S-2016

Problem 1: Clebsh-Gordon coefficients (10 pts)

A system of two particles with spins $s_1 = \frac{3}{2}$ and $s_2 = \frac{1}{2}$ is described by the Hamiltonian

$$\mathcal{H} = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2$$

with α a constant and $\mathbf{S}_i$ $(i = 1, 2)$ is the spin operator of the i-th particle.

a) What are the allowed values for the quantum numbers of the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$? (2 Points)

b) Calculate the energy levels of the Hamiltonian. (2 Points)

c) Let us define the basis of eigenstates of the $\mathbf{S}_1^2$, $\mathbf{S}_2^2$, S_{1z}, S_{2z} operators, $|s_1 s_2; m_1 m_2\rangle$, where m_1 and m_2 are the quantum numbers of the projection operators S_{1z} and S_{2z} respectively. The system at time $t = 0$ is initially in the state

$$\left| s_1 s_2; \frac{1}{2}, \frac{1}{2} \right\rangle .$$

Find the state of the system at times $t > 0$. (4 Points)

d) Assuming the initial state above, what is the probability of finding the system in the state

$$\left| s_1 s_2; \frac{3}{2}, -\frac{1}{2} \right\rangle$$

at $t > 0$? (2 Points)

S-2016

1 $S_1 = \frac{3}{2} \quad S_2 = \frac{1}{2} \qquad H = \alpha S_1 \cdot S_2$

A $|S_1 - S_2| \le S \le S_1 + S_2$

$\rightarrow \quad 1 \le S \le 2$

$$\boxed{S = 1, 2}$$

B $S_1 \cdot S_2$

$S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2$

$\rightarrow \quad S_1 \cdot S_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2)$

$\rightarrow H = \frac{\alpha}{2}(S^2 - S_1^2 - S_2^2)$

$\rightarrow$ S can be 1 or 2 $\rightarrow \quad \underbrace{1(1+1)\hbar^2}_{=2\hbar^2}$ or $\underbrace{2(2+1)\hbar^2}_{=6\hbar^2}$

$S_1^2 = \frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2 = \frac{15}{4}\hbar^2$

$S_2^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = \frac{3}{4}\hbar^2$

$\rightarrow \quad 2\hbar^2 - \frac{15}{4}\hbar^2 - \frac{3}{4}\hbar^2 = -\hbar^2\frac{\alpha}{2}$

$\rightarrow \quad 6\hbar^2 - 3\hbar^2 = 3\hbar^2\frac{\alpha}{2}$

$\boxed{-\hbar^2\frac{\alpha}{2},\ 3\hbar^2\frac{\alpha}{2}}$ > energy levels

c) $S_1 \cdot S_2 = S_{1x} S_{2x} + S_{1y} S_{2y} + S_{1z} S_{2z}$

$$S_{+1} S_{-1} = (S_{x_1} + i S_{y_1})(S_{x_2} - i S_{y_2})$$

$$= S_{x_1} S_{x_2} + i \cancel{S_{y_1} S_{x_2}} - i \cancel{S_{y_2} S_{x_1}} + S_{y_1} S_{y_2}$$

$$S_{-1} S_{+1} = (S_{x_1} - i S_{y_1})(S_{x_2} + i S_{y_2})$$

$$= S_{x_1} S_{x_2} + i \cancel{S_{y_2} S_{x_1}} - i \cancel{S_{y_1} S_{x_2}} + S_{y_1} S_{y_2}$$

$$\Rightarrow S_1 \cdot S_2 = \frac{1}{2} (S_{+1} S_{-2} + S_{-1} S_{+2}) + S_{1z} S_{2z}$$

$|\frac{3}{2}\ \frac{1}{2}\rangle$ $|\frac{1}{2}\ \frac{1}{2}\rangle$ $|-\frac{1}{2}\ \frac{1}{2}\rangle$ $|-\frac{3}{2}\ \frac{1}{2}\rangle$

