Relativistic

FALL 2011

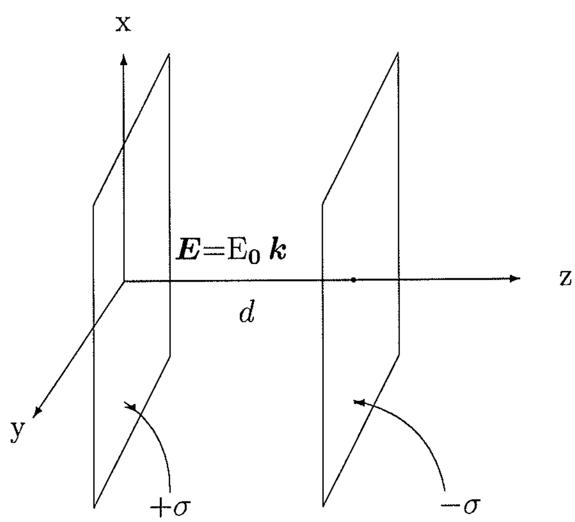
6. A uniform static electric field $\mathbf{E} = E_0\mathbf{k}$ exists between two large thin conducting metal plates. The positive plate is at $z = 0$ and the negative plate is at $z = a$. You can assume the plates are infinitely large in the x-y directions.

   (a) {1 pts} Use Maxwell's equations to relate the value of E_0 to the surface charge density $\pm\,\sigma$ on the plates.

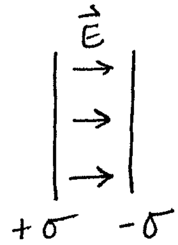
   (b) {4 pts} Lorentz transform $F^{\alpha\beta}$ to obtain the $\mathbf{E}$ and $\mathbf{B}$ fields seen by an observer moving between the plates with velocity c/2 $\hat{\imath}$?

   (c) {2 pts} What are the 4-current densities J^σ of the plates in the rest frame and in a frame moving with the observer?

   (d) {3 pts} Show that Maxwell's inhomogeneous equations are satisfied by your fields and charge-currents in the moving frame. (Hint: Using $\mathbf{E}$ and $\mathbf{B}$ rather than $F^{\alpha\beta}$ is probably easier.)



Part (a)



The electric field in this region is simply twice that of a single infinite sheet of charge. Using Gauss's law,

$$\oint \vec{E} \cdot d\vec{a} = 4\pi q_{enc}$$

$$E \oint da = 4\pi \sigma A$$

$$2EA = 4\pi \sigma A$$

$$E = 2\pi \sigma .$$

Then from both plates,

$$\vec{E}_{tot} = 2(2\pi\sigma)\,\hat{z}$$
$$= 4\pi\sigma\,\hat{z}$$

and thus,

$$\boxed{E_0 = 4\pi\sigma}$$

in

$$\vec{E} = E_0\,\hat{z} .$$

Part (b)

The transformation matrix is

$$L_x = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the electromagnetic field tensor is

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -b_z & b_y \\ E_y & b_z & 0 & -b_x \\ E_z & -b_y & b_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix}.$$

So

$$F'^{\alpha\beta} = L_x F^{\alpha\beta} L_x^T$$

$$F'^{\alpha\beta} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ \gamma E_z & -\beta\gamma E_z & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & -b_z' & b_y' \\ E_y' & b_z' & 0 & -b_x' \\ E_z' & -b_y' & b_x' & 0 \end{pmatrix}$$

Part (b), cont'd

Then since $v = \frac{c}{2}\hat{x}$, we have

$$\beta = \frac{v}{c} = \frac{1}{2}$$

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{2}{\sqrt{3}}$$

and

$$E_x' = 0 \qquad B_x' = 0$$

$$E_y' = 0 \qquad B_y' = \beta\gamma E_z = \frac{1}{2}\frac{2}{\sqrt{3}}E_z = \frac{E_z}{\sqrt{3}}$$

$$E_z' = \gamma E_z = \frac{2E_z}{\sqrt{3}} \qquad B_z' = 0$$

$$\vec{E}' = \frac{2}{\sqrt{3}}4\pi\sigma\hat{z} \qquad \vec{B}' = \frac{1}{\sqrt{3}}4\pi\sigma\hat{y}$$

$$\boxed{\vec{E}' = \frac{8\sqrt{3}\pi\sigma}{3}\hat{z}} \qquad \boxed{\vec{B}' = \frac{4\sqrt{3}\pi\sigma}{3}\hat{y}}.$$

Part (c)

We know

$$J^\alpha = \begin{pmatrix} \rho c \\ J_x \\ J_y \\ J_z \end{pmatrix}$$

where ρ is the charge density. In this case, we can write ρ as

$$\rho = \sigma \delta(z) - \sigma\delta(z-a)$$
$$\rho = \sigma[\delta(z) - \delta(z-a)].$$

In the rest frame, there is no current, so

$$\boxed{J^\alpha = \begin{pmatrix} c\sigma[\delta(z) - \delta(z-a)] \\ 0 \\ 0 \\ 0 \end{pmatrix}}.$$

