S-2016

## Problem 5: Magnetic Moments and Spin (10 pts)

Consider a spin 1/2 particle with a magnetic moment. We can write the interaction between the spin and an external magnetic field using the Hamiltonian:

$$H = -\gamma \vec{B} \cdot \vec{S} \tag{1}$$

where $\vec{B}$ is the external field, $\vec{S}$ is the spin operator for the particle, and $\gamma$ is a real positive constant. In this problem, use the usual basis states that are eigenstates of $S_z$

$$S_z \chi_{\pm} = \pm \frac{\hbar}{2} \chi_{\pm}, \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

For this problem, assume the magnetic field lies in the x-z plane:

$$\vec{B} = B_x \hat{e}_x + B_z \hat{e}_z \tag{3}$$

(a) [1 pt] Solve for the eigenenergies for the Hamiltonian, showing your work. Explain the physics of your results.

(b) [2 pts] Any state of the spin can be written in the $\chi_{\pm}$ basis as:

$$\Psi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \tag{4}$$

Using the Hamiltonian, derive the first-order coupled differential equations that give the time dependence for $\alpha(t)$ and $\beta(t)$. In other words, derive the equations for $\dot{\alpha}(t)$ and $\dot{\beta}(t)$.

(c) [2 pts] Show that you can re-write your results from part (b) as two uncoupled second-order differential equations:

$$\begin{aligned} \ddot{\alpha}(t) &= -\frac{\gamma^2 B_T^2}{4} \alpha(t) \\ \ddot{\beta}(t) &= -\frac{\gamma^2 B_T^2}{4} \beta(t) \end{aligned} \tag{5}$$

where $B_T = \sqrt{B_x^2 + B_z^2}$ is the magnitude of the total magnetic field. How is this result related to what you found in part (a)?

Of course, the solutions to these equations are:

$$\begin{aligned} \alpha(t) &= C_1 \cos(\omega t) + C_2 \sin(\omega t) \\ \beta(t) &= C_3 \cos(\omega t) + C_4 \sin(\omega t) \end{aligned} \tag{6}$$

with $\omega = \frac{\gamma B_T}{2}$.

(d) [3 pts] Consider the situation where the spin is in the spin-up $S_z$ state $\chi_+$ at time $t = 0$. Using the boundary conditions at time $t = 0$, determine the values for the constants $C_1$, $C_2$, $C_3$, $C_4$ that will solve for the time-dependence of the state. Remember that the equations in part (c) are second-order, so you need two boundary conditions at $t = 0$ for each.

(e) [2 pt] Write down the time-dependent probabilities, $P_{\pm}$ of the spin being in the spin-up and spin-down $S_z$ states. Show that your results are correct in the two cases where $B_x = 0$ and $B_z = 0$.

S-2016

5 $\vec{B} = B_x \hat{e}_x + B_z \hat{e}_z$

$H = -\gamma \vec{B} \cdot \vec{S}$

A $H = -\gamma \left[ B_x S_x + B_z S_z \right]$

$= -\frac{\gamma \hbar}{2} \left[ B_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$

$= -\frac{\gamma \hbar}{2} \left[ \begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix} \right]$

$\begin{vmatrix} B_z - \lambda & B_x \\ B_x & -B_z - \lambda \end{vmatrix} = \phi$

$\rightarrow (B_z - \lambda)(-B_z - \lambda) - B_x^2 = \phi$

$-B_z^2 + \lambda^2 - B_x^2 = \phi$

$\rightarrow \boxed{\lambda = \pm \sqrt{B_z^2 + B_x^2}}$

$\times$ This

(B) This does ~~NOT~~ involve transitions between 

states → $i\hbar \underbrace{\frac{\partial}{\partial t}\psi = H\psi}$

Time dependent Schrodinger equation

$$\rightarrow H\psi = \frac{-\gamma\hbar}{2}\begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \frac{-\gamma\hbar}{2}\begin{pmatrix} B_z\alpha + B_x\beta \\ B_x\alpha - \beta B_z \end{pmatrix} = i\hbar\begin{pmatrix} \dot\alpha \\ \dot\beta \end{pmatrix}$$

$$\rightarrow \boxed{\begin{aligned} \dot\alpha &= \frac{i\gamma}{2}(B_z\alpha + B_x\beta) \\ \dot\beta &= \frac{i\gamma}{2}(B_x\alpha - \beta B_z) \end{aligned}}$$

(C) $\ddot\alpha = \frac{i\gamma}{2}(B_z\dot\alpha + B_x\dot\beta)$

$$\rightarrow \ddot\alpha = \frac{i\gamma}{2}\left(B_z\dot\alpha + B_x\frac{i\gamma}{2}(B_x\alpha - \beta B_z)\right)$$

$$= \frac{i\gamma}{2}\frac{i\gamma}{2}\left(B_z(B_z\alpha + B_x\beta) + B_x(B_x\alpha - \beta B_z)\right)$$

$$= -\frac{\gamma^2}{4}\left(B_z^2\alpha + B_z\cancel{B_x\beta} + B_x^2\alpha - \cancel{\beta B_x}B_z\right)$$

$$\boxed{\ddot\alpha = -\frac{\gamma^2}{4}(B_T^2\alpha)}$$

C Continued

$\boxed{\text{The other Follows}}$

D $\alpha(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$

$t = \phi \rightarrow \boxed{1 = C_1}$

~~$\dot{\alpha}(\phi) = \phi = -C_1 \omega \sin(\phi) + \omega C_2 \cos(\phi)$~~ ~~$\boxed{C_2 = \phi}$~~ ~~$\boxed{C_2 = \phi}$~~

~~$\phi = C_3 \cos(\phi) + \phi$~~

~~$\ddot{\alpha}(\phi) = \phi = \omega^2 C_1 \cos(\omega t) - C_2 \omega^2 \sin(\omega t)$~~

at $t = \phi$ $\boxed{\phi = C_3}$ ~~$\boxed{C_4 = \phi}$~~

$\alpha(t) = \cos \omega t + C_2 \sin \omega t$

$\beta(t) = C_4 \sin \omega t$

$\rightarrow \dot{\alpha} = \frac{i\gamma}{2}(B_z \cos \omega t + B_z C_2 \sin \omega t + B_x C_4 \sin \omega t)$

$= -\sin(\omega t) + \omega C_2 \cos \omega t$

$t = \phi \rightarrow \frac{i\gamma}{2} B_z = \omega C_2 \rightarrow \boxed{C_2 = \frac{i\gamma B_z}{2\omega}}$

