F-2011

PROBLEM 2: Harmonic Oscillator

A particle of mass m is confined to one dimension. Its potential energy is

$$V(x) = \frac{1}{2} m\,\omega^2 x^2,$$

where $\omega > 0$ is a real parameter. At time $t = 0$, the state of the particle is represented by the real wave function

$$\Psi(x,0) = \frac{1}{\sqrt{2}}\left(1 - \frac{x}{|x|}\right)\phi(x),$$

where $\phi(x)$ is a normalized function of odd parity.

On each question, to receive *any* credit you must fully justify your answer.

(a) At $t = 0$, what is value of the *position probability density* $\mathcal{P}(x,0)$ at the origin, $x = 0$? [2 points]

(b) **Describe** the *parity* of the wave function at $t = 0$ and at any $t > 0$. [2 points]

(c) The *region probability* $\mathcal{P}\big([a,b],t\big)$ denotes the probability that a position measurement at time t would detect the particle in the finite region $x \in [a,b]$. What are the *initial values* of this quantity for the left and right halves of the x axis: $\mathcal{P}\big((-\infty,0],0\big)$ and $\mathcal{P}\big([0,\infty),t\big)$? [2 points]

(d) At what time $t_{\text{right}} > 0$, *if any*, is $\mathcal{P}\big([0,\infty),t_{\text{right}}\big) = 1$? [1 point]

(e) At what time $t_{\text{left}} > 0$, *if any*, is $\mathcal{P}\big((-\infty,0],t_{\text{left}}\big) = 1$? [1 point]

(f) At what time $t_{\text{same}} > 0$, *if any*, are the two region probabilities equal:
$\mathcal{P}\big((-\infty,0],t_{\text{same}}\big) = \mathcal{P}\big([0,\infty),t_{\text{same}}\big)$? [2 points]

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(2) $\psi(x,0) = \frac{1}{\sqrt{2}}\left(1 - \frac{x}{|x|}\right)\phi(x)$

$= \frac{1}{\sqrt{2}}\left(1 - \mathrm{Sgn}(x)\right)\phi(x)$

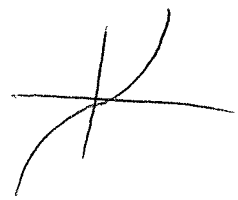
(A) This Function is 0 when $x > 0$ since

$\mathrm{Sgn}(x) = 1$

$\rightarrow (1 - 1) = 0$

$\psi(-x,0) = \sqrt{2}\,\phi(-x) = -\sqrt{2}\,\phi(x)$

$\psi^2 = 2\,\phi^2(x)$



Therefore $\phi(0) = 0$

so $\boxed{\psi^2 = 0}$

(B) $\psi(x,0) = \frac{1}{\sqrt{2}}\left(1 - \frac{x}{|x|}\right)\phi(x)$

$\psi(-x,0) = \frac{1}{\sqrt{2}}\left(1 + \frac{x}{|x|}\right) - \phi(x)$

$\boxed{\text{Indefinite parity}}$

c $P((-\infty,0],0) = 1$

$P([0,\infty],\phi) = \cancel{0}$

D Never

E All

F Never

F-2013

Problem 6: Harmonic Oscillators in 1D

A quantum harmonic oscillator is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

where p is momentum, x is position, m is mass, and ω is the oscillation frequency.

The Hamiltonian has the usual eigenstates and energies:

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle, \quad n = 0, 1, 2, ... \tag{2}$$

Let the system be perturbed by a potential in the form $V = Ax^2$ where A is a real constant.

(a) [2 pt] What is the change in the energy of the unperturbed eigenstates $|n\rangle$ to first order in A? Show your work.

(b) [2 pt] If the perturbation is time-dependent, $V(t) = A(t)x^2$, it can cause transitions between the harmonic oscillator states. To study these transitions, it is helpful to use the time-dependent expansion:

$$|\psi(t)\rangle = \sum_{n'} c_{n'}(t) e^{-\frac{i}{\hbar}E_{n'}t}|n'\rangle \tag{3}$$

The $c_{n'}(t)$ are time-dependent probability amplitudes for the states $|n'\rangle$ and the energies $E_{n'}$ are the unperturbed eigenenergies. Use the Schroedinger equation to show that the expansion amplitudes satisfy a set of coupled equations:

$$i\hbar\frac{\partial}{\partial t}c_n(t) = \sum_{n'} c_{n'}(t) e^{-\frac{i}{\hbar}(E_{n'}-E_n)t}\langle n|V(t)|n'\rangle \tag{4}$$

(c) [3 pt] Consider the case where the oscillator starts at time $t = 0$ in the ground state, $c_n(t = 0) = \delta_{n,0}$. Use the result from (b) to write down the time dependence of the excited state probability amplitudes to first order in V, $c_n^{(1)}(t)$, $n > 0$. This will be an integral equation, as we have not yet defined $A(t)$.

Show that, to first order, there is a transition only to the $n = 2$ excited state.

(d) [3 pt] Finally, consider a time dependent perturbation with $A(t)$ of the form

$$A(t) = Ae^{-i\Omega t}e^{-\Gamma t} \tag{5}$$

Ω and Γ being real and positive.

Compute the probability that the $n = 2$ state is populated for $t \to \infty$, and explain the dependence of your result on Ω and Γ.

Note: In this problem, it is useful to use

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - i\frac{\lambda}{\hbar}p\right), \quad a = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} + i\frac{\lambda}{\hbar}p\right) \tag{6}$$

where $\lambda = \sqrt{\frac{\hbar}{m\omega}}$ is the length scale in the problem.

You do not need to derive the properties of these two operators, but you should state the results you are using.

F-2013

$V = Ax^2$

6

A $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^*) \rightarrow x^2 = \frac{\hbar}{2m\omega}(a^2 + a^{+2} + a^+a + aa^+)$

$$\Rightarrow E_0^{(1)} = \langle n | Ax^2 | n\rangle = \frac{A\hbar}{2m\omega}\Big[\underbrace{\langle n|a^2|n\rangle}_{\emptyset} + \underbrace{\langle n|a^{+2}|n\rangle}_{\emptyset}$$

$$+ \underbrace{\langle n|a^+a|n\rangle}_{\sqrt{n}\sqrt{n}} + \underbrace{\langle n|aa^+|n\rangle}_{\sqrt{n+1}\sqrt{n+1}}\Big]$$

$$= \frac{A\hbar}{2m\omega}(n + n + 1) = \boxed{\frac{A\hbar}{m\omega}\left(n + \frac{1}{2}\right)}$$

B $|\psi(t)\rangle_I = e^{iH_0t/\hbar}|\psi(t)\rangle_S$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_I = i\hbar\frac{\partial}{\partial t}e^{iH_0t/\hbar}|\psi(t)\rangle_S$$

$$= -H_0e^{iH_0t/\hbar}|\psi(t)\rangle_S + i\hbar e^{iH_0t/\hbar}\frac{\partial}{\partial t}|\psi(t)\rangle_S$$

$\Rightarrow i\hbar\frac{\partial}{\partial t}|\alpha\rangle = H|\alpha\rangle$

$$= \cancel{-H_0e^{iH_0t/\hbar}|\psi(t)\rangle_S} + \cancel{e^{iH_0t/\hbar}H_0} + e^{iH_0t/\hbar}V_0|\psi(t)\rangle_S$$

$V(t)$

$$= e^{iH_0t/\hbar}\overset{V(t)}{V}|\psi(t)\rangle_S$$

$$= \underbrace{e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar}}_{V_I \Rightarrow V(t)}\underbrace{e^{iH_0t/\hbar}|\psi(t)\rangle_S}_{|\psi(t)\rangle_I}$$

V

(B) Continued

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_I = \underbrace{V_I |\psi(t)\rangle_I}$$

$$= \sum_{n'} C_{n'}(t) |n'\rangle_I$$

$$= \sum_{n'} c_{n'}(t) V_{\cancel{A}} |n'\rangle_I$$

$$\rightarrow {}_{A}\langle n| \rightarrow \boxed{i\hbar \frac{\partial}{\partial t} c_n = \sum_{n'} c_{n'}(t)\, \underbrace{{}_{A}\langle n|V|n'\rangle_{A}} \quad \langle n| e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar} |n'\rangle \quad = \sum_{n'} c_{n'}(t)\, e^{-\frac{i}{\hbar}(E_{n'}-E_n)t} V_{nn'}}$$

