F-2007

PROBLEM 1

The wave function of a particle of mass m in free space is approximated by $\phi(\vec{r}) = Ne^{i\vec{k}\cdot\vec{r}}$ where $\vec{k}$ is a constant and N is a normalization constant.

[a] (4 pts) What would be the result of a measurement of the momentum of the particle? Explain your answer.

[b] (4 pts) What would be the result of a measurement of the energy of the particle? Explain your answer.

[c] (2 pts) What would be the result of a measurement of the position of the particle? Explain your answer.

F-2007

1 $\phi(\vec{r}) = Ne^{i\vec{k}\cdot\vec{r}}$

A) most general solution is

$$\phi(r) = Ae^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

~~[illegible]~~ a probability density is

$|A|^2$ which does NOT depend on x or t

Therefore, we have infinite uncertainty on x & t (complete loss of information).

Therefore, a momentum measurement would yield $\boxed{p = \hbar k}$

B) Energy would yield $\boxed{\frac{k^2\hbar^2}{2m} = E}$

$\frac{1}{m^2}\, kg^2 \frac{m^2}{s^2}$

C) x can be anything

S-2016

Problem 3: Identical particles (10 pts)

Two non-interacting particles of mass m are trapped in a 1-dimensional infinite box of length L situated between $x = 0$ and $x = L$. (In the cases you are considering fermions, assume them to all be spin up.)

(a) [1 points] Write down the single particle energy eigenvalues and wavefunctions.

(b) [1 points] Write down the energy eigenvalues and wavefunctions for two distinguishable particles. Label the states by n_1 for particle 1 and n_2 for particle 2.

(c) [2 points] An energy measurement of the *two identical particle* system yields $E = \hbar^2\pi^2/mL^2$. Write down the state vector/wave function of the system.

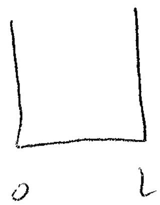
(d) [2 points] Suppose instead the energy of the two identical particle system is measured to be $E = 5\hbar^2\pi^2/mL^2$. What is the wave function? *Hint: there are two possibilities.*

(e) [2 points] Show that the fermion state you found in part (d) is an eigenfunction of the Hamiltonian, with the appropriate eigenvalue.

(f) [1 points] Write down the wavefunction for two identical spin-up fermions in the $n_1 = 2$ and $n_2 = 2$ state.

(g) [1 points] If instead you had three particles in the orthonormal states Ψ_1,Ψ_2, and Ψ_3, construct the three particle state for identical fermions.

S-2016

(13) 

Spin up Fermions.

(A) $\boxed{\psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}}$ $\boxed{E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}}$

(B)
$$\boxed{\psi_{n_1 n_2} = \frac{2}{L} \sin \frac{n_1 \pi x_1}{L} \sin \frac{n_2 \pi x_2}{L} \qquad E_{n_1 n_2} = \frac{\pi^2 \hbar^2}{2mL^2} (n_1^2 + n_2^2)}$$

(C) $E = \frac{\hbar^2 \pi^2}{mL^2} \quad (n_1^2 + n_2^2) = 1$

$\rightarrow \boxed{\psi_{11} = \frac{2}{L} \sin \frac{\pi x_1}{L} \sin \frac{\pi x_2}{L}}$

(D) $E = \frac{5\hbar^2 \pi^2}{mL^2} \quad (n_1^2 + n_2^2) = 10$

1 9 3

$$\boxed{\psi_{13} = \frac{2}{L} \sin \frac{\pi x_1}{L} \sin \frac{3\pi x_2}{L} \quad \text{or} \quad \psi_{31} = \frac{2}{L} \sin \frac{3\pi x_1}{L} \sin \frac{\pi x_2}{L}}$$

(c) Assuming Fermions?

ψ = (Antisymmetric)(Symmetric spin)

This is a weird problem
only because you don't
understand when
This is a Fermion
or Not

S-2016

Problem 4: Matrix Mechanics (10 pts)

Consider a system governed by a Hamiltonian H, with an observable C. The Hamiltonian is represented in the $|e_i\rangle$ basis as:

$$H = \hbar\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{Where } |e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvalues and eigenvectors of H are

$$|E_1 = -\hbar\omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, |E_2 = \hbar\omega, 1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, |E_2 = \hbar\omega, 2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let C be represented in the $|e_i\rangle$ basis as

$$C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

At t=0, the system is in the state: $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$

a) At time t=0, the observable C is measured. What results are possible and with what probabilities? (2 pts)

b) Determine the representation of the time evolution operator $U(t, t_0 = 0)$ in the $|e_i\rangle$ representation. (2 pts)

c) Determine $|\Psi(t)\rangle$ in the $|e_i\rangle$ basis. (2 pts)

d) If C is measured at some later time t, what results are possible and with what probabilities? (2 pts)

e) Are your probabilities time dependent or time independent? Explain (2 pts)

S-2016

4 A $C = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = \emptyset$$

$$\rightarrow -\lambda(-\lambda(1-\lambda)) + 2(0 - 2(1-\lambda)) = \emptyset$$

$$-\lambda(-\lambda+\lambda^2) + -4(1-\lambda) = \emptyset$$

$$\rightarrow \lambda^2(1-\lambda) - 4(1-\lambda) = \emptyset$$

$$\rightarrow (\lambda^2-4)(1-\lambda) = \emptyset$$

$$\rightarrow \lambda = 1, \pm 2$$

$\rightarrow \underline{\lambda = 1} \rightarrow \begin{pmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} =$

$B = 1$

$$\rightarrow |C=1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

A Continued

$$|\lambda=2\rangle = \begin{pmatrix} -2 & 0 & 2 \\ 0 & \cancel{AB}\,-1 & 0 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$A = C$$

$$\Rightarrow |c=2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$|\lambda=-2\rangle \Rightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\Rightarrow |c=-2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$|\psi\rangle = \sum_i |c_i\rangle\langle c_i|\psi\rangle$$

$$= \left[\frac{1}{\sqrt{2}}(0\ 1\ 0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}(0\ 1\ 0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]|c=1\rangle$$

$$+ \left[\frac{1}{2}(1\ 0\ 1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}(1\ 0\ 1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]|c=2\rangle$$

$$+ \left[\frac{1}{2}(1\ 0\ -1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}(1\ 0\ -1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]|c=-2\rangle$$

A Continued

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}|c=1\rangle + \frac{1}{2}|c=2\rangle + \frac{1}{2}|c=-2\rangle}$$

B
$$U = \sum_{ij} |E_i\rangle\langle E_i| e^{-iHt/\hbar} |E_j\rangle\langle E_j|$$

$$= |-\hbar\omega\rangle e^{i\omega t}\langle -\hbar\omega| + |\hbar\omega, 1\rangle e^{-i\omega t}\langle \hbar\omega, 1|$$

$$+ |\hbar\omega, 2\rangle e^{-i\omega t}\langle \hbar\omega, 2|$$

$$\boxed{\begin{aligned} U &= \frac{1}{2}\Big[|e_2\rangle - |e_3\rangle\Big] e^{i\omega t}\Big[\langle e_2| - \langle e_3|\Big] \\ &+ \frac{1}{2}\Big[|e_2\rangle + |e_3\rangle\Big] e^{-i\omega t}\Big[\langle e_2| + \langle e_3|\Big] \\ &+ |e_1\rangle e^{-i\omega t}\langle e_1| \end{aligned}}$$

