Sphere

SPRING 2009

4 Conducting Sphere

In this problem you are to prove that a perfectly conducting sphere acquires a magnetic dipole moment when placed in a uniform magnetic field $\vec{\mathbf{B}}_0$. Let the sphere have radius a. By perfectly conducting, we mean that there is no magnetic field in the interior of the sphere. As a result of the induced dipole moment, the magnetic field $\vec{\mathbf{B}}$ exterior to the sphere is no longer $\vec{\mathbf{B}}_0$. Determine the dipole moment as follows.

a) What are the boundary conditions on $\hat{\mathbf{n}} \cdot \vec{\mathbf{B}}$ and $\hat{\mathbf{n}} \times \vec{\mathbf{B}}$ at the surface of the sphere, where $\hat{\mathbf{n}}$ is the outward unit normal to the spherical surface? These boundary conditions involve the surface current density $\vec{\mathbf{K}}$, which will be determined below. **(1-point)**

b) Assume that the induced magnetic field is a pure magnetic dipole field, that is, in a spherical cordinate system with origin at the center of the sphere,

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 + \frac{3\vec{\mu} \cdot \vec{\mathbf{r}}\vec{\mathbf{r}} - \vec{\mu} r^2}{r^5}, \qquad r > a.$$

Use the boundary condition on $\hat{\mathbf{r}} \cdot \vec{\mathbf{B}}$ at $r = a_+$ to determine $\vec{\mu}$ in terms of a and $\vec{\mathbf{B}}_0$. **(3-points)**

c) Use the boundary condition on $\hat{\mathbf{r}} \times \vec{\mathbf{B}}$ at $r = a_+$ to determine the surface current density $\vec{\mathbf{K}}$ in terms of $\vec{\mathbf{r}}$ and $\vec{\mathbf{B}}_0$. **(3-points)**

d) Compute the magnetic dipole moment from the surface current according to

$$\vec{\mu} = \frac{1}{2c} \oint dS\, \hat{\mathbf{r}} \times \vec{\mathbf{K}}$$

where the integral extends over the surface of the sphere, and show that $\vec{\mu}$ coincides with the result found in part b. **(3-points)**

Part (a)

At the surface of the sphere, we must have

$$\hat{n} \cdot \vec{B} = 0$$

and

$$\hat{n} \times \vec{H} = \vec{K} \times \hat{n}$$

where $\vec{K}$ is the surface current density.

Part (b)

We know $\hat{n} \cdot \vec{B} = \hat{r} \cdot \vec{B} = 0$, so...

$$\hat{r} \cdot \vec{B} = \hat{r} \cdot \left[\vec{B}_0 + \frac{3\vec{\mu} \cdot \vec{r}\,\vec{r} - \vec{\mu} r^2}{r^5} \right] \Big|_{r=a_+}$$

$$= \hat{r} \cdot \vec{B}_0 + \frac{3(\vec{\mu} \cdot \vec{r})(\hat{r} \cdot \vec{r}) - (\hat{r} \cdot \vec{\mu}) r^2}{r^5} \Big|_{r=a_+}$$

$$= \hat{r} \cdot \vec{B}_0 + \frac{3(\vec{\mu} \cdot r\hat{r})(\hat{r} \cdot r\hat{r}) - (\hat{r} \cdot \vec{\mu}) r^2}{r^5} \Big|_{r=a_+}$$

$$= \hat{r} \cdot \vec{B}_0 + \frac{3a_+^2 (\vec{\mu} \cdot \hat{r}) - a_+^2 (\hat{r} \cdot \vec{\mu})}{a_+^5} = 0$$

$$\hat{r} \cdot \vec{B}_0 + \frac{3(\hat{r} \cdot \vec{\mu}) - (\hat{r} \cdot \vec{\mu})}{a_+^3} = 0$$

$$2(\hat{r} \cdot \vec{\mu}) = -a_+^3 \hat{r} \cdot \vec{B}_0$$

$$\hat{r} \cdot \vec{\mu} = -\frac{1}{2} a_+^3 \hat{r} \cdot \vec{B}_0$$

$$\hat{r} \cdot \vec{\mu} = \hat{r} \cdot \left(-\frac{1}{2} a_+^3 \vec{B}_0\right),$$

so

$$\boxed{\vec{\mu} = -\frac{1}{2} a_+^3 \vec{B}_0}.$$

Part (c)

We know

$$\hat{n} \times \vec{H} = \vec{K} \times \hat{n}$$

$$\hat{n} \times \left(\frac{1}{\mu_0}\vec{B}\right) = \vec{K} \times \hat{n}$$

$$\hat{n} \times \vec{B} = \mu_0 \vec{K} \times \hat{n}.$$

So

$$\hat{r} \times \vec{B} = \hat{r} \times \left[\vec{B}_0 + \frac{3\vec{\mu}\cdot\vec{r}\vec{r} - \vec{\mu}r^2}{r^5}\right]\Big|_{r=a_+}$$

$$= \hat{r} \times \vec{B}_0 + \frac{3(\vec{\mu}\cdot\vec{r})(\hat{r}\times\vec{r}) - (\hat{r}\times\vec{\mu})r^2}{r^5}\Big|_{r=a_+}$$

$$= \hat{r} \times \vec{B}_0 + \frac{3(\vec{\mu}\cdot\vec{r})(\hat{r}\times r\hat{r}) - (\hat{r}\times\vec{\mu})r^2}{r^5}\Big|_{r=a_+}$$

$$= r \times \vec{B}_0 - \frac{(\hat{r}\times\vec{\mu})}{r^3}\Big|_{r=a_+}$$

$$= \hat{r} \times \vec{B}_0 - \frac{(\hat{r}\times\vec{\mu})}{a_+^3} = \mu_0 \vec{K} \times \hat{r}$$

$$\hat{r} \times \vec{B}_0 - \frac{1}{a_+^3}\left(\hat{r} \times -\frac{1}{2}a_+^3\vec{B}_0\right) = -\mu_0 \hat{r} \times \vec{K}$$

$$\frac{3}{2}\hat{r} \times \vec{B}_0 = -\mu_0 \hat{r} \times \vec{K}$$

$$\hat{r} \times \left(\frac{3}{2}\vec{B}_0\right) = \hat{r} \times \left(-\mu_0 \vec{K}\right),$$

So

$$-\mu_0 \vec{K} = \frac{3}{2}\vec{B}_0$$

$$\boxed{\vec{K} = -\frac{3}{2\mu_0}\vec{B}_0}.$$