D (Continued)

$$\dot{\beta} = \frac{i\gamma}{2}\left(B_x \cos\omega t + \frac{i\gamma B_z}{2\omega}\sin\omega t\right) - \frac{i\gamma}{2} B_z C_y \sin\omega t$$

$$= \omega C_y \cos\omega t$$

at $T=\emptyset \rightarrow \frac{i\gamma B_x}{2} = \omega C_y \rightarrow \boxed{C_y = \frac{i\gamma B_x}{2\omega}}$

$$\boxed{\begin{aligned}\alpha(t) &= \cos\omega t + \frac{i\gamma B_z}{2\omega}\sin\omega t \\ \beta(t) &= \frac{i\gamma B_x}{2\omega}\sin\omega t\end{aligned}}$$

E $\quad P_+ = \alpha^2 = \left(\cos\omega t + \frac{i\gamma B_z}{2\omega}\sin\omega t\right)\left(\cos\omega t - \frac{i\gamma B_z}{2\omega}\sin\omega t\right)$

$$\boxed{\cos^2\omega t + \frac{\gamma^2 B_z^2}{4\omega^2}\sin^2\omega t}$$

$$\boxed{P_- = \beta^2 = \frac{\gamma^2 B_x^2}{4\omega}\sin^2\omega t}$$

F-2007

PROBLEM 4

A beam of spin-1/2 particles traveling in the $y$ direction is sent through a Stern-Gerlach apparatus in which the magnetic field is inhomgeneous in the $z$ direction, with $g\,\mu_B\,\partial B/\partial z < 0$. Here $g$ is the g-factor of the particles, $\mu_B$ the Bohr magneton, and $B$ the magnetic field. Two beams emerge from the apparatus. The beam that emerges with a velocity whose $z$ component is positive (the beam traveling upward) enters a second Stern-Gerlach apparatus. The inhomgeneity in this second magnet is aligned along the unit vector $\hat{\mathbf{e}} = \sin\boldsymbol{\theta}\,\hat{\mathbf{x}} + \cos\boldsymbol{\theta}\,\hat{\mathbf{y}}$

[a] (2pts) Derive an expression for the elements of the $2 \times 2$ matrix

$$\mathbf{S} \;= (\hbar/2)\boldsymbol{\sigma}\cdot\hat{\mathbf{e}} = \frac{\hbar}{2}(\sigma_x e_x + \sigma_y e_y + \sigma_z e_z)$$

where $\sigma_x$, $\sigma_y$, and $\sigma_z$ are the Pauli Spin matrices. Expression each matrix element in terms of $\hbar$ and the angle $\theta$ of the unit vector $\hat{\mathbf{e}}$.

[b] (2pts) What are the eigenvalues of $\mathbf{S}$? Justify your answer.

[c] (3pts) Determine the normalized eigenvectors of the matrix $\mathbf{S}$.

[d] (3pts) Derive expressions for the relative probability that the particles will be deflected into each of the two beams that emerge from the second Stern-Gerlach apparatus.

F-2007

4 A $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\cancel{\hat{e} = \cos\phi \sin\theta\, \hat{x} + \sin\phi \sin\theta\, \hat{y}\, \hat{z}}$$

$$\hat{e} = \cos\phi \sin\theta\, \hat{x} + \sin\phi \sin\theta\, \hat{y} + \cos\theta\, \hat{z}$$

$$= \begin{pmatrix} 0 & \cos\phi\sin\theta \\ \cos\phi\sin\theta & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\phi\sin\theta \\ i\sin\phi\sin\theta & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} \frac{\hbar}{2} = S}$$

B + C you've done

D write new $\vec{S}$ matrix

Find eigenvalues + eigenvectors

goes into second as $|+\rangle_z$

write $|+\rangle_z$ as superposition of eigenvectors from new $\vec{S}$ matrix

$$|+\rangle_z = A|\ \rangle_+ + B|\ \rangle_-$$

$A^2$ + $B^2$ are relative probabilities

F-2012



**PROBLEM 2: Generalized Uncertainty Principle**

Consider the spin 1/2 operator

$$\mathbf{S} = \frac{\hbar}{2}\vec{\sigma},$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli matrices, which are defined in the basis of the $S_z$ operator eigenvectors,

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) Compute the commutator $[S_i, S_j]$, with $i, j = x, y, z$. [2 Points]

(b) Compute the expectation values $\langle(\delta S_x)^2\rangle$ and $\langle(\delta S_y)^2\rangle$ for the state

$$|\alpha\rangle = \cos(\alpha)|+\rangle + \sin(\alpha)|-\rangle,$$

where $\delta\mathbf{S} = \mathbf{S} - \langle\mathbf{S}\rangle$. Show explicitly that the relation

$$\langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle \geq \frac{1}{4}\left|\langle[S_x, S_y]\rangle\right|^2$$

is satisfied. What does it physically mean? [4 Points]

(c) Find the states that maximize and minimize the product $\langle(\delta S_x)^2\rangle\langle(\delta S_y)^2\rangle$. Interpret the results. [2 Points]

(d) Suppose one performs an experiment which filters the $+\hbar/2$ eigenstate of the $S_z$ operator from the initially prepared state $|\alpha\rangle$. Then the $S_x$ component of the spin is measured. Compute the expectation value of this measurement in the state $|\alpha\rangle$. [2 Points]

F-2012

2 $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

A $[S_x, S_y] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$

$= \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right]\frac{\hbar^2}{4}$

$= \frac{\hbar^2}{4}\begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix} = \frac{2i\hbar^2}{4}(S_z)\left(\frac{2}{\hbar}\right)$

$= i\hbar S_z$

$S_x, S_z$ + $S_y, S_z$ + $S_x S_x$ Follow
^error

B $\delta S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \langle S_x \rangle$

$\rightarrow \langle S_x \rangle = (\cos\alpha \ \ \sin\alpha)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$

$= (\cos\alpha \ \ \sin\alpha)\begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} = 2\cos\alpha\sin\alpha$