(C)

$$\boxed{C_n^{(1)} = \frac{-i}{\hbar} \int e^{\frac{i}{\hbar}(E_n - E_{n'})t} V_{nn'}\, dt}$$

$$\rightarrow \cancel{\langle n|A x^2|\emptyset\rangle} \quad \langle n|Ax^2|\emptyset\rangle = \frac{A\hbar}{2m\omega}\Big[\langle n|a^2|\emptyset\rangle^{\to\emptyset}$$

$$+ \langle n|a^{\dagger 2}|\emptyset\rangle + \langle n|a^\dagger a|\emptyset\rangle^{\to\emptyset} + \underbrace{\langle n|a a^\dagger|\emptyset\rangle}$$

$\sqrt{1}\sqrt{n}$ $\qquad\qquad$ $\sqrt{1}\sqrt{n+1}$

$\delta_{n-1,0+1}$ $\qquad\qquad$ $\delta_{0+1,n+1}$

$\underbrace{n-1=0+1}$ $\qquad\qquad$ $0+1=n+1$

$\boxed{n=2}$ $\qquad\qquad$ $n=\emptyset$

D $$C_n^{(1)} = \sqrt{2}\left(\frac{-i}{\hbar}\right)\frac{\hbar}{2m\omega}\int_0^\infty e^{2i\omega t} e^{-i\Omega t} e^{-\Gamma t}\, dt$$

$$E_2 - E_0 = \hbar\omega\left(2+\tfrac{1}{2}\right) - \hbar\omega\left(\tfrac{1}{2}\right) = 2\hbar\omega$$

$$\int = \int_0^\infty \exp\left(-t\left(i\Omega + \Gamma - 2i\omega\right)\right) dt$$

$$= -\frac{\exp^{\cancel{max}\, -t(\sim)}}{i\Omega + \Gamma - 2i\omega} = \frac{1}{i\Omega + \Gamma - 2i\omega}$$

$$\rightarrow C_n^{(1)^2} = \frac{1}{2m^2\omega^2}\,\frac{1}{\underbrace{(i\Omega + \Gamma - 2i\omega)(-i\Omega + \Gamma + 2i\omega)}}$$

$$= \Omega^2 + \cancel{i\Omega\Gamma} - 2\omega\Omega\omega$$
$$- \cancel{\Gamma i\Omega} + \Gamma^2 + \cancel{2i\omega\Gamma}$$
$$- 2\omega\Omega - \cancel{2i\omega\Gamma} + 4\omega^2$$

$$= \Omega^2 - 4\Omega\omega + 4\omega^2 + \Gamma^2$$

$$\rightarrow \boxed{C_n^{(1)^2} = \frac{1}{2m^2\omega^2\left(\Omega^2 - 4\Omega\omega + 4\omega^2 + \Gamma^2\right)}}$$

F-2007

PROBLEM 3

A particle of mass m has a potential energy represented by two one-dimensional harmonic oscillator potentials centered at $\pm a$:

$$V(x) = \frac{1}{2}K(x-a)^2 + \frac{1}{2}K(x+a)^2$$

.

[a] (3 pts) What are the eigenvalues of the particle given this potential V? You may derive this result from first principles or deduce the result from the well known eigenvalues of a particle moving in a single harmonic-oscillator potential.

[b] (3 pts) The normalized ground-state eignefunction of the particle is given by

$$\phi(x) = \frac{1}{\pi^{1/4}\Delta^{1/2}} \exp\left(-\frac{x^2}{2\Delta^2}\right)$$

Use Schrodinger's equation to determine the constant Δ in terms of K, m, and fundamental constants.

[c] (4 pt) The potential well at $x = -a$ suddenly disappears, leaving the particle in a new potential

$$U(x) = \frac{1}{2}K(x-a)^2$$

Suppose that *before the sudden change,* the particle was in the ground state of the double-well potential $V(x)$. Derive an expression for the probability that after the sudden change the particle will be in the ground state of the single well potential $U(x)$. Express your answer in terms of a and Δ.

3 $V(x) = \frac{1}{2}k(x-a)^2 + \frac{1}{2}k(x+a)^2$

A $\frac{1}{2}kx^2 + \frac{1}{2}ka^2 - \cancel{kax} + \frac{1}{2}kx^2 + \frac{1}{2}ka^2 + \cancel{kax}$

$= kx^2 + ka^2 \quad \rightarrow \quad k = \frac{1}{2}m\omega^2$

$= \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2a^2$

$\rightarrow H = \underbrace{\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2}_{H_0} + \underbrace{\frac{1}{2}m\omega^2a^2}_{H_1}$

$\rightarrow H_0|n\rangle + H_1|n\rangle = E|n\rangle$

$\rightarrow \hbar\omega(n+\frac{1}{2})|n\rangle + \frac{1}{2}m\omega^2a^2|n\rangle = E|n\rangle$

$$\boxed{E_n = \hbar\omega(n+\tfrac{1}{2}) + \underbrace{\tfrac{1}{2}m\omega^2a^2}_{k}}$$

(A) Continued   explicitly

$$p = -i\sqrt{\frac{\hbar m\omega}{2}}\,(a - a^\dagger) \to p^2 = -\frac{\hbar m\omega}{2}(a - a^\dagger)^2$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger) \to x^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2$$

$$\to H = -\frac{\hbar \cancel{m}\omega}{4\cancel{m}}\left(a^2 + a^{\dagger 2} - aa^\dagger - a^\dagger a\right) + kx^2 + ka^2$$

$$= -\frac{\hbar\omega}{4}\left(\cancel{a^2} + \cancel{a^{\dagger 2}} - aa^\dagger - a^\dagger a\right) + \frac{1}{2}m\omega^2\left[\frac{\hbar}{2m\omega}\left(\cancel{a^2} + \cancel{a^{\dagger 2}} + aa^\dagger + a^\dagger a\right)\right] + ka^2$$

$$= \frac{\hbar\omega}{4}\left(aa^\dagger + a^\dagger a\right) + \frac{1}{4}\hbar\omega\left(aa^\dagger + a^\dagger a\right) + ka^2$$

$$= \frac{\hbar\omega}{2}\left(aa^\dagger + a^\dagger a\right) + ka^2$$

$$\to H|n\rangle = \frac{\hbar\omega}{2}\left[aa^\dagger|n\rangle + a^\dagger a|n\rangle\right] + ka^2|n\rangle$$

$a\sqrt{n+1}\,|n+1\rangle$ $a^\dagger\sqrt{n}\,|n-1\rangle$

$\sqrt{n+1}\sqrt{n+1}\,|n\rangle$ $\sqrt{n}\sqrt{n}\,|n\rangle$

$$\frac{\hbar\omega}{2}\left[n + 1 + n\right] + ka^2|n\rangle$$

$$= \boxed{\hbar\omega\left(n + \frac{1}{2}\right) + ka^2}$$

B $H|0\rangle = \left(\frac{\hbar\omega}{2} + \kappa a^2\right)|\phi\rangle$

$$\Rightarrow H = -\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + \frac{1}{2}m\omega^2 x^2\phi + \cancel{\kappa a^2\phi} = \frac{\hbar\omega}{2}\phi + \cancel{\kappa a^2\phi}$$

$$\Rightarrow H = -\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + \frac{1}{2}m\omega^2 x^2\phi = \frac{\hbar\omega}{2}\phi$$

$$\Rightarrow \frac{d^2\phi}{dx^2} = -\frac{2m}{\hbar^2 \cancel{2}}(\hbar\omega - m\omega^2)\phi$$

$$= -\frac{(\hbar\omega m - m^2\omega^2)}{\hbar^2}\phi$$

$$\Rightarrow \frac{d\phi}{dx} = -\frac{x}{\Delta^2}\frac{1}{\pi^{1/4}\Delta^{1/2}}\exp(\sim)$$

$$\frac{d^2\phi}{dx^2} = -\frac{1}{\Delta^2\pi^{1/4}\Delta^{1/2}}\exp(\sim) + \left(\frac{-x}{\Delta^2}\right)\left(\frac{-x}{\Delta^2}\right)\frac{1}{\pi^{1/4}\Delta^{1/2}}\exp\sim$$

$$\Rightarrow \cancel{-\frac{\hbar^2}{2m}\left(-\frac{1}{\Delta^{1/2}\pi^{1/4}}\right) - \frac{\hbar^2}{2m}\left(\frac{x^2}{\Delta^{1/2}\pi^{1/4}}\right) + \frac{1}{2}m\omega^2 x^2\left(\frac{1}{\pi^{1/4}\Delta}\right)}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(-\frac{1}{\Delta^2} + \frac{x^2}{\Delta^2}\right) + \frac{1}{2}m\omega^2 x^2 + \kappa a^2 = \frac{\hbar\omega}{2} + \kappa a^2$$

$$= \frac{\hbar^2}{2m\Delta^2} - \underbrace{\frac{\hbar x^2}{\Delta^2} + \underbrace{\frac{1}{2}m\omega^2}_{\kappa} x^2}_{} + \kappa a^2 = \frac{\hbar\omega}{2} + \kappa a^2$$

