Relativistic

FALL 2010

4. (25 points) A 50 MeV electron ($m\gamma c^2 = 50$ MeV, $mc^2 = 0.5$ MeV) moving along the z-axis is decelerated and brought to a stop after traveling 10 cm in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{k}}$. (Recall $\gamma \equiv 1/\sqrt{1-\frac{v^2}{c^2}}$.)

(a) {3 pts} Compute $\gamma(t)$ assuming the electron starts its deceleration at t=0.

(b) {3 pts} How long does it take the electron to stop?

(c) {3 pts} Compute the total energy radiated by the electron during the 10 cm stopping process.

(d) {1 pts} What fraction of the electrons initial energy was lost to radiation?

Hint: The general Larmor formula for power radiated by an accelerating point charge is

$$P(t) = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0 c}\gamma^6[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2], \qquad (SI)$$

$$P(t) = \frac{2}{3}\frac{q^2}{c}\gamma^6[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2]. \qquad (Gaussian)$$

It might be useful to use $\dot{(\gamma\beta)} = \gamma^3\dot{\beta}$.

$$\begin{aligned} 1e &= 4.8\times10^{-10}\text{statcoul} = 1.6\times10^{-19}\text{coul}, \\ 1eV &= 1.6\times10^{-12}\text{ergs} = 1.6\times10^{-19}\text{J}. \end{aligned}$$

Part (a)

By Newton's second law for relativistic charge,

$$\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta$$

$$m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta$$

$$m\left(\gamma\frac{d}{dt}U^\alpha\right) = \frac{q}{c}F^{\alpha\beta}U_\beta .$$

Let $\alpha = 0$. We know $U_\beta = \gamma(c,-\vec{v})$ and $U^\alpha = \gamma(c,\vec{v})$, so

$$m\left(\gamma\frac{d}{dt}(\gamma c)\right) = \frac{q}{c}F^{0\beta}U_\beta$$

$$\gamma mc^2\frac{d\gamma}{dt} = q\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} \gamma c \\ -v_x\gamma \\ -v_y\gamma \\ -v_z\gamma \end{pmatrix}$$

$$\gamma mc^2\frac{d\gamma}{dt} = q\left(E_xv_x + E_yv_y + E_zv_z\right)$$

$$\gamma mc^2\frac{d\gamma}{dt} = qE_zv_z(t)$$

$$\gamma mc^2\frac{d\gamma}{dt} = qE_0v_z(t) .$$

but $\beta = \frac{v}{c}$, so $v(t) = \beta(t)\,c$ and

$$\gamma mc\frac{d\gamma}{dt} = \gamma qE_0\beta(t) .$$

We know

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} ,$$

so...

Part (a), cont'd

$$\gamma^2 = \frac{1}{1-\beta^2}$$

$$1-\beta^2 = \frac{1}{\gamma^2}$$

$$\beta^2 = 1 - \frac{1}{\gamma^2}$$

$$= \frac{\gamma^2 - 1}{\gamma^2}$$

$$\beta = \frac{(\gamma^2-1)^{1/2}}{\gamma}.$$

Then

$$mc\frac{d\gamma}{dt} = qE_0 \frac{(\gamma^2-1)^{1/2}}{\gamma}$$

$$\frac{\gamma}{(\gamma^2-1)^{1/2}}\,d\gamma = \frac{qE_0}{mc}\,dt$$

$$\int_{\gamma_0}^{\gamma} \frac{\gamma}{(\gamma^2-1)^{1/2}}\,d\gamma = \int_0^t \frac{qE_0}{mc}\,dt$$

Let $u = \gamma^2 - 1$. Then $du = 2\gamma\,d\gamma$ and so

$$\frac{1}{2}\int_{\gamma_0^2-1}^{\gamma^2-1} \frac{du}{u^{1/2}} = \int_0^t \frac{qE_0}{mc}\,dt$$

$$\frac{1}{2}\left(2u^{1/2}\right)\Big|_{\gamma_0^2-1}^{\gamma^2-1} = \frac{qE_0}{mc}t$$

$$(\gamma^2-1)^{1/2} - (\gamma_0^2-1)^{1/2} = \frac{qE_0}{mc}t$$

$$(\gamma^2-1)^{1/2} = \frac{qE_0}{mc}t + (\gamma_0^2-1)^{1/2}$$

$$\gamma^2 = \left[\frac{qE_0}{mc}t + (\gamma_0^2-1)^{1/2}\right]^2 + 1$$

Part (a), cont'd

and

$$\boxed{\gamma(t) = \left[\left(\frac{qE_0}{mc} t + (\gamma_0^2 - 1)^2 \right)^2 + 1 \right]^{1/2}} .$$

We are told

$$m\gamma c^2 = 50 \text{ MeV}$$
$$mc^2 = 0.5 \text{ MeV},$$

but I am going to leave my answer general. We can also note that $q = -e$.

Part (b)

We want to determine the amount of time it takes the electron to stop. We know this will be when $V = 0$, so

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$\gamma = 1.$$

Then

$$\gamma(t) = 1 = \left[\left(\frac{qE_0}{mc}t + (\gamma_0^{\,2}-1)^2\right)^2 + 1\right]^{1/2}$$

$$1 = \left(\frac{qE_0}{mc}t + (\gamma_0^{\,2}-1)^{1/2}\right)^2 + 1$$

$$\frac{qE_0}{mc}t = -(\gamma_0^{\,2}-1)^{1/2}$$

$$-\frac{eE_0}{mc}t = -(\gamma_0^{\,2}-1)^{1/2}$$

$$\boxed{t = \frac{mc}{eE_0}(\gamma_0^{\,2}-1)^{1/2}}$$

Part (c)

We know the power radiated by an accelerating point charge is

$$P(t) = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right].$$

but $\vec{\beta}\times\dot{\vec{\beta}} = 0$ since the velocity and acceleration have the same direction, so

$$\begin{aligned} P(t) &= \frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot{\vec{\beta}})^2\right] \\ &= \frac{2}{3}\frac{q^2}{c}\left(\gamma^3\dot{\vec{\beta}}\right)^2 \\ &= \frac{2}{3}\frac{q^2}{c}\left(\dot{\gamma\vec{\beta}}\right)^2 \end{aligned}$$

since $(\dot{\gamma\vec{\beta}}) = \gamma^3\dot{\vec{\beta}}$.