C $|\psi(t)\rangle = U|\psi(0)\rangle$

$$= \Big[\frac{1}{2}\big[|e_2\rangle - |e_3\rangle\big] e^{i\omega t}\big[\langle e_2| - \langle e_3|\big]$$

distribute — get a #

$$\frac{\#}{2} e^{i\omega t}\big[|e_2\rangle - |e_3\rangle\big] \ldots$$

$$+ \frac{1}{2}\big[|e_2\rangle + |e_3\rangle\big] e^{-i\omega t}\big[\langle e_2| + \langle e_3|\big]$$

issue is up here

$$+ |e_1\rangle e^{-i\omega t}\langle e_1|\Big]\frac{1}{\sqrt{2}}\big[|e_1\rangle + |e_2\rangle\big]$$

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega t}|e_1\rangle + \underbrace{\frac{1}{2}|e_2\rangle e^{-i\omega t} + \frac{1}{2} e^{i\omega t}|e_2\rangle}_{\cos\omega t|e_2\rangle}}$$

D $|\psi(t)\rangle = \sum_i |c_i\rangle\langle c_i|\psi(t)\rangle$

$$\cancel{= \Big[(0\ 1\ 0)\begin{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t}\\ \cos\omega t\\ 0\end{pmatrix} + \frac{1}{\sqrt{2}}(1\ 0\ 1)\begin{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t}\\ \cos\omega t\\ 0\end{pmatrix}}$$

$$\cancel{+ \frac{1}{\sqrt{2}}(1\ 0\ -1)\begin{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t}\\ \cos\omega t\\ 0\end{pmatrix}\Big]|c=1\rangle}$$

$$= (0\ 1\ 0)\begin{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t}\\ \cos\omega t\\ 0\end{pmatrix}|c=1\rangle + \frac{1}{\sqrt{2}}(1\ 0\ 1)\begin{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t}\\ \cos\omega t\\ 0\end{pmatrix}|c=2\rangle$$

$$+ \frac{1}{\sqrt{2}}(1\ 0\ -1)\begin{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t}\\ \cos\omega t\\ 0\end{pmatrix}|c=-2\rangle$$

$$= \cos\omega t|c=1\rangle + \frac{1}{2}e^{-i\omega t}|c=2\rangle + \frac{1}{2}e^{-i\omega t}|c=-2\rangle$$

$\boxed{\text{No}}$

F-2011

PROBLEM 1: Postulates of Quantum Mechanics

A physical system consists of three distinct physical states. For this system, an operator Λ has eigenvalues λ_1, λ_2 and λ_3.

(a) Write down the completeness relation for this system. [2 points]

(b) Apply the completeness relation, then write down the expansion of a general state $|\psi\rangle$ in terms of eigenvectors of Λ [1 point]

(c) What is the probability that a measurement Λ of the state $|\psi\rangle$ yields the value λ_1? [2 points]

(d) A measurement of Λ on the state $|\psi\rangle$ is found to give a value λ_2. What is the state of the system immediately after the measurement? [1 point]

(e) A second measurement of Λ on the system is immediately performed. What is the probability of finding $\langle\Lambda\rangle = \lambda_1$? What is the probability of finding $\langle\Lambda\rangle = \lambda_2$? [2 points]

(f) Let us assume that the Hamiltonian H is time independent. Write down an equation that determines the time evolution of the state $|\psi(t)\rangle$ in the Schrödinger picture. Write down an equation that determines the time evolution of $\Lambda(t)$ in the Heisenberg picture. [2 points]

F-2011

1. $\lambda_1, \lambda_2, \lambda_3 \quad \Lambda$

A. $\sum_n |c_n\rangle\langle c_n| = \boxed{|\lambda_1\rangle\langle\lambda_1| + |\lambda_2\rangle\langle\lambda_2| + |\lambda_3\rangle\langle\lambda_3| = 1}$

B. $|\psi\rangle = \sum_n |c_n\rangle\langle c_n|\psi\rangle = \boxed{c_1|\lambda_1\rangle + c_2|\lambda_2\rangle + c_3|\lambda_3\rangle = |\psi\rangle \text{ where } c_i = \langle\lambda_i|\psi\rangle}$

C. $\boxed{\text{Probability} = |\langle\lambda_1|\psi\rangle|^2 = |c_1|^2}$

D. $\boxed{|\psi'\rangle = |\lambda_2\rangle}$

E. $|\langle\psi|\psi'\rangle|^2$

$\rightarrow |\langle\lambda_1|\lambda_2\rangle|^2 = \phi$

$\rightarrow |\langle\lambda_2|\lambda_2\rangle|^2 = 1$

F. Shrodinger $\rightarrow$ $\boxed{i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle}$

Heisenberg $\rightarrow$ $\boxed{\frac{d\langle\Lambda(t)\rangle}{dt} = \left\langle \frac{1}{i\hbar}[H, \Lambda]\right\rangle}$

S-2001

Problem 5: Measurement and Probability (10 points) 5

Consider the following two observables, H and C, whose representation in the unit basis $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$ is:

$$H = \hbar\omega \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

where:

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Assume that at time t=0 the ensemble of particles is in the state:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$$

The eigenvalues of H are given by $\lambda = 2, 1, -1$ with normalized eigenvectors given by $(1,1,1)/\sqrt{3}$, $(1,0,-1)/\sqrt{2}$ and $(1,-2,1)/\sqrt{6}$ respectively.

The eigenvalues of C are given by $\lambda = 1, 1, -1$ with normalized eigenvectors given by $(1,0,-1)/\sqrt{2}$, $(0,1,0)$ and $(1,0,1)/\sqrt{2}$ respectively.

a) What is the probability of measuring H and obtaining $E = \hbar\omega$? What state is the particle in after the measurement? (2 pts)

b) If one immediately measures C after the measurement of H in part b), what is the probability of obtaining $c = 1$? (1 pt)

c) What is the probability of measuring H first and getting $E = \hbar\omega$, then measuring C and getting $c = 1$, i.e. what is $P_{|\Psi(0)\rangle}(E = \hbar\omega, c = 1)$? (1 pt)

d) If the system is allowed to evolve in time after the measurement of H and before C is measured, will your answer to part c) change? Explain your reasoning. (1 pt)

e) With the ensemble of particles all in the original state: $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$, reverse the order of the above measurements and answer the same questions:

i) What is the probability of obtaining $c = 1$ if C is measured first? What state is the particle in after C is measured? (1 pt)

ii) If one immediately measures H after C is measured in part i), what is the probability of obtaining $E = \hbar\omega$? (1 pt) **(question continues on next page...)**

iii) What is the composite probability $P_{|\Psi(0)\rangle}(c = 1, E = \hbar\omega)$? (1 pt)

iv) If the system had been allowed to evolve in time after the measurement of C and before H is measured, would your answer to part ii) be different? Explain. (1 pt)

f) Are H and C compatible observables? Why?