In the moving frame,

$$J'^\alpha = L_x J^\alpha$$
$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\sigma[\delta(z) - \delta(z-a)] \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$J'^\alpha = \begin{pmatrix} c\gamma\sigma[\delta(z) - \delta(z-a)] \\ -\beta\gamma c\sigma[\delta(z) - \delta(z-a)] \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c\sigma' \\ J_x' \\ J_y' \\ J_z' \end{pmatrix}$$

so there is a current in the $-\hat{x}$-direction in the moving frame. Then

$$\boxed{J'^\alpha = \begin{pmatrix} \frac{2}{\sqrt{3}} c\sigma[\delta(z) - \delta(z-a)] \\ -\frac{1}{\sqrt{3}} c\sigma[\delta(z) - \delta(z-a)] \\ 0 \\ 0 \end{pmatrix}}.$$

Part (d)

We want to check that Maxwell's equations are satisfied in the moving frame.

$$① \quad \vec{\nabla}\cdot\vec{D}' = 4\pi\rho_F'$$

$$② \quad \vec{\nabla}\times\vec{H}' = \frac{4\pi}{c}\vec{J}_F' + \frac{1}{c}\frac{\partial D'}{\partial t}$$

$$③ \quad \vec{\nabla}\cdot\vec{B}' = 0$$

$$④ \quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

① Gauss's Law

$\vec{D} = \varepsilon_0\vec{E} = \vec{E}$ in Gaussian units

$$\vec{\nabla}\cdot\vec{E}' \stackrel{?}{=} 4\pi\rho_F'$$

$$\vec{\nabla}\cdot\vec{E}' = \vec{\nabla}\cdot\left(\frac{8\sqrt{3}\pi\sigma}{3}\hat{z}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\frac{8\sqrt{3}\pi\sigma}{3}\hat{z}\right)$$

$$= \frac{\partial}{\partial z}\frac{8\sqrt{3}\pi\sigma}{3}\hat{z}\cdot\hat{z}$$

But we can write σ as a step function,

$$\sigma\left[\theta(z) - \theta(z-a)\right]$$

and so

$$\vec{\nabla}\cdot\vec{E}' = \frac{\partial}{\partial z}\frac{8\sqrt{3}\pi}{3}\sigma\left[\theta(z)-\theta(z-a)\right]$$

$$= \frac{8\sqrt{3}\pi}{3}\sigma\left[\delta(z)-\delta(z-a)\right]$$

$$= 4\pi\,\frac{2\sqrt{3}}{3}\sigma\left[\delta(z)-\delta(z-a)\right]$$

$$\stackrel{\checkmark}{=} 4\pi\rho_F'.$$

Gauss's law is satisfied.

Part (d), cont'd

② Ampère-Maxwell Law

$\vec{H} = \frac{1}{\mu_0}\vec{B} = \vec{B}$ in Gaussian units and $\vec{D} = \varepsilon_0 \vec{E} = \vec{E}$.

$$\vec{\nabla} \times \vec{B}' \stackrel{?}{=} \frac{4\pi}{c} J_F' + \frac{1}{c}\frac{\partial E'}{\partial t}$$

$$\vec{\nabla} \times \vec{B}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & B' & 0 \end{vmatrix} = \hat{x}\left(-\frac{\partial}{\partial z}B'\right) - \hat{y}(0) + \hat{z}\left(\frac{\partial}{\partial x}B'\right)$$

$$= -\frac{\partial}{\partial z}B'\,\hat{x}$$

$$= -\frac{\partial}{\partial z}\left(\frac{4\sqrt{3}}{3}\pi\sigma[\theta(z) - \theta(z-a)]\right)\hat{x}$$

$$= -\frac{4\sqrt{3}}{3}\pi\sigma[\delta(z) - \delta(z-a)]\,\hat{x}$$

$$= \frac{4\pi}{c}\left(-\frac{\sqrt{3}}{3}c\sigma[\delta(z) - \delta(z-a)]\right)\hat{x}$$

$$\stackrel{\checkmark}{=} \frac{4\pi}{c} J_F'.$$

Clearly $\frac{\partial E'}{\partial t} = 0$, so indeed, the Ampère-Maxwell law is satisfied.

③ Gauss's law for magnetic fields

$$\vec{\nabla} \cdot \vec{B}' \stackrel{?}{=} 0$$

$$\vec{\nabla} \cdot \vec{B}' = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \frac{4\sqrt{3}}{3}\pi\sigma[\theta(z) - \theta(z-a)]\,\hat{y}$$

$\vec{B}' = \vec{B}'(z)$, but $\hat{z} \cdot \hat{y} = 0$ so, indeed, $\vec{\nabla} \cdot \vec{B}' \stackrel{\checkmark}{=} 0$. Gauss's law for magnetic fields is satisfied.

Part (d), cont'd

④ Faraday's law of induction

$$\vec{\nabla} \times \vec{E}' \overset{?}{=} -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{E}' \overset{?}{=} 0$$

$$\vec{\nabla} \times \vec{E}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E' \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}E'\right) - \hat{y}\left(\frac{\partial}{\partial x}E'\right) + \hat{z}(0)$$

$$= \frac{\partial E'}{\partial y}\hat{x} + \frac{\partial E'}{\partial x}(-\hat{y})$$

$$\overset{\checkmark}{=} 0$$

Thus, all of Maxwell's equations are satisfied.