S-2007

PROBLEM 6

Consider an ensemble of identical particles whose state space is spanned by the basis

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Assume that the Hamiltonian H and an observable A are represented by

$$H = \hbar\omega_o \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues of H are $\hbar\omega$, $2\hbar\omega$, and $-\hbar\omega$ with eigenvectors given by

$$|\hbar\omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad |2\hbar\omega\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } |-\hbar\omega\rangle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvalues of A are $-1, 1,$and 1 with eigenvectors given by

$$|a_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad |a_{1,1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad |a_{1,2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For all times $t < 0$, the particles are in a state given by

$$|\psi_o\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix}.$$

a) [1pt] Write down an expression for the time evolution operator $U(t, t_o = 0)$ in Dirac notation

b) [2 pt] Determine $|\psi(t)\rangle$, the state vector at an arbitrary time.

c) [2 pt] What is the probability that a measurement of A at a time $t = 0$ yeilds $a = -1$?

d) [2 pt] What is the probability that a measurement of A at an arbitrary time t yields a value $a = -1$?

e) [3 pt] Assume that at $t = 0$ the operator A is observed to be 1. What is the probability that a short time later ($0 < t << 1/\omega$), the eigenenergy of the system is observed to be $-\hbar\omega$?

5-2007

1 A $U = \sum_i \sum_j |i\rangle\langle i|U|j\rangle\langle j|$

$= \exp(-iE_i t/\hbar)\, |i\rangle \langle i|$

$$\boxed{U = e^{-i\omega t}|\hbar\omega\rangle\langle\hbar\omega| + e^{-2i\omega t}|2\hbar\omega\rangle\langle 2\hbar\omega| + e^{i\hbar\omega}|-\hbar\omega\rangle\langle-\hbar\omega|}$$

B $\psi = \sum_i |i\rangle\langle i|\psi\rangle$

$= \frac{1}{2\sqrt{2}}(1\ 0\ -1)\begin{pmatrix}1\\ \sqrt{2}\\ -1\end{pmatrix}|\hbar\omega\rangle + \frac{1}{2}(0\ 1\ 0)\begin{pmatrix}1\\ \sqrt{2}\\ -1\end{pmatrix}$

$+ \frac{1}{2\sqrt{2}}(1\ 0\ 1)\begin{pmatrix}1\\ \sqrt{2}\\ -1\end{pmatrix}$

$\frac{1}{2} - \frac{1}{2} -$

$= \frac{\cancel{2}}{\cancel{2}\sqrt{2}}|\hbar\omega\rangle + \frac{\sqrt{2}}{2}|2\hbar\omega\rangle + \emptyset$

$= \frac{1}{\sqrt{2}}\left[|\hbar\omega\rangle + |2\hbar\omega\rangle\right]$

$$\rightarrow \boxed{U|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\omega t}|\hbar\omega\rangle + e^{-2i\omega t}|2\hbar\omega\rangle\right]}$$

(c) $|\langle a_{-1}|\psi_0\rangle|^2$

$$|\psi_0\rangle = \sum_i |a_i\rangle\langle a_i|\psi_0\rangle$$

$$= \frac{1}{2\sqrt{2}}(1\ -1\ 0)\begin{pmatrix}1\\ \sqrt{2}\\ -1\end{pmatrix}|a_{-1}\rangle + \frac{1}{2\sqrt{2}}(1\ 1\ 0)\begin{pmatrix}1\\ \sqrt{2}\\ -1\end{pmatrix}|a_{1,1}\rangle$$

$$+ \frac{1}{2}(0\ 0\ 1)\begin{pmatrix}1\\ \sqrt{2}\\ -1\end{pmatrix}|a_{1,2}\rangle$$

$$= \frac{1-\sqrt{2}}{2\sqrt{2}}|a_{-1}\rangle + \frac{1+\sqrt{2}}{2\sqrt{2}}|a_{1,1}\rangle - \frac{1}{2}|a_{1,2}\rangle$$