$|\frac{3}{2}\ -\frac{1}{2}\rangle$ $|\frac{1}{2}\ -\frac{1}{2}\rangle$ $|-\frac{1}{2}\ -\frac{1}{2}\rangle$ $|-\frac{3}{2}\ -\frac{1}{2}\rangle$

will ~~have~~ be in either $|S=2, m=1\rangle$
or $|S=1, m=1\rangle$

$$|2\ 2\rangle = |\tfrac{3}{2}\ \tfrac{1}{2}\rangle$$

$$J_-|2\ 2\rangle = J_{-1}|\tfrac{3}{2}\ \tfrac{1}{2}\rangle + J_{-2}|\tfrac{3}{2}\ \tfrac{1}{2}\rangle$$

$$\underbrace{\sqrt{(2+2)(2-2+1)}}_{\sqrt{4}}\ |2\ 1\rangle = \underbrace{\sqrt{(\tfrac{3}{2}+\tfrac{3}{2})(\tfrac{3}{2}-\tfrac{3}{2}+1)}}_{\sqrt{3}}\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \underbrace{\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}}_{\sqrt{1}}\ |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle$$

$$|2\ 1\rangle = \sqrt{\tfrac{3}{4}}\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \frac{1}{\sqrt{4}}\ |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle$$

(c) Continued

$$|1\,1\rangle = \frac{1}{\sqrt{4}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \sqrt{\frac{3}{4}}\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle$$

$$\rightarrow\ |2\,1\rangle = \sqrt{\frac{3}{4}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{4}}\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle \qquad \times\sqrt{3}$$

$$|1\,1\rangle + \sqrt{3}\,|2\,1\rangle = \left[\frac{1}{\sqrt{4}} + \frac{3}{\sqrt{4}}\right]\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle$$

$$\frac{4}{\sqrt{4}} = \sqrt{4}$$

$$\rightarrow\ \boxed{\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{4}}\,|1\,1\rangle + \sqrt{\frac{3}{4}}\,|2\,1\rangle}$$

(d)

$$\boxed{U\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{4}}\exp\left(\frac{i\hbar\alpha t}{2}\right)|1\,1\rangle + \sqrt{\frac{3}{4}}\exp\left(\frac{-3i\hbar\alpha t}{2}\right)|2\,1\rangle}$$

(1)

$$-\sqrt{3}\,|11\rangle = -\sqrt{\frac{3}{4}}\,\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \frac{3}{\sqrt{4}}\,\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle$$

$$\to \sqrt{4}\,\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle = -\sqrt{3}\,|11\rangle + |21\rangle$$

$$\to \left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{4}}\,|21\rangle - \sqrt{\frac{3}{4}}\,|11\rangle$$

$$\to \left|\left\langle \tfrac{3}{2}\,-\tfrac{1}{2}\right|\tfrac{1}{2}\tfrac{1}{2}(t)\right\rangle\right|^2 = \left(\frac{\sqrt{3}}{4}\exp\left(\frac{-3i\hbar a t}{2}\right) - \frac{\sqrt{3}}{4}\exp\left(\frac{i\hbar a t}{2}\right)\right)^2$$

$$\to \frac{3}{16}\left[\left(\exp\left(\frac{3i\hbar a t}{2}\right) - \exp\left(\frac{-i\hbar a t}{2}\right)\right)\left(\exp\left(-\frac{3i\hbar a t}{2}\right) - \exp\left(\frac{i\hbar a t}{2}\right)\right)\right]$$

$$= \frac{3}{16}\left[1 - \underbrace{\exp(2i\hbar a t) - \exp(-2i\hbar a t)}_{-2\cos 2\hbar a t} + 1\right]$$

$$= \boxed{\frac{3}{8}\left[1 - \cos(2\hbar a t)\right]}$$

S-2014

Problem 4: Clebsh-Gordon Coefficients

Consider a system with two distinguishable spinless particles with angular momentum $j_1 = 1$ and $j_2 = 1$. Suppose the system is prepared in a state with total angular momentum $j = 2$ and total angular momentum projection $m = m_1 + m_2 = 0$. The state in the total j basis $|j_1, j_2; j, m\rangle$ is

$$|\psi\rangle \equiv |1, 1; j = 2, m = 0\rangle.$$

a) Express $|\psi\rangle$ in terms of products of single particle states, namely in the direct product basis $|j_1 = 1, m_1\rangle|j_2 = 1, m_2\rangle$. (4 Points).