$\frac{\hbar\kappa}{2m}$ $\quad \kappa = \frac{1}{2}m\omega^2$ $\quad = \frac{\hbar\omega^2}{4}$

$$\frac{\hbar \cancel{x^2}}{\Delta^2} = \kappa \cancel{x^2} \Rightarrow \boxed{\Delta = \left(\frac{\hbar}{\kappa}\right)^{1/2}}$$

c $U(x) = \frac{1}{2}k(x-a)^2$

$y = x - a$

$dy = dx$

$$H = -\hbar^2 \frac{d^2\psi}{dy^2} + \frac{1}{2}ky^2$$

$$H = \hbar\omega(a^+a + \frac{1}{2})$$

$$y = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)$$

you can Solve For $\psi(y)$

then convert to x

it will be instead of

$e^{\frac{t^2}{\dots}}$

$\Downarrow$

$(x-a)^2$

$$\text{Probability} = |C_n|^2 = \left[\int_{-\infty}^{\infty} \psi(x)\,\psi(y)\,dx\right]^2$$

double single

F-2015

Problem 2: Confined Harmonic Oscillator

Consider a particle of mass m confined in the potential

$$\begin{aligned} V(\vec{r}) &= \frac{m}{2}\omega^2\left(x^2+y^2\right) + V_z(z) \\ V_z(z) &= 0, \quad 0 \le z \le a, \qquad V_z(z) = \infty, \qquad z<0,\ z>a \end{aligned} \tag{1}$$

(a) [2 pts] Show that the energy eigenstates for this potential can be separated into a product of three functions, each depending on a single coordinate: $X(x)$, $Y(y)$, and $Z(z)$. Using this product, determine the energy eigenvalues for the Hamiltonian, and the general form for the corresponding eigenstates. Show your work, although you don't need to solve the three 1D problems giving all the details.

(b) [1 pt] Define the energy:

$$E_a = \frac{\pi^2\hbar^2}{2ma^2} \tag{2}$$

What are the first four energy eigenvalues and their degeneracies for this potential in the case that $E_a = \frac{1}{2}\hbar\omega$? Give your answer in terms of the parameters in the problem.

(c) [3 pts] Using standard cylindrical polar coordinates, ρ, ϕ, and z, where $x = \rho\cos(\phi)$ and $y = \rho\sin(\phi)$, show that the eigenstates of this potential can also be written as a product of three functions, $R(\rho)$, $F(\phi)$, and $Z(z)$. Hint: Consider the ϕ dependence of the system.

(d) [2 pts] Show that the energy eigenstates of this Hamiltonian can be also be eigenstates of the z-component of the angular momentum, $L_z = -i\hbar\frac{\partial}{\partial\phi}$.

What is the angular dependence, $F(\phi)$, for the simultaneous eigenstates of H and L_z?

(e) [2 pts] The ground state you found in part (b) is an eigenstate of L_z, but the first excited states are not eigenstates of L_z. Write down two eigenstates of L_z from linear combinations of the first excited states from part (b).

What possible values of L_z can be measured for a particle in the ground state?

What possible values of L_z can be measured for a particle in the first excited states?

F-2015

(2) $V(\vec{r}) = \frac{m}{2}\omega^2(x^2+y^2) + V_z(z)$

$V_z(z) = \phi, \quad 0 \le z \le a$ & ∞, For $z<0,\ z>a$

$H = \frac{p^2}{2m} + V \quad \rightarrow \quad p = -i\hbar\nabla$

$p^2 = -\hbar^2\nabla^2$

$\rightarrow H = -\hbar^2\left[\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right] + \frac{m}{2}\omega^2x^2 + \frac{m}{2}\omega^2y^2 + V_z$

$\rightarrow \psi = XYZ$

& $E = E_x + E_y + E_z$

$\rightarrow \frac{1}{\psi}\left[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right]\psi + \frac{m}{2}\omega^2x^2\frac{\psi}{\psi} + \frac{m\omega^2}{2}y^2\frac{\psi}{\psi} = E\frac{\psi}{\psi}$

$\rightarrow \frac{1}{X}\frac{-\hbar^2\partial^2 X}{2m\,\partial x^2} + \frac{m}{2}\omega^2x^2 = E_x$ & $\underbrace{\frac{1}{Z}\frac{-\hbar^2}{2m}\frac{\partial^2 Z}{\partial z^2} = E_z}$

$\frac{\partial^2 Z}{\partial z^2} = \frac{-2mE_z}{\hbar^2}Z$

$\rightarrow \frac{1}{Y}\frac{-\hbar^2}{2m}\frac{\partial^2 Y}{\partial y^2} + \frac{m}{2}\omega^2y^2 = E_y$

$E_x = \hbar\omega(n_x + \tfrac{1}{2})$

$E_y = \hbar\omega(n_y + \tfrac{1}{2})$

X & Y have Forms of $A\,h_n(\beta x)\,e^{-\beta^2x^2/2}$

Front page

$\hookrightarrow Z = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$

$E_z = \frac{n_z^2\pi^2\hbar^2}{2ma^2}$

A Continued

$$\boxed{E = \hbar\omega(n_x + n_y) + \frac{n_z^2 \pi^2 \hbar^2}{2ma^2}}$$

B $E_a = \frac{\pi^2 \hbar^2}{2ma^2} \rightarrow n_z^2 E_a = \frac{n_z^2 \hbar\omega}{2}$

$$\rightarrow E = \hbar\omega\left(n_x + n_y + \frac{n_z^2}{2}\right)$$

n_x & n_y can be ϕ, n_z cannot be ϕ

$E_{001} = \frac{\hbar\omega}{2}$ — 1st, Deg = 1

$E_{101} = \hbar\omega(1 + 0 + \frac{1}{2}) = \frac{3\hbar\omega}{2}$ — 2nd, Deg = 2

$E_{011} = \hbar\omega(0 + 1 + \frac{1}{2}) = \frac{3}{2}\hbar\omega$

$E_{111} = \hbar\omega(1 + 1 + \frac{1}{2}) = \hbar\omega(2 + \frac{1}{2}) = \frac{5}{2}\hbar\omega$ — 3rd, Deg = 1

Should have Tried 002

$E_{211} = \hbar\omega(2 + 1 + \frac{1}{2}) = \hbar\omega(3 + \frac{1}{2}) = \frac{7}{2}\hbar\omega$ — 4th, Deg = 2