S-2004

5 $\underline{H} \rightarrow \quad |\lambda=2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$|\lambda=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$|\lambda=-1\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

$\underline{C} \rightarrow \quad |\lambda=1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$$|\lambda=1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$|\lambda=-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

A $\begin{pmatrix}\hbar\omega & \hbar\omega & 0\\ \hbar\omega & 0 & \hbar\omega\\ 0 & \hbar\omega & \hbar\omega\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}\frac{1}{\sqrt{2}} = \begin{pmatrix}\hbar\omega\\0\\-\hbar\omega\end{pmatrix}\frac{1}{\sqrt{2}}$

$$= \hbar\omega\begin{pmatrix}1\\0\\-1\end{pmatrix}\frac{1}{\sqrt{2}}$$

A Continued

$$|\langle \lambda = 1 | \psi(\phi) \rangle|^2 = ?$$

$$|\psi\rangle = \sum_i |E_i\rangle\langle E_i|\psi\rangle = \frac{1}{\sqrt{3}}(1\;1\;1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}|2\rangle$$

$$+ \frac{1}{\sqrt{2}}(1\;0\;-1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}|1\rangle$$

$$+ \frac{1}{\sqrt{6}}(1\;-2\;1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}|-1\rangle$$

$$= \frac{2}{\sqrt{6}}|2\rangle + \frac{1}{2}|1\rangle - \frac{1}{2\sqrt{3}}|-1\rangle$$

$\cancel{\frac{4}{6} + \frac{3}{4} + \frac{1}{12}}$

$\cancel{\frac{4}{6} + \frac{1}{4} + \frac{1}{12}}$

$$\rightarrow \boxed{|\langle \lambda = 1|\psi\rangle|^2 = \frac{1}{4}}$$

$$\boxed{|\psi'\rangle = |1\rangle} = \frac{1}{\sqrt{2}}\left[|e_1\rangle - |e_3\rangle\right]$$

B $|\langle C=1|\psi'\rangle|^2$

$$|\psi'\rangle = \sum_{\text{all } i} |c_i\rangle\langle c_i|\psi'\rangle$$

$$= \frac{1}{2}(1\ 0\ {-1})\begin{pmatrix}1\\0\\-1\end{pmatrix}|1,1\rangle$$

$$+ \frac{1}{\sqrt{2}}(0\ 1\ 0)\begin{pmatrix}1\\0\\-1\end{pmatrix}|1,2\rangle$$

$$+ \frac{1}{2}(1\ 0\ 1)\begin{pmatrix}1\\0\\-1\end{pmatrix}|-1\rangle$$

$$= |1,1\rangle$$

$$\rightarrow \boxed{P=1}$$

C $\boxed{P = \frac{1}{4} = (1)(\frac{1}{4})}$

D No, Stationary state (No dependence on time in terms of expectation values)

E (i) $|\psi\rangle = \frac{1}{2}(1\ 0\ {-1})\begin{pmatrix}1\\1\\0\end{pmatrix}|1,1\rangle + \frac{1}{\sqrt{2}}(0 1 0)\begin{pmatrix}1\\1\\0\end{pmatrix}|1,2\rangle$

$$+ \frac{1}{2}(1\ 0\ 1)\begin{pmatrix}1\\1\\0\end{pmatrix}$$

E 1 (continued)

$$|\psi\rangle = \frac{1}{2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,2\rangle + \frac{1}{2}|-1\rangle$$

$$\rightarrow |\langle c=1|\psi\rangle|^2 = \underbrace{|\langle c=1,1|\psi\rangle|^2}_{\frac{1}{4}} + \underbrace{|\langle c=1,2|\psi\rangle|^2}_{\frac{1}{2}=\frac{2}{4}}$$

$$= \boxed{\frac{3}{4}}$$

$$\boxed{|\psi'\rangle = \frac{1}{\sqrt{2}}\left[|1,1\rangle + |1,2\rangle\right]}$$

(ii)

ii $|2\rangle_E = \sum_i |c_i\rangle\langle c_i|2\rangle_E$

$= \frac{1}{\sqrt{6}} (1\ 0\ -1)\begin{pmatrix}1\\1\\1\end{pmatrix} |1,1\rangle +$

$\frac{1}{\sqrt{3}} (0\ 1\ 0)\begin{pmatrix}1\\1\\1\end{pmatrix} |1,2\rangle$

$+ \frac{1}{\sqrt{6}} (1\ 0\ 1)\begin{pmatrix}1\\1\\1\end{pmatrix} |-1\rangle$

$= \frac{1}{\sqrt{6}} \cancel{0} + \frac{1}{\sqrt{3}} |1,2\rangle + \frac{2}{\sqrt{6}} |-1\rangle$

$\frac{1}{3} + \frac{4}{6}$ ✓

$\rightarrow |{}_E\langle 2|\psi'\rangle|^2 = \boxed{\frac{1}{6}}$

iii $= \boxed{\frac{1}{8}}$

iv No

f $\boxed{\text{No}}$

S-2014

PROBLEM 3: Matrix Mechanics

Let A, B and C be three ensembles that are represented in the orthonormal basis $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$,

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of A are doubly degenerated, $a = 1,\ 1,\ -1$, with eigenvectors

$$|a=1,1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad |a=1,2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |a=-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The eigenvalues of C are also doubly degenerate, $c = 2,\ 1,\ 1$, with eigenvectors:

$$|c=2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |c=1,1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |c=1,2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Assume that all particles in the ensemble are in the state $|\psi\rangle$,

$$|\psi\rangle = \frac{1}{2}|e_1\rangle - \frac{1}{2}|e_2\rangle + \frac{1}{\sqrt{2}}|e_3\rangle.$$

Answer the following questions:

a) Find the probability of measuring C and obtaining a value $c = 2$; then immediately measuring A and getting $a = 1$, *i.e.* find $P_{|\psi\rangle}(c = 2, a = 1)$. Identify the intermediate state $|\psi'\rangle$ after C is measured. (2 Points)

b) Now find the probability if those measurements are performed in the reverse order, *i.e.*, find $P_{|\psi\rangle}(a = 1, c = 2)$. Identify the intermediate state $|\psi''\rangle$ after A is measured. (2 Points)

c) Compare the results of parts a) and b) and explain why this happened. (1 Point)

d) If you are told that the eigenvalues of B are $b = -2, -2, 4$, justify whether or not the following 2 probabilities $P_{|\psi\rangle}(a = -1, b = 4)$ and $P_{|\psi\rangle}(b = 4, a = -1)$ will be equal (do NOT explicitly calculate the probabilities). Will the final states be the same or different? Explain. (2 Points)

e) Does $\{A, B\}$ constitute a complete set of commuting observables? Demonstrate explicitly. (3 Points)

S-2014

①

③ Ⓐ $|c=2\rangle = |e_1\rangle$

$$\Rightarrow |\langle c=2|\psi\rangle|^2 = \boxed{\frac{1}{4} = P(c=2)}$$

$$\boxed{|\psi'\rangle = |e_1\rangle}$$

$P(a=1) = P(a=1,2)$ or $P(a=1,1)$

$$P(a=1,2) = |\langle a=1,2|\psi'\rangle|^2$$

$$|\psi'\rangle = \sum_i |a_i\rangle\langle a_i|\psi'\rangle$$

$$= \frac{1}{\sqrt{2}}(1\,1\,0)\begin{pmatrix}1\\0\\0\end{pmatrix}|a=1,1\rangle + (0\,0\,1)\begin{pmatrix}1\\0\\0\end{pmatrix}|a=1,2\rangle$$

$$+ \frac{1}{\sqrt{2}}(1\,-1\,0)\begin{pmatrix}1\\0\\0\end{pmatrix}|a=-1\rangle$$

$$= \frac{1}{\sqrt{2}}|a=1,1\rangle + \frac{1}{\sqrt{2}}|a=-1\rangle$$

$$\Rightarrow p(a=1,2) = |\langle a=1,2|\psi'\rangle|^2 = \emptyset$$

$$\Rightarrow \boxed{P(a=1,1) = \frac{1}{2}}$$

$$\Rightarrow \boxed{P(c=2, a=1) = \frac{1}{8}}$$

B $|\psi\rangle = \sum_i |a_i\rangle\langle a_i|\psi\rangle$

$$= \frac{1}{\sqrt{2}}(1\ 1\ 0)\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} |a=1,1\rangle$$

$$+ (0\ 0\ 1)\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} |a=1,2\rangle$$

$$+ \frac{1}{\sqrt{2}}(1\ -1\ 0)\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} |a=-1\rangle$$

$$= \frac{1}{\sqrt{2}}|a=1,2\rangle + \frac{1}{\sqrt{2}}|a=-1\rangle$$

$\rightarrow \quad P(a=1) = \frac{1}{2}$

$\rightarrow \quad \boxed{|\psi'\rangle = |a=1,2\rangle = |e_3\rangle}$

$\rightarrow \quad \boxed{P(a=1, c=2) = \emptyset}$

C These operators do NOT commute.

