S-2010

PROBLEM 6: Hyperfine Splitting

The hyperfine splitting in hydrogen comes from a spin-spin interaction between the electron and the proton. The total Hamiltonian can be written as

$$H = \frac{P_p^2}{2m_p} + \frac{P_e^2}{2m_e} - \frac{e^2}{r} + H_{HF}$$

where $H_{HF} = A\vec{S}_e \cdot \vec{S}_p$, and A is a real constant.

(a) [1 points] What are the spin quantum numbers s and m_s of the electron?

(b) [1 points] What are the spin quantum numbers s and m_s of the proton?

(c) [1 points] What are the spin quantum numbers s and m_s of the combined electron-proton system?

(d) [5 points] Diagonalize H_{HF} in the total $\vec{S} = \vec{S}_e + \vec{S}_p$ basis and compute the energy eigenvalues.

(e) [2 points] Write an expression for the energy of a photon that would be emitted from a hyperfine transition in terms of A, $\hbar$, and any other relevant constants.

S-2010

6 1 $\boxed{S_e = \frac{1}{2} \quad m_s = \pm\frac{1}{2}}$

2 $\boxed{S_p = \frac{1}{2} \quad m_s = \pm\frac{1}{2}}$

3 $|S_e - S_p| \leq S \leq S_e + S_p$

$$\boxed{S = 0, 1}$$

$m_s = m_p + m_e = \boxed{1, 0, 1 = m_s}$

4 $H_{HF} = A(\vec{S_e} \cdot \vec{S_p})$

$$(S_e + S_p)^2 = S_e^2 + S_p^2 + 2 S_e \cdot p_p = S^2$$

$$\rightarrow S_e \cdot S_p = (S^2 - S_e^2 - S_p^2)\frac{1}{2}$$

basis optns $= |S_p\, m_p\, S_e\, m_e\rangle$

$$\boxed{|S\ m\ S_e\ S_p\rangle}$$

$S^2|S m \tfrac{1}{2}\tfrac{1}{2}\rangle = \hbar^2 S(S+1)|S m \tfrac{1}{2}\tfrac{1}{2}\rangle$

$S_e^2|S m \tfrac{1}{2}\tfrac{1}{2}\rangle = \hbar^2\frac{3}{4}|S m \tfrac{1}{2}\tfrac{1}{2}\rangle$

$= S_p^2|S m \tfrac{1}{2}\tfrac{1}{2}\rangle$

$S = 1 = 2\hbar^2 \quad \frac{8}{4} - \frac{3}{4} - \frac{3}{4} = \frac{1}{4}\hbar^2$

$S = 0 = \emptyset \quad = -\frac{3}{4}\hbar^2$

4 continued

$$
\begin{array}{cccccc}
 & & \multicolumn{4}{c}{|s\, m\rangle} \\
 & & |1\,1\rangle & |1\,0\rangle & |1\,{-1}\rangle & |0\,0\rangle \\
H_{HF} \doteq A\hbar^2 & \left(\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \begin{array}{c} \frac{1}{4} \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ -\frac{3}{4} \end{array}\right)
\end{array}
$$

$$\boxed{\text{eigenvalues are } \underbrace{\frac{A\hbar^2}{4}}_{\text{degeneracy of 3}} \ \& \ \underbrace{\frac{-3\hbar^2}{4}}_{\text{degeneracy of 1}}}$$

5 ~~[illegible]~~ $\frac{A\hbar^2}{4} + \frac{3\hbar^2 A}{4} = \boxed{\hbar^2 A}$

S-2010

PROBLEM 2: Hydrogenic Atoms with One Electron

In terms of the first Bohr radius, $a_0 \equiv \hbar/(c\alpha m_e)$, where α is the fine-structure constant, the ground-state eigenfunction of a hydrogen atom is

$$\psi_{1,0,0}(r,\theta,\varphi) = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0}.$$

(a) [3 points] Evaluate the probability of finding an electron in the ground-state of a hydrogen atom in the classically forbidden region. The classically forbidden region is the region of space where the classical kinetic energy is negative.

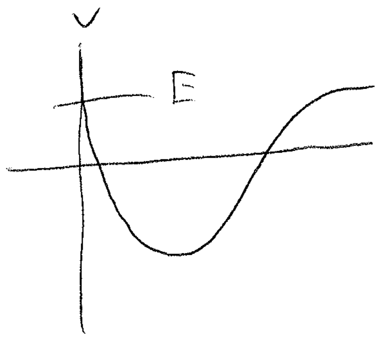
(b) [4 points] For the ground state, evaluate the uncertainty in the Cartesian coordinate x and the uncertainty in the corresponding component of the linear momentum, p_x. *Hint: You need not use the explicit form of the operator for the linear momentum to evaluate Δp_x.*

(c) [3 points] Show explicitly that the product of your uncertainties, $\Delta x\, \Delta p_x$, is consistent with the Heisenberg uncertainty principle.

S-2010

2 $\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$

A Classically $\rightarrow$ $T + V = E \rightarrow T = E - V$; T is + which means $E > V$



So its where

$E < V$

E for Hydrogen is $\frac{-13.6\,eV}{n^2} = -13.6\,eV$

$E = \frac{-Ryd}{n^2} \rightarrow \boxed{Ryd = \frac{e^2}{2a_0} \quad \text{remember this} ☆☆☆☆}$

~~Lets see if you can find it~~

$\cancel{\frac{-\hbar^2}{2m}\frac{d^2\psi}{dr^2} - \frac{e^2\psi}{r} = E\psi}$

$\cancel{\frac{-\hbar^2}{2m}\frac{1}{\sqrt{\pi a_0^3}}\frac{1}{a_0^2} - \frac{e^2}{r} = E}$

$\cancel{\frac{-\hbar^2}{2m} - \frac{e^2}{r} = E}$

$\Rightarrow \frac{-e^2}{2a_0} < \frac{-e^2}{r} \rightarrow \boxed{r > 2a_0}$

A $P = \iiint_{2a_0}^{\infty} \frac{1}{\pi a_0^3} e^{-2r/a_0} r^2 \cancel{dr} \sin\theta \, d\theta \, d\phi \, dr$

$$= \frac{4\pi}{\pi a_0^3} \int_{2a_0}^{\infty} e^{-2r/a_0} r^2 \, dr$$

$$\frac{e^{-2r/a_0}}{(-2/a_0)} \left(r^2 - \frac{2r}{(-2/a_0)} + \frac{2}{(-2/a_0)^2} \right) \Bigg|_{2a_0}^{\infty}$$

$$= -\left(\frac{e^{-4}}{(-2/a_0)} \left(4a_0^2 - \frac{4a_0}{(-2/a_0)} + \frac{2}{(-2/a_0)^2} \right) \right)$$