$E_{121} = \frac{7}{2}\hbar\omega$

$E_{221} = \hbar\omega(4 + \frac{1}{2}) = \frac{9}{2}\hbar\omega$

$E_{222} = \hbar\omega(4 + \frac{4}{2}) = 6\hbar\omega$

(c) $R(\rho)$, $F(\phi)$, $Z(z)$

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$

$$\rightarrow p^2 = -\hbar^2\nabla^2 = -\hbar^2\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z}\right]$$

$$V = \frac{m\omega^2}{2}(\rho^2\cos^2\phi + \rho^2\sin^2\phi) = \frac{m\omega^2}{2}\rho^2$$

$$\rightarrow \frac{-\hbar^2}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z}\right]\psi + \frac{m\omega^2}{2}\rho^2\psi = E\psi$$

$$\rightarrow \psi = RFZ$$

→ substitute & divide by $\frac{1}{\psi}$

$$\rightarrow \frac{-\hbar^2}{2m}\left[\frac{1}{R}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{F\rho^2}\frac{\partial^2 F}{\partial\phi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right] + \frac{m\omega^2\rho^2}{2} = E$$

$$-\frac{2mE_z}{\hbar^2} \rightarrow \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} = Z(z)$$

$$\rightarrow \frac{-\hbar^2}{2m}\left[\frac{1}{R}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho\partial R}{\partial\rho}\right) + \frac{1}{F\rho^2}\frac{\partial^2 F}{\partial\phi^2}\right] + E_z + \frac{m\omega^2\rho^2}{2} = E$$

$$\rightarrow \frac{-\hbar^2}{2m}\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + (E_z - E)\rho^2 + \frac{m\omega^2\rho^4}{2}$$

$$-\frac{\hbar^2}{2mF}\frac{\partial^2 F}{\partial\phi^2} = E_\phi$$

$$\hookrightarrow F = Ae^{ik\phi}$$

$$k^2 = \frac{2mE_\phi}{\hbar^2}$$

$$\rightarrow \underbrace{\frac{-\hbar^2}{2m}\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\frac{\rho\partial R}{\partial\rho}\right) + (E_z - E)\rho^2 + \frac{m\omega^2}{2}\rho^4}_{\text{only depends on } R} + E_\phi = \emptyset$$

(D) $[H, L_z] = \phi$

$$(HL_z - L_zH)\psi = \phi$$

$$\rightarrow HL_z\psi = H\left(-i\hbar \frac{\partial}{\partial\phi} RZAe^{ik\phi}\right) = +H\hbar k RZF$$

$$= \hbar k E\psi$$

$$\rightarrow L_zH\psi = -i\hbar\frac{\partial}{\partial\phi} ERZAe^{ik\phi} = k\hbar ERZF = k\hbar E\psi$$

The same

(E) ~~I think use $2A^2(A_x + A_y + A_z^2)$~~

I'll have to think about this

S-2012

PROBLEM 3: Harmonic Oscillator

A particle of mass m is under the influence of the following potential

$$V(x) = V_0\sqrt{A^2 + x^2}$$

where V_0 and A are constants. For small displacements $x \ll A$ this potential can be approximated by a simple harmonic oscillator.

(a) Determine the lowest energy this particle can have in terms of $\hbar$, m, V_0 and a for $x \ll A$. (2 Points)

Now consider the Hamiltonian describing the true one-dimensional harmonic oscillator

$$\mathbf{H} = \frac{\mathbf{P}^2}{2m} + \frac{1}{2}k\mathbf{X}^2$$

with eigenstates

$$\mathbf{H}|n\rangle = E_n|n\rangle \quad n = 0, 1, 2, \cdots .$$

(b) Using commutation relations, calculate the equations of motion for $\mathbf{P}$ and $\mathbf{X}$ in the Heisenberg picture. (Find $\dot{X}$ and $\dot{P}$.) (2 Points)

(c) Solve for $P(t)$ and $X(t)$ in terms of $P(0)$ and $X(0)$ and show that $[X(t), X(0)] \neq 0$ for $t \neq 0$. (2 Points)

A harmonic oscillator system is known to be in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle)$$

where $|0\rangle$ and $|3\rangle$ are the normalized ground state and the third excited state of the harmonic oscillator respectively.

(d) What is the value of $n > 0$ for the first non-zero value of $\langle X^n\rangle$ with the state vector $|psi\rangle$? (2 Points)

(e) What is the expectation value $\langle X^3\rangle$ with the state vector $|\psi\rangle$? (2 Points)

S-2012

3) $V(x) = V_0\sqrt{A^2+x^2}$

A) $x \ll A \rightarrow V(x) = V_0 A\left(1+\frac{x^2}{A^2}\right)^{1/2}$

$$= V_0 A\left(1+\frac{1}{2}\frac{x^2}{A^2}\right)$$

$$= V_0 A + \frac{1}{2}\frac{x^2}{A}$$

$$\rightarrow \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}\frac{x^2\psi}{A} + V_0 A\psi = E\psi$$

$$\rightarrow A = \frac{1}{m\omega^2}$$

$$\rightarrow \underbrace{\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi}_{\hbar\omega(n+\frac{1}{2})} + \underbrace{\frac{V_0\psi}{m\omega^2}}_{E_n - A_0V_0} = E_n\psi$$

$$\boxed{E_n = \frac{\hbar\omega}{2} + AV_0}$$

↑ For ground state

B

Any operator, $A \rightarrow \dot{A} = \frac{i}{\hbar}[H, A]$

$$\rightarrow \dot{x} = \frac{i}{\hbar}[H,x] = \frac{i}{\hbar}\left[\left[\frac{p^2}{2m}, x\right] + \cancel{\left[\frac{1}{2}kx^2, x\right]}^{\rightarrow 0}\right]$$

$$= \frac{i}{2m\hbar}[pp, x] = \frac{i}{2m\hbar}\Big[p\underbrace{[p,x]}_{-i\hbar} + \underbrace{[p,x]}_{-i\hbar}p\Big]$$

$$= \frac{i}{\cancel{2m\hbar}}(-(\cancel{2}i\cancel{\hbar}p)) = \boxed{\frac{p}{m} = \dot{x}}$$

Note

$$\dot{A} = \frac{1}{i\hbar}[H, A]$$

$$\rightarrow \dot{p} = \frac{i}{\hbar}[H,p] = \frac{i}{\hbar}\left[\cancel{\left[\frac{p^2}{2m}, p\right]}^{\rightarrow 0} + \left[\frac{1}{2}kx^2, p\right]\right]$$

$$= \frac{ik}{2\hbar}\Big[x\underbrace{[x,p]}_{i\hbar} + \underbrace{[x,p]}_{i\hbar}x\Big]$$

$$= \boxed{-kx = \dot{p}}$$

$$F = \frac{dp}{dt} = -\frac{\partial V}{\partial x} = -kx \checkmark$$

Answers are correct but your commutation relations are wrong

C $\frac{dx}{dt} = \frac{p(0)}{m} \rightarrow x(t) - x(0) = \frac{p(0)}{m}t$

$\rightarrow \boxed{x(t) = x(0) + \frac{p(0)t}{m}}$

$\frac{dp}{dt} = -kx(\phi) \rightarrow \boxed{p(t) = p(\phi) - kx(\phi)t}$

$\rightarrow [x(t), x(\phi)] = \cancel{[x(\phi), x(\phi)]}^{\phi} + [x(\phi), \frac{p(\phi)t}{m}]$

$= \boxed{\frac{t}{m} i\hbar = [x(t), x(\phi)]}$

D $|\psi\rangle = \frac{1}{\sqrt{2}}(|\phi\rangle + |3\rangle)$

$x^n = (a+a^\dagger)^N \left(\sqrt{\frac{\hbar}{2m\omega}}\right)^N$

$\hookrightarrow a^n + na^{n-1}a^\dagger + \frac{n(n-1)}{2!}a^{n-2}a^{\dagger 2}$