D $\cdot b = -2, -2, 4$ $\qquad B = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

$\rightarrow \underline{b=-2} \rightarrow \begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \emptyset$

$A = B$

& $\quad C = 1$

$\rightarrow |b=-2,1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\rightarrow |b=-2,2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\rightarrow \underline{b=4} \rightarrow \begin{pmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$\rightarrow |b=4\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

D $P(a=1, b=4)$

~~$P(a=1)/2$~~ $P(a=-1) = \frac{1}{2}\left| (1\ -1\ 0) \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2$

$= \frac{1}{2}$

$P(b=4) = \frac{1}{2}\left| (1\ -1\ 0) \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2$

$= \frac{1}{2}$

They will be the same since

$\boxed{|b=4\rangle = |a=-1\rangle}$

E $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A \rightarrow A^{+} = A$

$B = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \rightarrow B^{+} = B$

$AB = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

E Continued

$$BA = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\boxed{AB = BA \checkmark \quad \text{Commute}}$$

S-2015

Problem 4: Finite Quantum System

Consider a quantum system that can be described by three basis states, $|n\rangle$, $n = 1, 2, 3$, and the Hamiltonian in this basis:

$$H = \frac{\hbar\omega}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix} \tag{1}$$

(a) [3 pts] Solve for the energy eigenvalues and eigenstates of this system.

(b) [2 pts] If the system starts in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \tag{2}$$

determine the time-dependence of the state $|\psi(t)\rangle$. You may write your answer in terms of either the states $|n\rangle$ or the eigenstates you found in part (a).

(c) [3 pts] Calculate the time dependent probabilities for measuring the system to be in each of the states $|1\rangle$, $|2\rangle$, and $|3\rangle$, if the system starts in the state given in part (b). Explain why the different states can or cannot be measured and the frequency of the oscillations you found.

(d) [2 pts] Finally, assume that the states $|n\rangle$ are the eigenstates of some observable O where

$$O|n\rangle = (-1)^n n|n\rangle \tag{3}$$

If, again, the system starts in the state given in part (b), what is the time dependent expectation value of O, $\langle O\rangle(t)$?

S-2015

(1)

(4) Basis $|1\rangle, |2\rangle, |3\rangle$

$$H = \frac{\hbar\omega}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}$$

(A) $$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & i \\ 0 & -i & 1-\lambda \end{vmatrix} \to (2-\lambda)\underbrace{\left[(1-\lambda)^2 - 1\right]}_{=\emptyset} = \emptyset$$

$\lambda = 2$

$(1-\lambda)^2 - 1 = \emptyset$

$\lambda^2 - 2\lambda = \emptyset$

$\lambda = 2, \emptyset$

$\times \frac{\hbar\omega}{2}$

$\to \boxed{\lambda = 2, 2, 0}$

$\to \underline{\lambda = \emptyset} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} \to A = \emptyset$

$B = -iC$

$\to \begin{pmatrix} 0 \\ -iC \\ C \end{pmatrix}$

$\to 1 = \begin{pmatrix} 0 & iC & C \end{pmatrix}\begin{pmatrix} 0 \\ -iC \\ C \end{pmatrix} = C^2 + C^2 = 2C^2$

A Continued

$$\boxed{|\lambda = 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}}$$

$\cancel{\rightarrow \text{check} \rightarrow \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}}$

$\underline{\lambda = 2}$ $\rightarrow$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & i \\ 0 & -i & -1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$A = 1$

$$\rightarrow \boxed{|\lambda = 2\rangle_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$\underline{\lambda = 2}$ $\rightarrow$ $B = iC$ $\rightarrow$ $\boxed{\underline{|\lambda = 2\rangle_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}}$

(5)

(B) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle + |2\rangle\right]$

$|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$|2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$|1\rangle + |2\rangle = \begin{pmatrix}1\\1\\0\end{pmatrix}\frac{1}{\sqrt{2}}$

$$\rightarrow |\psi\rangle = \sum_{E_i} |E_i\rangle\langle E_i|\psi\rangle$$

$$= \frac{1}{2}(0 \; +i \; 1)\begin{pmatrix}1\\1\\0\end{pmatrix}|0\rangle + \frac{1}{\sqrt{2}}(1\,0\,0)\begin{pmatrix}1\\1\\0\end{pmatrix}|2\rangle_1$$

$$+ \frac{1}{2}(0 \; -i \; 1)\begin{pmatrix}1\\1\\0\end{pmatrix}|2\rangle_2$$

$$= \frac{+i}{2}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle_1 \; \frac{i}{2}|2\rangle_2$$

~~Absolute phase are unimportant~~

$$\frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1 \checkmark$$

~~$\rightarrow |\psi(t)\rangle = \frac{+i}{2} \ldots |0\rangle + \frac{1}{\sqrt{2}}$~~

remember eigenvalues are actually $0 \quad \hbar\omega \quad \& \quad \hbar\omega$

$$\rightarrow \boxed{|\psi(t)\rangle = \frac{i}{2}|0\rangle + \frac{1}{\sqrt{2}}e^{-i\omega t}|2\rangle_1 - \frac{i}{2}e^{-i\omega t}|2\rangle_2}$$

(c) $|\langle n|\psi(t)\rangle|^2$

$$|\psi(t)\rangle = \sum_n |n\rangle\langle n|\psi(t)\rangle$$

$$\rightarrow \psi(t) = \frac{i}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-i\\1\end{pmatrix} + \frac{1}{\sqrt{2}}e^{-i\omega t}\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{i}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\i\\1\end{pmatrix}$$

$$\rightarrow \langle 1|\psi(t)\rangle = \frac{i}{2}\frac{1}{\sqrt{2}}(1\ 0\ 0)\begin{pmatrix}0\\-i\\1\end{pmatrix} + \frac{1}{\sqrt{2}}e^{-i\omega t}(1\,0\,0)\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$-\frac{i}{2}\frac{1}{\sqrt{2}}(1\,0\,0)\begin{pmatrix}0\\i\\1\end{pmatrix}$$

$$= \emptyset + \frac{1}{\sqrt{2}}e^{-i\omega t} + \emptyset$$

$$\rightarrow \boxed{|\langle 1|\psi(t)\rangle|^2 = \frac{1}{2}}$$

$$\rightarrow \langle 2|\psi(t)\rangle = \frac{i}{2}\frac{1}{\sqrt{2}}(0\,1\,0)\begin{pmatrix}0\\-i\\1\end{pmatrix} + \frac{1}{\sqrt{2}}e^{-i\omega t}(0\,1\,0)\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$-\frac{i}{2}\frac{1}{\sqrt{2}}(0\,1\,0)\begin{pmatrix}0\\i\\1\end{pmatrix}e^{-i\omega t}$$

$$= \frac{1}{2\sqrt{2}} + \emptyset + \frac{1}{2\sqrt{2}}e^{-i\omega t}$$

$$\rightarrow \boxed{|\langle 2|\psi(t)\rangle|^2} = \frac{1}{8}\left(1+e^{-i\omega t}\right)\left(1+e^{i\omega t}\right)$$

$$= \frac{1}{8}\left(1+e^{i\omega t}+e^{-i\omega t}+1\right)$$

$$\boxed{= \frac{1}{4}+\frac{1}{4}\cos(\omega t)}$$

C Continued

$$\boxed{P_2 = P_3}$$

Stationary states

D $\langle \psi(t) | O | \psi(t) \rangle = \sum_{nm} \langle \psi(t) | n \rangle \langle n | O | m \rangle \langle m | \psi(t) \rangle$

