in Cartesian coordinates x and y.

a) What is the energy of the *three* lowest energy levels and their respective degeneracies? (2 Points)

b) Consider a perturbative potential of the form:

$$V(x, y) = Am\omega^2 xy.$$

Compute the energy correction of the lowest level in the lowest order in perturbation theory where the result is non-zero. (3 Points)

c) Compute the energy splitting of the first excited energy level (which is degenerate), due to the perturbation. Compute the split ket states in terms of the original unperturbed kets. (3 Points)

d) Suppose that there are three indistinguishable spin 1/2 particles in the system. Compute the total energy of the ground state in first order in perturbation theory. (2 Points)

5-2014

(6) (A) $E = \hbar\omega\left[n_x + \frac{1}{2} + n_y + \frac{1}{2}\right]$

$= \hbar\omega\left[n_x + n_y + 1\right]$

$$\boxed{\begin{array}{l} E_{00} = \hbar\omega \quad \Big\rangle \quad 0 \text{ degeneracy} \\ E_{10} = E_{01} = 2\hbar\omega \\ E_{11} = 3\hbar\omega \end{array}}$$

degenerate

(B) $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$

$V = Am\omega^2 \frac{\hbar}{2m\omega}(a_x + a_x^\dagger)(a_y + a_y^\dagger)$

$= \frac{A\omega\hbar}{2}(a_x a_y + a_x a_y^\dagger + a_x^\dagger a_y + a_x^\dagger a_y^\dagger)$

$\rightarrow E_{00}^{(1)} = \langle n_x, n_y | V | n_x, n_y \rangle$

$= \frac{A\omega\hbar}{2}\Big[\langle 0,0 | \cancel{a_x a_y} | 00 \rangle^{\to 0} + \langle 00 | \cancel{a_x a_y^\dagger} | 00 \rangle^{\to 0}$

$+ \langle 00 | \cancel{a_x^\dagger a_y} | 00 \rangle^{\to 0} + \langle 00 | \cancel{a_x^\dagger a_y^\dagger} | 00 \rangle^{\to 0}\Big]$

go. to $E_{00}^{(2)} = \sum_{n \neq k} \frac{|V_{kn}|^2}{(E_k^0 - E_n^0)}$

B Continued

$$E_{00}^{2} = \frac{\langle 00|V|01\rangle^2}{E_{00}-E_{01}} + \frac{\langle 00|V|10\rangle|^2}{E_{00}-E_{10}} + \frac{\langle 00|V|11\rangle^2}{E_{00}-E_{11}}$$

$\downarrow \; \phi \qquad \downarrow \; \phi$

$$\rightarrow \langle 00|a_x a_y|11\rangle + \langle 00|a_x a_y^{+}|00\rangle + \langle 00|a_x^{+} a_y|00\rangle$$

$$+ \langle 00|a_x^{+} a_y^{+}|00\rangle$$

$$\rightarrow \frac{A^2 \omega^2 \hbar^2}{2(-(2\hbar\omega))} \rightarrow \boxed{E_{00}^{(2)} = -\frac{A^2}{4}\hbar\omega}$$

look at this again

C $E_{10}^{(1)} = \langle n_1 n_2 | V' | n_1 n_2\rangle$

$$E^{(1)} = \begin{array}{c|cc} & |01\rangle & |10\rangle \\ \hline |01\rangle & \emptyset & 1 \\ |10\rangle & 1 & \emptyset \end{array} \; \frac{A\omega\hbar}{2}$$

$$\langle 01|V'|01\rangle = \langle 01|a_x a_y|01\rangle + \langle 01|a_x a_y^{+}|01\rangle$$

$$+ \langle 01|a_x^{+} a_y|01\rangle + \langle 01|a_x^{+} a_y^{+}|01\rangle$$

$$\langle 10|a_x a_y|10\rangle + \langle 10|a_x a_y^{+}|01\rangle + \underbrace{\langle 10|a_x^{+} a_y|01\rangle}_{\frac{A\omega\hbar}{2}}$$

$$+ \langle 10|a_x^{+} a_y^{+}|00\rangle$$

(2) Continued

$$\frac{A\omega\hbar}{2}\begin{pmatrix}-E & 1\\ 1 & -E\end{pmatrix} = \emptyset$$

$$E^2 - 1 = \qquad \underline{E = \pm \frac{A\omega\hbar}{2}}$$

$$\underline{E_1 = \frac{A\omega\hbar}{2}} \rightarrow \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}A'\\ B\end{pmatrix} = \emptyset$$

$$\hookrightarrow \quad -A' + B = \emptyset$$

$$\rightarrow A' = B$$

$$\left|E = \frac{A\omega\hbar}{2}\right\rangle = \begin{pmatrix}A'\\ A'\end{pmatrix}$$

$$\boxed{\left|E = \frac{A\omega\hbar}{2}\right\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right]}$$

$$\underline{E_2 = -\frac{A\omega\hbar}{2}} \rightarrow \boxed{\left|E = -\frac{A\omega\hbar}{2}\right\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]}$$

(D) Finish