$a^n + a^{n-1}a^\dagger + a^{n-2}a^2 + \cdots a^1 a^{n-1} + a^{\dagger n}$

$\langle 0 | 0\rangle$ $\quad n=1 \rightarrow a + a^\dagger$

$\langle 0 | 3\rangle$

$\langle 3 | 0\rangle$

$\langle 3 | 3\rangle$

$\boxed{n = 2}$

$\langle 0 | a a^\dagger | 0\rangle \neq \phi$

E $\langle \psi | x^3 | \psi \rangle$

$\hookrightarrow x^3 = a^3 + a^2 a^\dagger + a a^{\dagger 2} + a^{\dagger 3}$

$$\rightarrow a^3 = \underbrace{\langle 0 | a^3 | 0 \rangle}_{\emptyset} + \underbrace{\langle 3 | a^3 | 3 \rangle}_{\emptyset} + \underbrace{\langle 3 | a^3 | 0 \rangle}_{\emptyset} + \underbrace{\langle 0 | a^3 | 3 \rangle}_{\sqrt{3}\langle 0 | a^2 | 2 \rangle}$$

$$\sqrt{3}\sqrt{2}\,\langle 0 | a | 1 \rangle$$

$$= \sqrt{6}$$

$$\rightarrow a^{\dagger 3} = \cancel{\emptyset}\ \langle 0 | \cancel{a^{\dagger 3}} | 0 \rangle \rightarrow \emptyset + \langle 3 | \cancel{a^{\dagger 3}} | 3 \rangle \rightarrow \emptyset$$

$$\rightarrow a^2 a^\dagger \rightarrow \langle 0 | \cancel{a^2 a^\dagger} | 0 \rangle \rightarrow \emptyset + \langle 3 | \cancel{a^2 a^\dagger} | 3 \rangle \rightarrow \emptyset + \langle 3 | \cancel{a^2 a^\dagger} | 0 \rangle \rightarrow \emptyset$$

$$\langle 2| \quad |1\rangle \qquad \langle 5| \quad |4\rangle \qquad \langle 5| \quad |1\rangle$$

$$+ \langle 0 | \cancel{a^2 a^\dagger} | 3 \rangle \rightarrow \emptyset$$

$$\langle 2| \quad |4\rangle$$

$$\rightarrow \cancel{\emptyset}\ a a^{\dagger 2} = \emptyset$$

$$a^{\dagger 3} \text{ con'd} = \underbrace{\langle 3 | a^{\dagger 3} | 0 \rangle}_{\langle 3 | a^{\dagger 2} | 1 \rangle} + \langle 0 | \cancel{a^{\dagger 3}} | 3 \rangle \rightarrow \emptyset$$

$$\sqrt{2}\ \underbrace{\langle 3 | a^\dagger | 2 \rangle}_{\langle 3 | 3 \rangle}$$

$$\sqrt{2}\sqrt{3}$$

$$= 2\sqrt{6}$$

$$\rightarrow \left(\frac{\hbar}{2m\omega}\right)^{3/2} \frac{1}{2}\, 2\sqrt{6} \rightarrow \boxed{\langle \psi | x^3 | \psi \rangle = \sqrt{6} \cdot \left(\frac{\hbar}{2m\omega}\right)^{3/2}}$$

S-2015

Problem 1: Solving the Harmonic Oscillator

Solving the differential equation form of the time-independent Schrödinger equation for the eigenstates of the harmonic oscillator Hamiltonian in 1D requires solving a second order differential equation. By using operator algebra, it is possible to simplify the solution to this problem.

The 1D harmonic oscillator is described by the Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m}{2}\omega^2 X^2. \tag{1}$$

Define the unitless variables

$$x = \frac{X}{\lambda}, \quad p = \frac{\lambda}{\hbar}P, \quad \lambda = \sqrt{\frac{\hbar}{m\omega}}. \tag{2}$$

such that the Hamiltonian has the form

$$H = \frac{\hbar\omega}{2}\left(p^2 + x^2\right). \tag{3}$$

Note that x and p are conjugate observables, $[x, p] = i$

(a) [2 pt] Using the harmonic oscillator operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(x + ip\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(x - ip\right), \quad \hat{n} = \hat{a}^\dagger \hat{a}, \tag{4}$$

and their commutation relations, show that the Hamiltonian can be written as

$$H = \hbar\omega(\hat{n} + \frac{1}{2}). \tag{5}$$

(b) [2 pts] Define the eigenstates of the operator $\hat{n}$:

$$\hat{n}|n\rangle = n|n\rangle, \tag{6}$$

with n some (unitless) numbers. Use the operator commutation relations to show that

$$\begin{aligned} \hat{a}|n\rangle &= c(n)|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= d(n)|n+1\rangle. \end{aligned} \tag{7}$$

Derive expressions for $c(n)$ and $d(n)$. Show your work.

(c) [3 pts] The potential, $V(x) = \frac{\hbar\omega}{2}x^2 \geq 0$ for all x. Explain why this implies that:

1. The eigenenergies of the Harmonic Oscillator must be positive
2. The eigenvalues of $\hat{n}$ must be non-negative integers
3. There is a lowest eigenstate of $\hat{n}$, $|0\rangle$ defined by $\hat{a}|0\rangle = 0$.

(d) [2 pts] Show that results above define a first order differential equation in X that can be solved for the ground state harmonic oscillator wavefunction $\psi_0(X)$. Determine this equation and solve for this wavefunction.

(e) [1 pt] Use the result from (e) and the operators to determine the first excited state wavefunction for the harmonic oscillator, $\psi_1(X)$.

~~[illegible]~~ 5-2015

1 A

$$\hat{n} = \hat{a}^{+}\hat{a} = \frac{1}{2}\left[(x-ip)(x+ip)\right]$$

$$= \frac{1}{2}\left[x^2+p^2+\underbrace{ixp-ipx}_{i[x,p]\,=\,-1}\right]$$

$$= \frac{1}{2}x^2+\frac{1}{2}p^2-\frac{1}{2}$$

$$\to \quad \hat{n}+\frac{1}{2} = \frac{1}{2}x^2+\frac{1}{2}p^2 = \frac{1}{2}\left[\frac{x^2}{\lambda^2}+\frac{\lambda^2}{\hbar^2}p^2\right]$$

$$= \frac{1}{2}\left[\frac{m\omega}{\hbar}x^2+\frac{1}{\hbar m\omega}p^2\right]$$

$$\to \quad \boxed{\hbar\omega\left(\hat{n}+\frac{1}{2}\right) = \frac{p^2}{2m}+\frac{m}{2}\omega^2x^2 = H}$$

B $\hat{a}|n\rangle$

$$[\hat{n},\hat{a}] = \hat{n}\hat{a} - \hat{a}\hat{n} \rightarrow \underbrace{\hat{n}[\hat{a}|n\rangle]}_{=([\hat{n},\hat{a}]+\hat{a}\hat{n})|n\rangle} = c(n)|n-1\rangle$$

$$\rightarrow [\hat{n},\hat{a}] = [a^{+}a,\hat{a}] = a^{+}\underbrace{[\hat{a},\hat{a}]}_{=\phi} + [a^{+},a]\,a$$

$$\rightarrow [a^{+},a] = \frac{1}{2}[x-ip,\, x+ip] = \frac{1}{2}\Big(\cancel{[x,x]}^{\phi} + i\underbrace{[x,p]}_{i} - i\underbrace{[p,x]}_{-i} - i^{2}\underbrace{[p,p]}_{\phi}\Big)$$

$$= \frac{1}{2}(-1-1) = -1$$

$$\rightarrow [\hat{n},\hat{a}] = -a$$

$$\rightarrow (-a+an)|n\rangle \boxed{\begin{aligned}&= (n-1)[a|n\rangle] \\ &= \hat{n}[a|n\rangle]\end{aligned}}$$

$$\rightarrow \langle n|a^{+}a|n\rangle = \begin{cases} c^{2}\langle n-1|n-1\rangle \\ n\langle n|n\rangle \end{cases}$$

$$\rightarrow n = c^{2} \rightarrow \boxed{c(n) = \sqrt{n}}$$

$\boxed{a^{+} \text{ Follows}}$

C 2 $\hat{n}|n\rangle = n|n\rangle$

$$\langle n|a^{\dagger}a|n\rangle = \underbrace{\left[\langle n|a^{\dagger}\right]\left[a|n\rangle\right]}_{\text{inner product}}$$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$ → inner products are $\geq \emptyset$ (positive definite)

$a|n-1\rangle = \sqrt{n}\sqrt{n-1}\,|n-2\rangle$

$a|n-2\rangle = \sqrt{n}\sqrt{n-1}\sqrt{n-2}\,|n-2\rangle$

The series must terminate to avoid $\sqrt{n-k}$ → $n-k < \emptyset$ → $\boxed{n \text{ is integer}}$

$$\boxed{\langle n|a^{\dagger}a|n\rangle = \sqrt{n}\sqrt{n}\,\underbrace{\langle n-1|n-1\rangle}_{1} \geq \emptyset \\ = n \geq \emptyset}$$