B Continued

$$\delta S_x = \begin{pmatrix} 0 & -2\cos\alpha\sin\alpha \\ -2\cos\alpha\sin\alpha & 0 \end{pmatrix}$$

$$\delta S_x^{\,2} = \begin{pmatrix} 0 & -2\cos\alpha\sin\alpha \\ -2\cos\alpha\sin\alpha & 0 \end{pmatrix}\begin{pmatrix} 0 & -2\cos\alpha\sin\alpha \\ -2\cos\alpha\sin\alpha & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4\cos^2\alpha\sin^2\alpha & 0 \\ 0 & 4\cos^2\alpha\sin^2\alpha \end{pmatrix}$$

$$\langle(\delta S_x)^2\rangle = (\cos\alpha \quad \sin\alpha)\begin{pmatrix} 4\cos^2\alpha\sin^2\alpha & 0 \\ 0 & 4\cos^2\alpha\sin^2\alpha \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$(\cos\alpha \quad \sin\alpha)\begin{pmatrix} 4\cos^3\alpha\sin^2\alpha \\ 4\cos^2\alpha\sin^3\alpha \end{pmatrix}$$

$$= 4\cos^4\alpha\sin^2\alpha + 4\cos^2\alpha\sin^4\alpha$$

$$= 4\left[(\cos^2\alpha\sin^2\alpha)\underbrace{(\cos^2\alpha + \sin^2\alpha)}_{1}\right]$$

$$= 4(\cos^2\alpha\sin^2\alpha)\frac{\hbar^2}{4}$$

B Continued

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\langle S_y \rangle = (\cos\alpha \ \sin\alpha)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= (\cos\alpha \ \sin\alpha)\begin{pmatrix} -i\sin\alpha \\ i\cos\alpha \end{pmatrix} = -i\sin\alpha\cos\alpha + i\sin\alpha\cos\alpha = \emptyset$$

$$\star \rightarrow \delta S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \rightarrow \delta S_y^{2} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Transpose

complex conjugate

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rightarrow \langle \delta S_y^{2} \rangle = (\cos\alpha \ \sin\alpha)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= (\cos\alpha \ \sin\alpha)\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= \frac{\hbar^2}{4}$$

$$\rightarrow \langle \delta S_x^{2} \rangle \langle \delta S_y^{2} \rangle = \frac{\hbar^4}{4}\cos^2\alpha\sin^2\alpha$$

B $[S_x, S_y] = i\hbar S_z$

$$= \frac{i\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rightarrow \langle [S_x, S_y] \rangle = \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= \frac{i\hbar^2}{2}\left(\cos^2\alpha - \sin^2\alpha\right)$$

$$\rightarrow \langle [S_x, S_y] \rangle^2 = -\frac{\hbar^4}{4}\underbrace{\left[\cos^4\alpha + \sin^4\alpha - 2\cos^2\alpha\sin^2\alpha\right]}_{\cos^4\alpha + \sin^2\alpha\left(\sin^2\alpha - 2\cos^2\alpha\right)}$$

Don't use This $\delta S = S - \langle S \rangle$

~~forget~~ garbage

That's what it is

B again

$$\langle \delta S_x^2 \rangle = \langle S_x^2 \rangle - \langle S_x \rangle^2$$

$$\rightarrow \quad S_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\hbar^2}{4}$$

$$\rightarrow \quad \langle S_x^2 \rangle = (\cos\alpha \;\; \sin\alpha)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}\frac{\hbar^2}{4}$$

$$= \frac{\hbar^2}{4}$$

$$\rightarrow \quad \langle S_x \rangle = (\cos\alpha \;\; \sin\alpha)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}\frac{\hbar}{2}$$

$$= (\cos\alpha \;\; \sin\alpha)\begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}\frac{\hbar}{2} = \hbar\cos\alpha\sin\alpha$$

$\uparrow$ $\hbar^2\cos\alpha^2\sin\alpha^2$ squared

$$\rightarrow \langle \delta S_x^2 \rangle \; \frac{\hbar^2}{4} - \hbar\cos\alpha\sin\alpha$$

$$\rightarrow \quad S_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\hbar^2}{4}$$

$$\rightarrow \quad \langle S_y^2 \rangle = \frac{\hbar^2}{4}(\cos\alpha \;\; \sin\alpha)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= \frac{\hbar^2}{4}$$

$$\rightarrow \quad \langle S_y \rangle = \frac{\hbar}{2}(\cos\alpha \;\; \sin\alpha)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= \phi$$

(B) Continued

$$\langle \delta S_x^2 \rangle \langle \delta S_y^2 \rangle = \frac{\hbar^4}{16} - \frac{\hbar^4}{4}\cos^2\alpha\sin^2\alpha$$

$$\geq -\frac{\hbar^4}{16}\left[\cos^4\alpha + \sin^4\alpha - 2\cos^2\alpha\sin^2\alpha\right]$$

$$\rightarrow \boxed{\frac{\hbar^4}{16} + \frac{\hbar^4}{16}\left[\cos^4\alpha + \sin^4\alpha\right] \geq \frac{\hbar^4}{4}\cos^2\alpha\sin^2\alpha + \frac{1\cos^2}{8\sin} \geq \frac{3\hbar^4\cos^2\alpha\sin^2\alpha}{8}}$$

(C) Maximized when $\alpha = 0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}$

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix}$$

Minimized $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$$\begin{pmatrix}\pm 1\\ \pm 1\end{pmatrix}$$

(D) $|a\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$\langle S_x \rangle = 0$$

F-2011

**PROBLEM 4: Spin Angular Momentum**

A Stern-Gerlach experiment is set up with the axis of the inhomogeneous magnetic field in the $x-y$ plane, at an angle $\theta$ relative to the $x$−axis. Let us call this direction $\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}$. Then the spin operator in the $\hat{r}$ direction is $S_r = \cos\theta S_x + \sin\theta S_y$. Let us describe the common eigenvectors for $S^2$ and $S_i$ as $|s, m_i\rangle$, e.g. $|s, m_x\rangle$ or $|s, m_z\rangle$.

(a) For a spin−1/2 particle, calculate the matrix corresponding to $S_r$. [1 point]

(b) Evaluate the eigenvalues of $S_r$. [1 point]

(c) Find the normalized eigenvectors of $S_r$. [2 points]

(d) Suppose a measurement of the spin of the particle in the $\hat{r}$ direction is made and it is determined that the spin is in the positive $\hat{r}$ direction, i.e. $S_r|\psi\rangle = (+\hbar/2)|\psi\rangle$. Now a second measurement is made to determine $m_x$ (the component of the spin in the $x$ direction). What is the probability that $m_x = -1/2$? [3 points]

(e) Suppose that the particle has spin in the positive $\hat{r}$ direction as in part (d). The $z$ component of the spin is measured and it is discovered that $m_z = +1/2$. Now a third measurement is made to determine $m_x$. What is the probability that $m_x = -1/2$? [3 points]

F-2011



4) $S_r = \cos\theta\, S_x + \sin\theta\, S_y$

A) $\sigma_x = \begin{pmatrix} \emptyset & 1 \\ 1 & \emptyset \end{pmatrix}$ $\quad \sigma_y = \begin{pmatrix} \emptyset & -i \\ i & \emptyset \end{pmatrix}$

