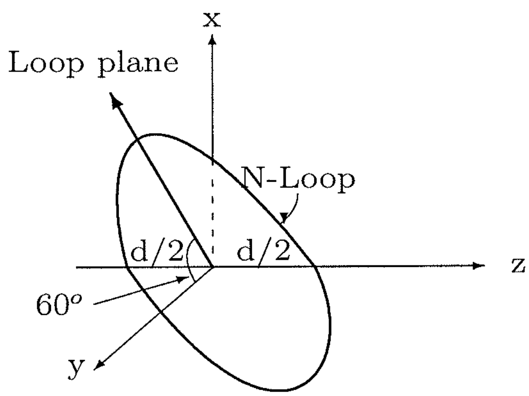
Waves

FALL 2011



4. In vacuum, a plane electromagnetic wave of angular frequency $\omega = c\,k = 2\pi\,c/\lambda$ travels parallel to the z-axis. The wave has an electric field given by

$$\mathbf{E} = \hat{\mathbf{j}}\; E_0 e^{i(kz-\omega t)},$$

A small N-loop (i.e., an N-turn circular coil) of diameter d very much smaller than the wavelength λ ($d << \lambda$) acting as an antenna is located with its center at the origin. It is oriented so that a diameter of the coil lies along the z-axis and the plane of the coil makes an angle $\theta = 60^o$ with the y-axis.

(a) {3 pts} Use Maxwell's equations to obtain the **B** field associated with the above wave?

(b) {3 pts} Compute the Magnetic flux through the N turn coil as a function of time.

(c) {4 pts} What is the peak EMF induced in the antenna?

Part (a)

From Faraday's law,

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}.$$

So

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E & 0 \end{vmatrix}$$

$$= \hat{x}\left(-\frac{\partial E}{\partial z}\right) - \hat{y}(0) + \hat{z}\left(\frac{\partial E}{\partial x}\right)$$

$$= \frac{\partial E}{\partial z}(-\hat{x}),$$

and

$$-\frac{\partial E}{\partial z}\hat{x} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$ikcE_0 e^{i(kz-\omega t)}\hat{x} = \frac{\partial \vec{B}}{\partial t}$$

$$\vec{B} = ikcE_0 \int e^{i(kz-\omega t)}dt\ \hat{x}$$

$$= -\frac{ikcE_0}{i\omega} e^{i(kz-\omega t)}\hat{x}$$

$$\boxed{\vec{B} = \frac{kc}{\omega}E_0 e^{i(kz-\omega t)}(-\hat{x})}.$$

Part (b)

Next we would like to calculate the magnetic flux through the coil as a function of time. The coil is angled at 60° with respect to the direction of propagation. The magnetic flux is given, in general, by

$$\Phi_B = \oint \vec{B} \cdot d\vec{a}$$

$$= \oint B \, da \cos\theta$$

$$= B \cos 60° \oint da$$

$$= B a \cos 60° .$$

This is for one loop of wire. For N-turns of coil,

$$\Phi_{B,tot} = N B a \cos 60°$$

$$= \frac{N}{2} \left(\pi \left(\frac{d}{2}\right)^2\right) B$$

$$= \frac{\pi d^2 N}{8} \frac{kc}{\omega} E_0 e^{i(kz - \omega t)} \Big|_{Re}$$

$$\boxed{\Phi_{B,tot} = \frac{\pi d^2 N k c}{8\omega} E_0 \cos(kz - \omega t)}$$

Part (c)

The EMF is given by

$$\mathcal{E} = -\frac{\partial \Phi_B}{\partial t}$$

$$= \frac{1}{8}\pi d^2 N k c E_0 \sin(kz - \omega t) .$$

The maximum occurs when

$$\sin(kz - \omega t) = 1,$$

so

$$\boxed{\mathcal{E}_{max} = \frac{1}{8}\pi d^2 N k c E_0} .$$