b) If the angular momentum projection of particle 1 is measured along the z direction, what is the probability of finding a non-zero result? (2 Points)

c) If $\mathbf{J}_i$ is the angular momentum operator of each particle (i = 1, 2), compute the expectation value of $\mathbf{J}_1 \cdot \mathbf{J}_2$ in the $|\psi\rangle$ state. (2 Points)

d) If the $|\psi\rangle$ state is rotated by an infinitesimal angle $\delta\theta$ around the x direction, compute the probability of measuring the $|1, 1; j = 2, m = 1\rangle$ state in leading order in $\delta\theta$. (2 Points)

Raising and lowering angular momentum operators:

$$J_{\pm}|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

S-2014 ①

④ $J_1 = 1 \quad J_2 = 1$

$J = 2 \;\; \& \;\; m_1 + m_2 = \emptyset$

$|J_1, J_2; J, m\rangle$

$|\psi\rangle = |11; 20\rangle$

Ⓐ $$|JM\rangle = \sum_{m_1 m_2} |m_1 m_2\rangle\langle m_1 m_2 | JM\rangle$$

$J_1 = 1 \quad m_1 = 1, 0, -1$

$J_2 = 1 \quad m_2 = 1, 0, -1$

$m_1 m_2 = \emptyset$

$m_1 = 1 \quad m_2 = -1$

$m_2 = 1 \quad m_1 = -1$

$$\to \; |2,0\rangle = \langle 0\,0|20\rangle\,|00\rangle + \langle 1\,{-1}|20\rangle\,|1\,{-1}\rangle + \langle -1\,1|20\rangle\,|{-1}\,1\rangle$$

A Continued

$$|2,2\rangle = |11\rangle$$

$$J_-|2,2\rangle = J_{1-}|11\rangle + J_{2-}|11\rangle$$

$$\underbrace{\sqrt{(2+2)(2-2+1)}}_{2} = \underbrace{\sqrt{(1+1)(1-1+1)}}_{\sqrt{2}}\,|01\rangle + \underbrace{\sqrt{(1+1)(1-1+1)}}_{\sqrt{2}}\,|10\rangle$$

$$\rightarrow |21\rangle = \frac{1}{\sqrt{2}}\left[\,|01\rangle + |10\rangle\,\right]$$

$$\rightarrow J_-|21\rangle = \frac{1}{\sqrt{2}}\left[\left(J_{1-}|01\rangle + J_{1-}|10\rangle\right) + \left(J_{2-}|01\rangle + J_{2-}|10\rangle\right)\right]$$

$$\cancel{= \underbrace{\sqrt{(2+1)(2-1+1)}}_{\sqrt{5}}\,|20\rangle = \frac{1}{\sqrt{2}}\left[\underbrace{\sqrt{(1+0)(1-0+1)}}_{\sqrt{2}}\,|-11\rangle + \underbrace{\sqrt{(1+1)(1-1+1)}}_{2}\,|00\rangle\right.}$$

$$\cancel{\left. + \underbrace{\sqrt{(1+0)(1+0+1)}}_{2}\,|00\rangle + \underbrace{\sqrt{(1+0)(1-0+1)}}_{\sqrt{2}}\,|1-1\rangle\right]}$$