$S_x = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\sigma_y$

$S_r = \frac{\hbar}{2}\begin{bmatrix} 0 & \cos\theta \\ \cos\theta & 0 \end{bmatrix} + \frac{i\hbar}{2}\begin{bmatrix} 0 & -\sin\theta \\ \sin\theta & \emptyset \end{bmatrix}$

$$\boxed{S_r = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & \emptyset \end{pmatrix}}$$

B) $\begin{pmatrix} 0-\lambda & \frac{\hbar}{2}e^{-i\theta} \\ \frac{\hbar}{2}e^{i\theta} & -\lambda \end{pmatrix} = \emptyset \rightarrow \lambda^2 - \frac{\hbar^2}{4} = \emptyset$

$$\boxed{\lambda = \pm\frac{\hbar}{2}}$$

C) $\underline{\lambda = \frac{\hbar}{2}} \rightarrow \frac{\hbar}{2}\begin{pmatrix} -1 & e^{-i\theta} \\ e^{i\theta} & -1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \emptyset$

$-A + Be^{-i\theta} = \emptyset$

$|\lambda = \frac{\hbar}{2}\rangle = \begin{pmatrix} A \\ Ae^{i\theta} \end{pmatrix} \qquad B = e^{i\theta}A$

$1 = (A \;\; Ae^{-i\theta})\begin{pmatrix} A \\ Ae^{i\theta} \end{pmatrix} = A^2 + A^2 = 1$

$A = \frac{1}{\sqrt{2}}$

$$\rightarrow \boxed{|\lambda = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\theta}|-\rangle\right)}$$

(C) Continued

$$\lambda = -\frac{\hbar}{2} \rightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & -1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \emptyset$$

$$A + Be^{-i\theta} = \emptyset \rightarrow B = -Ae^{i\theta}$$

$$\boxed{|\lambda = -\tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - e^{i\theta}|-\rangle\right)}$$

(D) $S_r|\psi\rangle = \frac{\hbar}{2}|\psi\rangle$

$$|\psi\rangle = |\lambda = \tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\theta}|-\rangle\right)$$

Assuming this means $m_x = -\frac{\hbar}{2}$

$$|S_x = -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

$$P_{S_x = -\frac{\hbar}{2}} = \left|\langle S_x = -\tfrac{1}{2}|\lambda = \tfrac{\hbar}{2}\rangle\right|^2$$

$$= \frac{1}{4}\left|\langle +|+\rangle - e^{i\theta}\langle +|-\rangle - \emptyset - e^{i\theta}\langle -|-\rangle\right|^2$$

$$= \frac{1}{4}\left|1 - e^{i\theta}\right|^2$$

$$= \frac{1}{4}(1 - e^{-i\theta})(1 - e^{i\theta}) = \frac{1}{4}\left[1 - e^{i\theta} - e^{-i\theta} + 1\right]$$

$$= \frac{1}{4}(2 - 2\cos\theta)$$

$$\boxed{P_{S_x = -\frac{\hbar}{2}} = \frac{1}{2} - \frac{1}{2}\cos\theta}$$

(D) $|x \text{ basis}\rangle = U|r \text{ basis}\rangle$ (3)

$$= U\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow U = \begin{pmatrix} \langle r=\tfrac{1}{2}|x=\tfrac{1}{2}\rangle & \langle r=\tfrac{1}{2}|x=-\tfrac{1}{2}\rangle \\ \langle r=-\tfrac{1}{2}|x=\tfrac{1}{2}\rangle & \langle r=-\tfrac{1}{2}|x=-\tfrac{1}{2}\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(1+e^{i\sigma}) & \frac{1}{2}(1-e^{i\sigma}) \\ \frac{1}{2}(1-e^{i\sigma}) & \frac{1}{2}(1+e^{i\sigma}) \end{pmatrix}$$

$$|x \text{ basis}\rangle = \frac{A}{2}\begin{pmatrix} 1+e^{i\sigma} \\ 1-e^{i\sigma} \end{pmatrix} \begin{matrix} |x=\tfrac{1}{2}\rangle \\ |x=-\tfrac{1}{2}\rangle \end{matrix}$$

$$1 = \frac{A^2}{4}\begin{pmatrix} 1+e^{-i\sigma} & 1-e^{-i\sigma} \end{pmatrix}\begin{pmatrix} 1+e^{i\sigma} \\ 1-e^{i\sigma} \end{pmatrix}$$

$$(1+e^{-i\sigma})(1+e^{i\sigma}) + (1-e^{-i\sigma})(1-e^{i\sigma})$$

$$= 1+\cancel{e^{i\sigma}+e^{-i\sigma}}+1 + \cancel{-e^{i\sigma}-e^{-i\sigma}}+1$$

$$A = 1 \qquad 4$$

$$P_{x=-\frac{1}{2}} = |\langle x=-\tfrac{1}{2}|r=\tfrac{1}{2}\rangle|^2$$

$$= \left|\frac{1}{2}(1-e^{i\sigma})\right|^2 = \boxed{\frac{1}{2}-\frac{1}{2}\cos\sigma = P_{x=-\frac{1}{2}}}$$

E $|+\rangle = \frac{1}{\sqrt{2}}\left(|\lambda = -\frac{\hbar}{2}\rangle + |\lambda = \frac{\hbar}{2}\rangle\right)$

$$P_{x=-\frac{\hbar}{2}} = |\langle S_x = -\tfrac{\hbar}{2} | +\rangle|^2$$

$$= \frac{1}{4}\left|\left[\langle +| - \langle -|\right]\left[\frac{1}{\sqrt{2}}|+\rangle - e^{i\omega}|-\rangle + \frac{1}{\sqrt{2}}|+\rangle + e^{i\omega}|-\rangle\right]\right|^2$$

$$= \frac{1}{4}\left|\frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}\right|^2 \cancel{+ \frac{1}{4\sqrt{2}}e^{i\omega} - \frac{1}{4\sqrt{2}}e^{i\omega}}$$

$$= \frac{1}{4}\left|\frac{2^2}{4^2(2)}\right|$$

$$\boxed{P_{x=\frac{\hbar}{2}} = \frac{1}{32}}$$

? are you sure?

