Method of Images

FALL 2007

1. A point charge Q is located a distance d from the center of a grounded sphere of radius R, as shown in the figure. The point charge is located outside the sphere, that is, $d > R$. Use the image method to answer the following questions.

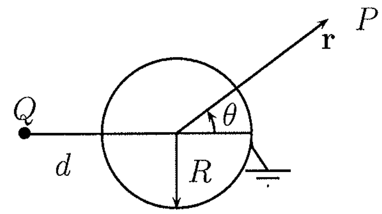


Figure 1: Point charge Q exterior to a grounded conducting sphere.

a) 2 pts. Find the position and magnitude of the image charge Q' that will make the potential zero on the surface of the sphere.

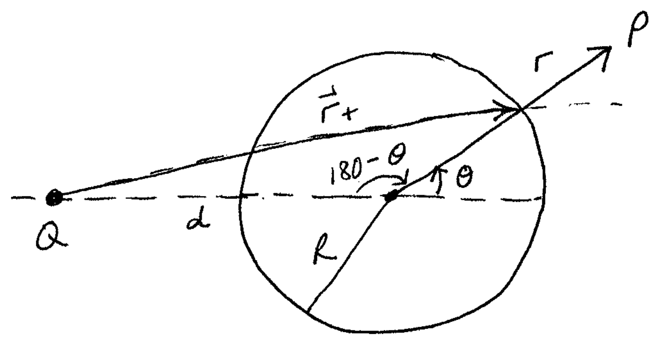
b) 2 pts. Show that the image method is applicable to this problem by proving that the result in a) will make the potential zero at an arbitrary point on the surface of the sphere.

c) 2 pts. Write down the expression for the potential at an arbitrary point $P(r, \theta)$ outside the sphere. Take the origin of the coordinate system to be the center of the sphere.

d) 2 pts. Use the result of part c) to calculate the radial component of the electric field, E_r, outside the sphere.

e) 2 pts. Use Gauss' law to find the total induced charge on the surface of the sphere.

Part (a)



Let Q' be an image charge within the sphere, at some distance d' from the center of the sphere. The vector pointing from Q' to the point where $\vec{r}$ intersects the surface is $\vec{r}'$. For a point charge, we know

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}.$$

Since the potential must be zero on the surface, we will have

$$\frac{1}{4\pi\varepsilon_0}\frac{Q}{r_+} + \frac{1}{4\pi\varepsilon_0}\frac{Q'}{r'} = 0.$$

We know

$$r_+ = \sqrt{d^2+R^2-2dR\cos(180-\theta)}$$
$$= \sqrt{d^2+R^2+2dR\cos\theta}$$

and

$$r' = \sqrt{d'^2+R^2-2d'R\cos(180-\theta)}$$
$$= \sqrt{d'^2+R^2+2d'R\cos\theta}\ .$$

So

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{d^2+R^2+2dR\cos\theta}} + \frac{Q'}{\sqrt{d'^2+R^2+2d'R\cos\theta}}\right] = 0$$

Part (a), cont'd

From Jackson, we know that

$$\boxed{d' = \frac{R^2}{d}}$$

and

$$\boxed{Q' = -\frac{RQ}{d}}.$$

Part (b)

Plugging these results into our potential expression,

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{d^2+R^2+2dR\cos\theta}} + \frac{-RQ/d}{\sqrt{(R^2/d)^2+R^2+2(R^2/d)R\cos\theta}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{d^2+R^2+2dR\cos\theta}} - \frac{Q}{\frac{d}{R}\sqrt{(\frac{R^2}{d})^2+R^2+2(\frac{R^2}{d})R\cos\theta}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{d^2+R^2+2dR\cos\theta}} - \frac{Q}{\sqrt{R^2+d^2+2dR\cos\theta}}\right]$$

and indeed, $V = 0$ on the surface.

Part (c)

Outside of the sphere, we no longer have $r=R$. Our potential is simply

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{d^2+r^2+2dr\cos\theta}} + \frac{Q'}{\sqrt{d'^2+r^2+2d'r\cos\theta}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{d^2+r^2+2dr\cos\theta}} + \frac{(-RQ/d)}{\sqrt{(R^2/d)^2+r^2+2(R^2/d)r\cos\theta}}\right]$$

and

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{d^2+r^2+2dr\cos\theta}} - \frac{Q}{\sqrt{R^2+\frac{r^2d^2}{R^2}+2dr\cos\theta}}\right]}$$

Part (d)

We know

$$\vec{E}_r = -\vec{\nabla}_r V = -\frac{\partial V}{\partial r}\,\hat{r}$$

so

$$\vec{E}_r = -\frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left[\frac{1}{\sqrt{R^2+\left(\frac{rd}{R}\right)^2+2dr\cos\theta}} - \frac{1}{\sqrt{d^2+r^2+2dr\cos\theta}}\right]\hat{r}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{2}\cdot\frac{\frac{2rd^2}{R^2}+2d\cos\theta}{\left(R^2+\left(\frac{rd}{R}\right)^2+2dr\cos\theta\right)^{3/2}} + \frac{1}{2}\cdot\frac{2r+2d\cos\theta}{\left(d^2+r^2+2dr\cos\theta\right)^{3/2}}\right]\hat{r}$$

$$\boxed{\vec{E}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{rd^2+d\cos\theta}{R^2\left(R^2+\left(\frac{rd}{R}\right)^2+2dr\cos\theta\right)^{3/2}} - \frac{r+d\cos\theta}{\left(d^2+r^2+2dr\cos\theta\right)^{3/2}}\right]\hat{r}}\,.$$

Part (e)

In differential form, Gauss's law says

$$\vec{\nabla} \cdot \vec{E} = \frac{\sigma}{\varepsilon_0}.$$

So

$$\sigma = \varepsilon_0 \vec{\nabla} \cdot \vec{E} \Big|_{r=R}$$

$$\sigma = \varepsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} \left[\frac{r^3 d^2 + r^2 d \cos\theta}{R^2 \left(R^2 + \left(\frac{rd}{R}\right)^2 + 2dr\cos\theta\right)^{3/2}} - \frac{r^3 + r^2 d \cos\theta}{\left(d^2 + r^2 + 2dr\cos\theta\right)^{3/2}} \right] \Bigg|_{r=R}$$

After taking the derivative, we would let $r = R$. After simplifying, we would have an expression for σ.