$$= + \frac{e^{-4} a_0}{2} \left(4a_0^2 + 2a_0^2 + \frac{1}{2} a_0^2 \right)$$

$$= \quad \frac{8}{2} \quad \frac{4}{2} + \frac{1}{2}$$

$$\frac{13 e^{-4} a_0^2}{4}$$

$$\rightarrow \boxed{P = 13 e^{-4}}$$

B $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle}$

$$\psi = \frac{1}{\sqrt{\pi a_0^3}} e^{-\sqrt{x^2+y^2+z^2}/a_0}$$

easier →

$x = r\sin\theta\cos\phi$

$$\rightarrow \langle x \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\pi a_0^3} e^{-2r/a_0} r^3 \sin^2\theta \cos\phi \, dr\, d\phi\, d\theta$$

$$\int_0^{2\pi} \cos\phi \, d\phi = \emptyset$$

$= \emptyset$

$$\rightarrow \langle x^2 \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\pi a_0^3} e^{-2r/a_0} r^4 \sin^3\theta \cos^2\phi \, dr\, d\phi\, d\theta$$

$$\int_0^{2\pi} \cos^2\phi \, d\phi$$

$$= \frac{2\pi}{2} \cancel{4} = \pi$$

$$\rightarrow \int_0^{\pi} \sin^3\theta \, d\theta = -\frac{\cos\theta}{1} + \frac{\cos^3\theta}{3}\Big|_0^{\pi}$$

$$= \left(1 - \frac{1}{3}\right) + \left(+1 - \frac{1}{3}\right)$$

$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

B Continued

$$\frac{4}{3}\pi \frac{1}{\pi a_0^3} \underbrace{\int_0^\infty e^{-2r/a_0} r^4 \, dr}_{\frac{\Gamma(4+1)}{(2/a_0)^5}} = \frac{4}{3}\pi \frac{1}{\pi a_0^3} \frac{24}{\cancel{8}} \frac{a_0^5}{32}$$

$\Gamma(4+1) = 4\cdot3\cdot2 = 24$

$2\times2\times2\times2\times2$
4 8 16 32

$$\frac{\cancel{4}}{\cancel{3}} a_0^5 \frac{\cancel{3}}{\cancel{4}}$$

$$= a_0^2$$

$\rightarrow$ ~~Δx~~ ~~dt~~ $\boxed{\Delta x = a_0}$

$$\underline{P_x} = -i\hbar \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial x} e^{-\sqrt{x^2+y^2+z^2}/a_0}$$

$\frac{1}{2} - \frac{3}{2}$

$$U = \sqrt{x^2+y^2+z^2}$$

$$\frac{dU}{dx} = \frac{\cancel{2}x}{\cancel{2}(x^2+y^2+z^2)^{1/2}} = \frac{r\cos\phi\sin\theta}{r}$$

$$\frac{de^{-U/a_0}}{dU} = -a_0 e^{-r/a_0}$$

$$\langle P_x \rangle = \emptyset \text{ since } \int \underline{\quad\quad} = \emptyset$$

$$\frac{\partial^2}{\partial x^2} e^{-r/a_0} = -a_0 \cos\phi \sin\theta \frac{\partial}{\partial x} e^{-r/a_0} = a_0^2 \cos^2\phi \sin^2\theta$$

$$\rightarrow \langle p_x^2 \rangle = \hbar^2 \frac{4}{3} \pi \frac{1}{\pi a_0^3} a_0^2 \underbrace{\int_0^\infty e^{-2r/a_0} r^2 \, dr}_{\frac{\Gamma(2+1)}{(2/a_0)^3}}$$

$$= \hbar^2 \frac{4(2)}{3 a_0 8} a_0^3 = \frac{1}{3} a_0^2 \hbar^2$$

$$p^2 = kg^2 \frac{m^2}{s^2}$$

$$\hbar = kg \frac{m^2}{s} = kg^2 \frac{m^4}{s^2}$$

you're very close, This is how to do it but the Math is wrong somewhere

F-2015

Problem 6: Hydrogen Atom Measurements

Consider a hydrogen atom, ignoring the spin of the electron, with the usual eigenstates of H, L^2, and L_z written as $|n, \ell, m_z\rangle$.

(a) [2 pts] If the hydrogen atom is in its ground state, $|1, 0, 0\rangle$, what is $\langle r\rangle$, the average distance of the electron from the proton?

(b) [3 pts] If the hydrogen atom is in its ground state, $|1, 0, 0\rangle$, what is the probability of measuring the electron's position to be in the classically forbidden region of space?

The forbidden region is where the energy of the atom is less than the potential energy, $V(r)$, corresponding to a negative value for the classical kinetic energy.

(c) [2 pts] Consider the first excited states of the atom with $\ell = 1$, $|2, 1, m\rangle$. Calculate the expectation value $\langle z\rangle$ for these states (where $z = r\cos\theta$ using standard spherical coordinates).

(d) [3 pts] The state $|2, l, 0\rangle$ has a rather different shape from the states $|2, 1, \pm 1\rangle$. This can be seen by considering the spread in z, $\Delta z = \sqrt{\langle z^2\rangle - \langle z\rangle^2}$, or the expectation value $\langle z^2\rangle$.

Compute the ratio of $\langle z^2\rangle$ in the state $|2, 1, 0\rangle$ to that in the state $|2, 1, 1\rangle$,

$$\frac{\langle z^2\rangle_{2,1,0}}{\langle z^2\rangle_{2,1,1}} \tag{1}$$

Hydrogen Atom States:

$$V(r) = -\frac{e^2}{r}, \quad a_0 = \frac{\hbar^2}{me^2}, \quad Ryd = \frac{e^2}{2a_0}, \quad \alpha = \frac{e^2}{\hbar c} \tag{2}$$

The spatial representation of the Hydrogen Atom energy eigenstates can be written:

$$\psi_{n,\ell,m}(r) = R_{n,\ell}(r) Y_{\ell,m}(\theta, \phi), \quad E_n = -\frac{Ryd}{n^2}$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}, \; Y_{1,0} = \sqrt{\frac{3}{4\pi}} cos\theta, \; Y_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}} sin\theta \; e^{\pm i\phi}$$

$$R_{10} = \frac{2}{(a_0)^{3/2}} e^{-r/a_0}, \; R_{20} = \frac{2}{(2a_0)^{3/2}}\left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0}, \; R_{21} = \frac{1}{(2a_0)^{3/2}} \frac{r}{\sqrt{3}a_0} e^{-r/2a_0}$$

A possibly useful integral:

$$\int_x^\infty t^n e^{-\alpha t} dt = \frac{n!}{\alpha^{n+1}} \, e^{-\alpha x} \sum_{k=0}^{n} \frac{(\alpha x)^k}{k!}$$

where α is real and positive.