$|x = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ $\quad |x = -\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|x = \tfrac{\hbar}{2}\rangle + |x = -\tfrac{\hbar}{2}\rangle\right)$$

$$\boxed{P_{x=-\frac{\hbar}{2}} = |\langle x = -\tfrac{\hbar}{2}|+\rangle|^2 = \frac{1}{2}}$$

S-2001



# Problem 1: Spin $\frac{1}{2}$ particles (10 points)

Consider a system made up of spin 1/2 particles. If one measures the spin of the particles, one can only measure spin up or spin down. The general spin state of a spin 1/2 particle can be expressed as a two-element column matrix.

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

The spin matrices are:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) Can one simultaneously measure $S_x$, $S_y$ and $S_z$? Explain your answer. (1 pt)

b) Can one simultaneously measure $S^2$ and $S_z$? Explain your answer. (1 pt)

c) Show $S_z$ is Hermetian. (1 pt)

d) Calculate the normalized eigenvectors and eigenvalues of $S_z$. (2 pts)

Suppose a spin 1/2 particle is in the state

$$\chi = A\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

e) Normalize the state in order to determine A (1 pt)

f) If one measures $S_z$, what is the probability of getting -$\hbar/2$ ? (1 pt)

g) If one measures $S_x$, what is the probability of getting $+\hbar/2$ ? (2 pts)

h) What is the expectation value of $S_y$ (1 pt)

S-2009

(1)

(1) (A) $\boxed{\text{No since } [S_i, S_j] = i\hbar \epsilon_{ijk} S_k \rightarrow \text{These operators do NOT commute}}$

(B) $\boxed{\text{Yes since } [S^2, S_z] = \phi}$

(C) $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow S_z^{+} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\boxed{S_z = S_z^{+}}$

(D) $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\rightarrow \begin{vmatrix} \frac{\hbar}{2} - \lambda & 0 \\ 0 & -\frac{\hbar}{2} - \lambda \end{vmatrix} = \left(\frac{\hbar}{2} - \lambda\right)\left(-\frac{\hbar}{2} - \lambda\right) = \phi$

$-\frac{\hbar^2}{4} \cancel{- \frac{\hbar}{2}\lambda + \frac{\hbar}{2}\lambda} + \lambda^2 = \phi$

$\rightarrow -\frac{\hbar^2}{4} + \lambda^2 = \phi$

$\lambda^2 = \frac{\hbar^2}{4}$

$\boxed{\lambda = \pm \frac{\hbar}{2}}$

D Continued

in $|\pm\rangle$ basis

$\underline{\lambda = \frac{\hbar}{2}} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -\hbar \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \emptyset$

$\rightarrow \emptyset = \emptyset \rightarrow$

$0 \; \cancel{A} - \hbar B = \emptyset$

B has to be $\emptyset$

A can be anything

$\rightarrow A = 1$

$$\boxed{|\lambda = \tfrac{\hbar}{2}\rangle = |+\rangle}$$

$\underline{\lambda = -\frac{\hbar}{2}} \rightarrow \begin{pmatrix} \hbar & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \Rightarrow A\hbar = \emptyset$

$\emptyset = \emptyset$

$A = \emptyset$

$B = 1$

$$\boxed{|\lambda = -\tfrac{\hbar}{2}\rangle = |-\rangle}$$

E $\langle X | X \rangle = 1 = A^2 \begin{pmatrix} 1-i & 2 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$

$= A^2 \left[ \underbrace{(1-i)(1+i)}_{1+i-i+1} + 4 \right] = 1$

$\rightarrow \boxed{A = \frac{1}{\sqrt{6}}}$

(F)



$$\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

$$\chi = \frac{1+i}{\sqrt{6}}|+\rangle + \frac{2}{\sqrt{6}}|-\rangle$$

$$\rightarrow P = |\langle + | \chi \rangle|^2 = \left|\frac{1+i}{\sqrt{6}}\right|^2$$

$$= \frac{1}{6}|(1-i)(1+i)|^2$$

$$(1+i)(1-i) = 1+1$$

$$= \boxed{\frac{2}{3} = P_{\frac{\hbar}{2}}}$$

(6) $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\rightarrow \begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} = \emptyset$$

$$\rightarrow \underline{\lambda = \pm \frac{\hbar}{2}}$$

$$\rightarrow \underline{\lambda = \frac{\hbar}{2}} \rightarrow \begin{pmatrix} -\frac{\hbar}{2} & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \emptyset$$

$$\rightarrow A = B \rightarrow |\lambda = \tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right]$$

G Continued

$$\lambda = -\frac{\hbar}{2} \rightarrow \begin{pmatrix} \frac{\hbar}{2} & \frac{\hbar}{2} \\ \frac{\hbar}{2} & \frac{\hbar}{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \phi$$

$$A = -B$$

$$|\lambda = -\tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\left[ |+\rangle - |\cancel{-}\rangle \right]$$

$$|\langle x = \tfrac{\hbar}{2} | \chi \rangle|^2 = \left| \frac{1}{\sqrt{12}} (1 \quad 1) \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{12}} (1 + i + 2) \right|^2 = \frac{9}{12} = \frac{3}{4}$$

H $\langle x | S_y | X \rangle$

$$= \frac{\hbar}{12} (1-i \quad 2) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

$$= \frac{\hbar}{12} (1-i \quad 2) \begin{pmatrix} -2i \\ i-1 \end{pmatrix}$$

$$= \frac{\hbar}{12} (-\cancel{2i} - 2 + \cancel{2i} - 2)$$

$$= \frac{-4}{12} \hbar = \boxed{-\frac{\hbar}{3}}$$

S-2010



## PROBLEM 4: Spin Physics

Spin-1/2 objects generally have magnetic moments that affect their energy levels and dynamics in magnetic fields. The interaction between the magnetic moment and a magnetic field, $\vec{B}$ can be written as:

$$H = -\mu \vec{S} \cdot \vec{B} \tag{1}$$

where $\vec{S}$ is the spin of the particle

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \tag{2}$$

where the $\sigma_i$'s are Pauli matrices.

In this problem we'll be using as our basis the eigenstates of $S_z$,

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

with eigenvalues $\pm\frac{\hbar}{2}$.

(a) [1 point] If a particle is in the spin state $|+\rangle$, compute the expectation values of $S_x$, $S_y$, and $S_z$.

(b) [1 point] If a particle is in the spin state $|+\rangle$, what are the uncertainties of $S_x$, $S_y$, and $S_z$? ($\Delta S_i^2 = \langle S_i^2\rangle - \langle S_i\rangle^2$.) Explain the physics of your results in terms of the eigenvalues and measurement probabilities of the spin in the x, y, and z directions.

(c) [3 points] A large ensemble of particles are all prepared to be in the spin state $|+\rangle$ at time $t = 0$ when a magnetic field in the x-direction is switched on, $\vec{B} = B_0\hat{e}_x$. Solve for the time-dependent probabilities, $P_\pm(t)$, of measuring $S_z$ to be $\pm\hbar/2$.

