F-2009

Problem 4: 3-d central-force problem (10 points)

A particle of mass m and spin $s = 0$ has a short-range potential energy $V(r)$. The particle is in a stationary state with Hamiltonian eigenfunction

$$\psi_E(\boldsymbol{r}) = N \frac{1}{r} \left(e^{-\alpha r} - e^{-\beta r} \right) , \tag{6}$$

where N is a normalization constant (which you need not determine), and α and β are real numbers such that $\beta > \alpha$.

1. Is the orbital angular momentum of the particle sharp in this state? (That is, does L^2 have zero uncertainty?) If not, explain why not. If so, justify your answer and give the value of L^2 for this state. (4 pts)
2. What is the stationary-state energy of this state? (4 pts)
3. What is the potential energy $V(r)$? (2 pts)

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4 $m = mass, \; s = \emptyset$ short range $V(r)$

$$\psi_E(r) = N\frac{1}{r}\left(e^{-\alpha r} - e^{-\beta r}\right) \qquad \beta > \alpha$$

A $[L^2, H] = \emptyset \rightarrow \boxed{(\Delta L) = \emptyset}$

~~$\langle L^2 \rangle = \emptyset$ since the eigenfunction is real~~

B $H = -\frac{\hbar^2}{2m}\nabla^2 + V$

$$\rightarrow \left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\psi = E\psi$$

$$\rightarrow \left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{L^2}{2mr^2} + V\right]\psi = E\psi$$

$$\rightarrow -\frac{\hbar^2}{2m}\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)} + \frac{\hbar^2 l(l+1)}{2mr^2} = (E-V)\psi$$

$$\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}r^2\left[-\frac{N}{r^2}\left(e^{-\alpha r} - e^{-\beta r}\right) + \frac{N}{r}\left[-\alpha e^{-\alpha r} + \beta e^{-\beta r}\right]\right]}$$

$$\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left[-N\left(e^{-\alpha r} - e^{-\beta r}\right) + Nr\left[-\alpha e^{-\alpha r} + \beta e^{-\beta r}\right]\right]}$$

$$\frac{1}{r^2}\left[-N\left(\cancel{-\alpha e^{-\alpha r} + \beta e^{-\beta r}}\right) + N\left[\cancel{-\alpha e^{-\alpha r}} + \beta e^{-\beta r}\right] + Nr\left(\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}\right)\right]$$

$$= \frac{N}{r}\left(\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}\right)$$

(B) Continued

This is actually ϕ since $L^2\psi = \phi$ ← No θ/ϕ dependence

(2)

$$-\frac{\hbar^2}{2m} N\left(\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}\right) + \frac{\hbar^2 \ell(\ell+1)}{2mr} = (E - V)\, N(e^{-\alpha r} - e^{-\beta r})$$

$$\rightarrow E - V = -\frac{\hbar^2}{2m}\frac{\left(\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}\right)}{e^{-\alpha r} - e^{-\beta r}} \cancel{+ \text{stuff}}$$

$$\beta > \alpha$$

$$= -\frac{\hbar^2}{2m}\frac{\alpha^2 - \beta^2 e^{-(\beta-\alpha)r}}{1 - e^{-\underbrace{(\beta-\alpha)}_{+}r}}$$

so overall $-$

$\rightarrow$ short range $V \rightarrow$ $V \rightarrow \phi$ at $r \rightarrow \infty$

$$\rightarrow \boxed{E = -\frac{\hbar^2}{2m}\alpha^2}$$

(3) $$-\frac{\hbar^2}{2m} N\left(\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}\right) = \left(-\frac{\hbar^2}{2m}\alpha^2 - V\right) N(e^{-\alpha r} - e^{-\beta r})$$

$$\boxed{\frac{\hbar^2}{2m}\frac{\left(\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}\right)}{e^{-\alpha r} - e^{-\beta r}} - \frac{\hbar^2}{2m}\alpha^2 = V}$$