Waves

FALL 2008

2. Consider a monochromatic plane electromagnetic wave of frequency ω propagating in a non-magnetic dielectric (with index of refraction n_1), traveling in the z direction and polarized in the x direction, which impinges normally upon a second non-magnetic semi-infinite dielectric material (with index of refraction n_2), where the boundary between the two media occurs at $z = 0$, as shown in Fig. 1. The incident electric field is

$$\mathbf{E}_I(z,t) = \hat{\mathbf{x}} E_{0I} e^{i(kz-\omega t)}.$$

There are no free charges or currents in either medium.

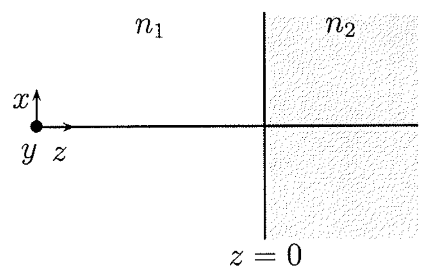

Figure 1: Plane wave normally incident on a surface separating two dielectric materials at $z = 0$. The medium in the the region $z < 0$ has index of refraction n_1 while the material in the region $z > 0$ has index of refraction n_2.

a) 1 pt. Use Maxwell's equations to determine the relation between k and ω in each region.

b) 1 pt. Use Maxwell's equations to determine the incident magnetic field, $\mathbf{B}_I(z,t)$, using the result of part b).

c) 1 pt. What are the forms of the reflected wave $\mathbf{E}_R(z,t)$, $\mathbf{B}_R(z,t)$ ($z < 0$), and of the transmitted wave $\mathbf{E}_T(z,t)$, $\mathbf{B}_T(z,t)$ ($z > 0$)?

d) 2pts. Apply the appropriate boundary conditions at the interface between the two media to obtain the equations determining the reflected amplitudes E_{0R} and B_{0R} and the transmitted amplitude E_{0T} and B_{0T} in terms of E_{0I}.

e) 2pts. Solve these equations for the reflection and transmission coefficients, $r = E_{0R}/E_{0I}$, $t = E_{0T}/E_{0I}$ in terms of the indices of refraction of the two media.

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~~f)~~ 2pts. Show that the averaged energy flux in a plane wave of amplitude E_0 moving in a medium with index of refraction n is given by (Gaussian units)

$$S = \frac{c}{8\pi} n |E_0|^2.$$

Show that the relative reflected and transmitted energy fluxes are

$$R = \frac{S_R}{S_I} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2, \quad T = \frac{S_T}{S_I} = \frac{4 n_1 n_2}{(n_1 + n_2)^2}.$$

~~g)~~ 1 pt. Show that $R + T = 1$. Why is this as expected?

Part (a)

We first want to find the relationship between k and ω in each region. Faraday's law tells us

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}.$$

We also know

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

since there is no current density. Then

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_0} = \frac{1}{c}\frac{\partial}{\partial t}(\varepsilon \vec{E})$$

where ε is either ε_1 or ε_2, depending on which material we consider. Since $n = \sqrt{\mu_0 \varepsilon}$ and $\mu_0 = 1$ in Gaussian units, $n_1 = \sqrt{\varepsilon_1}$ and $n_2 = \sqrt{\varepsilon_2}$, so

$$\varepsilon_1 = n_1^2$$
$$\varepsilon_2 = n_2^2,$$

and

$$\vec{\nabla} \times \vec{B} = \frac{n^2}{c}\frac{\partial}{\partial t}\vec{E}.$$

Taking the curl of Faraday's law,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}).$$

Substituting our above result,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{n^2}{c}\frac{\partial}{\partial t}\vec{E}\right)$$
$$= -\left(\frac{n}{c}\right)^2\frac{\partial^2}{\partial t^2}\vec{E}$$
$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\left(\frac{n}{c}\right)^2\frac{\partial^2}{\partial t^2}\vec{E}.$$