$$\cancel{= \sqrt{5}\,|20\rangle = \frac{1}{\sqrt{2}}\left[\sqrt{2}\,|-11\rangle + \sqrt{2}\,|1-1\rangle + \frac{4}{\sqrt{2}}\,|00\rangle\right]}$$

$$\cancel{\rightarrow |20\rangle = \frac{1}{\sqrt{5}}\,|-11\rangle + \frac{1}{\sqrt{5}}\,|1-1\rangle + \frac{4}{\sqrt{10}}\,|00\rangle}$$

$$\cancel{\frac{2}{5} + \frac{2}{5} + \frac{16}{10}}$$

A Continued

$$\sqrt{(2+1)(2-1+1)}\,|20\rangle = \frac{1}{\sqrt{2}}\Big[\sqrt{(1+0)(1-0+1)}\,|{-1}1\rangle + \sqrt{(1+1)(1-1+1)}\,|00\rangle + \sqrt{(1+1)(1-1+1)}\,|00\rangle + \sqrt{(1+0)(1-0+1)}\,|1{-1}\rangle\Big]$$

$\sqrt{6}$; $\sqrt{2}$, $\sqrt{2}$, $\sqrt{2}$, $\sqrt{2}$

$$= |{-1}1\rangle + 2\,|00\rangle + |1{-1}\rangle$$

$$\rightarrow |20\rangle = \frac{1}{\sqrt{6}}|{-1}1\rangle + \frac{1}{\sqrt{6}}|1{-1}\rangle + \frac{2}{\sqrt{6}}|00\rangle$$

$$\rightarrow \boxed{|20\rangle = \frac{1}{\sqrt{6}}|{-1}1\rangle + \frac{1}{\sqrt{6}}|1{-1}\rangle + \frac{2}{\sqrt{6}}|00\rangle}$$

$12 = 4\,3$

$$\frac{1}{6} + \frac{1}{6} + \frac{4}{3} =$$

$\frac{4}{2}$

$4+1+1$ $\quad \frac{2+2}{12} + 16$

B $\boxed{\frac{1}{6}}$

C $J_1 \cdot J_2 = (J_{1x} + J_{1y} + J_{1z})(J_{2x} + J_{2y} + J_{2z})$

$$= \frac{1}{2}\left(J_{1+}J_{2-} + J_{1-}J_{2+}\right) + 2J_{1z}J_{2z}$$

or $\quad J^2 = J_1^2 + S_2^2 + 2J_1J_2$

$$\longrightarrow J_1J_2 = \frac{1}{2}\left(J^2 - J_1^2 - J_2^2\right)$$

(C) Continued

$$\frac{1}{2}\langle 2\,0|\cancel{\hbar}(J^2-J_1^2-J_2^2)|2\,0\rangle$$

$$= \frac{1}{2}\Big[\underbrace{\langle 20|J^2|20\rangle}_{\substack{2(2+1)\hbar^2 \\ 6\hbar^2}} - \underbrace{\langle 20|J_1^2|20\rangle}_{\substack{1(1+1)\hbar^2 \\ 2\hbar^2}} - \underbrace{\langle 20|J_2^2|20\rangle}_{\substack{1(1+1)\hbar^2 \\ 2\hbar^2}}\Big]$$

$$= \boxed{\hbar^2}$$

(D) $\exp\left(-\frac{i}{\hbar}\vec{J}\cdot\hat{n}\phi\right)$ or $1-\frac{i\vec{J}\cdot\hat{n}}{\hbar}\delta\phi$

around x direction $R(\phi)$

$$= 1-\frac{iJ_x\delta\phi}{\hbar}$$

$$J^2 = \cancel{J_x^2+J_y^2+J_z^2}$$

$$\cancel{(J_1+J_2)(J_1+J_2) = J_1^2+J_2^2-2J_1J_2}$$

$$\cancel{J_1J_2 = J_{1x}+J_{1y}+J}$$

$$\boxed{J_x = \frac{J_+ + J_-}{2}}$$

$$\Rightarrow 1-\frac{i}{2\hbar}\delta\phi\left[J_+ + J_-\right]$$

(D) Continued

$$D|11,20\rangle = |11,20\rangle - \frac{i\delta\phi}{2\hbar}\left[\sqrt{(2-0)(2+0+1)}\,|21\rangle + \sqrt{(2+0)(2-0+1)}\,|2\,-1\rangle\right.$$