1 Since $n \geq \emptyset$, The Hamiltonian For $H = \frac{p^2}{2m} + \frac{1}{2}\hbar\omega x^2$

$$= \frac{\hbar\omega}{2}(p^2 + x^2)$$

whose derivation to

$$H = \hbar\omega(\hat{n} + \tfrac{1}{2})$$

was made using this form

($x = \frac{iX}{?}$ would be correct way to derive a negative potential)

→ $\boxed{H|n\rangle = \underbrace{\hbar\omega(n+\tfrac{1}{2})}|n\rangle}$

positive since n is positive

assuming follows from this

IF $-\frac{1}{2}$, eigen energy could be $-$ for $n = \emptyset$

C Continued

Since $a|n\rangle = \sqrt{n}\, \cancel{}\, |n-1\rangle$

$a|n-1\rangle = \sqrt{n}\sqrt{n-1}\, \cancel{}\, |n-2\rangle$

$\mapsto$ ~~than~~ we proved that

n is an integer $\geq \emptyset$,

therefore $a|1\rangle = \sqrt{1}\, \underbrace{|1-1\rangle}_{|0\rangle}$

$a|\emptyset\rangle = \sqrt{n}\,|0-1\rangle$

$\uparrow$

$n=\emptyset \rightarrow \boxed{\emptyset}$

D $a|\emptyset\rangle = \emptyset$

$\rightarrow \frac{1}{\sqrt{2}}(x+ip)|\emptyset\rangle = \emptyset \rightarrow \langle x|\frac{1}{\sqrt{2}}(x+ip)|\emptyset\rangle = \emptyset$

$\rightarrow \frac{1}{\sqrt{2}}[x\,\psi_0 + ip\, \psi_0] = \emptyset$

where $\langle x|\emptyset\rangle = \psi_0$

$\rightarrow x\psi_0 + i\left(-i\hbar\frac{\partial}{\partial x}\psi_0\right) = \emptyset$

$x\psi_0 + \hbar\frac{\partial}{\partial x}\psi_0 = \emptyset$

$$\boxed{\psi_0 = A\exp\left(\frac{-\hbar}{2}x^2\right)}$$

only thing to notice is that it wants it in terms of $\underline{X}$ not x

(5)

(E) $\frac{x}{\lambda}\psi_0 + \frac{\lambda\hbar}{\hbar}\frac{\partial}{\partial x}\psi_0 = \frac{x}{\lambda}\psi_0 + \lambda\frac{\partial}{\partial x}\psi_0 = \emptyset$

$= x\psi_0 + \lambda^2\frac{\partial}{\partial x}\psi_0 = \emptyset$

$\Rightarrow \boxed{\psi_0 = A\exp\left(-\frac{x^2}{2\lambda^2}\right)}$

For D

$a^{+}\psi_0 = \sqrt{1}\,\psi_1$

$= \frac{1}{\sqrt{2}}\left[\frac{x}{\lambda} - i\left(-i\hbar\frac{\partial}{\partial x}\right)\right]\psi_0 = \psi_1$

$\times\frac{\lambda}{\hbar}$

$= \frac{1}{\sqrt{2}}\left[x - \lambda^2\frac{\partial}{\partial x}\right]\psi_0 = \psi_1$

$= \frac{1}{\sqrt{2}}\left[x\psi_0 - \cancel{\lambda^2}\left(-\frac{2x}{2\cancel{\lambda^2}}\right)\right]\psi_0 = \psi_1$

$= \frac{1}{\sqrt{2}}\left[x\psi_0 + x\psi_0\right] = \psi_1$

$\Rightarrow \boxed{\psi_1 = \frac{2}{\sqrt{2}}xAe^{-x^2/2\lambda^2}}$

S-2013

Problem 4: Operator Solutions to the Harmonic Oscillator

Consider the Harmonic Oscillator Hamiltonian in one dimension:

$$H_{ho} = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2 \tag{1}$$

To simplify this problem, define the new observables:

$$p = \sqrt{\frac{1}{m\hbar\omega}}P\ , \quad q = \sqrt{\frac{m\omega}{\hbar}}X \tag{2}$$

This gives the dimensionless Hamiltonian,

$$H = \frac{1}{\hbar\omega}H_{ho} = \frac{1}{2}\left(p^2 + q^2\right) \tag{3}$$

(a) [1 pt] Calculate the commutation relation for these new variables, $[q, p]$. Be sure to show your work.

(b) [1 pt] Define the non-Hermitian operators $a = \frac{1}{\sqrt{2}}(q + ip)\ ,\quad a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$ and the Hermitian operator $n = a^\dagger a$. Compute $[a, a^\dagger]$, $[n, a^\dagger]$, and $[n, a]$

(c) [1 pt] Write the dimensionless Hamiltonian H in terms of a and $a^\dagger$. Write the dimensionless Hamiltonian H in terms of n.

(d) [3 pts] Define the eigenvalues and eigenvectors of n as:

$$n|\lambda\rangle = \lambda|\lambda\rangle. \tag{4}$$

and assume that these eigenvectors form a complete set.

Show that

$$\begin{aligned} a^\dagger|\lambda\rangle &= A|\lambda + 1\rangle \\ a|\lambda\rangle &= B|\lambda - 1\rangle \end{aligned} \tag{5}$$

Determine the normalization constants A and B.

(e) [2 pts.] Show that $n = a^\dagger a$ must have non-negative eigenvalues, $\lambda \geq 0$. Explain why this implies that there must be a state where $a|0\rangle = 0$ and that the eigenvalues of n must be non-negative integers.

(f) [2 pts.] Write the definition for the state $|0\rangle$

$$a|0\rangle = 0 \tag{6}$$

as a differential equation, in q, for the ground state wavefunction of H. Solve this expression for the normalized ground state wavefunction.

5-2013

4 A $\cancel{[q,p] = \left[\sqrt{\frac{1}{m\hbar\omega}}\,p, \sqrt{\frac{m\omega}{\hbar}}\,x\right] = \sqrt{\frac{m\omega}{m\omega\hbar^2}}\,(px - xp)}$

$$[q,p] = \left[\sqrt{\frac{m\omega}{\hbar}}\,x, \sqrt{\frac{1}{m\hbar\omega}}\,p\right]$$

$$= \frac{1}{\hbar}\underbrace{[x,p]}_{i\hbar}$$

$$\rightarrow \boxed{[q,p] = i}$$

B $$[a,a^\dagger] = \frac{1}{2}[q+ip, q-ip] = \frac{1}{2}\Big[\underbrace{[q,q]}_{\emptyset} - i\underbrace{[q,p]}_{i} + i\underbrace{[p,q]}_{-i} - i^2\underbrace{[p,p]}_{\emptyset}\Big]$$

$$= \frac{1}{2}[1+1] \rightarrow \boxed{[a,a^\dagger] = 1}$$

$$[n,a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger[a,a^\dagger] + \underbrace{[a^\dagger,a^\dagger]}_{\emptyset}a$$

$$= \boxed{a^\dagger = [n,a^\dagger]}$$

B Continued

$$[n, a] = [a^\dagger a, a] = a^\dagger \cancel{[a, a]}^{\to 0} + [a^\dagger, a] a$$

$$= \underbrace{[a^\dagger, a]}_{-1} a \rightarrow \boxed{[n, a] = -a}$$

C $a^\dagger a = \frac{1}{2}\left[(q - ip)(q + ip) \right]$

$$= \frac{1}{2}\left[q^2 + \underbrace{\underbrace{qip - ipq}_{i[qp - pq]}}_{i} + p^2 \right]$$

$$= \frac{1}{2}\left[q^2 + p^2 - 1 \right] \rightarrow \boxed{\frac{1}{2}(p^2 + q^2) = a^\dagger a + \frac{1}{2} = H}$$

$$\boxed{a^\dagger a = n \rightarrow H = n + \frac{1}{2}}$$

D $n|\lambda\rangle = \lambda|\lambda\rangle$

$\rightarrow [a^\dagger|\lambda\rangle] \rightarrow [n, a^\dagger] = a^\dagger = na^\dagger - a^\dagger n$

$\rightarrow na^\dagger|\lambda\rangle = (a^\dagger + a^\dagger n)|\lambda\rangle = a^\dagger(1 + n)|\lambda\rangle = a^\dagger(1 + \lambda)|\lambda\rangle$