But $\vec{\nabla} \cdot \vec{E} = 0$, so

$$\vec{\nabla}^2 \vec{E} = \left(\frac{n}{c}\right)^2\frac{\partial^2}{\partial t^2}\vec{E}$$

Part (a), cont'd

Then

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{E} = \left(\frac{n}{c}\right)^2\frac{\partial^2}{\partial t^2}\vec{E}$$

$$-k^2 E_{0_I} e^{i(kz-\omega t)} = -\left(\frac{n}{c}\right)^2\omega^2 E_{0_I} e^{i(kz-\omega t)}$$

and thus,

$$k^2 = \left(\frac{n}{c}\right)^2\omega^2$$

$$k = \frac{n\omega}{c}.$$

In the first medium,

$$\boxed{k_1 = \frac{n_1\omega}{c}},$$

and in the second medium,

$$\boxed{k_2 = \frac{n_2\omega}{c}}.$$

Part (b)

Now we want to determine the incident magnetic field. We can use Faraday's law and our definition of k from part (a).

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \left(E_{0_I} e^{i(kz-\omega t)} \hat{x} \right) = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E & 0 & 0 \end{vmatrix} = \hat{x}(0) - \hat{y}\left(-\frac{\partial}{\partial z}E\right) + \hat{z}\left(-\frac{\partial}{\partial y}E\right)$$

$$\frac{\partial E}{\partial z}\hat{y} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$-ikc E_{0_I} e^{i(kz-\omega t)} \hat{y} = \frac{\partial \vec{B}}{\partial t}$$

and

$$\begin{aligned} \vec{B} &= -ikc E_{0_I} \int e^{i(kz-\omega t)} dt \ \hat{y} \\ &= -ikc E_{0_I} \left(-\frac{1}{i\omega} e^{i(kz-\omega t)} \right) \hat{y} \\ &= \frac{kc}{\omega} E_{0_I} e^{i(kz-\omega t)} \hat{y} \\ &= \left(\frac{n_1 \omega}{c}\right)\frac{c}{\omega} E_{0_I} e^{i(kz-\omega t)} \hat{y} \end{aligned}$$

$$\boxed{\vec{B} = n_1 E_{0_I} e^{i(kz-\omega t)} \hat{y}}$$

Part (c)

Since $\vec{E}_I$ is in solely the $\hat{x}$ direction...

$$\vec{E}_T(z,t) = \hat{x}\, E_{0_T} e^{i(k_T z - \omega t)}$$
$$\vec{B}_T(z,t) = \hat{y}\, n_2 E_{0_T} e^{i(k_T z - \omega t)}$$

for the transmitted wave, and

$$\vec{E}_R(z,t) = \hat{x}\, E_{0_R} e^{i(k(-z) - \omega t)}$$
$$\vec{B}_R(z,t) = (-\hat{y})\, E_{0_R} e^{i(k(-z) - \omega t)}\, n_1$$

for the reflected wave, where $k_T = k_2 = \frac{n_2 \omega}{c}$.

Part (d)

We want to find E_{0_R}, B_{0_R}, E_{0_T}, and B_{0_T} in terms of E_{0_I} by applying boundary conditions. We know

$$① \quad \vec{B}_{1,\perp} - \vec{B}_{2,\perp} = 0$$

$$② \quad \vec{E}_{1,\parallel} - \vec{E}_{2,\parallel} = 0$$

and

$$\vec{D}_{1,\perp} - \vec{D}_{2,\perp} = 4\pi\sigma_f \hat{n}$$

$$(n_1^2 \vec{E}_{1,\perp}) - (n_2^2 \vec{E}_{2,\perp}) = 4\pi\sigma_f \hat{n}$$

$$③ \quad (n_1^2 \vec{E}_{1,\perp}) - (n_2^2 \vec{E}_{2,\perp}) = 0$$

$$\vec{H}_{1,\parallel} - \vec{H}_{2,\parallel} = 4\pi \vec{K}_f \times \hat{n}$$

$$\left(\frac{1}{\mu_0} \vec{B}_{1,\parallel}\right) - \left(\frac{1}{\mu_0} \vec{B}_{2,\parallel}\right) = 4\pi \vec{K}_f \times \hat{n}$$

$$④ \quad \vec{B}_{1,\parallel} - \vec{B}_{2,\parallel} = 0 .$$

We can determine our amplitudes by using conditions ② and ④.