$n = m$

otherwise ϕ

$$\boxed{= \sum_n |\langle n | \psi(t) \rangle|^2 \langle n | O | n \rangle}$$

$O|1\rangle = -1|1\rangle$

$O|2\rangle = 2|2\rangle$

$O|3\rangle = -3|3\rangle$

rest of the problem is easy

S-2014

PROBLEM 1: Rigid Rotator

A free molecule of NaCl can be approximated as a dumbell, or rigid rotator. Attach a reference frame to its center of mass, with z-axis oriented in an arbitrary direction. The Hamiltonian can be taken to be $H = \frac{L^2}{2I}$ where $\vec{L}$ is angular momentum and I is the (fixed) moment of inertia.

a) Write the Schroedinger equation for the molecule. (1 Point)

b) What are the energy eigenvalues? (2 points)

c) What are the steady-state eigenfunctions? (2 points)

d) Sketch an energy level diagram for the rotator. Note any possible degeneracies. (2 points)

e) The rotator, with electric dipole moment $\vec{D}$ oriented along the dumbell symmetry axis, is placed in an electric field $\vec{E} = E\hat{z}$. The dipole interaction is $H_D = -\vec{D} \cdot \vec{E}$. What is the first order perturbative correction to the lowest energy level? (3 points)

5-2014

1. $H = \dfrac{L^2}{2I}$ $\qquad I = \mu R^2 \rightarrow \mu = \dfrac{m_1 m_2}{m_1 + m_2}$

A. $H\psi = E\psi$

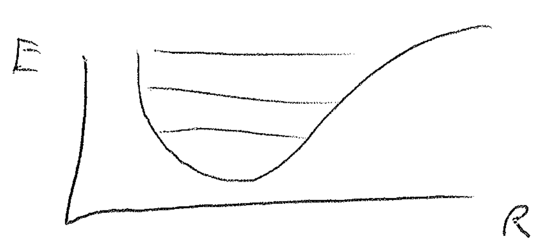
$$\rightarrow \boxed{\frac{L^2}{2I}\psi = E\psi}$$

B. $\dfrac{L^2\psi}{2I} = \dfrac{l(l+1)\hbar^2}{2I}\psi$

$$\boxed{E_l = \frac{l(l+1)\hbar^2}{2I}}$$

C. eigenfunctions of the L^2 operator are the spherical harmonics, $Y_l^m(\theta, \phi)$

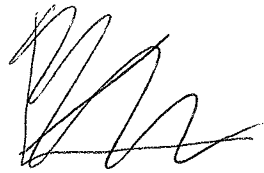
D. $(2l+1)$ degenerate



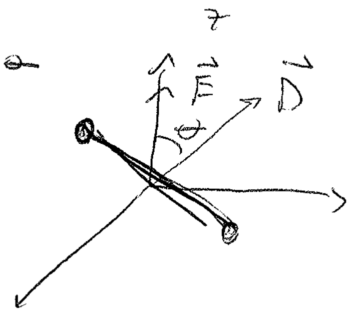
(E) $E_0^{(1)} = \langle \psi_0 | H_p | \psi_0 \rangle$

$\rightarrow \psi_0 = \frac{1}{\sqrt{4\pi}}$ $\qquad \rightarrow H_p = -DE\cos\theta$

$$\rightarrow E_0^{(1)} = \int_0^{2\pi}\int_0^{\pi} \frac{-DE}{4\pi}\cos\theta\sin\theta\, d\theta\, d\phi$$



$$= \frac{-DE}{2}\int_0^{\pi}\cos\theta\sin\theta\, d\theta$$



$u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$$= \int_0^{\pi} u\, du = \frac{DE}{4} u^2\Big|_0^{\pi} = \frac{DE}{4}\left[\cos^2\pi - \cos^2 0\right]$$

1 $\qquad$ 1

$$= \boxed{\varnothing}$$

S-2012

PROBLEM 2: Dirac Notation in Quantum Mechanics

Consider the kets $|a_n\rangle$ as the eigenstates of an observable operator $\mathbf{A}$

$$\mathbf{A}|a_n\rangle = a_n|a_n\rangle\,.$$

Assume that $|a_n\rangle$ form a discrete orthonormal basis in the vector space. Define an operator $U(m,n)$ as

$$U(m,n) = |a_m\rangle\langle a_n|.$$

(a) Show that $U(m,n)$ is an Hermitian operator. Calculate the commutator $[A, U(m,n)]$. [2 Points]

(b) For a generic operator with matrix elements $B_{mn} = \langle a_m|B|a_n\rangle$, show that

$$B = \sum_{mn} B_{mn} U(m,n).$$

[2 Points]

(c) Assume the Hamiltonian of a three-level system

$$\mathbf{H} = H_{12}U(1,2) + H_{21}U(2,1) + H_{23}U(2,3) + H_{32}U(3,2)$$

where $H_{12} = H_{23}$, and $H_{21} = H_{32}$ are complex numbers with dimension of energy. Find the eigenvectors and the eigenvalues of the Hamiltonian in the $|a_n\rangle$ basis. [4 Points]

(d) Assuming the Hamiltonian above, and $n = 1, 2, 3$, find the condition where the observable operator A is time independent. [2 Points]

5-2012

2 $A|a_n\rangle = a_n|a_n\rangle$

$U(m,n) = |a_m\rangle\langle a_n|$

A $\langle a_m|U|a_n\rangle = \langle a_m|a_m\rangle\langle a_n|a_n\rangle$

$= 1$

$\langle a_m|a_n\rangle \longleftrightarrow \langle a_n|a_m\rangle^*$

$U|a_n\rangle \mapsto \langle a_n|U^+$

$\rightarrow \langle a_n|U^+|a_m\rangle^*$

$\rightarrow \langle a_n|a_n\rangle\langle a_m|a_m\rangle$

$= 1$

$\rightarrow$ $\boxed{\langle a_m|U|a_n\rangle = \langle a_n|U^+|a_m\rangle^* \quad \text{Hermitian}}$

$\boxed{[A,U]} = AU|a_m\rangle - UA|a_m\rangle$

$= A|a_m\rangle\langle a_n|a_m\rangle - |a_m\rangle\langle a_n|A|a_m\rangle$

$= a_m\delta_{nm}|a_m\rangle - a_m\delta_{nm}|a_m\rangle$

$= 0$

(2)