(d) [2 points] For the experiment described in part (c), what are the probabilities for measuring $S_x$ to be $\pm\hbar/2$? Explain the differences between the results for $S_z$ and $S_x$.

(e) [3 points] Consider the case where the magnetic field is $\vec{B} = \frac{B_0}{\sqrt{2}}(\hat{e}_x + \hat{e}_z)$. In this case what is the time-dependent probability of measuring $S_z$ to be $+\hbar/2$?

S-2010



4 $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

A $\langle + | S_x | + \rangle$

$= S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\to \langle S_x \rangle = (1\ 0)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \frac{\hbar}{2}(1\ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \boxed{\langle S_x \rangle = \phi}$

$\langle + | S_y | + \rangle = (1\ 0)\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \boxed{\langle S_y \rangle = \phi}$

$\langle + | S_z | + \rangle = (1\ 0)\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= (1\ 0)\frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \boxed{\frac{\hbar}{2} = \langle S_z \rangle}$

B $S_x^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\to \langle S_x^2 \rangle = \frac{\hbar^2}{4}(1\ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \frac{\hbar^2}{4}(1\ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}$

$\to \boxed{\Delta S_x = \frac{\hbar}{2}}$ Same for $\boxed{\Delta S_y = \frac{\hbar}{2}}$

B-Continued



$$S_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar^2}{4}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\hbar^2}{4}$$

$$\rightarrow \langle S_z^2 \rangle = (1\ 0)\frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (1\ 0)\frac{\hbar^2}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}$$

$$\rightarrow \Delta S_z = \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{4}} = \phi$$

$$\rightarrow \boxed{\Delta S_z = \phi}$$

C

$$H = -\mu\frac{\hbar}{2}B_0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = \phi \rightarrow \lambda = \pm 1$$

For $\underline{\lambda = +1}$ $\rightarrow$

$$-\mu_0\frac{\hbar}{2}B_0\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \phi$$

$$A = B$$

$$\rightarrow \left|E = \frac{\mu_0 \hbar B_0}{2}\right\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

C Continued

$$\lambda = -1 \rightarrow -\frac{\mu_0 \hbar B_0}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \emptyset$$

$$\rightarrow A = B$$

$$\rightarrow \left| E = -\frac{\mu_0 \hbar B_0}{2} \right\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle - |-\rangle \right)$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( \left| E = \frac{\mu_0 \hbar B_0}{2} \right\rangle + \left| E - \frac{\mu_0 \hbar B_0}{2} \right\rangle \right)$$

$$|\psi(t)\rangle = \exp\left(-\frac{iHt}{\hbar}\right)|\psi\rangle = \frac{1}{\sqrt{2}} \left( \exp\left(-\frac{i\mu_0 B_0 t}{2}\right) |E = +\rangle + \exp\left(\frac{i\mu_0 B_0 t}{2}\right) |E = -\rangle \right)$$

$$\rightarrow P_{+\frac{\hbar}{2}} = |\langle + | \psi(t) \rangle|^2 = \left| \frac{1}{2} \left( e^{-\frac{i\mu_0 B_0 t}{2}} + e^{\frac{i\mu_0 B_0 t}{2}} \right) \right|^2$$

$$= \boxed{\cos^2\left(\frac{\mu_0 B_0 t}{2}\right) = P_{+\frac{1}{2}}}$$

$$\rightarrow \boxed{P_{-\frac{1}{2}} = \sin^2\left(\frac{\mu_0 B_0 t}{2}\right)}$$

D You can find $|S_x; +\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ Diagonalizing $S_x$ & $S_y$

Same procedure as above For $P$ answer is $\frac{1}{2}$

S-2013

## Problem 3: Spin Measurements and Uncertainty

Define the operator $S_\alpha = \vec{S} \cdot \hat{n}_\alpha$ where $\vec{S}$ is the vector spin operator and $\hat{n}_\alpha$ is a unit vector in the $x - z$ plane that makes an angle $\alpha$ with the z-axis. So $\hat{n}_\alpha = \hat{z}$ for $\alpha = 0$ and $\hat{n}_\alpha = \hat{x}$ for $\alpha = \pi/2$.

Consider a spin 1/2 system initially prepared to be in the eigenstate of $S_\alpha$ with eigenvalue $+\hbar/2$,

$$S_\alpha|\alpha, +\rangle = \frac{\hbar}{2}|\alpha, +\rangle \tag{1}$$

(a) [3 pts] Compute the eigenstates of $S_\alpha$ in the basis of the $S_z$ operator, $|0, \pm\rangle \equiv |\pm\rangle$.

(b) [2 pts] If the spin is in the state $|\alpha, +\rangle$ and $S_x$ is measured, what is the probability of measuring $-\hbar/2$?

(c) [3 pts] Compute the expectation value $\langle(\delta S_x)^2\rangle$ for the state $|\alpha, +\rangle$, where $\delta S_x = S_x - \langle S_x\rangle$.

If one measures $S_x$, what are the values of $\alpha$ that minimize the uncertainty of the measurement for the state $|\alpha, +\rangle$? Interpret the physical meaning of those states.

(d) [2 pts] Finally, define $\mathcal{P}_{x,+}$ to be the projection operator for the spin 1/2 state of $S_x$, $|\pi/2, +\rangle$. Compute the matrix element $\mathcal{P}_{x,+}$ in the initial state, $\langle +, \alpha|\mathcal{P}_{x,+}|\alpha, +\rangle$. Explain the behavior of the resultant expression as a function of the angle $\alpha$.