Actually, you want

to expand $\exp\left(\frac{-iJ_n\phi}{\hbar}\right)$

in Taylor ~~series~~ Series

around ϕ

Then apply on

ket

$\rightarrow$ This method

gives you an i

F-2012

PROBLEM 3: Clebsch-Gordan Coefficients

Consider a system of 2 spin 1/2 particles, i.e. $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}$ where:

$$S_{1z}|s_1, m_{s1}\rangle = m_{s1}\hbar|s_1, m_{s1}\rangle$$

$$S_1^2|s_1, m_{s1}\rangle = s_1(s_1+1)\hbar^2|s_1, m_{s1}\rangle = 3/4\hbar^2|s_1, m_{s1}\rangle$$

and similarly for S_{2z} and S_2^2.

Initially, the 2 spin particles are uncoupled and subject to a Hamiltonian:

$$H_0 = \omega_1 S_{1z} + \omega_2 S_{2z}$$

The eigenvectors $|s_1, s_2; m_{s1}, m_{s2}\rangle$, for this Hamiltonian can be written in compact notation as: $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ where the + and – denote the sign of m_{s1} and m_{s2} respectively.

Answer the following questions:

(a) Set up the matrix representation for H_0 in this uncoupled basis. [1 point]

Now add an interaction term: $A\vec{S}_1 \cdot \vec{S}_2$ to H_0:

$$H = H_0 + A\vec{S}_1 \cdot \vec{S}_2$$

(b) Determine the commutator : $[H, S_{1z}]$. Will the uncoupled basis be an eigenbasis for H? Explain. [2 points]

(c) Determine a coupled basis for this system: $|S, M\rangle$ where S is the value of the total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ and M is its component, i.e.

$$S^2|S, M\rangle = S(S+1)\hbar^2|S, M\rangle, \;\; S_z|S, M\rangle = M\hbar|S, M\rangle.$$

by setting up the matrix for $S^2 = (\vec{S}_1 + \vec{S}_2)^2$ in the uncoupled basis and diagonalizing it. List the eigenvectors of S^2 with the correct values of S and M i.e. as $|S, M\rangle$ states. [3 points]

(d) Identify the Clebsch-Gordan coefficients: $\langle s_1, s_2, m_{s1}, m_{s2}|S, M\rangle$ from the expansions you found in part c). Fill in values for all the quantum numbers in the Dirac braket for each Clebsch-Gordan coefficient and give the numerical value for all the Clebsch-Gordan coefficients you have found. There should be 6 Clebsch-Gordan coefficients. [4 points]

F-2012

③ $S_1 = S_2 = \frac{1}{2}$

$H_0 = \omega_1 S_{1z} + \omega_2 S_{2z}$

$|S_1 S_2 ; m_{s1}, m_2\rangle$

Ⓐ $H_0|++\rangle = \frac{\omega_1 \hbar}{2} + \frac{\omega_2 \hbar}{2} |++\rangle$

$H_0|+-\rangle = \frac{\omega_1 \hbar}{2} - \frac{\omega_2 \hbar}{2} |+-\rangle$

$H_0|-+\rangle = -\frac{\omega_1 \hbar}{2} + \frac{\omega_2 \hbar}{2} |-+\rangle$

$H_0|--\rangle = -\frac{\omega_1 \hbar}{2} - \frac{\omega_2 \hbar}{2}$

$$\boxed{H_0 \doteq \frac{\hbar}{2} \begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{pmatrix} \omega_1+\omega_2 & 0 & 0 & 0 \\ 0 & \omega_1-\omega_2 & 0 & 0 \\ 0 & 0 & -\omega_1+\omega_2 & 0 \\ 0 & 0 & 0 & -\omega_1-\omega_2 \end{pmatrix} \end{array}}$$