(B) $B_{mn} = \langle a_m | B | a_n \rangle$

$$B = \sum_{mn} B_{mn} U(m,n)$$

$$B = \sum_m \sum_n |a_m\rangle \underbrace{\langle a_m | B | a_n \rangle}_{B_{mn}} \langle a_n |$$

$$= \sum_{mn} B_{mn} \underbrace{|a_m\rangle\langle a_n|}_{U(m,n)}$$

$$\rightarrow \boxed{B = \sum_{mn} B_{mn} U(m,n)}$$

(C) $H = H_{12}|1\rangle\langle 2| + H_{21}|2\rangle\langle 1| + H_{23}|2\rangle\langle 3| + H_{32}|3\rangle\langle 2|$

$$H = \begin{pmatrix} \emptyset & H_{12} & \emptyset \\ H_{21} & \emptyset & H_{23} \\ \emptyset & H_{32} & \emptyset \end{pmatrix}$$

$$\rightarrow \begin{vmatrix} -\lambda & H_{12} & \emptyset \\ H_{21} & -\lambda & H_{23} \\ \emptyset & H_{32} & -\lambda \end{vmatrix} = \emptyset$$

C Continued

$$-\lambda\left[\lambda^2 - H_{32}H_{23}\right] - H_{12}\left[-\lambda H_{21}\right] = \emptyset$$

$$-\lambda\left[\lambda^2 - H_{32}H_{23} - H_{12}H_{21}\right] = \emptyset$$

$H_{12} = H_{23}$

$H_{21} = H_{32}$

$$-\lambda\left[\lambda^2 - H_{21}H_{23} - H_{23}H_{21}\right] = \emptyset$$

$$-\lambda\left[\lambda^2 - 2H_{21}H_{23}\right] = \emptyset \rightarrow \boxed{\lambda = \emptyset, \pm\sqrt{2H_{21}H_{23}}}$$

$$\begin{pmatrix} -\lambda & H_{23} & \emptyset \\ H_{21} & -\lambda & H_{23} \\ \emptyset & H_{21} & -\lambda \end{pmatrix}$$

$$\lambda = \emptyset \rightarrow \begin{pmatrix} 0 & H_{23} & 0 \\ H_{21} & 0 & H_{23} \\ 0 & H_{21} & 0 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\boxed{|\lambda = 0\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle - |3\rangle\right]} \qquad B = \emptyset \rightarrow A = -C$$

C Continued

$\underline{\lambda = \emptyset}$

$$\rightarrow \begin{pmatrix} 0 & H_{23} & 0 \\ H_{21} & 0 & H_{23} \\ 0 & H_{21} & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \emptyset$$

$B = \emptyset$

$H_{21} A = -H_{23} C$

$\rightarrow C = -\frac{H_{21}}{H_{23}} A$

$$\rightarrow \begin{pmatrix} A \\ \emptyset \\ -\frac{H_{21}}{H_{23}} A \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} A & \emptyset & -\frac{H_{21}}{H_{23}} A \end{pmatrix} \begin{pmatrix} A \\ 0 \\ -\frac{H_{21}}{H_{23}} A \end{pmatrix} = 1$$

$$A^2 + \frac{H_{21}^2}{H_{23}^2} A^2 = 1 \rightarrow A^2 \underbrace{\left(1 + \frac{H_{21}^2}{H_{23}^2} \right)} = 1$$

$$A^2 \frac{H_{23}^2 + H_{21}^2}{H_{23}^2} = 1$$

$$\rightarrow \boxed{A = \frac{H_{23}}{\sqrt{H_{23}^2 + H_{21}^2}}}$$

$$\boxed{|\lambda = \emptyset\rangle = \frac{H_{23}}{\sqrt{H_{23}^2 + H_{21}^2}} \left[|1\rangle - \frac{H_{21}}{H_{23}} |3\rangle \right]}$$

(c) Continued

(5)

$\lambda = \sqrt{2H_{21}H_{23}}$ $\rightarrow$ $\begin{pmatrix} -\sqrt{2H_{21}H_{23}} & H_{23} & 0 \\ H_{21} & -\sqrt{2H_{21}H_{23}} & H_{23} \\ 0 & H_{21} & -\sqrt{2H_{21}H_{23}} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \emptyset$

$B = \frac{\sqrt{2H_{21}H_{23}}}{H_{23}} A$

$\rightarrow \begin{pmatrix} A \\ \frac{\sqrt{2H_{21}H_{23}}}{H_{23}} A \\ 0 \end{pmatrix}$

$\begin{pmatrix} A^* & \frac{\sqrt{2H_{21}H_{23}}}{H_{23}} A & 0 \end{pmatrix} \begin{pmatrix} A \\ \frac{\sqrt{2H_{21}H_{23}}}{H_{23}} A \\ 0 \end{pmatrix} = 1$

$A^2 + \frac{2H_{21}H_{23}}{H_{23}^2} A^2 = 1 \rightarrow A^2\left(1 + \frac{2H_{21}}{H_{23}}\right) = 1$

$\frac{H_{23} + 2H_{21}}{H_{23}}$

$\rightarrow A = \sqrt{\frac{H_{23}}{H_{23}+2H_{21}}}$

$|\lambda = \sqrt{2H_{21}H_{23}}\rangle$

$$\boxed{= \frac{\sqrt{H_{23}}}{\sqrt{H_{23}+2H_{21}}}\left[|1\rangle + \sqrt{\frac{2H_{21}H_{23}}{H_{23}^2}}\,|2\rangle\right]}$$

(c) ~~MM~~ $\boxed{\lambda = -\sqrt{2 H_{21} H_{23}} \quad \text{follows}}$

Finish

F-2010

PROBLEM 6: Radioactive Decay

In this problem you will calculate the transmission and reflection coefficients for a simple potential step. Then you will use this result to estimate the tunneling probability through an arbitrary potential. This evaluated tunneling probability is called the Gamow Factor. Finally, you will use the Gamow Factor to explain radioactive decay by calculating the decay probability for an α-particle being emitted from a radioactive nuclei and the mean lifetime for that process.

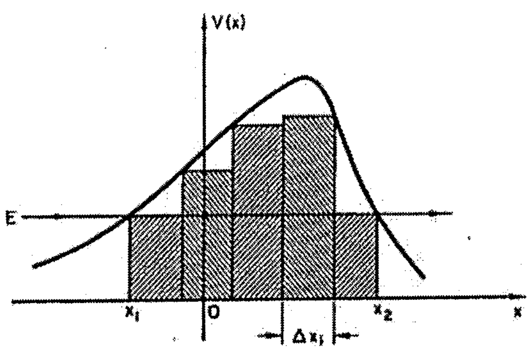
(a) [4 points] **Potential Step:** Calculate the transmission and reflection coefficients for a particle with total energy E interacting with a potential barrier that is a simple potential step ($V_0 > 0$):

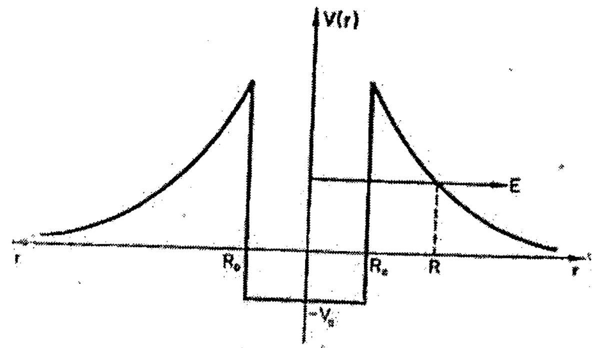
$$V(x) = \begin{cases} 0, & \text{if } x < 0 \\ V_0, & \text{if } 0 < x < a \\ 0, & \text{if } x > a. \end{cases}$$

(b) [3 points] **Arbitrary Potential:** A particle of total energy E interacts with an arbitrary potential barrier $V = V(x)$. The classical turning points are $x = x_1$ and $x = x_2$. Assume the potential curve $V(x)$ is sufficiently smooth, then divide the interval $[x_1, x_2]$ into intervals of length Δx_i, large compared with the relative penetration depth $d_i = \hbar\,[8m(v(x_i) - E)]^{-1/2}$ of a particle in the rectangular barriers. Find an expression for the transmission coefficient T (the Gamow Factor) in this approximate way for the barrier $V = V(x)$, knowing that

$$T_i \approx e^{\left[-\frac{1}{\hbar}\sqrt{8m(V(x_i)-E)}\right]}$$

for the *ith* rectangular barrier.

