Relativistic

SPRING 2013

3. (a) [1 pts] Give the 4-current $J^\alpha(x^\beta)$ for the static surface charge density σ_0 shown in the figure (a thin uniform and infinite sheet of charge located at $z = 0$ in the lab).

   (b) [2 pts] Give the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ caused by the static surface charge.

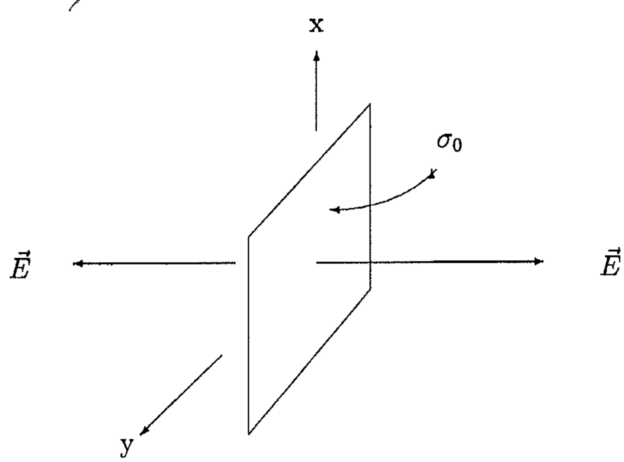
   (c) [2 pts] Compute the 4-current $J'^\alpha(x'^\beta)$ in a frame that moves with speed $v < c$ in the positive x-direction relative to the lab (v isn't necessarily small).

   (d) [1 pts] What is the surface charge density σ' in the moving frame?

   (e) [1 pts] What is the electric field $\mathbf{E}'$ in the moving frame?

   (f) [1 pts] What is the surface current density $\mathbf{K}'$ in the moving frame?

   (g) [2 pts] What is the magnetic induction $\mathbf{B}'$ in the moving frame?



Part (a)

In general,

$$J^\alpha = \begin{pmatrix} c\sigma \\ J_x \\ J_y \\ J_z \end{pmatrix}.$$

Since the charge on the sheet is static, we have $\vec{J} = 0$, and because the sheet is thin and infinite,

$$\sigma = \sigma_0 \, \delta(z).$$

Thus,

$$\boxed{J^\alpha = \begin{pmatrix} c\sigma_0 \, \delta(z) \\ 0 \\ 0 \\ 0 \end{pmatrix}}.$$

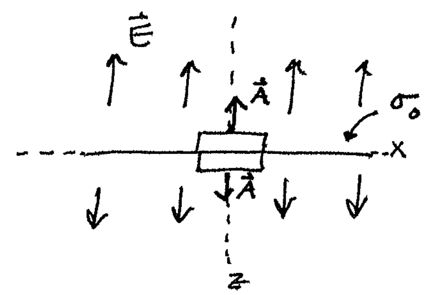
Part (b)

From Gauss's law,

$$\oint \vec{E} \cdot d\vec{a} = 4\pi q_{enc}$$

$$E \oint da = 4\pi \sigma_0 A$$

$$2EA = 4\pi \sigma_0 A$$



and

$$E = 2\pi\sigma_0.$$

Then

$$\boxed{\vec{E} = \begin{cases} 2\pi\sigma_0 \hat{z}, & z > 0 \\ -2\pi\sigma_0 \hat{z}, & z < 0 \end{cases}}$$

Since $\vec{J} = 0$, we know $\boxed{\vec{B} = 0}$.

Parts (c) and (d)

We want to boost our 4-current in the "lab" frame to the moving frame. The transformation matrix is

$$L_x = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$J'^{\alpha} = L_x J^{\alpha}$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\sigma_0 \delta(z) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c\sigma' \\ J_x' \\ J_y' \\ J_z' \end{pmatrix}$$

$$= \begin{pmatrix} c\gamma\sigma_0 \delta(z) \\ -\beta c\gamma\sigma_0\delta(z) \\ 0 \\ 0 \end{pmatrix}$$

So we have current in the $-\hat{x}$-direction since

$$\boxed{J_x' = \beta c \gamma \sigma_0 \delta(z)\,(-\hat{x})}.$$

We also have

$$c\sigma' = c\gamma\sigma_0\delta(z)$$

$$\boxed{\sigma' = \gamma\sigma_0\delta(z)}.$$

Parts (e) and (g)

The electromagnetic field tensor is

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$F'^{\alpha\beta} = L_x F^{\alpha\beta} L_x^T$$

$$F'^{\alpha\beta} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ \gamma E_z & \gamma\beta E_z & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & -B_z' & B_y' \\ E_y' & B_z' & 0 & -B_x' \\ E_z' & -B_y' & B_x' & 0 \end{pmatrix}$$

So

$$E_x' = 0 \qquad B_x' = 0$$
$$E_y' = 0 \qquad B_y' = -\gamma\beta E_z$$
$$E_z' = \gamma E_z \qquad B_z' = 0$$

$$\boxed{\vec{E}' = \begin{cases} \gamma 2\pi\sigma_0 \ \hat{z}, & z>0 \\ -\gamma 2\pi\sigma_0 \ \hat{z}, & z<0 \end{cases}} \qquad \boxed{\vec{B}' = \begin{cases} \gamma\beta 2\pi\sigma_0 \ (-\hat{y}), & z>0 \\ \gamma\beta 2\pi\sigma_0 \ \hat{y}, & z<0 \end{cases}}$$

Part (f)

In this case, we know that the surface current density is given by

$$\vec{K}' = \sigma' v$$

$$\boxed{K' = \gamma v \sigma_0 \delta(z)},$$

moving in the $-\hat{x}$-direction. This is equivalent to J_x', so

$$\vec{K} = J_x'.$$