Method of Images

SPRING 2007

5. A very long (infinitely long) and very thin uniformly charged rod is parallel to the z axis and located at $x = a$ and $y = 0$. An infinite grounded conducting sheet is located in the plane $x = 0$, as shown in the figure.

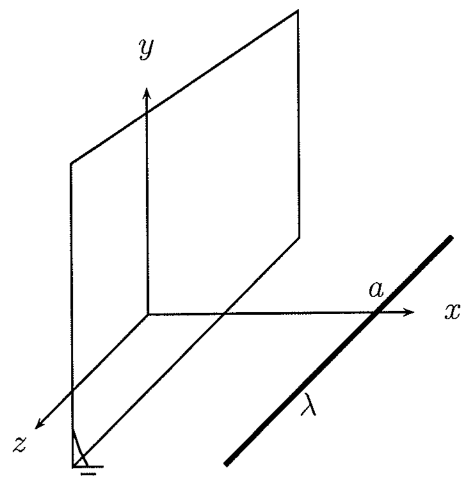


Figure 3: Charged rod, carrying uniform charge of density λ per unit length, parallel to, and a distance a from a grounded plane at $x = 0$.

(a) [2 pts.] What are the boundary conditions on the electric field at the conducting plane?

(b) [5 pts.] Compute the charge density (charge per unit area) on the conducting sheet $\sigma(y)$.

(c) [3 pts.] Evaluate the integral

$$-\lambda = \int_{-\infty}^{\infty} dy\, \sigma(y)$$

to check your value of σ.

Part (a)

Since the plane is conducting, we have the following two boundary conditions on the electric field,

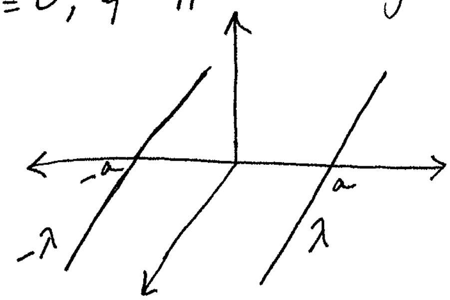
$$E_{left,\perp} - E_{right,\perp} = \frac{\sigma_f}{\varepsilon_0}$$

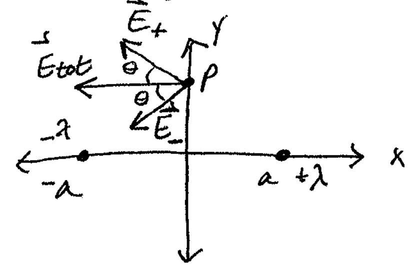
$$E_{left,\parallel} - E_{right,\parallel} = 0 .$$

This means that the parallel, or tangential, components of the electric field are continuous across the surface, whereas the normal components are discontinuous by an amount $4\pi\sigma_f$, where σ_f is the free surface charge density.

Part (b)

Now we want to compute the charge density on the conducting sheet. In order to do so, we need the electric field. We can use the method of images to determine the total field at the position of the conducting sheet. So we essentially have another long wire at $x = -a, y = 0$, of opposite charge density per unit length.





The electric field at a point P on the y-axis should be directed in the $-x$ direction since the wire of positive charge density creates a field directed away from that wire, and the wire of negative charge density creates a field directed toward that wire. Using Gauss's law, with a cylindrical Gaussian surface, each field has a magnitude of

$$\oint \vec{E} \cdot d\vec{a} = \frac{|q_{encl}|}{\varepsilon_0}$$

$$E(2\pi r l) = \frac{\lambda l}{\varepsilon_0}$$

$$E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r},$$

where r is the distance of the wire from point P. Given the symmetry, this distance is the same for both wires. If θ is the angle $\hat{r}$ makes with respect to the x-axis, then the total $\vec{E}$ field at point P is

$$\vec{E}_{tot} = \frac{\lambda}{\pi\varepsilon_0 r}\cos\theta\,(-\hat{x}).$$

Then by our boundary condition on $E_\perp$,

$$-\frac{\lambda}{\pi\varepsilon_0 r}\cos\theta = \sigma/\varepsilon_0$$

$$\sigma = \frac{-\lambda}{\pi r}\left(\frac{a}{r}\right)$$

$$\boxed{\sigma = -\frac{\lambda a}{\pi r^2}}, \text{ where } r = \sqrt{a^2+y^2}.$$

Part (c)

We have

$$-\int_{-\infty}^{\infty} dy \frac{\lambda a}{\pi(a^2+y^2)} = -\frac{\lambda}{\pi} \tan^{-1}\left(\frac{y}{a}\right)\Big|_{-\infty}^{\infty}$$

$$= -\frac{\lambda}{\pi}\left[\tan^{-1}\left(\frac{y\to\infty}{a}\right) - \tan^{-1}\left(\frac{y\to-\infty}{a}\right)\right]$$

$$= -\frac{\lambda}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= -\lambda,$$

as expected.