$$\langle\psi_0|\psi_0\rangle = \frac{(1-\sqrt{2})^2}{\cancel{4}\ 8} + \frac{(1+\sqrt{2})^2}{8} + \frac{1}{4}$$

$$= \frac{1+2-\cancel{2\sqrt{2}}}{8} + \frac{1+2+\cancel{2\sqrt{2}}}{8} + \frac{2}{8}$$

$$= \frac{8}{8} \checkmark$$

$$\rightarrow \boxed{|\langle a_{-1}|\psi_0\rangle|^2 = \frac{(1-\sqrt{2})^2}{8}}$$

(3)

~~(2)~~ (b) $|\psi(t)\rangle = \sum_i |a_i\rangle\langle a_i|\psi(t)\rangle$

$\cancel{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(1\ 0\ 0)}$

$$|\psi(t)\rangle = \sum_{e_i} |e_i\rangle\langle e_i|\psi(t)\rangle$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} e^{-i\omega t}(1\ 0\ 0)\overset{e_1}{\begin{pmatrix}1\\0\\-1\end{pmatrix}} + \frac{1}{\sqrt{2}} e^{-2i\omega t}(1\ 0\ 0)\overset{e_1}{\begin{pmatrix}0\\1\\0\end{pmatrix}}$$

$$+ \frac{1}{\sqrt{2}} e^{-i\omega t}(0\ 1\ 0)\frac{1}{\sqrt{2}}\overset{e_2}{\begin{pmatrix}1\\0\\-1\end{pmatrix}} + \frac{1}{\sqrt{2}} e^{-2i\omega t}(0\ 1\ 0)\overset{e_2}{\begin{pmatrix}0\\1\\0\end{pmatrix}}$$

$$+ \frac{1}{\sqrt{2}} e^{-i\omega t}\frac{1}{\sqrt{2}}(0\ 0\ 1)\overset{e_3}{\begin{pmatrix}1\\0\\-1\end{pmatrix}} + \frac{1}{\sqrt{2}} e^{-2i\omega t}(0\ 0\ 1)\overset{e_3}{\begin{pmatrix}0\\1\\0\end{pmatrix}}$$

$$= \frac{1}{2} e^{-i\omega t}|e_1\rangle + \emptyset\ \cancel{\frac{1}{2}e^{-i\omega t}}\ \emptyset + \frac{1}{\sqrt{2}} e^{-2i\omega t}|e_2\rangle$$

$$-\frac{1}{2} e^{-i\omega t}|e_3\rangle$$

$$\frac{1}{4} + \frac{1}{4} + \frac{2}{4} = \checkmark$$

$\rightarrow$ $|\psi(t)\rangle = \sum_a |a_i\rangle\langle a_i|\psi(t)\rangle$

$\cancel{= a_1\ \frac{1}{2} e^{-i\omega t}\left[\frac{1}{\sqrt{2}}(1\ 0\ 0)\begin{pmatrix}1\\-1\\0\end{pmatrix} + \frac{1}{\ }\right.}$

D) (4)

$$\frac{1}{\sqrt{2}}(1\ -1\ 0)\left[\frac{1}{2}e^{-i\omega t}\begin{pmatrix}1\\0\\0\end{pmatrix}+\frac{1}{\sqrt{2}}e^{-2i\omega t}\begin{pmatrix}0\\1\\0\end{pmatrix}-\frac{1}{2}e^{-i\omega t}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]|a_{-1}\rangle$$

$$+\frac{1}{\sqrt{2}}(1\ 1\ 0)\left[\frac{1}{2}e^{-i\omega t}\begin{pmatrix}1\\0\\0\end{pmatrix}+\frac{1}{\sqrt{2}}e^{-2i\omega t}\begin{pmatrix}0\\1\\0\end{pmatrix}-\frac{1}{2}e^{-i\omega t}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]|a_{1,1}\rangle$$