Ⓑ $\vec{S}_1 \cdot \vec{S}_2 = S_{1x} S_{2x} + S_{1y} S_{2y} + S_{1z} S_{2z}$

$\rightarrow S_{-1} S_{+2} =$

$(S_{x_1} - i S_{y_1})(S_{x_2} + i S_{y_2})$

$= S_{x_1} S_{x_2} + i S_{x_1} S_{y_2} - i S_{y_1} S_{x_2} + S_{y_1} S_{y_2}$

$= \underbrace{\left(S_{-1} S_{+2} + S_{+1} S_{-2} \right)}_{2} + S_{1z} S_{2z}$

$S_{+1} S_{-2} = (S_{x_1} + i S_{y_1})(S_{x_2} - i S_{y_2})$

$= S_{x_1} S_{x_2} - i S_{x_1} S_{y_2} + i S_{y_1} S_{x_2} + S_{y_2}^2$

B Continued

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{(S_{-1}S_{+2} + S_{+1}S_{-2})}{2} + S_{1z}S_{2z}$$

$$\rightarrow H = W_1 S_{1z} + W_2 S_{2z} + \frac{A}{2}(S_{-1}S_{+2} + S_{+1}S_{-2}) + A S_{1z} S_{2z}$$

$$\rightarrow [H, S_{1z}]\,|++\rangle = \Big(W_1 S_{1z} + W_2 S_{2z} + \frac{A}{2}(S_{-1}S_{+2} + S_{+1}S_{-2})$$

$$+ A S_{1z} S_{2z}\Big) S_{1z}\,|++\rangle$$

$$= \frac{W_1 \hbar}{2} + \frac{W_2 \hbar}{2} + \frac{A}{2}\Big[\emptyset + \emptyset\Big]$$

$$+ A \frac{\hbar^2}{4}\Big] \frac{\hbar}{2}\ |++\rangle$$

$$= \frac{\hbar^2}{4}\left[\frac{2W_1}{\hbar} + \frac{2W_2}{\hbar} + A\frac{\hbar}{2}\right] |++\rangle$$

reverse $\rightarrow \frac{\hbar}{2}\left[\frac{W_1 \hbar}{2} + \frac{W_2 \hbar}{2} + A\frac{\hbar^2}{4}\right] |++\rangle$

$$\rightarrow \boxed{[H, S_{1z}] = \emptyset}$$

$S_- = \sqrt{(S+m)(S-m+1)}$ (3)

(c) $S^2 = S_1^2 + S_2^2 + S_1 S_2 + S_2 S_1$

$= S_1^2 + S_2^2 + S_{-1}S_{+2} + S_{+1}S_{-2} + 2S_{1z}S_{2z}$

$\rightarrow S^2|++\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 + \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 + 2\left(\frac{\hbar}{2}\frac{\hbar}{2}\right)$

$= \frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + \frac{2\hbar^2}{4} = \underline{2\hbar^2}$

$\rightarrow S^2|--\rangle = \underline{2\hbar^2}$

$\rightarrow S^2|+-\rangle = \left(\frac{6}{4}\hbar^2 - \frac{\hbar^2}{4\cdot 2}\right)|+-\rangle + \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)(1)}\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)}|-+\rangle$

$= \hbar^2|+-\rangle + \hbar^2|-+\rangle$

$\rightarrow S^2|-+\rangle = \hbar^2|-+\rangle + \hbar^2|+-\rangle$

$$\rightarrow S^2 \doteq \begin{array}{c|cccc} & ++ & +- & -+ & -- \\ \hline ++ & 2\hbar^2 & 0 & 0 & 0 \\ +- & 0 & \hbar^2 & \hbar^2 & 0 \\ -+ & 0 & \hbar^2 & \hbar^2 & 0 \\ -- & 0 & 0 & 0 & 2\hbar^2 \end{array}$$