3 $S_\alpha |\alpha,+\rangle = \frac{\hbar}{2}|\alpha,+\rangle$

$\hat{n}_\alpha$ = unit vector in $xz$


$$\hat{n}_\alpha = \cos\alpha\,\hat{z} + \sin\alpha\,\hat{x}$$

$$\vec{S}\cdot\hat{n}_\alpha = S_z\cos\alpha + S_x\sin\alpha$$

$$= \frac{\hbar}{2}\cos\alpha\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{\hbar}{2}\sin\alpha\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\alpha & \sin\alpha\\ \sin\alpha & -\cos\alpha\end{pmatrix}$$

A $\frac{\hbar}{2}\begin{vmatrix}\cos\alpha-\lambda & \sin\alpha\\ \sin\alpha & -\cos\alpha-\lambda\end{vmatrix} = \emptyset \rightarrow$

$$(\cos\alpha-\lambda)(-\cos\alpha-\lambda) - \sin^2\alpha = \emptyset$$

$$-\cos^2\alpha - \cancel{\cos\alpha\lambda} + \cancel{\cos\alpha\lambda} + \lambda^2 - \sin^2\alpha = \emptyset$$

$$\lambda^2 - 1 = \emptyset \rightarrow \lambda = \pm 1$$

$$\rightarrow \lambda = \pm\frac{\hbar}{2}$$

$\lambda = +\frac{\hbar}{2} \rightarrow \frac{\hbar}{2}\begin{pmatrix}\cos\alpha - 1 & \sin\alpha\\ \sin\alpha & -\cos\alpha - 1\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} = \emptyset$

$$(\cos\alpha - 1)A + \sin\alpha B = \emptyset$$

$$\cos\alpha A + \sin\alpha B = A$$

A Continued

$$A\sin\alpha - B(\cancel{\phantom{x}}\ \underbrace{\cos\alpha+1}_{2\cos^2\left(\frac{\alpha}{2}\right)}) = \emptyset$$

$\uparrow$ $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$$\rightarrow A\sin\frac{\alpha}{2} = \cos\left(\frac{\alpha}{2}\right)B$$

$$A = \cos\left(\frac{\alpha}{2}\right) \qquad B = \sin\left(\frac{\alpha}{2}\right)$$

$$\rightarrow \boxed{|\lambda = \tfrac{\hbar}{2}\rangle = \begin{pmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} \end{pmatrix}}$$

$\underline{\lambda = -\frac{\hbar}{2}}$ $\rightarrow$ $\frac{\hbar}{2}\begin{pmatrix} \cos\alpha+1 & \sin\alpha \\ \sin\alpha & -\cos\alpha+1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \emptyset$

$$A(\underbrace{\cos\alpha+1}_{2\cos^2\left(\frac{\alpha}{2}\right)}) + B\sin\alpha = \emptyset$$

$\uparrow$ $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$$\boxed{|\lambda = -\tfrac{\hbar}{2}\rangle = \cancel{\begin{pmatrix} \cos\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} \end{pmatrix}} \begin{pmatrix} -\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2} \end{pmatrix}}$$

B $|\alpha, +\rangle$ want $S_x = -\frac{\hbar}{2}$

$$|S_x = -\tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

$$P_{x=-\frac{\hbar}{2}} = \frac{1}{2}\left|\cos\tfrac{\alpha}{2} - \sin\tfrac{\alpha}{2}\right|^2 = \frac{1}{2}\left(\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} - 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\right)$$

$$= \boxed{\frac{1}{2} - \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = P_{x=-\frac{\hbar}{2}}}$$

(c) $\delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle}$

$\rightarrow \langle S_x^2 \rangle = \langle \alpha, + | S_x^2 | \alpha, + \rangle$

$|x \text{ basis}\rangle = U |\alpha \text{ basis}\rangle \qquad U = \begin{pmatrix} \langle \alpha, + | S_x + \rangle & \langle \alpha + | S_x - \rangle \\ \langle \alpha - | S_x + \rangle & \langle \alpha - | S_x - \rangle \end{pmatrix}$

$= U \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2} \end{pmatrix}$

$\rightarrow |x \text{ basis}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} \end{pmatrix} \begin{matrix} |S_x + \rangle \\ |S_x - \rangle \end{matrix}$

$S_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\rightarrow \langle S_x^2 \rangle = \frac{\hbar^2}{8} \left( \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \quad -\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} \end{pmatrix}$

$= \frac{\hbar^2}{8} \left( \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \quad -\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} \right) \begin{pmatrix} \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} \end{pmatrix}$

$= \frac{\hbar^2}{8} \left[ \underbrace{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}}_{1} + \cancel{2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}} + \underbrace{\sin^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}}_{1} - \cancel{2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}} \right]$

$= \frac{\hbar^2}{4}$

$\langle S_x \rangle \rightarrow \frac{\hbar}{4} \left( \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \quad -\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} \right) \begin{pmatrix} -\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} \\ \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \end{pmatrix}$

$= \frac{\hbar}{4} \left[ -\cancel{\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}} + \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} + \cancel{\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}} - \cancel{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} - \sin^2\frac{\alpha}{2} \right.$
$\left. + \cos^2\frac{\alpha}{2} + \cancel{\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}} \right]$

$= \frac{\hbar}{2} \cos\alpha$

C Continued

$$\langle S_x \rangle^2 = \frac{\hbar^2}{4}\cos^2\alpha$$

$$\rightarrow \boxed{\delta S_x = \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{4}\cos^2\alpha} = \frac{\hbar}{2}\sin\alpha}$$

$\alpha = \frac{\pi}{2}$, $\delta S_x = \phi$

at $\frac{\pi}{2}$, aligned with $x$ axis

D $\rho_{x,+} = |S_x +\rangle\langle S_x +|$

or $|+\alpha\rangle\langle+\alpha|$ where $\alpha = \frac{\pi}{2}$

$$\rightarrow |\langle \alpha, + | \alpha, \tfrac{\pi}{2} \rangle|^2$$

$$= \left| \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{pmatrix} \right|^2$$

$$= \left| \cos\frac{\alpha}{2}\cos\frac{\pi}{4} + \sin\frac{\alpha}{2}\sin\frac{\pi}{4} \right|^2$$

$$= \frac{2}{4}\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)^2$$

$$= \frac{1}{2} + \cos\frac{\alpha}{2}\sin\frac{\alpha}{2}$$

$$= \boxed{\frac{1}{2} + \frac{1}{2}\sin\alpha}$$

F-2009



# Problem 6: Spin $\frac{1}{2}$ System (10 points)

Consider a spin $\frac{1}{2}$ particle in the state space $E_s$. This space can be spanned by the 2 eigenvectors of $S_x$, $S_y$, or $S_z$, the components of the spin operator $S = S_x\hat{i} + S_y\hat{j} + S_z\hat{k}$. The matrix representation of $S_x$, $S_y$ and $S_z$ in the eigenbasis $|+\rangle_z$, $|-\rangle_z$ of $S_z$ are given below:

$$S_x = \hbar/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \hbar/2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \hbar/2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $S_z|+\rangle_z = \hbar/2|+\rangle_z$ and $S_z|-\rangle_z = -\hbar/2|-\rangle_z$ .