F-2015 (1)

(6) $|n, l, m_l\rangle$

(A) $|1,0,0\rangle = R_{1,0} Y_{00}$

$$= \frac{2}{(a_0)^{3/2}} e^{-r/a_0} \frac{1}{\sqrt{4\pi}}$$

→ ~~Mean~~ average = expectation value

$$\rightarrow \langle r \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{4}{a_0^3} e^{-2r/a_0} \frac{1}{4\pi}\, r\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{4}{a_0^3} \underbrace{\int_0^{\infty} r^3 e^{-2r/a_0}\, dr}_{\frac{\Gamma(3+1)}{\left(\frac{2}{a_0}\right)^4} = \frac{6 a_0^4}{16}}$$

$$= \boxed{\frac{3}{2} a_0 = \langle r \rangle}$$

(B) $-\frac{\cancel{e^2}}{r} > -\frac{\cancel{e^2}}{2a_0 n^2}$ → $\boxed{2a_0 < r}$

$n = 1$

Multiply both sides by (-1)

→ Multiply or divide

- #'s in inequality → Flip

(B) Continued

Probability $= \frac{4}{a_0^3} \int_{2a_0}^{\infty} r^2 e^{-2r/a_0}\, dr$

$$\underbrace{\qquad}_{\frac{2!}{\left(\frac{2}{a_0}\right)^3} e^{-2/a_0} \sum_0^2 \frac{(4)^k}{k!}}$$

$$\underbrace{\sum_0^2 \frac{(4)^k}{k!}}_{1 + 4 + \frac{16}{2}}$$

$= 13$

$\frac{13}{2}$, 26

$\frac{16}{4}$, 64

$= \frac{4 \cdot 26\, a_0^3}{a_0^3\, 8 \cdot 2} e^{-4}$

$= \boxed{13 e^{-4} = P}$

(C) $|2,1,m\rangle = R_{21} Y_1^m$

$m = 0 \rightarrow R_{21} Y_1^0 = \frac{1}{(2a_0)^{3/2}} \frac{r}{\sqrt{3}\, a_0} e^{-r/2a_0} \sqrt{\frac{3}{4\pi}} \cos\theta$

$\rightarrow z = r\cos\theta$

$\langle z \rangle = \frac{1}{(2a_0)^3} \frac{3}{4\pi} \frac{1}{3a_0^2} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^4 e^{-r/a_0} \cos^3\theta \sin\theta\, dr\, d\theta\, d\phi$

~~This will be ϕ, $\int_0^{\pi} \cos\theta \sin\theta\, d\theta = \phi$~~

$$= \frac{1}{2} \frac{a_0^3}{8 a_0^2} \int_0^{\infty} e^{-r/a_0} r^4 \int_0^{\pi} \cos^3\theta \sin\theta\, d\theta$$

$u = \cos\theta$

$\frac{du}{d\theta} = \sin\theta$

$$\int_0^{\pi} u^3\, du = \frac{1}{4} u^4 = \phi$$

$$\rightarrow \boxed{\langle z \rangle_{210} = \phi}$$

$$R_{21} Y_1^{\pm 1} = \frac{1}{(2a_0)^{3/2}} \frac{r}{\sqrt{3}\, a_0} e^{-r/2a_0} \mp \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}$$

$$\langle z \rangle_{+} = \frac{1}{(2a_0)^3} \frac{1}{3} \frac{1}{a_0^2} \frac{3}{8\pi} \int_0^{\infty} r^5 e^{-r/2a_0}\, dr \underbrace{\int_0^{\pi} \cos\theta \sin^3\theta\, d\theta}_{\phi} \int d\phi$$

$$\boxed{\langle z \rangle_{21\pm1} = \phi}$$

(D) $\langle z^2 \rangle_{2,1,\phi}$

$$= \frac{1}{2}\frac{a_0^3}{8a_0^2}\underbrace{\int_0^\infty e^{-r/a_0} r^2 r^2 r^2\,dr}_{\frac{6!}{\left(\frac{1}{a_0}\right)^7}}\underbrace{\int_0^\pi \cos^2\theta\cos^2\theta\sin\theta\,d\theta}$$

↑ actually $\frac{1}{a_0^5}$ made mistake

$u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$-\frac{1}{5}\cos^5\theta\Big|_0^\pi$

$-\frac{1}{5}\underbrace{(-1-1)}_{-2} = \frac{2}{5}$

$$= \frac{1}{5}\frac{a_0^2}{8}6!$$

$$\Rightarrow \langle z^2\rangle_{211} = \frac{1}{8a_0^5}\frac{2\pi}{2\pi}\underbrace{\int_0^\infty r^6 e^{-r/a_0}}_{\frac{6!}{\left(\frac{1}{a_0}\right)^7}}\underbrace{\int_0^\pi \cos^2\theta\sin^3\theta\,d\theta}$$

$\int_0^\pi (1-\sin^2\theta)\sin^3\theta\,d\theta$

$\int_0^\pi \sin^3\theta\,d\theta - \int_0^\pi \sin^5\theta\,d\theta$

$-\cos\theta + \frac{\cos^3\theta}{3}\Big|_0^\pi$

$\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)$

$\frac{6}{3} - \frac{2}{3} = \frac{4}{3}$

$\left[\cancel{-\frac{\sin^4\theta\cos\theta}{5}} + \underbrace{\frac{4}{5}\frac{4}{3}}_{\frac{16}{15}}\right]$

$\Rightarrow \frac{20}{15} - \frac{16}{15} = \frac{4}{15}$

1) Continued

$$\frac{1}{8a_0^5}\,\frac{1}{\cancel{4}}\,6!\,a_0^7\,\frac{\cancel{4}}{15} = 6!\,\frac{a_0^2}{(8)(15)}$$

$$\rightarrow \frac{\langle r^2\rangle_{210}}{\langle r^2\rangle_{211}} = \frac{1}{5}\,\frac{\cancel{a_0^2}}{\cancel{8}}\,\cancel{6!}\,\frac{1}{\cancel{6!}}\,\frac{1}{\cancel{a_0^2}}\,(\cancel{8})(15)$$

$$= \boxed{3}$$

S-2014

PROBLEM 5: Zeeman Field

Consider the eight $n = 2$ states of Hydrogen. This problem is on the *strong* field Zeeman effect with spin-orbit interaction. Assume that the constant magnetic field B lies along the z-direction. The spin orbit coupling term is

$$H_{SO} = \frac{1}{2m_l^2 c^2} \frac{1}{r} \frac{\mathrm{d}V}{\mathrm{d}r} \mathbf{L} \cdot \mathbf{S},$$

where $V(r)$ is the Coulomb potential, c is the speed of light and m_l is the angular momentum projection quantum number. Remember:

$$\langle n, l, m_l | \frac{1}{r^3} | n, l, m_l \rangle = \frac{1}{a_0^3 n^3 l(l + \frac{1}{2})(l+1)}$$

for $l \neq 0$.