$\rightarrow (1 + \lambda)[a^\dagger|\lambda\rangle] \rightarrow$ implies n on a vector $[a^\dagger|\lambda\rangle]$

give $\lambda + 1$

$\rightarrow$

1 Continued

$$\langle \lambda | a a^{\dagger} | \lambda \rangle = A^2 \underbrace{\langle \lambda+1 | \lambda+1 \rangle}_{1}$$

$$[a, a^{\dagger}] = a a^{\dagger} - \underbrace{a^{\dagger} a}_{n} = 1 \rightarrow a a^{\dagger} = 1 + n$$

$$\rightarrow \langle \lambda | 1 + n | \lambda \rangle = 1 + \lambda$$

$$\rightarrow \boxed{A = \sqrt{1+\lambda}}$$

$$[a|\lambda\rangle] \rightarrow [n, a] = -a = na - an$$

$$\rightarrow n a|\lambda\rangle = an - a|\lambda\rangle = a(n-1|\lambda\rangle = \underbrace{(n-1)}[a|\lambda\rangle]$$

implies $n[a|\lambda\rangle]$ returns $n-1$

$$\langle \lambda | a^{\dagger} a | \lambda \rangle = \langle \lambda | n | \lambda \rangle = \lambda \underbrace{\langle \lambda | \lambda \rangle}_{1}$$

$$= \langle \lambda-1 | B^2 | \lambda-1 \rangle = B^2$$

$$\rightarrow \boxed{B = \sqrt{\lambda}}$$

E $\langle \lambda | a^{+} a | \lambda \rangle$

$$= \left[\langle \lambda | a^{+}\right]\left[a|\lambda\rangle\right]$$

inner product.

By the postulates of quantum mechanics, inner products must be $\geq \phi$ (positive definite)

$$\boxed{\langle \lambda | a^{+} a | \lambda \rangle = \sqrt{\lambda}\sqrt{\lambda}\underbrace{\langle \lambda | \lambda \rangle}_{1} \geq \phi = \lambda \geq 0}$$

$a|\lambda\rangle = \sqrt{\lambda}\,|\lambda - 1\rangle$

$a^{2}|\lambda\rangle = \sqrt{\lambda}\, a|\lambda-1\rangle = \sqrt{\lambda(\lambda-1)}\,|\lambda-2\rangle$

$a^{3}|\lambda\rangle = \sqrt{\lambda(\lambda-1)(\lambda-2)}\,|\lambda-3\rangle$

~~If This~~

If this continues to $a^{k+1}|\lambda\rangle$

$$= \sqrt{\ldots(\lambda-k)}\,|\lambda-k\rangle$$

If $\lambda - k < \phi$, Then A has an imaginary value

with violates $\langle \lambda | a^{+} a | \lambda \rangle \geq \phi$,

Therefore, there must be $\lambda - k = \phi$ in order to terminate the series.

F $a|\phi\rangle = \phi$

$$\frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,x + i\sqrt{\frac{1}{m\hbar\omega}}\,p\right]\underbrace{\langle x|\phi\rangle}_{\psi_0(x)} = \phi$$

$$\rightarrow \sqrt{\frac{m\omega}{\hbar}} = x_0$$

$$\rightarrow \sqrt{\frac{1}{m\hbar\omega}} = p_0$$

$$\rightarrow \frac{1}{\sqrt{2}}\left[x_0 x + i p_0 p\right]\psi_0(x) = \phi$$

$$\left[x + \frac{p_0 i}{x_0}\left(-i\hbar\frac{\partial}{\partial x}\right)\right]\psi_0(x) = \phi$$

$$= x\,\psi_0(x) + \frac{p_0\hbar}{x_0}\frac{\partial}{\partial x}\psi_0(x) = \phi$$

$$\psi_0(x) = A\exp\left(-\frac{x_0 x^2}{2p_0\hbar}\right) \qquad \text{Call} \quad \frac{x_0}{2p_0\hbar} = C$$

$$\rightarrow = A\exp(-Cx^2)$$

$$\rightarrow 1 = A^2\int_{-\infty}^{\infty}\exp(-2Cx^2)\,dx = A^2\sqrt{\frac{\pi}{2C}} = 1$$

$$\rightarrow A = \left(\frac{2C}{\pi}\right)^{1/4}$$

$$\rightarrow \frac{x_0}{p_0} = m\omega \qquad = \left(\frac{x_0}{p_0\pi\hbar}\right)^{1/4}$$

$$\rightarrow \boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)} \qquad = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

F-2008

Problem 3: The Harmonic Oscillator(10 Points)

A one dimensional harmonic oscillator has a potential given by

$$V(x) = m\omega^2 x^2/2.$$

where ω is the oscillator frequency and m is its mass. Derive all results.

a. Write the Schrodinger equation for a single particle in a one dimensional harmonic oscillator potential. **(1 Point)**

b. Consider the raising and lowering operators

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\hbar\omega}}$$

and

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\hbar\omega}},$$

respectively, where p is the momentum operator. If Ψ_E is an eigenvector of the Hamiltonian with energy eigenvalue E, find the energy eigenvalues of $a^\dagger\Psi_E$ and $a\Psi_E$. (You may need to use the fact that $[x, p] = i\hbar$). **(2 Points)**

c. Using the raising and lowering operators find the energy eigenvalues for a single particle in a one dimensional harmonic oscillator potential. **(2 Points)**

d. Find the normalized ground state wave function. **(2 Points)**

e. The harmonic oscillator models a particle attached to an ideal spring. If the spring can only be stretched, and not compressed, so that $V = \infty$ for $x < 0$, what will be the energy levels of this system? **(3 Points)**

F-2008

3 $V(x) = \frac{m\omega^2 x^2}{2}$

A $\left[\frac{p^2}{2m} + V(x)\right]\psi(x) = E\,\psi(x)$

$p = i\hbar \frac{d}{dx} \to -\hbar^2 \frac{d^2}{dx^2}$

$\to \boxed{\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2}\right]\psi(x) = E\psi(x)}$

B $H|\psi_E\rangle = E|\psi_E\rangle$

$\to [H, a^\dagger] = Ha^\dagger - a^\dagger H$

$= [a^\dagger a\hbar\omega, a^\dagger] = a^\dagger\hbar\omega$

$\to H a^\dagger|\psi_E\rangle = [a^\dagger H + a^\dagger\hbar\omega]|\psi_E\rangle$

$= E + \hbar\omega\,[a^\dagger|\psi_E\rangle]$

$\to \boxed{\text{eigenvalue} = E + \hbar\omega}$

$\to [H, a] = Ha - aH$

$= [a^\dagger a\hbar\omega, a]$

$= a\hbar\omega\underbrace{[a^\dagger, a]}_{-1}$

$= -a\hbar\omega$

$\to H a|\psi_E\rangle = [aH - a\hbar\omega]|\psi_E\rangle$

$= E - \hbar\omega\,[a|\psi_E\rangle]$

$\to \boxed{\text{eigenvalue} = E - \hbar\omega}$

C $a^\dagger a = \frac{m\omega}{2\hbar}x^2 + \frac{1}{2m\hbar\omega}p^2 + \underbrace{\frac{i}{2\hbar}\underbrace{[x, p]}_{i\hbar}}_{-\frac{1}{2}}$

$\to \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega = \frac{m\omega^2 x^2}{2} + \frac{p^2}{2m}$

$= H$

$\to \boxed{E = \left(n + \frac{1}{2}\right)\hbar\omega}$

D $a|\emptyset\rangle = \emptyset$

$\rightarrow \langle x|a|\emptyset\rangle = \emptyset \rightarrow \langle x\left|\sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{ip}{\sqrt{2m\hbar\omega}}\right|\emptyset\rangle = \emptyset$

$\rightarrow \sqrt{\frac{m\omega}{2\hbar}}\,x\,\psi(x) + \frac{\hbar}{\sqrt{2m\hbar\omega}}\frac{d}{dx}\psi(x) = \emptyset$

$\rightarrow \frac{1}{\sqrt{2}}x_0 x\,\psi(x) + \frac{1}{\sqrt{2}}\frac{1}{x_0}\frac{d}{dx}\psi(x) = \emptyset$