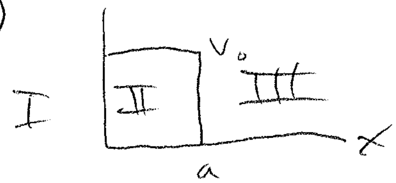




(c) [3 points] **α-emission of radioactive nuclei:** Now show that α-particles with energies of a few MeV can leave potential wells with depths of tens of MeV. Use a simplified model potential, *i.e.* let $V(r) = -V_0$ if $r < R_0$, and $V(r) = \frac{e_1 e_2}{r}$ if $r > R_0$. Now calculate Gamow's factor for this barrier, *i.e.* the decay probability for emission of α-particles of energy E through the barrier. Express the result in terms of the final velocity of the α-particle, and estimate the mean lifetime of an α-emitting nucleus.

F-2010

6



A For $E < V_0$ Since we care about Tunneling (mainly?)

$$\left[\frac{p^2}{2m} + V\right]\psi(x) = E\psi(x) \qquad p = -i\hbar\frac{\partial}{\partial x}$$

$$\rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx}\psi + V\psi = E\psi$$

$$\rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = (E-V)\psi \rightarrow \frac{d^2}{dx^2}\psi = \frac{2m}{\hbar^2}(V-E)\psi$$

$$\kappa = \frac{\sqrt{2m(V-E)}}{\hbar}$$

$$\rightarrow \frac{d^2}{dx^2}\psi = \kappa^2\psi$$

$$\rightarrow \psi_{II} = Ce^{\kappa_2 x} + De^{-\kappa_2 x}$$

$\rightarrow$ For I $\rightarrow$ $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \psi_I = Ae^{ik_1x} + Be^{-ik_1x}$

For III $=$ $\psi_{III} = Ee^{ik_1x}$

A, continued

at $\emptyset$ $\Rightarrow$ $\psi_I(\emptyset) = \psi_{II}(\emptyset) \Rightarrow A+B = C+D$

$$\frac{d\psi_I(\emptyset)}{dx} = \frac{d\psi_{II}(\emptyset)}{dx} \Rightarrow ik_1 A - ik_1 B = k_2 C - k_2 D$$

$$\rightarrow R = \frac{|B|^2}{|A|^2}$$

$\rightarrow$ $k_2(A+B) = k_2(C+D)$

$ik_1(A-B) = k_2(C-D)$ $\rightarrow$ $k_2(A+B) + ik_1(A-B) = 2k_2 C$ ①

$\rightarrow$ $-ik_1(A-B) = k_2(D-C)$

$k_2(A+B) = k_2(D+C)$ $\rightarrow$ $-ik_1(A-B) + k_2(A+B) = 2k_2 D$ ②

$$\rightarrow A+B = \frac{A+B}{2} + \frac{ik_1}{2k_2}(A-B) + \frac{A+B}{2} - \frac{ik_1}{2k_2}(A-B)$$

A - Continued

$\psi_{II}(a) = \psi_{III}(a) = \cancel{\quad} \; Ce^{k_2 a} + De^{-k_2 a} = Ee^{ik_1 a}$

$\frac{d\psi_{II}(a)}{dx} = \frac{d\psi_{III}(a)}{dx} = k_2 C e^{k_2 a} - k_2 D e^{-k_2 a} = ik_1 F e^{ik_1 a}$

$\to \cancel{\quad} \; \cancel{\quad} \; k_2 C e^{k_2 a} = \frac{Ee^{ik_1 a}(ik_1 + k_2)}{2}$

$\to C = \frac{Ee^{ik_1 a}}{2k_2 e^{k_2 a}}(ik_1 + k_2)$

$\to \frac{Ee^{ik_1 a}}{2k_2}(ik_1 + k_2) + De^{-k_2 a} = Ee^{ik_1 a}$

$\to D = \frac{Ee^{ik_1 a}}{e^{-k_2 a}}\left(1 - \left(\frac{ik_1 + k_2}{2k_2}\right)\right)$

$\to A + B = \frac{Ee^{ik_1 a}}{2k_2 e^{k_2 a}}(ik_1 + k_2) + Ee^{ik_1 a}e^{k_2 a}$

$- Ee^{ik_1 a}e^{k_2 a}\left(\frac{ik_1 + k_2}{2k_2}\right)$

$= Ee^{ik_1 a}\left(\frac{ik_1 + k_2}{2k_2 e^{k_2 a}}(ik_1 + k_2) + e^{k_2 a} - e^{k_2 a}\left(\frac{ik_1 + k_2}{2k_2}\right)\right)$

F-2010

PROBLEM 3: Dirac formulation of quantum mechanics

Let $\mathcal{E}_3$ be a three-dimensional Hilbert space that is spanned by the orthonormal basis $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$. The operator Ω acts in $\mathcal{E}_3$ as follows:

$$\Omega|u_1\rangle = 3|u_1\rangle \tag{1}$$
$$\Omega|u_2\rangle = 2|u_2\rangle - |u_3\rangle \tag{2}$$
$$\Omega|u_3\rangle = -|u_2\rangle + 2|u_3\rangle \tag{3}$$

(a) [5 pt] Prove that Ω is Hermitian. Find its eigenvalues, ω_1, ω_2, and ω_3, and write down each of the corresponding eigenvectors in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis.

(b) [1 pt] Does $\{\Omega\}$ constitute a complete set of commuting operators for $\mathcal{E}_3$? Why or why not?

(c) [2 pt] According to Eq. (1), $\mathcal{E}_3$ can be partitioned into eigensubspaces by letting $\mathcal{E}_a$ be the subspace spanned by $\{|u_1\rangle\}$ and $\mathcal{E}_b$ be its orthogonal supplement. Construct an orthonormal basis $\{|t_2\rangle, |t_3\rangle\}$ of $\mathcal{E}_b$, and write each basis vector in $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis. (Choose $|t_3\rangle$ to correspond to the *smallest* eigenvalue of Ω.)

(d) [2 pt] With $|t_1\rangle = |u_1\rangle$, the set $\{|t_1\rangle, |t_2\rangle, |t_3\rangle\}$ constitutes an alternate basis of $\mathcal{E}_3$. Find the matrix S, with elements $S_{i,k} = \langle u_i \mid t_k\rangle$, that transforms between $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ and $\{|t_1\rangle, |t_2\rangle, |t_3\rangle\}$.