a) Find a general expression for the energy due to the spin-orbit term in the physical limit of strong magnetic field, where the strong field Zeeman splitting expressions are valid. Express your answer in terms of the good quantum numbers in this problem. Recall that because of the strong magnetic field, the good quantum numbers in this regime are n, l, m_l and m_s and not j and m_j. (Hint: compute $\langle H_{SO} \rangle$ in the proper basis) (3 Points)

b) Explicitly write down the quantum numbers for all eight $n = 2$ states. Find the energy of each state under strong field Zeeman splitting. Express the energy of each state as the sum of 3 terms: the Bohr energy, the spin-orbit interaction, and the Zeeman contribution. (4 Points)

c) If you ignore the spin-orbit interaction, how many distinct energy levels are there and what are their degeneracies? (3 Points)

S-2014

(5) $n=2$ Strong Field Zeeman $\vec{B} = B\hat{z}$

$$H_{SO} = \frac{1}{2m_e^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{L}\cdot\vec{S}$$

(A) $\vec{L}\cdot\vec{S} =$

$$L_xS_x + L_yS_y + L_zS_z$$

$$L_+ = L_x + iL_y \qquad L_- = L_x - iL_y$$

$$S_+ = S_x + iS_y \qquad S_- = S_x - iS_y$$

$$L_+S_+ = L_xS_x + iL_xS_y + iL_yS_x - L_yS_y$$

$$L_-S_- = L_xS_x +$$

$$\Rightarrow L_+S_- = L_xS_x - iL_xS_y + iL_yS_x + L_yS_y$$

$$L_-S_+ = L_xS_x + iS_yL_x - iL_yS_x + L_yS_y$$

$$\Rightarrow \vec{L}\cdot\vec{S} = \frac{1}{2}(L_+S_- + L_-S_+)$$

A Continued

good quantum #'s are n, ℓ, m_ℓ, m_s

$$V = \frac{k}{r} \qquad \frac{dV}{dr} = -\frac{k}{r^2}$$

$$\rightarrow H_{so} = \frac{-k}{4 m_e^2 c^2} \frac{1}{r^3} \left(\underbrace{L_+ S_- + L_- S_+}_{\text{give } \phi} + 2 L_z S_z \right)$$

$$\rightarrow \langle n, \ell, m_\ell, m_s | H_{so} | n, \ell, m_\ell, m_s \rangle$$

$$= -\frac{k}{2 m_e^2 c^2} \langle n, \ell, m_\ell, m_s | \frac{L_z S_z}{r^3} | n, \ell, m_\ell, m_s \rangle$$

$$= -\frac{m_s m_\ell k \hbar^2}{2 m_e c^2} \underbrace{\langle n, \ell, m_\ell, m_s | \frac{1}{r^3} | n, \ell, m_\ell, m_s \rangle}_{\frac{1}{a_0^3 n^3 \ell(\ell+\frac{1}{2})(\ell+1)}}$$

$$\rightarrow \boxed{\Delta E_{so} = \frac{-m_s k \hbar^2}{2 m_e c^2 a_0^3 n^3 \ell(\ell+\frac{1}{2})(\ell+1)}}$$

Technically a perturbation.

B

n	ℓ	m_ℓ	m_s
$n=2$	$\ell=0$	$m_\ell=0$	$m_s=\frac{1}{2}$
2	0	0	$-\frac{1}{2}$
2	1	-1	$\frac{1}{2}$
2	1	-1	$-\frac{1}{2}$
2	1	0	$\frac{1}{2}$
2	1	0	$-\frac{1}{2}$

③

(B) Continued

2	1	1	$\frac{1}{2}$
2	1	1	$-\frac{1}{2}$

Weak Field $\rightarrow$ $E_{Zeeman} = \underbrace{\mu_B}_{\text{Bohr Mag}} m_j B$

Strong Field $\Rightarrow$ $E_{Zeeman} = \mu_B (m_\ell + 2m_s) B$

Bohr energy $= -\frac{13.6 \text{ eV}}{n^2}$

~~But~~ Simply add all These

(C) $E_B + E_Z$

~~$\ell = 0$ $m_s = -\frac{1}{2}$ $E_B - \mu_B B$~~

~~$m_s = \frac{1}{2}$ $E_B + \mu_B B$~~

~~$\ell = 1$ $m_s = -$~~

Should be Simple

$E_B + E_Z$

$\ell = 0$ $m_\ell = 0$

$\ell = 1$ $m_\ell = -1, 0, 1$

$m_s = \pm\frac{1}{2}$ For each

F-2013

Problem 4: Properties of the Hydrogen Atom

The wavefunctions for the ground state and first excited states of the hydrogen atom are given on the first page of this test.

(a) [2 pt] For the ground state of the hydrogen atom, determine the expectation value for the radial position of the electron, $\langle 1,0,0|r|1,0,0\rangle$.

(b) [3 pt] Define the radial probably density for the electron in a hydrogenic eigenstate: $P_{n,\ell,m}(r)dr$ as the probability of the electron being measured in the spherical shell between r and $r+dr$.

Write down expressions for $P_{1,0,0}(r)$ and $P_{2,1,1}(r)$, and sketch these as functions of r.

Hint: Remember that the integral of the probability density over r must be equal to one,

$$\int_0^\infty P_{n,\ell,m}(r)dr = 1 \tag{1}$$

(c) [3 pt] For the ground state of the hydrogen atom, determine the most probable radius for the electron. Compare your result to part (a) and explain the similarities and differences.

(d) [1 pt] What is the functional form for $P_{1,0,0}(r)$ in the limit as $r \to 0$? Explain your result considering that the ground state wavefunction is non-zero at $r=0$.

(e) [1 pt] What are the functional forms of $P_{1,0,0}(r)$, $P_{2,1,1}(r)$, and $P_{200}(r)$ as $r \to 0$? Explain the similarities and differences.