(b) $\boxed{|S=1, m=1\rangle = |++\rangle}$

(c) ~~[illegible]~~

Continued

$\boxed{|S=1, m=-1\rangle = |--\rangle}$

$2\hbar^2 = s(s+1)\hbar^2$

S=1 when $2\hbar^2$ is eigenvalue

S=0 when $\lambda = 0$

$$\cancel{\begin{array}{c|cc} & +- & -+ \\ \hline +- & \hbar^2-\lambda & \hbar^2 \\ -+ & \hbar^2 & \hbar^2-\lambda \end{array}} \rightarrow \cancel{(\hbar^2-\lambda)^2 - \hbar^4 = 0}$$

$$\cancel{\hbar^4 + \lambda^2 - 2\hbar^2\lambda - \hbar^4 = 0}$$

$$\cancel{\lambda =}$$

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = 0$$

$$\rightarrow 1 + \lambda^2 - 2\lambda - 1 = 0$$

$$\lambda^2 = 2\lambda$$

$$\rightarrow \lambda = 2, 0$$

$\underline{\lambda = 2}$ $\rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow |\lambda=2\rangle = \boxed{\frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right] = |S=1, m=0\rangle}$

$\underline{\lambda = 0}$ $\rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow |\lambda=0\rangle = \boxed{\frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right] = |S=0, m=0\rangle}$

(D) $\boxed{1}$

$$\boxed{|11\rangle = |++\rangle}$$

$$J_-|11\rangle = J_{-1}|++\rangle + J_{-2}|++\rangle$$

$$\underbrace{\sqrt{(1+1)(1-1+1)}}_{\sqrt{2}}\,|10\rangle = \sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\,|-+\rangle + \sqrt{1}\,|+-\rangle$$

$$\rightarrow \boxed{|10\rangle = \boxed{\frac{1}{\sqrt{2}}}|-+\rangle + \boxed{\frac{1}{\sqrt{2}}}|+-\rangle}$$

$$J_-|10\rangle = \frac{1}{\sqrt{2}}\left[J_{1-}\cancel{|-+\rangle}^{0} + J_{1-}|+-\rangle + J_{2-}|-+\rangle + J_{2-}\cancel{|+-\rangle}^{0}\right]$$

$$\rightarrow \underbrace{\sqrt{(1+0)(1-0+1)}}_{\sqrt{2}}\,|1-1\rangle = \frac{1}{\sqrt{2}}\left[|--\rangle + |--\rangle\right]$$

$$\rightarrow \boxed{|1-1\rangle = \cancel{\tfrac{2}{2}}\,|--\rangle}$$

$|00\rangle$ orthogonal to $|10\rangle$

$$\rightarrow \boxed{\frac{1}{\sqrt{2}}\left[|-+\rangle - |+-\rangle\right] = |00\rangle}$$

F-2014

PROBLEM 5: Addition of angular momenta

Consider an electron. We know its orbital angular momentum $\ell = 1$ and the z component $m = 1/2$ of its total angular momentum j.

a) What are the possible values of j? (2 Points).

b) Write down the kets $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$ in terms of products of spin and orbital angular momentum states (3 Points)

c) Calculate the expectation value of the spin operator $\mathbf{S}$ in the state $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$. Consider all possible values of j. (3 Points).

d) The magnetic dipole moment of the electron is

$$\boldsymbol{\mu} = \frac{e}{2m_e c}(\mathbf{L} + 2\mathbf{S}),$$

with $\mathbf{L}$ the orbital angular momentum operator, e the electron charge, m_e the mass and c the speed of light. Calculate the expectation value of $\boldsymbol{\mu}$ in the states $|\ell = 1, \frac{1}{2}; j, m = \frac{1}{2}\rangle$. (2 Points)