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(3) (A) $\Omega|u_1\rangle = 3|u_1\rangle \rightarrow \Omega = 3|u_1\rangle\langle u_1|$

$\Omega|u_2\rangle = 2|u_2\rangle - |u_3\rangle \rightarrow \Omega = 2|u_2\rangle\langle u_2| - |u_3\rangle\langle u_2|$

$\Omega|u_3\rangle = -|u_2\rangle + 2|u_3\rangle \rightarrow \Omega = -|u_2\rangle\langle u_3| + 2|u_3\rangle\langle u_3|$

$$\Omega \doteq \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

$$\boxed{\Omega^\dagger = \left(\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right) = \Omega \quad , \text{hermitian}}$$

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = \emptyset \rightarrow (3-\lambda)\left[(2-\lambda)^2 - 1\right] = \emptyset$$

$(2-\lambda)(2-\lambda)$

$4 + \lambda^2 - 4\lambda - 1$

$= \lambda^2 - 4\lambda + 3$

$(\lambda - 3)(\lambda - 1)$

$$\boxed{\begin{aligned} \omega_1 &= 3 \\ \omega_2 &= 3 \\ \omega_3 &= 1 \end{aligned}}$$

A Continued

$\underline{\omega_1 = 3} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{matrix} 0 = \phi \\ -b - c = \phi \\ -b - c = \phi \end{matrix} \rightarrow a = 1$

$$\boxed{\rightarrow |\omega_1 = 3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |u_1\rangle}$$

$\underline{\omega_2 = 3} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{matrix} 0 = \phi \\ -b - c = \phi \\ -b - c = \phi \end{matrix} \rightarrow c = -b$

$$\boxed{|\omega_2 = 3\rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(|u_2\rangle - |u_3\rangle\right)}$$

$\underline{\omega_3 = 1} \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{matrix} a = \phi \\ b = c \\ b = c \end{matrix}$

$$\boxed{|\omega_3 = 1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|u_2\rangle + |u_3\rangle\right)}$$

B No, Ω is degenerate

C $\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \phi \rightarrow \lambda = 3, 1$

$\underline{t_2 = 3} \rightarrow \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \rightarrow \begin{matrix} -b-c=\phi \\ -b-c=\phi \end{matrix}$

$$\boxed{|\tau_2 = 3\rangle = \frac{1}{\sqrt{2}} [\,|u_2\rangle - |u_3\rangle\,]}$$

$\underline{t_3 = 1} \rightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \rightarrow \begin{matrix} b=c \\ b=c \end{matrix}$

$$\boxed{|\tau_3 = 1\rangle = \frac{1}{\sqrt{2}} (\,|u_2\rangle + |u_3\rangle\,)}$$

D

$$S_{ij} = \begin{array}{c} \\ \left(\begin{matrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{matrix}\right) \\ \begin{matrix} |u_1\rangle & |u_2\rangle & |u_3\rangle \end{matrix} \end{array} \begin{matrix} \langle u_1| \\ \langle u_2| \\ \langle u_3| \end{matrix}$$

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PROBLEM 1: Eigenvalues and Eigenvectors

Suppose the Hamiltonian for a system is given by

$$H = \hbar\omega_0(\sigma_x + \sigma_y)$$

where σ_x and σ_y are two of the Pauli matrices.

(a). Calculate the eigenvalues and eigenvectors for this Hamiltonian. (*5 points*)

(b). In the Schrödinger picture, the state vector is

$$|\psi(t)\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

At $t = 0$, the state of this system has $\alpha(0) = 0$ and $\beta(0) = 1$. Evaluate $\alpha(t)$ and $\beta(t)$ for $t > 0$. (*5 points*)

① $H = \hbar\omega_0(\sigma_x + \sigma_y)$

$$= \hbar\omega_0 \begin{pmatrix} 0 & 1-i \\ 1+i & \phi \end{pmatrix}$$

Ⓐ $\hbar\omega_0 \left[\begin{vmatrix} -\lambda & 1-i \\ 1+i & -\lambda \end{vmatrix} \right] = \phi = \lambda^2 - \underbrace{(1-i)(1+i)}_{1+1+i-i} = \phi$

$$= \lambda^2 - 2 = \phi \rightarrow \boxed{\lambda = \pm\sqrt{2}\,\hbar\omega_0 = \text{eigenvalues}}$$

$\rightarrow \underline{\lambda = \sqrt{2}} \rightarrow \hbar\omega_0 \begin{pmatrix} -\sqrt{2} & 1-i \\ 1+i & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \phi$

$$\rightarrow -\sqrt{2}\,a + (1-i)b = \phi$$

$$a(1+i) - \sqrt{2}\,b = \phi$$

$$\rightarrow b = \frac{\sqrt{2}}{1-i}\,a$$

$\rightarrow |E = \sqrt{2}\,\hbar\omega_0\rangle = \hbar\omega_0 \begin{pmatrix} a \\ \frac{\sqrt{2}}{1-i}\,a \end{pmatrix}$

$\rightarrow \langle E = \sqrt{2}\,\hbar\omega_0 | E = \sqrt{2}\,\hbar\omega_0\rangle = \hbar^2\omega_0^2 \left(a \quad \sqrt{2}/1+i\, a \right) \begin{pmatrix} a \\ \frac{\sqrt{2}}{1-i}\,a \end{pmatrix}$

$\rightarrow \hbar^2\omega_0^2 \left[a^2 + \frac{2}{(1+i)(1-i)}\,a^2 \right] = 1$

$\rightarrow \hbar^2\omega_0^2 a^2 = \frac{1}{2} \rightarrow \hbar\omega_0 a = \frac{1}{\sqrt{2}}$

$\rightarrow \boxed{|E = \sqrt{2}\,\hbar\omega_0\rangle = \frac{\hbar\omega_0}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{1-i} \end{pmatrix}}$

A (Continued)

$$\underline{\lambda = -\sqrt{2}} \rightarrow \hbar\omega_0 \begin{pmatrix} \sqrt{2} & 1-i \\ 1+i & \sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \emptyset$$

$$\rightarrow \sqrt{2}\, a + (1-i)\, b = \emptyset$$

$$(1+i)\, a + \sqrt{2}\, b = \emptyset$$

$$\hookrightarrow b = -\frac{\sqrt{2}}{(1+i)}\, a$$

$$\rightarrow |E = -\sqrt{2}\,\hbar\omega_0\rangle = \hbar\omega_0 \begin{pmatrix} a \\ -\frac{\sqrt{2}}{(1+i)}\, a \end{pmatrix}$$

$$\rightarrow \langle E = -\sqrt{2}\,\hbar\omega_0 | E = -\sqrt{2}\,\hbar\omega_0\rangle = \hbar^2\omega_0^2 \begin{pmatrix} a & \frac{-\sqrt{2}}{(1-i)}\, a \end{pmatrix} \begin{pmatrix} a \\ \frac{-\sqrt{2}}{(1+i)}\, a \end{pmatrix}$$

$$= \hbar^2\omega_0^2 \left[a^2 + a^2 \frac{2}{(1-i)(1+i)} \right] = 1$$

$$\rightarrow \hbar\omega_0\, a = \frac{1}{\sqrt{2}}$$

$$\rightarrow \boxed{|E = -\sqrt{2}\,\hbar\omega_0\rangle = \frac{\hbar\omega_0}{\sqrt{2}} \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{(1+i)} \end{pmatrix}}$$

③

B $|\psi(t)\rangle = U(t)|\psi(\emptyset)\rangle$

$$= \exp\left(-\frac{i\hbar H}{t}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\rightarrow \quad \hbar\omega_0 \begin{pmatrix} 0 & 1-i \\ 1+i & \emptyset \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\omega_0 \begin{pmatrix} 1-i \\ 0 \end{pmatrix}$$

$$= \exp\left(\frac{-i\hbar E_0}{t}\right)\begin{pmatrix} 1-i \\ \emptyset \end{pmatrix}$$

where $E_0 = \hbar\omega_0$ & $\alpha(t) = e^{\frac{-i\hbar E_0}{t}}(1-i)$

$\beta(t) = \emptyset$