F-2013

1

4 A $\langle 1,0,0|r|1,0,0\rangle$

$2\times 2\times = 4\times 2 = 8?$

$$\Rightarrow \psi_{1,0,0} = \frac{2}{(a_0)^{3/2}} e^{-r/a_0} \frac{1}{\sqrt{4\pi}}$$

$$\Rightarrow \langle 1,0,0|r|1,0,0\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi_{1,0,0}^{*}\, r\, \psi_{1,0,0}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{4\pi}{4\pi}\int_0^{\infty} \frac{2^2}{(a_0)^{3/2\cdot 2}} e^{-r/a_0\cdot 2} r^3 dr$$

$$= \frac{4}{a_0^3}\int_0^{\infty} e^{-2r/a_0} r^3 dr$$

$$= \frac{4}{a_0^3}\frac{\Gamma(3+1)}{(2/a_0)^{3+1}} = \frac{4}{a_0^3}\frac{3!\, a_0^4}{2^4}$$

$$= \frac{6\times 4}{2^4} a_0$$

$$= \frac{24}{16} a_0$$

$$= \frac{12}{8} a_0 = \boxed{\frac{3}{2} a_0}$$

(B) $1 = A^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi^2 r^2\, dr\, d\theta\, d\phi$ (2)

$= 4\pi A^2 \int_0^{\infty} \psi^2 r^2\, dr$

$\underline{P_{1,00}(r)} \rightarrow \frac{4\pi}{4\pi} A^2 \int_0^{\infty} \frac{4}{(a_0)^3} e^{-2r/a_0} r^2\, dr = 1$

$\rightarrow A^2 \frac{4}{(a_0)^3} \frac{\Gamma(2+1)}{(2/a_0)^{2+1}} = 1$

$\rightarrow A^2 \cancel{\frac{4}{(a_0)^3}} \frac{\cancel{2}\, \cancel{a_0^3}}{2^{\cancel{3}\,2}} = 1 \rightarrow \underline{A^2 = 1}$

$\rightarrow \boxed{P_{1,0,0}(r) = \frac{4}{(a_0)^3} e^{-2r/a_0} r^2\, dr}$

$\underline{P_{2,1,1}(r)} \rightarrow \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} A^2 \frac{1}{(2a_0)^3} \frac{r^2}{3a_0^2} e^{-r/a_0} \frac{3}{8\pi} \sin^2\theta \; r^2 \sin\theta\, dr\, d\theta\, d\phi$

$\rightarrow \int_0^{\pi} \sin^3\theta\, d\theta$

$= \left(-\cos\theta + \frac{\cos^3\theta}{3}\right)\Big|_0^{\pi}$

$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)$

$= 1 + 1 - \frac{1}{3} - \frac{1}{3}$

$= 2 - \frac{2}{3}$

$= \frac{6-2}{3} = \frac{4}{3}$

$\rightarrow 2\cancel{\pi}\left(\frac{4}{3}\right) \frac{A^2}{(2a_0)^3} \frac{1}{(\cancel{3}a_0^2)} \frac{\cancel{3}}{8\cancel{\pi}} \underbrace{\int_0^{\infty} r^4 e^{-r/a_0}\, dr}_{\Gamma(4+1)\, a_0^5 = 4\times3\times2\times1 = 24 a_0^5} = 1$

$\rightarrow \frac{8 \times 24}{3} A^2 = 1$

$\rightarrow 64 A^2 = 1 \rightarrow A = \frac{1}{8}$

(B) Continued

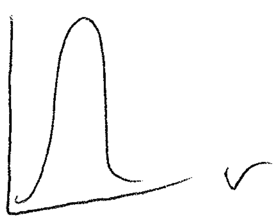
$$1 = \frac{1}{64}\,\frac{1}{\cancel{\beta} a_0^3}\,\frac{1}{\beta a^2}\,\frac{\cancel{\pi}}{8\pi}\,\cancel{\beta\pi}\,\frac{\cancel{4}}{3}\int_0^\infty r^4 e^{-r/a_0}\,dr$$

= This can't be right since this
dose't equal 1

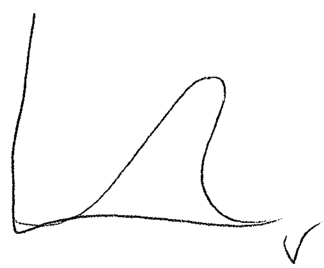
→ $\boxed{\text{has to be } \frac{1}{24 a_0^5}\, r^4 e^{-r/a_0}\, dr}$

If you run into this
just do $\int_0^\infty R^2\, r^2 dr = 1$

P_{100}



$P_{2,11}$



(C) $\frac{dP}{dr} = \emptyset \rightarrow \frac{\cancel{4}}{(\cancel{a_0})^3}\left[-\frac{2}{a_0} r^2 + 2r\, e^{-2r/a_0}\right] = \emptyset$

$\rightarrow r^2 \frac{\cancel{2}}{a_0} = \cancel{2} r$

$\rightarrow \boxed{r = a_0}$

(C) Continued

(4)

expectation value is $\frac{3}{2}a_0 = r$ which is the average measurement from multiple measurements.
The most probable radius is where the probability is at a maximum. The p.d.f has a ~~flat~~ tail which skews the average.

(D) $\lim_{r \to 0} P_{1,0,0} = \frac{0}{\infty} \Rightarrow \phi$

↳ The probability changes to a point.

(E) $\lim_{r \to 0} P_{2,0,0} \to \left(\frac{2}{(2a_0)^{3/2}}\right)^2$

$\boxed{\text{All } 3 \to \phi}$

$P_{2,11} \to \phi$

$P_{20} \to$ $\to \phi$

$\frac{r^2}{2a_0^3} e^{-r/a_0}\left(1 - \frac{r}{2a_0}\right)^2$

$\frac{r^2 e^{-2r/a_0}}{2a_0^3}\left(1 + \frac{r^2}{4a_0} - \frac{r}{a_0}\right)$

$\to \phi$

Problem 6: The hydrogen atom (10 points)

The figure below shows the radial function $R_{n,\ell}(r)$ for a stationary state of atomic hydrogen. The normalized Hamiltonian eigenfunction for this state, in atomic units, is

$$\psi_{n,\ell,m_\ell}(\boldsymbol{r}) = \frac{1}{81}\sqrt{\frac{2}{\pi}}(6 - r)\, e^{-r/3} \cos\theta. \tag{1}$$

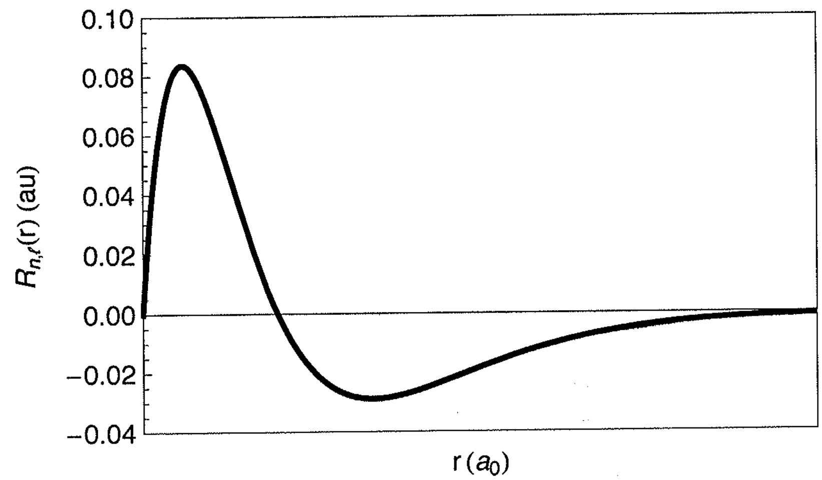


Figure 1: A radial function for a stationary state of atomic hydrogen.