Assume that the state of the system at time $t = 0$ is: $|\Psi(0)\rangle = |-\rangle_z$.

a) If the observable $S_x$ is measured at time $t = 0$, what results can be found and with what probabilities? (1 pt)

Now assume that a magnetic field is applied in the $x$ **direction**: $\vec{B} = B_0\hat{i}$. The original wave function $|\Psi(0)\rangle = |-\rangle_z$ is allowed to evolve in time. The Hamiltonian governing the evolution is:

$$H_{spin} = \vec{S} \cdot \vec{B}$$

b) Set up the time evolution operator for this system, $U(t, 0)$. (1 pt)

c) Find $|\Psi(t)\rangle$, the wave function at a later time $t$. (1 pt)

d) At time $t > 0$ after $|\Psi(0)\rangle$ has evolved, $S_x$ is measured. What is the probability of obtaining $+\hbar/2$? Is your answer time dependent or time independent? Explain correctly for credit. (1 pt)

e) Now let $|\Psi(0)\rangle$ evolve again and measure $S_z$ at time $t$. Determine the probability of measuring $S_z$ at time $t$ and obtaining $-\hbar/2$. Is your answer time dependent or time independent? Explain correctly for credit. (1 pt)

f) Without explicitly finding the probabilities, discuss whether you expect the following probabilities to be equal or not. Give a brief explanation of your reasoning for any credit. The symbol $P_{|\Psi(t)\rangle}(a, c)$ represents the probability of starting with an ensemble in the state $|\Psi(t)\rangle$, measuring $A$ first and getting eigenvalue "$a$" and then measuring $C$ and getting eigenvalue "$c$". Assume that the eigenvalues of $H_{spin}$ are $E_+$ and $E_-$. (1 pt)

i) Is $P_{|\Psi(0)\rangle}(+\hbar/2 \text{ for } S_y, -\hbar/2 \text{ for } S_x) = P_{|\Psi(0)\rangle>}(-\hbar/2 \text{ for } S_x, +\hbar/2 \text{ for } S_y)$? All measurements are taken at $t = 0$, i.e. the second measurement is taken immediately after the first measurement in each case. (1 pt)

ii) Is $P_{|\Psi(0)\rangle}(E_+, -\hbar/2 \text{ for } S_x) = P_{|\Psi(0)\rangle}(-\hbar/2 \text{ for } S_x, E_+)$? The first measurement in each case is taken at $t = 0$; the second measurement is taken immediately after the first measurement in each case. (1 pt)

iii) Is the probability $P_{|\Psi(0)\rangle}(+\hbar/2 \text{ for } S_x \text{ at } t, -\hbar/2 \text{ for } S_y \text{ at } t')$ time dependent or time independent in regards to the time $t$ of the first measurement? Same question for the time $t'$ of the second measurement. Discuss your reasoning in each case. (2 pts)

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(b) $|\psi(\emptyset)\rangle = |-\rangle$

(A) $S_x|\psi(0)\rangle$

$\langle \alpha|U|\alpha\rangle = \sum \langle \alpha|\beta\rangle \langle \alpha|\alpha\rangle$

Matrix element

Change of basis $\rightarrow$ $|\beta\rangle = U|\alpha\rangle \rightarrow U = |\beta\rangle\langle\alpha|$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = ?$$

eigen values $\rightarrow$ $\frac{\hbar}{2}\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \emptyset$

$\lambda^2 - 1 = \emptyset \rightarrow \lambda = \pm 1$

$\lambda = \frac{\hbar}{2} \rightarrow \frac{\hbar}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \emptyset \rightarrow A = B$

$-A + B = \emptyset$

$|S_x = +\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$\lambda = -\frac{\hbar}{2} \rightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \emptyset$

$|S_x = -\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

$$\rightarrow U \doteq \begin{pmatrix} \langle \alpha_1|\beta_1\rangle & \langle \alpha_1|\beta_2\rangle \\ \langle \alpha_2|\beta_1\rangle & \langle \alpha_2|\beta_2\rangle \end{pmatrix} =$$

or $|-\rangle = \frac{1}{\sqrt{2}}(|S_x = +\rangle - |S_x = -\rangle)$ Just add

A Continued

Probability That in Spin $\frac{\hbar}{2}$ (in x-basis Since That's what we Measure)

$$P_{\frac{1}{2}} = |\langle S_x = + | \psi(\phi) \rangle|^2 = \frac{1}{2}\left[(1 \;\; 0)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]^2$$

in $S_x$ basis

$$= \boxed{\frac{1}{2} = P_{S_x = \frac{1}{2}}}$$

$$P_{-\frac{1}{2}} = |\langle S_x = - | \psi(\phi) \rangle|^2 = \frac{1}{2}\left[(0 \;\; 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]^2$$

$$= \boxed{\frac{1}{2} = P_{S_x = -\frac{1}{2}}}$$

B $H = \vec{S} \cdot \vec{B}$

$= S_x B_0$

$U(t,0) = \exp\left( -\frac{iHt}{\hbar} \right)$

$\rightarrow \boxed{U(t,0) = \exp\left( -\frac{i S_x B_0 t}{\hbar} \right)}$

C $|\psi(t)\rangle = U(t,0)|\psi(0)\rangle$

$= \frac{1}{\sqrt{2}} \left[ \exp\left( -\frac{i B_0 t}{2} \right) |S_x = +\rangle - \exp\left( \frac{i B_0 t}{2} \right) |S_x = -\rangle \right]$

$= |\langle S_x = + | \psi(t) \rangle|^2$

$P_{S_x = \frac{\hbar}{2}} = \frac{1}{2} \exp\left( +\frac{i B_0 t}{2} \right) \exp\left( -\frac{i B_0 t}{2} \right)$

$\boxed{P_{S_x = \frac{\hbar}{2}} = \frac{1}{2}}$ since in eigenbasis of $S_x$

D $P_{S_z = -\frac{1}{2}} = |\langle - | \psi(t) \rangle|^2 \rightarrow \psi(t) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \exp\left( \frac{i B_0 t}{2} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

in z basis

$- \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \exp\left( -\frac{i B_0 t}{2} \right) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\rightarrow P_{S_z = -\frac{1}{2}} = \left| \frac{1}{2} \exp\left( \frac{i B_0 t}{2} \right) (0 \ 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \exp\left( -\frac{i B_0 t}{2} \right) (0 \ 1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right|^2$

$= \Big| \underbrace{\frac{1}{2} \exp\left( \frac{i B_0 t}{2} \right) + \frac{1}{2} \exp\left( -\frac{i B_0 t}{2} \right)}_{\cos\left( B_0 t / 2 \right)} \Big|^2$

$\rightarrow \boxed{P_{S_z = -\frac{1}{2}} = \cos^2\left( \frac{B_0 t}{2} \right)}$

Finish