$$+(0\ 0\ 1)\left[\frac{1}{2}e^{-i\omega t}\begin{pmatrix}1\\0\\0\end{pmatrix}+\frac{1}{\sqrt{2}}e^{-2i\omega t}\begin{pmatrix}0\\1\\0\end{pmatrix}-\frac{1}{2}e^{-i\omega t}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]|a_{1,2}\rangle$$

$$=\frac{1}{2\sqrt{2}}e^{-i\omega t}-\frac{1}{2}e^{-2i\omega t}\,|a_{-1}\rangle \quad + \ldots\ldots$$

$$\rightarrow |\langle a_{-1}|\psi(t)\rangle|^2=\left(\frac{1}{2\sqrt{2}}e^{-i\omega t}-\frac{1}{2}e^{-2i\omega t}\right)\left(\frac{1}{2\sqrt{2}}e^{i\omega t}-\frac{1}{2}e^{2i\omega t}\right)$$

$$=\frac{1}{8}\underbrace{-\frac{1}{4\sqrt{2}}e^{i\omega t}-\frac{1}{4\sqrt{2}}e^{-i\omega t}}_{-\frac{\cos\omega t}{2\sqrt{2}}}+\frac{1}{4}$$

$$\rightarrow \boxed{P=\frac{3}{8}-\frac{\cos\omega t}{2\sqrt{2}}}$$

E ~~[illegible]~~

Put $|\psi_0\rangle$ in Terms of $A=1$

Then get $|\psi_0'\rangle$

expand $|\psi_0'\rangle$ in Terms of $|E_i\rangle$

Then evolve in time.

~~Then~~ get ~~[illegible]~~ Probability

Probably have a $\sin\omega t$
or $\cos\omega t$

term

might have to expand
Term.

F-2012

PROBLEM 6: Neutron Evolution

A polarized beam of neutrons with energy E_0 and spin projection along the positive z-axis enters abruptly at $t = 0$ a region where there is a uniform magnetic field $\vec{B}$. If we ignore the spatial degrees of freedom the Hamiltonian for the neutron interacting with the magnetic field is

$$H = -\vec{B} \cdot \vec{\mu}_n = 2\omega\hat{n} \cdot \vec{S}$$

where $\hat{n}$ is a unit vector in the direction of the magnetic field and $\omega = B\mu_n/\hbar$.

(a) **Hamiltonian:** Express $\hat{n}$ in spherical coordinates $\{\theta, \phi\}$ and then find an expression for $\hat{n} \cdot \vec{S}$. [2 points]

(b) **Time Evolution Operator:** Write down an explicit expression for the time-evolution operator in terms of $\{\theta, \phi, t\}$. [3 points]

(c) **Evolved State:** Find the state of the time evolved system for any time $t > 0$. [2 points]

(d) **Expectations:** Find the expectation value of the spin $\vec{S}$. [2 points]

(d) **A Special Case:** Determine and describe the motion for a system where $\vec{B} = B\hat{x}$ [1 point]

F-2012

(8) $H = -\vec{B}\cdot\vec{\mu}_n = 2\omega\hat{n}\cdot\vec{S}$

(A) $\cancel{\vec{V} = 2\omega ...}$

$$\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

$$\boxed{\hat{n} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}$$

$$\Rightarrow \vec{S}\cdot\hat{n} = \begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} = \vec{S}\cdot\hat{n}}$$