1. **3 points.** What are the values of the quantum numbers n, ℓ, and m_ℓ for this state? To receive any credit, you must fully justify your answer.
2. **1 points.** What is the energy (in eV) of this state?
3. **2 points.** What are the mean value and uncertainty in r (in atomic units) for this state?
4. **2 points.** Calculate the value of r (in atomic units) at which a position measurement would be most likely to find the electron if the atom is in this state.
5. **2 points.** From Eq. 1, generate the normalized eigenfunction $\psi_{n,\ell,m_\ell+1}(\boldsymbol{r})$.

Hint:

$$\int_0^\infty e^{-2r/3} r^n \, dr = n! \left(\frac{3}{2}\right)^{n+1} \tag{2}$$

Hint: *The following table gives the orbital-angular-momentum operators in Cartesian and spherical coordinates.*

Component	Cartesian coordinates	Spherical coordinates
$\widehat{L}_x$	$-\imath\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)$	$\imath\hbar\left(\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$
$\widehat{L}_y$	$-\imath\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)$	$-\imath\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$
$\widehat{L}_z$	$-\imath\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$	$-\imath\hbar\frac{\partial}{\partial\varphi}$
$\widehat{\boldsymbol{L}}^2$	$\widehat{L}_x^2+\widehat{L}_y^2+\widehat{L}_z^2$	$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$

Table 1: Components and square of the orbital angular momentum operator in Cartesian and spherical coordinates.

S-2009

1

(6) $$\psi_{n,\ell,m}(r) = \frac{1}{81}\overbrace{\sqrt{\frac{2}{\pi}}\,(6-r)e^{-r/3}}^{A}\cos\theta$$

(A) $L^2\psi = \hbar^2(\ell(\ell+1))\psi$

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi$$

$$= -\hbar^2 A\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta(-\sin\theta)\right)\right]$$

$$= \hbar^2 A\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\right] = \hbar^2 A\left[\frac{1}{\sin\theta}(2\cos\theta\sin\theta)\right]$$

$$= 2\cancel{A}\cancel{\hbar^2}\cancel{\cos\theta} = \cancel{\hbar^2}(\ell(\ell+1))\cancel{A\cos\theta}$$

$\ell(\ell+1) = 2$

$\boxed{\ell = 1}$

$\rightarrow L_z\psi = m\hbar\psi \longrightarrow -i\hbar\frac{\partial}{\partial\phi}\psi = \emptyset \rightarrow \boxed{m = \emptyset}$

# of Nodes $= (n-\ell-1) = 1$

$(n-2) = 1$

$\boxed{\cancel{n=4}}$ $\boxed{\cancel{n=4}}$ $\boxed{n=3}$

(B) $\cancel{E = -13.6\,eV / n^2 = -(13.6\,eV)(9)}$

$$\boxed{E = \frac{-13.6\,eV}{9}}$$

(C) $\langle r \rangle = \int_{\text{All space}} r\,\psi^2\, d^3r = \cancel{\frac{1}{81}\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3(6-r)e^{-r/3}\cos\theta\sin\theta\, d\theta\, d\phi\, dr}$

$\cancel{= \frac{1}{81}\sqrt{\frac{2}{\pi}}(2\pi)}$

$$= \frac{1}{81^2}\left(\frac{2}{\pi}\right)\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3(6-r)^2 e^{-2r/3}\cos^2\theta\,\sin\theta\, d\theta\, d\phi\, dr$$

$$= \frac{1}{81^2}(4)\int_0^\infty\int_0^\pi r^3(6-r)^2 e^{-2r/3}\cos^2\theta\,\sin\theta\, d\theta\, dr$$

$$= \frac{1}{81^2}\,4\int_0^\infty r^3(6-r)^2 e^{-2r/3}\ \underbrace{\left[\frac{-\cos^{n+1}\theta}{n+1}\Big|_0^\pi\right]}_{\frac{-\cos^3\theta}{3}\Big|_0^\pi = -\frac{1}{3}[-1-1] = \frac{2}{3}}$$

$(6-r)(6-r)$
$= 36 - 12r + r^2$

$5\times4\times3\times2\times1$
$20\times3\times2\times1$
$60\times2 = 120$

$$= \frac{8}{3(81^2)}\left[\underbrace{\int_0^\infty 36r^3 e^{-2r/3}\,dr}_{36(6)\left(\frac{3}{2}\right)^4} - \underbrace{\int_0^\infty 12r^4 e^{-2r/3}\,dr}_{12(24)\left(\frac{3}{2}\right)^5} + \underbrace{\int_0^\infty r^5 e^{-2r/3}}_{120\left(\frac{3}{2}\right)^6}\right]$$

(C) Continued

$36 \times 6 = 216$

$3\times3\times3\times3 = 81$

$2\times2\times2\times2 = 16$

(3)

24×12

48

240

288

$81 \times 3 = 243$

$16 \times 2 = 32$

$243 \times 3 = 729$

$32 \times 2 = 64$

$$= \frac{8}{3(81^2)}\left[\frac{(216)(81)}{16} - \frac{(288)(243)}{32} + \frac{(120)(729)}{64}\right]$$

$\times 3$ $\times 2$

$= 216 \times 81$

216

2240

2456

$+ \quad 3$

7368

$- 288 \times 243$

864

11520

57600

69984

Blah Blah

$$\boxed{\begin{array}{l}\langle r\rangle \\ \text{\&}\quad (\Delta r) = \sqrt{\langle r^2\rangle - \langle r\rangle}\end{array}}$$

(D) $\boxed{\dfrac{d(\psi^2)}{dr} = \phi}$

(E) $L_+\psi = \sqrt{(l+m)(l-m+1)}\,\psi_{m+1}$

[illegible] 5-2009

Problem 6: Spherically Symmetric States: (10 Points)

Consider eigenfunctions of the Hamiltonian of a particle in a three-dimensional central potential. In particular, consider those eigenfunctions that depend only on the electron's radial coordinate r, that is $\Psi_E = \Psi_E(r)$. States represented by such eigenfunctions are called "spherically symmetric states".

1. Derive an equation for a function $\chi_E(r)$ defined by:

$$\Psi_n(r) \equiv \frac{1}{r}\chi_n(r),$$

where n is the principle quantum number. **(2 Points)**

The remainder of this problem concerns a hydrogen atom in the approximation that we neglect all interactions except the Coulomb interaction and treat the proton as an infinitely massive point particle at the origin.