$$\vec{E}_{1,\parallel} - \vec{E}_{2,\parallel} = 0$$

$$\left[(\vec{E}_I + \vec{E}_R) - \vec{E}_T\right] \times \hat{n} = 0$$

$$\left[\hat{x} E_{0_I} e^{i(kz-\omega t)} + \hat{x} E_{0_R} e^{i(k(-z)-\omega t)} - \hat{x} E_{0_T} e^{i(k_2 z-\omega t)}\right] \times \hat{n} = 0 .$$

In this case, $\hat{n} = \hat{z}$, so

$$E_{0_I} e^{i(kz-\omega t)} \hat{y} + E_{0_R} e^{i(k(-z)-\omega t)} \hat{y} = E_{0_T} e^{i(k_2 z-\omega t)} \hat{y}$$

Part (d), cont'd

At the interface, $z=0$ so we end up with

$$ⓐ \quad E_{0_I} + E_{0_R} = E_{0_T}.$$

Now applying condition ④...

$$\vec{B}_{1,\parallel} - \vec{B}_{2,\parallel} = 0$$

$$\left[\left(\vec{B}_I + \vec{B}_R\right) - \vec{B}_T\right] \times \hat{z} = 0$$

$$\left[n_1 E_{0_I} e^{i(kz-\omega t)} \hat{y} + n_1 E_{0_R} e^{i(k(-z)-\omega t)} (-\hat{y}) - n_2 E_{0_T} e^{i(k_2 z-\omega t)} \hat{y}\right] \times \hat{z} = 0$$

$$n_1 E_{0_I} e^{i(kz-\omega t)} \hat{x} - n_1 E_{0_R} e^{i(k(-z)-\omega t)} \hat{x} = n_2 E_{0_T} e^{i(k_2 z-\omega t)}$$

Again, $z=0$ so we have

$$ⓑ \quad n_1 \left(E_{0_I} - E_{0_R}\right) = n_2 E_{0_T}.$$

Rearranging ⓑ for E_{0_R} and plugging into ⓐ...

$$E_{0_R} = E_{0_I} - \frac{n_2}{n_1} E_{0_T}$$

$$E_{0_I} + \left(E_{0_I} - \frac{n_2}{n_1} E_{0_T}\right) = E_{0_T}$$

$$2E_{0_I} - \frac{n_2}{n_1} E_{0_T} = E_{0_T}$$

$$E_{0_T}\left(1 + \frac{n_2}{n_1}\right) = 2E_{0_I}$$

$$\boxed{E_{0_T} = \frac{2n_1 E_{0_I}}{n_1 + n_2}}$$

Part (d), cont'd

Rearranging ⓑ for E_{0_T} and plugging into ⓐ ...

$$E_{0_T} = \frac{n_1}{n_2}\left(E_{0_I} - E_{0_R}\right)$$

$$E_{0_I} + E_{0_R} = \frac{n_1}{n_2}\left(E_{0_I} - E_{0_R}\right)$$

$$E_{0_R}\left(1 + \frac{n_1}{n_2}\right) = \left(\frac{n_1}{n_2} - 1\right)E_{0_I}$$

$$E_{0_R}\left(\frac{n_1+n_2}{n_2}\right) = \left(\frac{n_1-n_2}{n_2}\right)E_{0_I}$$

and

$$\boxed{E_{0_R} = \frac{n_1-n_2}{n_1+n_2}E_{0_I}}.$$

We know

$$\vec{B}_0 = n\hat{k}\times\vec{E}_0,$$

in general. For the reflected wave...