(B) $H = 2\omega \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$

$$U = e^{-iHt/\hbar} = \exp\left(\frac{-iHt}{\hbar}\right)$$

$$\Rightarrow U = \sum_i \sum_j |i\rangle\langle i|U|j\rangle\langle j| = \cancel{\begin{pmatrix} \langle +|U|+\rangle & \langle +|U|-\rangle \\ \langle -|U|+\rangle & \langle -|U|-\rangle \end{pmatrix}}$$

when i, j are

eigenvalues of H

B Continued

$$\begin{vmatrix} \cos-\lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta-\lambda \end{vmatrix} = \phi$$

$$\rightarrow \boxed{\lambda = \pm 2\omega}$$

$\underline{\lambda = 2\omega} \rightarrow \begin{pmatrix} \cos\theta-1 & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta-1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \phi$

$$A(\cos\theta-1) + B\sin\theta e^{-i\phi} = \phi$$

$$\rightarrow B = \frac{A(1-\cos\theta)e^{i\phi}}{\sin\theta}$$

$$\rightarrow \begin{pmatrix} A & \frac{A(1-\cos\theta)e^{-i\phi}}{\sin\theta} \end{pmatrix}\begin{pmatrix} A \\ \frac{A(1-\cos\theta)e^{i\phi}}{\sin\theta} \end{pmatrix} = 1$$

$$\rightarrow A^2\left(1 + \frac{(1-\cos\theta)^2}{\sin^2\theta}\right) \qquad \rightarrow (1-\cos\theta)(1-\cos\theta)$$

$$1 - 2\cos\theta + \cos^2\theta$$

$$\rightarrow A^2\left(\frac{2-2\cos\theta}{\sin^2\theta}\right) = 1 \rightarrow A = \frac{\sin\theta}{\sqrt{2}\sqrt{1-\cos\theta}}$$

$$\uparrow$$

$$\sqrt{2}\sin\frac{\theta}{2}$$

$$\rightarrow A = \frac{\sin\theta}{2\sin\frac{\theta}{2}}$$

B Continued

$$\rightarrow \quad 1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$\rightarrow \quad \frac{A\, 2\sin^2\frac{\theta}{2}\, e^{+i\phi}}{\sin\theta} = \frac{\cancel{\sin\theta}}{\cancel{2\sin\frac{\theta}{2}}}\;\frac{\cancel{2\sin^2\frac{\theta}{2}}}{\cancel{\sin\theta}}\; e^{+i\phi}\,\sin\frac{\theta}{2}$$

$$= e^{-i\phi}$$

$$\rightarrow \quad \cancel{\boxed{|\chi = 2w\rangle = \frac{\sin\theta}{2\sin\frac{\theta}{2}}\; e^{-i\phi}}}$$

$$\frac{\sin\theta}{2\sin\frac{\theta}{2}} = \frac{\cancel{2\sin\frac{\theta}{2}}\cos\frac{\theta}{2}}{\cancel{2\sin\frac{\theta}{2}}} = \cos\frac{\theta}{2}$$

$$\rightarrow \quad \boxed{|2w\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{+i\phi} \end{pmatrix}}$$

B Continued

$\underline{\lambda = -2\omega}$ $\rightarrow$ $\begin{pmatrix} \cos\theta + 1 & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta + 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \phi$

$\rightarrow A \underbrace{(\cos\theta + 1)}_{2\cos^2\frac{\theta}{2}} + B \sin\theta e^{-i\phi} = \phi$

$\rightarrow B = -\dfrac{2\cos^2\frac{\theta}{2}}{\sin\theta} A e^{i\phi}$

$\rightarrow A^2 \left(1 + \cancel{4\cos^4\frac{\theta}{2}} \dfrac{(\cos\theta + 1)^2}{\sin^2\theta} \right) = 1$

$\cos^2\theta + 1 + 2\cos\theta$

$A^2 \left(\dfrac{2 + 2\cos\theta}{\sin^2\theta} \right) = 1 \rightarrow \dfrac{\sin\theta}{\sqrt{2}\underbrace{\sqrt{1+\cos\theta}}_{\sqrt{2}\cos\frac{\theta}{2}}} = A = \dfrac{\sin\theta}{2\cos\frac{\theta}{2}}$

$\rightarrow B = -\cos\frac{\theta}{2} e^{i\phi}$

$A = \dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = \sin\frac{\theta}{2}$

$$\boxed{|-2\omega\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{i\phi} \end{pmatrix}}$$