2. Sketch $\chi_n(r)$ for the lowest three spherical bound states of the hydrogen atom. Justify the qualitative features of each function. **(2 Points)**

3. **(2 Points)**. Consider the eigenfunction for the ground state. Prove that to be physically admissible this function must decay exponentially as r becomes infinite.

$$\chi_1(r) \rightarrow e^{-\alpha r}, \text{when r} \rightarrow \infty$$

where α is a constant, and that therefore $\chi_1(r)$ must have the form.

$$\chi_1(r) = f(r)e^{-\alpha r}.$$

4. Use $f(r) = r$. Justify why this is an appropriate choice and show that the above equation is a solution of the equation you derived for $\chi_1(r)$ and determine the corresponding eigenvalue E_1. **(2 Points)**

5. Derive an expression for the constant α in terms of fundamental constants. **(2 Points)**

⑥ $$\left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r}\right]\psi(\theta,\phi,r) = E(\theta,\phi,r)$$

Ⓐ $$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2}$$

$$\to -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{e^2}{r}\psi = E\psi$$

$$\to L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\to -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{L^2}{2mr^2}\psi - \frac{e^2}{r}\psi = E\psi$$

$$\to \psi = \psi_n(r)\,Y_{ml}(\theta,\phi)$$

$$\to -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi_n(r)}{\partial r}Y\right) + \frac{L^2\psi_n Y}{2mr^2} - \frac{e^2}{r}\psi_n Y = E\psi_n Y$$

$$\to -\frac{\hbar^2}{2mr^2}\frac{1}{\psi(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi_n(r)}{\partial r}\right) + \frac{1}{Y}\frac{L^2 Y}{2mr^2} - \frac{e^2}{r} = E$$

$$\uparrow$$

$$L^2 Y = l(l+1)\hbar^2 Y$$

$$\to -\frac{\hbar^2}{2mr^2}\frac{1}{\psi(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi_n(r)}{\partial r}\right) + \frac{\hbar^2 l(l+1)}{2mr^2} - \frac{e^2}{r} = E$$

$$\to -\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}r^2\frac{\partial\psi_n(r)}{\partial r} + \left[\frac{\hbar^2 l(l+1)}{2mr^2} - \frac{e^2}{r}\right]\psi_n(r) = E\psi_n(r)$$

A - continued

$U = rR$

$\chi = r\psi \rightarrow \psi = \frac{\chi}{r}$

$$\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left[\frac{\chi}{r}\right] = \frac{\partial}{\partial r} r^2 \left[\frac{1}{r}\frac{d\chi}{dr} - \frac{\chi}{r^2}\right]$$

$$= \frac{\partial}{\partial r}\left[r\frac{d\chi}{dr} - \chi\right]$$

$$= \cancel{\frac{d\chi}{dr}} + r\frac{d^2\chi}{dr^2} \cancel{- \frac{d\chi}{dr}}$$

$$\rightarrow -\frac{\hbar^2}{2mr^2} r \frac{d^2\chi}{dr^2} + \left[\frac{\hbar^2 \ell(\ell+1)}{2mr^2} - \frac{e^2}{r}\right]\frac{\chi}{r} = E\frac{\chi}{r}$$

$$\rightarrow \boxed{-\frac{\hbar^2}{2m}\frac{d^2\chi}{dr^2} + \left[\frac{\hbar^2 \ell(\ell+1)}{2mr^2} - \frac{e^2}{r}\right]\chi = E\chi}$$

$\rightarrow$ remember $m \rightarrow \mu$

C Skip B $\rightarrow$ I don't think you can do this without finishing the problem

$$\rightarrow r \rightarrow \infty \rightarrow \frac{d^2\chi}{dr^2} = -\frac{2mE}{\hbar^2}\chi$$

$\rightarrow$ For bound states E is $-$

$$\frac{d^2\chi}{dr^2} = \alpha^2 \chi$$

where $\alpha = \frac{\sqrt{2mE}}{\hbar}$

$\rightarrow$ Solution is $\chi_1 = e^{-\alpha r}$

The $e^{\alpha r}$ is not physically acceptable

D $\chi_1(r) = f(r)e^{-\alpha r} = r e^{-\alpha r}$

lets not plug in $f(r) = r$ yet

$$-\frac{\hbar^2}{2\mu}\underbrace{\frac{d^2}{dr^2}\left[f(r)\cancel{e^{-\alpha r}}\right]} + \cancel{\left[\frac{\hbar^2 \ell(\ell+1)}{2\mu r^2}\right]} \cancel{\left[-\frac{e^2}{r}\right]}\cancel{f(r)e^{-\alpha r}} = E f(r)\cancel{e^{-\alpha r}}$$

$$\underbrace{\frac{d}{dr}\left[\frac{df}{dr}e^{-\alpha r} - \alpha f(r)e^{-\alpha r}\right]}$$

$$\frac{d^2 f}{dr^2}\cancel{e^{-\alpha r}} - \frac{df}{dr}\alpha \cancel{e^{-\alpha r}} - \frac{df}{dr}\alpha \cancel{e^{-\alpha r}} + \alpha^2 f(r)\cancel{e^{-\alpha r}}$$

$$-\frac{\hbar^2}{2\mu}\left[\frac{d^2 f}{dr^2} - \frac{df}{dr}\alpha - \frac{df}{dr}\alpha + \alpha^2 f(r)\right]$$

$\emptyset$ $-\alpha$ $-\alpha$ $\alpha^2 r$

$$-2\alpha + \boxed{\alpha^2 r = \alpha^2 r}$$

assuming you made a mistake and these should cancel

$$\boxed{E_1 = \frac{\alpha^2\hbar^2}{2m}}$$

E $E_1 = \dfrac{\alpha^2 \hbar^2}{2m}$

$E = kg\dfrac{m^2}{s^2} = \alpha^2 \dfrac{kg^2 m^4}{kg\, s^2}$

$kg\dfrac{m^2}{s}$

$= \alpha^2 kg \dfrac{m^4}{s^2}$

$$\boxed{\alpha = \frac{1}{a_0}}$$

F-2007

Problem 6: Hydrogen Atom (10 Points)

The spatial component of the ground state wavefunction for the hydrogen atom is

$$\phi(r,\theta,\phi) = Ae^{-\left(\frac{r}{a_o}\right)}$$

where A and a_o (the Bohr radius) are constants.

a) Find A by normalizing the wavefunction. Express your answer in terms of a_o. **(2 Points)**

b) Calculate the expectation value of the potential energy. **(2 Points)**

c) Calculate the expectation value of r and the most probable value for r. **(2 Points)**

d) What is the expectation value for L, the magnitude of the angular momentum? How does this value compare to the prediction of the Bohr model? **(2 Points)**

e) Many solutions to the Schrodinger equation for the hydrogen atom are related to a z-axis despite the fact that the potential energy is spherically symmetric. What defines the z-axis? Explain your answer. **(2 Points)**