$\rightarrow \frac{1}{\sqrt{2}}x_0\left[\frac{1}{x_0^2}\frac{d}{dx} + x\right]\psi(x) = \emptyset$

$\rightarrow \frac{d\psi(x)}{dx} + x\,x_0^2\,\psi(x)$

$\rightarrow \psi(x) = A\exp\left(-\frac{x^2}{2}x_0^2\right)$

$\rightarrow \int_{-\infty}^{\infty} A^2\exp(-x^2x_0^2)\,dx$

$\rightarrow z = x x_0 \rightarrow \frac{dz}{dx} = x_0 \rightarrow \frac{A^2}{x_0}\underbrace{\int_{-\infty}^{\infty} e^{-z^2}dz}_{\sqrt{\pi}} = 1$

$\rightarrow A^2 = \frac{x_0}{\sqrt{\pi}} = \sqrt{\frac{m\omega}{\hbar\pi}}$

$\rightarrow A = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}$

$\rightarrow \boxed{\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)}$

E $\psi_0(x=\emptyset) = \emptyset \rightarrow$ $V(x) \neq V(-x)$ odd function,

n should be ~~odd~~

$\rightarrow \boxed{E = (n+\tfrac{1}{2})\hbar\omega,\ \text{odd}}$

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PROBLEM 2: Harmonic Oscillator with Two Particles

Consider a Hamiltonian for two non-interacting particles:

$$\begin{aligned} H(1,2) &= \frac{P_1^2}{2m} + \frac{1}{2}m\omega_1^2 X_1^2 + \frac{P_2^2}{2m} + \frac{1}{2}m\omega_2^2 X_2^2 \\ &= H_1 + H_2 \end{aligned}$$

where $\omega_2 = 2\omega_1 = 2\omega$.

Defining the raising and lowering operators:

$$\begin{aligned} a_n &= \frac{1}{\sqrt{2}}(\bar{X}_n + i\bar{P}_n) \\ a_n^\dagger &= \frac{1}{\sqrt{2}}(\bar{X}_n - i\bar{P}_n) \end{aligned}$$

where $n = 1, 2$ and

$$\begin{aligned} \bar{X}_n &= \left(\frac{m\omega_n}{\hbar}\right)^{1/2} X_n \\ \bar{P}_n &= \left(\frac{1}{\hbar m\omega_n}\right)^{1/2} P_n \end{aligned}$$

such that $[a_m, a_n^\dagger] = \delta_{mn}, m, n = 1, 2$.

Answer the following questions:

(a) [2 points] Write the Hamiltonian in terms of raising and lowering operators.

(b) [2 points] Write the eigenvector $|\psi_{n_1,n_2}\rangle$ in terms of the ground state $\psi_{0,0}\rangle = |\phi_{n_1=0}\rangle|\phi_{n_2=0}\rangle$ where $|\phi_{n_1}\rangle$ is the eigenvector for particle 1, i.e.,

$$H_1|\phi_{n_1}\rangle = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1|\phi_{n_1}\rangle$$

and similarly for particle 2.

(c) [1 points] Write a formula for the energy levels of this oscillator, E_n with n defined in terms of n_1 and n_2.

(d) [1 points] Determine a formula for the degeneracy, g_n , of an energy level E_n.

(e) [2 points] Using your results from part (d) determine the degeneracy g_n for the energy, $E = 15/2\hbar\omega$ and list all the eigenfunctions $|\psi_{n_1,n_2}\rangle$ that have this energy.

(f) [2 points] Determine ΔX_1, the uncertainty in X_1 for the state $|\psi_{n_1=1,n_2=2}\rangle$ using raising and lowering operators. Discuss the dependence of ΔX_1, on the frequency ω_1 and explain why it makes sense physically.

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2 A $a_n^\dagger a_n = \frac{1}{2}(\bar{X}_n - i\bar{P}_n)(\bar{X}_n + i\bar{P}_n)$

$$= \frac{1}{2}\left[\bar{X}_n^2 + \bar{P}_n^2 + i\bar{X}_n\bar{P}_n - i\bar{P}_n\bar{X}_n\right]$$

$$= \frac{1}{2}\left[\bar{X}_n^2 + \bar{P}_n^2 + i[\bar{X}_n, \bar{P}_n]\right]$$

$$\rightarrow [\bar{X}_n, \bar{P}_n] = \left(\frac{m\omega_n}{\hbar}\right)^{1/2}\left(\frac{1}{\hbar m\omega_n}\right)^{1/2}[X_n, P_n]$$

$$= \frac{1}{\hbar}[X_n, P_n] = i$$

$$\rightarrow a_n^\dagger a_n = \frac{1}{2}\left[\bar{X}_n^2 + \bar{P}_n^2 - 1\right] = \frac{1}{2}\left[\frac{m\omega_n}{\hbar}X_n^2 + \frac{1}{\hbar m\omega_n}P_n^2 - 1\right]$$

$$= \frac{P_n^2}{2\hbar m\omega_n} + \frac{m\omega_n X_n^2}{2} - \frac{1}{2}$$

$$\rightarrow \frac{P_n^2}{2m} + \frac{1}{2}m\omega_n^2 X_n^2 = \hbar\omega_n\left(a_n^\dagger a_n + \frac{1}{2}\right)$$

$$\rightarrow H(1,2) = \hbar\omega\left(a_1^\dagger a_1 + \frac{1}{2}\right) + 2\hbar\omega\left(a_2^\dagger a_2 + \frac{1}{2}\right)$$

$$= \hbar\omega\left(a_1^\dagger a_1 + 2a_2^\dagger a_2 + \frac{3}{2}\right)$$

B $a^\dagger = a_1^\dagger a_2^\dagger$

$$(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|\psi_{0,0}\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\left[|\phi_{n_1}=\phi\rangle \otimes |\phi_{n_2}=\phi\rangle\right]$$

$$= \sqrt{n_1!}\sqrt{n_2!}\left[|\psi_{n_1,n_2}\rangle\right]$$

$$\rightarrow \boxed{|\psi_{n_1,n_2}\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}}|\psi_{0,0}\rangle}$$

②

(C) $H(1,2)|n_1,n_2\rangle = \boxed{\hbar\omega\left[n_1 + 2n_2 + \frac{3}{2}\right]}|n_1,n_2\rangle$

where $n_1 = a_1^\dagger a_1$ & $n_2 = a_2^\dagger a_2$

call $\underline{n = n_1 + 2n_2}$

(D) I have no idea how to do this.

$\frac{3}{2} - \frac{1}{2} = \frac{1}{2}$

(E) $\frac{15}{2} = n + \frac{3}{2} \Rightarrow n = 6 \rightarrow$

n_1	n_2
0	3
2	2
4	1
6	0

(F) $\Delta x_1 = \sqrt{\langle x_1^2\rangle - \langle x_1\rangle^2}$

$a_n + a_n^\dagger = \frac{2}{\sqrt{2}}\bar{x}_n$

$\rightarrow \frac{2}{\sqrt{2}}\left(\frac{m\omega_1}{\hbar}\right)^{1/2} x_1 = (a_1 + a_1^\dagger) \Rightarrow x_1 = \frac{\hbar^{1/2}}{(2m\omega_1)^{1/2}}(a_1 + a_1^\dagger)$

$\sqrt{2}$

$\Rightarrow \langle x_1\rangle = \emptyset \qquad \rightarrow x_1^2 = \frac{\hbar}{2m\omega_1} a_1^2 + a_1^{\dagger 2} + a_1 a_1^\dagger + a_1^\dagger a$

$\rightarrow \langle x_1^2\rangle = \frac{\hbar}{2m\omega_1}\left[\langle n=1|a_1 a_1^\dagger|n=1\rangle + \langle n=1|a_1^\dagger a|n=1\rangle\right]$

$2\langle n=2|n=2\rangle \qquad 1\langle n=0|n=0\rangle$

$= \frac{3\hbar}{2m\omega_1} \rightarrow$ $\Delta x_1 = \sqrt{\frac{3\hbar}{2m\omega_1}}$

$\hookrightarrow V = \omega R$

$\Rightarrow m\omega_1 \propto p \Rightarrow \uparrow p$

$\downarrow \Delta x$