B Continued

$$\langle 2\omega | -2\omega \rangle = \cos\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \phi \checkmark$$

$$\langle -2\omega | -2\omega \rangle = 1$$

$$\boxed{U = \begin{pmatrix} e^{2i\omega t/\hbar} & \phi \\ \phi & e^{-2i\omega t/\hbar} \end{pmatrix}}$$

$$UU^{\dagger} = \begin{pmatrix} e^{2i\omega t/\hbar} & 0 \\ 0 & e^{-2i\omega t/\hbar} \end{pmatrix}\begin{pmatrix} e^{-2i\omega t/\hbar} & 0 \\ 0 & e^{2i\omega t/\hbar} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\tilde{1}}$$ unitary

C $|+\rangle$ For neutrons

$$U = \sum_i \sum_j |i\rangle\langle i|U|j\rangle\langle j|$$

$$|+\rangle = \sum_i |i\rangle\langle i|+\rangle$$

$$= \left(\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}e^{-i\phi}\right)\begin{pmatrix}1\\0\end{pmatrix} |2\omega\rangle$$

$$+ \left(\sin\frac{\theta}{2} \quad -\cos\frac{\theta}{2}e^{-i\phi}\right)\begin{pmatrix}1\\0\end{pmatrix} |-2\omega\rangle$$

$$= \cos\frac{\theta}{2}|2\omega\rangle + \sin\frac{\theta}{2}|-2\omega\rangle$$

$$= \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$$

$$\rightarrow \boxed{U|+\rangle = \cos\frac{\theta}{2}e^{2i\omega t/\hbar}|2\omega\rangle + \sin\frac{\theta}{2}e^{-2i\omega t/\hbar}|-2\omega\rangle}$$

Could put back into I if you want

D $\langle +|S|+\rangle$ easy

E $S_x \rightarrow$ easy

F-2012

PROBLEM 1: Eigenvalue Equation and Time Evolution

The Hamiltonian for a certain three-level system is represented by the matrix

$$H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix},$$

where a, b, and c are real numbers and $a - c \neq \pm b$.

(a) Find the eigenvalues E_n and normalized eigenvectors $|E_n\rangle, n = 1, 2, 3$ of H. [4 points]

(b) If the system starts out in the state

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

what is $|\psi(t)\rangle$? [3 points]

(c) If the system starts out in the state

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

what is $|\psi(t)\rangle$? [3 points]

F-2012

1 $H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}$

A $\begin{pmatrix} a-\lambda & 0 & b \\ 0 & c-\lambda & 0 \\ b & 0 & a-\lambda \end{pmatrix} \rightarrow (a-\lambda)\left[(c-\lambda)(a-\lambda)\right]$

$- \emptyset + b\left[-b(c-\lambda)\right] = \emptyset$

$\rightarrow (c-\lambda)(a-\lambda)^2 - b^2(c-\lambda) = \emptyset$

$(a-\lambda)^2 = b^2$

$a - \lambda = b$

$\lambda = a + b$

$\boxed{\lambda = c \qquad \lambda = a \pm b}$

$\underline{\lambda = c} \rightarrow \begin{pmatrix} a-c & 0 & b \\ 0 & 0 & 0 \\ b & 0 & a-c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \emptyset$

y = anything since always multiplied by $\emptyset$ x + z have values

~~$(a-c)x + bz = \emptyset \quad y = \emptyset$~~

$\boxed{|\lambda = c\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

~~$\Rightarrow z = \left(\frac{c-a}{b}\right)x$~~

~~$\rightarrow \begin{pmatrix} x & 0 & \left(\frac{c-a}{b}\right)x \end{pmatrix} \begin{pmatrix} x \\ 0 \\ \left(\frac{c-a}{b}\right)x \end{pmatrix} = 1$~~

~~$x^2 + \left(\frac{c-a}{b}\right)^2 x^2 = 1 \rightarrow x^2 = \frac{1}{1 + \left(\frac{c-a}{b}\right)^2}$~~