F-2008

$$\phi(r,\theta,\phi) = Ae^{-(r/a_0)}$$

A $$1 = \int_0^\infty\int_0^{2\pi}\int_0^\pi A^2 e^{-(2r/a_0)} r^2 \sin\theta \, d\phi\, d\theta\, dr = \int \phi \phi$$

$$= 4\pi A^2 \int_0^\infty e^{-(2r/a_0)} r^2 dr$$

$$= 4\pi A^2 \frac{\Gamma(2+1)}{(2/a_0)^{2+1}} = 4\pi A^2 \frac{2}{\left(\frac{2}{a_0}\right)^3} = 1$$

$$\rightarrow 8\pi A^2 \frac{a_0^3}{2^3} = 1 \rightarrow \boxed{A = \frac{1}{\sqrt{a_0^3 \pi}}}$$

B $$V(r) = \frac{e^2}{r}$$ Negative?

$$\rightarrow \langle V \rangle = \int_0^\infty\int_0^{2\pi}\int_0^\pi \frac{1}{a_0^3\pi} e^2 r e^{-2r/a_0} \sin\theta \, d\phi\, d\theta\, dr$$

$\frac{2}{2} - \frac{1}{2} = \frac{1}{2}$

$a^{4/2} - a^{3/2}$

$= a^{1/2}$

$$= \frac{4\pi e^2}{a_0^3\pi} \underbrace{\int_0^\infty r e^{-2r/a_0} dr}$$

$$\frac{\Gamma(1+1)}{(2/a_0)^{1+1}} = \frac{(1)a_0^2}{4}$$

$$= \frac{\cancel{4}\pi e^2 a_0^2}{\cancel{4} a_0^3 \pi}$$

$$\boxed{\langle V \rangle = \frac{e^2}{a_0}}$$

(c) $$\langle r \rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{a_0^3 \pi} r^3 e^{-2r/a_0} \sin\theta \, d\phi \, d\theta \, dr$$ ②

$$= \frac{4\pi}{a_0^3 \pi} \underbrace{\int_0^\infty r^3 e^{-2r/a_0} dr}_{\frac{\Gamma(3+1)}{\left(\frac{2}{a_0}\right)^{3+1}}}$$

$$= \frac{3! \, a_0^4}{2^4}$$

$$= \frac{6 a_0^4}{16}$$

$$= \frac{3 a_0^4}{8}$$

$$\rightarrow \boxed{\frac{3}{2} a_0 = \langle r \rangle}$$

Most probable $\rightarrow$ $P = \frac{1}{a_0^3 \pi} \int_0^\pi \int_0^{2\pi} \int_0 r^2 e^{-2r/a_0} \sin\theta \, d\phi \, d\theta \, dr$

$$= \frac{4\pi}{a_0^3 \pi} \int_0^\infty r^2 e^{-2r/a_0} dr$$

$$\frac{e^{ar}}{a}\left(r^2 - \frac{2r}{a} + \frac{2}{a^2}\right)$$

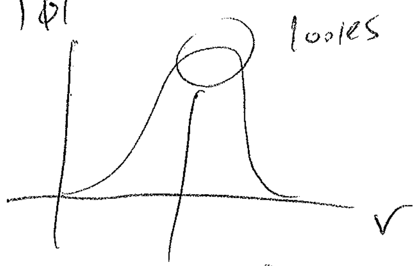
$$\frac{1}{a}\left(\frac{2}{a^2}\right)$$

$$\frac{dP}{dr} = 0 = \frac{4}{a_0^3}\left[e^{ar}\left(r^2 - \frac{2r}{a} + \frac{2}{a^2}\right) + \frac{e^{ar}}{a}\left(2r - \frac{2}{a}\right)\right]$$

$$= \frac{4}{a_0^3}\left[e^{ar} r^2 - e^{ar}\frac{2r}{a} + e^{ar}\frac{2}{a^2} + \frac{e^{ar}}{a} 2r - \frac{e^{ar}}{a}\frac{2}{a}\right]$$

Most probable

Probability density distribution $= |\phi|^2 = \frac{1}{a^3 \pi} r^2 e^{-2r/a_0} \sin\theta$



looks something like this

This is the most probable

$\rightarrow \quad \frac{dP}{dr} = \phi = \frac{d}{dr}\left[\frac{1}{a_0^3 \pi} \sin\theta \, r^2 e^{-2r/a_0}\right]$

$= \frac{\sin\theta}{a_0^3 \pi}\underbrace{\left[2re^{-2r/a_0} - \frac{2r^2}{a_0}e^{-2r/a_0}\right]}_{=\phi}$

$\cancel{2}r - \frac{\cancel{2}r^{\cancel{2}}}{a_0} = \phi$

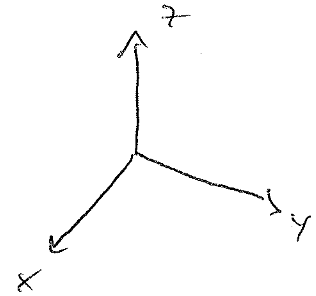
$1 - \frac{r}{a_0} = \phi$

$\boxed{r = a_0}$

(D) $\langle L \rangle = \int_{\text{All space}} \phi^* L \phi \, d^3 r$

real

$= \int L \phi^2 \, d^3 r$



$\rightarrow \quad L = r \times p$

$= r p \sin\theta$

$= -i\hbar \nabla \sin\theta$

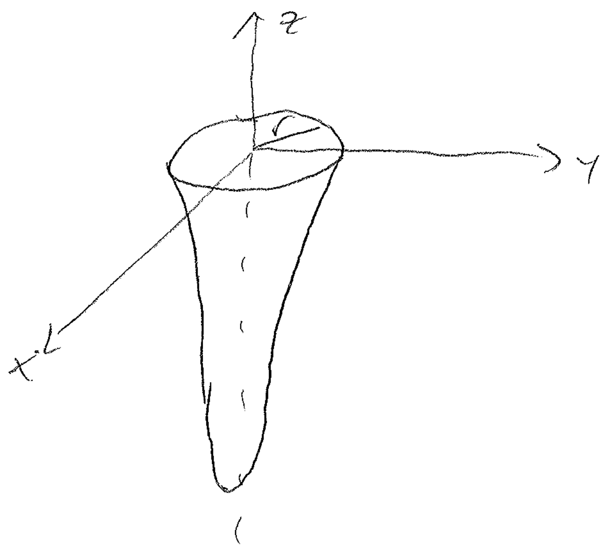
without doing the integral, the answer will be imaginary ; since the ~~...~~ expectation value is observable,

$$\langle L \rangle = \emptyset$$

Compared to the bohr model where

$$\langle L \rangle = \hbar$$

(e)



The potential is spherically symmetric in the xy plane

z corresponds to the direction of the measurable angular momentum in the z direction

$L_z = r \times p \sin\phi$