$$\vec{B}_{0_R} = -n_1\hat{z}\times\vec{E}_{0_R}$$

$$= n_1(-\hat{z})\times E_{0_R}\hat{x}$$

$$= n_1 E_{0_R}(-\hat{y})$$

so

$$B_{0_R} = n_1 E_{0_R}$$

$$\boxed{B_{0_R} = \frac{n_1(n_1-n_2)}{n_1+n_2}E_{0_I}}.$$

For the transmitted wave...

$$\vec{B}_{0_T} = n_2\hat{z}\times\vec{E}_{0_T}$$

$$= n_2 E_{0_T}\hat{y}$$

so

$$B_{0_T} = n_2 E_{0_T}$$

$$\boxed{B_{0_T} = \frac{2n_1n_2E_{0_I}}{n_1+n_2}}$$

Part (e)

The reflection coefficient is given by

$$r = \frac{E_{0R}}{E_{0I}}$$

$$= \frac{1}{E_{0I}}\left(\frac{n_1 - n_2}{n_1 + n_2} E_{0I}\right)$$

$$\boxed{r = \frac{n_1 - n_2}{n_1 + n_2}}.$$

The transmission coefficient is

$$t = \frac{E_{0T}}{E_{0I}}$$

$$= \frac{1}{E_{0I}}\left(\frac{2n_1 E_{0I}}{n_1 + n_2}\right)$$

$$\boxed{t = \frac{2n_1}{n_1 + n_2}}.$$

Part (f)

We know

$$\vec{S} = \frac{c}{8\pi} Re\left(\frac{n}{\mu}\right)(\vec{E}\cdot\vec{E}^*)\,\hat{k},$$

where $\hat{k}$ is the direction of propagation. Then

$$\vec{S} = \frac{c}{8\pi}\frac{1}{2}(n+n^*)|E_0|^2\,\hat{z}$$

$$= \frac{c}{8\pi}\frac{1}{2}(n+n)|E_0|^2\,\hat{z}$$

and

$$\boxed{S = \frac{c}{8\pi} n |E_0|^2},$$

as expected. We want to determine S_I, S_R, and S_T so we can find R and T.

① $$S_I = \frac{c}{8\pi} n_1 |E_{0_I}|^2$$

$$S_R = \frac{c}{8\pi} n_1 |E_{0_R}|^2$$

$$= \frac{c}{8\pi} n_1 \left| \frac{n_1 - n_2}{n_1 + n_2} E_{0_I} \right|^2$$

② $$S_R = \frac{c}{8\pi} n_1 \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 |E_{0_I}|^2$$

$$S_T = \frac{c}{8\pi} n_2 |E_{0_T}|^2$$

$$= \frac{c}{8\pi} n_2 \left| \frac{2n_1 E_{0_I}}{n_1 + n_2} \right|^2$$

③ $$S_T = \frac{c}{8\pi} n_2 \left(\frac{2n_1}{n_1 + n_2}\right)^2 |E_{0_I}|^2.$$

Part (f), cont'd

Then dividing ② by ① ...

$$R = \frac{S_R}{S_I}$$

$$= \frac{c}{8\pi} n_1 \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 |E_{0I}|^2 \cdot \frac{8\pi}{c} \frac{1}{n_1 |E_{0I}|^2}$$

$$\boxed{R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}.$$

Dividing ③ by ① ...

$$T = \frac{S_T}{S_I}$$

$$= \frac{c}{8\pi} n_2 \left(\frac{2n_1}{n_1 + n_2}\right)^2 |E_{0I}|^2 \cdot \frac{8\pi}{c} \frac{1}{n_1 |E_{0I}|^2}$$

$$\boxed{T = \frac{4n_1 n_2}{(n_1 + n_2)^2}}.$$

Part (g)

We want to show $R + T = 1$.

$$R + T = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 + \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

$$= \frac{n_1^2 - 2n_1 n_2 + n_2^2 + 4n_1 n_2}{(n_1 + n_2)^2}$$

$$= \frac{n_1^2 + 2n_1 n_2 + n_2^2}{n_1^2 + 2n_1 n_2 + n_2^2}$$

✓ $= 1$.

This makes sense because the reflected and transmitted fluxes are the only fluxes present after the wave is incident on the surface.