A Continued

$\underline{\lambda = a+b}$

$$\rightarrow \begin{pmatrix} -b & 0 & b \\ 0 & c-a-b & 0 \\ b & 0 & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\rightarrow -bx + bz = \emptyset \rightarrow x = z$$

$$\rightarrow \boxed{|\lambda = a+b\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}}$$

$\underline{\lambda = a-b}$

$$\begin{pmatrix} b & 0 & b \\ 0 & c-a+b & 0 \\ b & 0 & b \end{pmatrix} \rightarrow \boxed{|\lambda = a-b\rangle = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}}$$

③

(B) $|\lambda = c\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |\psi(0)\rangle$

$$\boxed{|\psi(t)\rangle = U(t)|\lambda=c\rangle = e^{-\frac{itc}{\hbar}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

$$U(t) = \sum_i e^{-\frac{iE_i t}{\hbar}} |E_i\rangle\langle E_i|$$

in Dirac Form

(C) $|\psi(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\rightarrow |\psi(0)\rangle = |z\rangle$

$\rightarrow |a+b\rangle = \frac{1}{\sqrt{2}}\left[|x\rangle + |z\rangle\right]$

$\rightarrow \underline{|a-b\rangle} = \frac{1}{\sqrt{2}}\left(|x\rangle - |z\rangle\right)$

$\rightarrow |z\rangle = \frac{1}{\sqrt{2}}\left(|a+b\rangle - |a-b\rangle\right)$

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-it(a+b)/\hbar}|a+b\rangle - e^{-it(a-b)/\hbar}|a-b\rangle\right]}$$

F-200P

Problem 5: Time Evolution (10 Points)

Consider the Hamiltonian and a second observable, B, for a system that can be represented in a 3-dimensional Hilbert space using the orthonormal basis: $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$
with

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

as:

$$H = \hbar\omega \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

The system at time t=0 is in the state:

$$|\Psi(0)\rangle = |e_2\rangle$$

a) Calculate the eigenvalues and normalized eigenvectors of H and B. **(2 Point)**

b) Determine $|\Psi(t)\rangle$, the wavefunction at a later time. **(1 Point)**

c) Determine $P_{|\Psi(t)\rangle}(b = 2)$, the probability of obtaining $b = 2$ if b is measured at an arbitrary time. **(1 Points)**

d) Is your probability in part c) time-dependent or time-independent? Discuss in detail. **(1 Point)**

e) Derive an expression for $\frac{\partial}{\partial t}\langle B\rangle$ where $\langle B\rangle = \langle\Psi(t)|B|\Psi(t)\rangle$ by explicit differentiation using the Time-Dependent Schrodinger Equation. **(2 Points)**

f) Use your expression in part b) to find $\frac{\partial}{\partial t}\langle B\rangle$ for this system using the $|\Psi(t)\rangle$ you found in part a). **(2 Points)**

g) Without doing further calculations describe what result you would expect for $\frac{\partial}{\partial t}\langle B\rangle$ if the initial wavefunction $|\Psi(0)\rangle = |e_2\rangle$ changes to:

$$|\Psi(0)\rangle = |e_1\rangle$$

Explain your answer in detail. **(1 Point)**

F-2008

5 [redacted] A $H = \hbar\omega \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$\rightarrow H = \left| \begin{pmatrix} 2\hbar\omega - \lambda & 0 & 0 \\ 0 & -\lambda & \hbar\omega \\ 0 & \hbar\omega & -\lambda \end{pmatrix} \right| = \emptyset$

$(2\hbar\omega - \lambda)(\lambda^2 - \hbar^2\omega^2) = \emptyset$

$\boxed{\lambda = 2\hbar\omega \ \& \ \pm\hbar\omega}$

$\rightarrow \underline{\lambda = 2\hbar\omega} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\hbar\omega & \hbar\omega \\ 0 & \hbar\omega & -2\hbar\omega \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \emptyset$

