Relativistic

SPRING 2008

2. Consider an infinite slab of thickness d, carrying uniform charge density ρ, centered on the origin and extending in the x-y plane. Assume both the electric permittivity ε and the magnetic permeability μ have their vacuum values.

~~a)~~ 2 pt. Find the electric field vector, $\mathbf{E}$, and magnetic flux density (magnetic induction), $\mathbf{B}$, everywhere. Do not just write the answer down, but in all cases clearly articulate your arguments and solution to receive credit.

For parts (b)–(e) consider an observer moving at velocity $\mathbf{v} = v_0\hat{\mathbf{x}}$. *Do not assume that* $v_0 \ll c$.

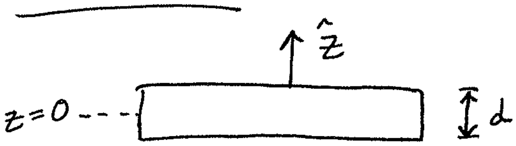
~~b)~~ 1 pt. What is the current density, $\mathbf{J}'$, in the observer's frame of reference? [Hint: How does the charge density transform?]

1/2 ~~c)~~ 3 pt. Find the electric field vector, $\mathbf{E}'$, and magnetic flux density, $\mathbf{B}'$, everywhere in the observer's frame of reference. Do this by solving Ampére's and Gauss' law in the observer's frame.

~~d)~~ 2 pt. Alternatively, obtain the same result by performing a Lorentz transformation on the fields found in part (a).

~~e)~~ 2 pt. Show explicitly that $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ have the same value in both the rest frame of the slab and the observer's frame. Why is that? Is it possible to find a frame where $\mathbf{E} = \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$?

Part (a)



We have two regions we must consider; within the slab and outside the slab. Using Gauss's law...

Inside ($|z| < d$)

Consider a cubical Gaussian surface centered on $z=0$, with thickness $2z$, and surface area on top and bottom of A. Then

$$\oint \vec{E} \cdot d\vec{a} = 4\pi q_{enc}$$

$$E(2A) = 4\pi \rho A (2z)$$

$$\boxed{\vec{E}_{in} = 4\pi \rho z \, \hat{z}}.$$

Outside ($|z| > d$)

Consider a cubical Gaussian surface centered on $z=0$. Then

$$\oint \vec{E} \cdot d\vec{a} = 4\pi q_{enc}$$

$$E(2A) = 4\pi \rho A d$$

$$\boxed{\vec{E}_{out} = 2\pi \rho d \, \hat{z}}.$$

In both cases, the field is directed away from the slab, so for values of z below $z=0$, the direction of the field is $(-\hat{z})$.

The magnetic induction is $\boxed{\vec{B} = 0}$ since there are no moving charges.

Part (b)

Let $\vec{v} = v_0 \hat{x}$ in the moving frame, and say $\beta = \frac{v_0}{c}$. Then

$$J'^{\alpha} = L_x J^{\alpha}$$

where

$$L_x = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$J^{\alpha} = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} c\rho \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So

$$J'^{\alpha} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\rho \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{J'^{\alpha} = \begin{pmatrix} \gamma c\rho \\ -\beta\gamma c\rho \\ 0 \\ 0 \end{pmatrix}} = \begin{pmatrix} c\rho' \\ J_x' \\ J_y' \\ J_z' \end{pmatrix}.$$

This means that there is a current in the $-\hat{x}$-direction.

Part (c)

First, we transform our coordinates.

$$x'^{\alpha} = L_x x^{\alpha}$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \gamma ct - \beta\gamma x \\ -\beta\gamma ct + \gamma x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}.$$

Since $z = z'$ and $\rho' = \gamma\rho$, we simply use our results from part (a) with this new density. So

$$\boxed{\vec{E}_{in} = 4\pi\gamma\rho z\,\hat{z}}$$

$$\boxed{\vec{E}_{out} = 2\pi\gamma\rho d\,\hat{z}},$$

where again, the field is directed away from the slab. So for values of $z<0$, the direction of the field is $(-\hat{z})$.

Part (d)

The electromagnetic field tensor in the "lab" frame is

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$F'^{\alpha\beta} = L_x F^{\alpha\beta} L_x^T$$

$$F'^{\alpha\beta} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ \gamma E_z & -\beta\gamma E_z & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & -B_z & B_y \\ E_y' & B_z & 0 & -B_x \\ E_z' & -B_y & B_x & 0 \end{pmatrix}$$

Part (d), cont'd

Then we have

$$E_x' = 0 \qquad B_x' = 0$$
$$E_y' = 0 \qquad B_y' = \beta\gamma E_z$$
$$E_z' = \gamma E_z \qquad B_z' = 0$$

$$\boxed{\begin{aligned}\vec{E}_{in}' &= \pm 4\pi\gamma\rho z\,\hat{z}\\ \vec{E}_{out}' &= \pm 2\pi\gamma\rho d\,\hat{z}\end{aligned}} \qquad \boxed{\begin{aligned}\vec{B}_{in}' &= \pm 4\pi\beta\gamma\rho z\,\hat{y}\\ \vec{B}_{out}' &= \pm 2\pi\beta\gamma\rho d\,\hat{y}\end{aligned}}$$

Part (e)

In the rest frame, $\vec{B} = 0$ so

$$\vec{E}\cdot\vec{B} = 0,$$

and

$$E_{in}^2 - B^2 = 16\pi^2\rho^2 z^2$$
$$E_{out}^2 - B^2 = 4\pi^2\rho^2 d^2.$$

In the moving frame,

$$\vec{E}\cdot\vec{B} = 0$$

since $\hat{z}\cdot\hat{y} = 0$, which is the same as in the rest frame, and

$$E_{in}'^2 - B_{in}'^2 = 16\pi^2\rho^2 z^2\gamma^2(1-\beta^2)$$
$$= 16\pi^2\rho^2 z^2$$
$$E_{out}'^2 - B_{out}'^2 = 4\pi^2\rho^2 d^2\gamma^2(1-\beta^2)$$
$$= 4\pi^2\rho^2 d^2,$$

which also agrees with the rest frame result. This is due to the fact that the $\vec{E}$ and $\vec{B}$ fields are Lorentz invariants.

We can never have $\vec{E} = 0$, but $\vec{B} \neq 0$, because moving charges create magnetic fields. If a magnetic field is present, there must be an electric field due to the charge.