Raising and lowering angular momentum operators:

$$J_\pm |j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

F-2014

5 $s=\frac{1}{2}$, $l=1$, $m=\frac{1}{2}$ or J

A $|l-s| \le s \le l+s$

$$\boxed{J=\frac{3}{2}, \frac{1}{2}}$$

B $|l=1, s=\frac{1}{2}; J, m=\frac{1}{2}\rangle \longrightarrow |J_1, J_2; J\, m\rangle$

$|J_1 J_2; m_1 m_2\rangle$

$\rightarrow |J\, m\rangle = \sum_{m_l}\sum_{m_s} |m_l\, m_s\rangle\langle m_l m_s | J\, m\rangle$

$J_{max} = \frac{3}{2}$

$m_{l\,max} = 1$
$m_{s\,max} = \frac{1}{2}$ } $m_{max} = \frac{3}{2}$

$\rightarrow$ at max, probability is one (only one way to build max state)

$\rightarrow |\frac{3}{2}\,\frac{3}{2}\rangle = |1\,\frac{1}{2}\rangle \rightarrow J_-|\frac{3}{2}\,\frac{3}{2}\rangle = (J_{-l}+J_{-s})|1\,\frac{1}{2}\rangle$

$\rightarrow \hbar\sqrt{(\frac{3}{2}+\frac{3}{2})(\frac{3}{2}-\frac{3}{2}+1)}\,|\frac{3}{2}\,\frac{1}{2}\rangle = \sqrt{(1+1)(1-1+1)}\,|0\,\frac{1}{2}\rangle$

$+\sqrt{(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}-\frac{1}{2}+1)}\,|1\,-\frac{1}{2}\rangle$

B Continued

$$\sqrt{3}\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \sqrt{2}\,|0\,\tfrac{1}{2}\rangle + |1\,-\tfrac{1}{2}\rangle$$

$$\rightarrow \boxed{|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|0\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1\,-\tfrac{1}{2}\rangle}$$

$|\tfrac{1}{2}\,,\tfrac{1}{2}\rangle$

$$\rightarrow \langle\tfrac{1}{2}\,\tfrac{1}{2}|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \emptyset$$

easiest way is to introduce −
& switch coefficients

$$\boxed{|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|0\,\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1\,-\tfrac{1}{2}\rangle}$$

Can also say $|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = a^2|0\,\tfrac{1}{2}\rangle + b^2|1\,-\tfrac{1}{2}\rangle$

$$a\sqrt{\tfrac{2}{3}} + b\sqrt{\tfrac{1}{3}} = \emptyset \rightarrow b = -\sqrt{2}\,a$$

$$\& \quad a^2+b^2=1 \rightarrow a^2+2a^2=1$$

$$3a^2=1 \rightarrow a=\frac{1}{\sqrt{3}}$$

$$\rightarrow b = -\sqrt{\tfrac{2}{3}}$$

c

I think take this in z-direction?

$$\langle \tfrac{3}{2}\, \tfrac{1}{2} | S_z | \tfrac{3}{2}\, \tfrac{1}{2} \rangle$$

$$= \left(\frac{2}{3}\right)\left(\frac{\hbar}{2}\right) - \left(\frac{1}{3}\right)\left(\frac{\hbar}{2}\right)$$

$$= \frac{\hbar}{3} - \frac{\hbar}{6} = \left(\frac{2}{6} - \frac{1}{6}\right)\hbar$$

$$= \frac{1}{6}\hbar$$

$$\langle \tfrac{1}{2}\, \tfrac{1}{2} | S_z | \tfrac{1}{2}\, \tfrac{1}{2} \rangle$$

$$= \frac{1}{3}\left(\frac{\hbar}{2}\right) - \frac{2}{3}\left(\frac{\hbar}{2}\right) = \frac{\hbar}{6} - \frac{2\hbar}{6}$$

$$= -\frac{\hbar}{6}$$

Not sure