$0 = 0 = 0$

$\rightarrow -B\,2\hbar\omega + \hbar\omega C = \emptyset$

$\rightarrow C = 2B$

$\rightarrow$ Say $A = 1$, $B = C = \emptyset$

$\rightarrow \boxed{|\lambda = 2\hbar\omega\rangle = |e_1\rangle}$

$\rightarrow \underline{\lambda = \hbar\omega} \rightarrow \begin{pmatrix} \hbar\omega & 0 & 0 \\ 0 & -\hbar\omega & \hbar\omega \\ 0 & \hbar\omega & -\hbar\omega \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \emptyset$

$\boxed{|\lambda = \hbar\omega\rangle = \frac{1}{\sqrt{2}}\left(|e_2\rangle + |e_2\rangle\right)}$

$A = \emptyset$

$\rightarrow B = C$

A) Continued

$\underline{\lambda = -\hbar\omega} \rightarrow \begin{pmatrix} 3\hbar\omega & 0 & 0 \\ 0 & \hbar\omega & \hbar\omega \\ 0 & \hbar\omega & \hbar\omega \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$\underline{A = \phi} \rightarrow B = -C$

$\rightarrow \boxed{|\lambda = -\hbar\omega\rangle = \frac{1}{\sqrt{2}}\left(|e_2\rangle - |e_3\rangle\right)}$

B) $B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = \phi$

$\cancel{(1-\lambda)[(2-\lambda)(1-\lambda)] + (\phi\ldots) = \phi}$

$\rightarrow (1-\lambda)\left[(2-\lambda)(1-\lambda)\right] + \phi - 1\underbrace{\left[\phi - (-1)(2-\lambda)\right]}_{+} = \phi$

$\rightarrow (1-\lambda)(2-\lambda)(1-\lambda) - (2-\lambda) = \phi$

$\rightarrow (2-\lambda)\left[\underbrace{(1-\lambda)(1-\lambda) - 1}_{1-\lambda-\lambda+\lambda^2-1}\right] = \phi$

$\underbrace{1-\lambda-\lambda+\lambda^2-1}$

$-2\lambda+\lambda^2 = \phi$

$(1-\lambda)^2 = 1$

$(1-\lambda) = \pm 1$

$1 = \pm 1 + \lambda$

$\lambda = \pm 2 \text{ or } \phi$

$\rightarrow \lambda = 2, 2, \phi$

A Continued

$\underline{\lambda = \emptyset} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \rightarrow \begin{matrix} A - C = \emptyset \\ B = \emptyset \\ -A + C = \emptyset \end{matrix}$

$$\rightarrow \boxed{|\lambda = \emptyset\rangle = \frac{1}{\sqrt{2}}\left(|e_1\rangle + |e_2\rangle\right)}$$

$\underline{\lambda = 2} \rightarrow \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \rightarrow B = 1$

$$\rightarrow \boxed{|\lambda = 2\rangle_1 = |e_2\rangle}$$

$\underline{\lambda = 2} \rightarrow \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \rightarrow \begin{matrix} -A - C = \emptyset \\ A = -C \end{matrix}$

$$\rightarrow \boxed{|\lambda = 2\rangle_2 = \frac{1}{\sqrt{2}}\left(|e_1\rangle - |e_3\rangle\right)}$$

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B $|\psi(t)\rangle = e^{-iHt/\hbar}|e_2\rangle$

$$\rightarrow \quad H|e_2\rangle = \begin{pmatrix} 2\hbar\omega & 0 & 0 \\ 0 & 0 & \hbar\omega \\ 0 & \hbar\omega & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \hbar\omega \\ 0 \end{pmatrix}$$

$$\boxed{|\psi(t)\rangle = \exp(-i\omega t)|e_2\rangle}$$

C $p(b=2) = |\langle b=2|\psi(t)\rangle|^2$

$= 1?$

go over this problem again