Sphere

FALL 2005

4. A dielectric sphere of radius R (with dielectric constant ϵ) is embedded with free charge so that it acquires a polarization of $\mathbf{P} = (K/r)\,\hat{\mathbf{r}}$ where K is a given constant, r is the distance from the center of the sphere, and $\hat{\mathbf{r}}$ is the radial unit vector.

   (a) [2 points] Calculate the bound charge volume and surface densities ρ_b and σ_b respectively.

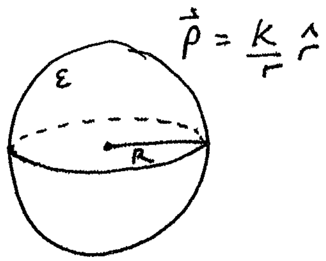
   (b) [1 point] Find the electric field $\mathbf{E}$ inside the dielectric.

   (c) [2 points] Calculate the volume density of free charge ρ_f.

   (d) [1 point] Find the electric field $\mathbf{E}$ outside the dielectric.

   (e) [4 points] Calculate the electric potential V, inside and outside the dielectric.

Part (a)



We want to calculate the bound charge volume and surface densities.

We know

$$\sigma_b = \vec{P} \cdot \hat{n} \Big|_{\text{surface}},$$

so

$$\sigma_b = \frac{K}{r} \hat{r} \cdot \hat{r} \Big|_{r=R}$$

$$\boxed{\sigma_b = \frac{K}{R}}.$$

We also have

$$\rho_b = -\vec{\nabla} \cdot \vec{P},$$

so

$$\rho_b = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P\right)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{K}{r}\right)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(rK\right)$$

$$\boxed{\rho_b = -\frac{K}{r^2}}.$$

Part (b)

We want to find $\vec{E}$ within the dielectric. We know

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$
$$= \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E}$$
$$= \varepsilon_0 (1+\chi_e)\vec{E}$$
$$\vec{D} = \varepsilon_0 \varepsilon \vec{E}.$$

So

$$\varepsilon_0 \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}$$
$$\vec{E} = \frac{\vec{P}}{\varepsilon_0(1-\varepsilon)}$$
$$\boxed{\vec{E} = \frac{K}{\varepsilon_0(1-\varepsilon)r}\,\hat{r}}\,.$$

Part (c)

Now we want to calculate the volume density of free charge, ρ_f. We know

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\varepsilon},$$

so

$$\rho_f = \varepsilon\left(\vec{\nabla}\cdot\vec{E}\right)$$

$$= \varepsilon\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E\right)\right]$$

$$= \frac{\varepsilon}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{K}{\varepsilon_0(1-\varepsilon)r}\right)$$

$$= \frac{\varepsilon}{r^2}\frac{\partial}{\partial r}\left(\frac{Kr}{\varepsilon_0(1-\varepsilon)}\right)$$

$$\boxed{\rho_f = \frac{\varepsilon}{\varepsilon_0(1-\varepsilon)}\frac{K}{r^2}}.$$

Part (d)

Now we want to find the electric field outside of the dielectric. We know

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}.$$

Then

$$\begin{aligned} q_{enc} &= \int \rho_{tot}\, d^3x \\ &= \int (\rho_f + \rho_b)\, r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \int \left(-\frac{K}{r^2} + \frac{\varepsilon}{\varepsilon_0(1-\varepsilon)}\frac{K}{r^2}\right) r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \left(\frac{\varepsilon}{\varepsilon_0(1-\varepsilon)} - 1\right) \int \frac{K}{r^2}\, r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= 4\pi \left(\frac{\varepsilon}{\varepsilon_0(1-\varepsilon)} - 1\right) KR. \end{aligned}$$

So

$$\oint \vec{E} \cdot d\vec{a} = \frac{4\pi}{\varepsilon_0}\left(\frac{\varepsilon}{\varepsilon_0(1-\varepsilon)} - 1\right) KR$$

$$E(4\pi r^2) = \frac{4\pi}{\varepsilon_0}\left(\frac{\varepsilon}{\varepsilon_0(1-\varepsilon)} - 1\right) KR$$

and

$$\boxed{\vec{E} = \left(\frac{\varepsilon}{\varepsilon_0^2(1-\varepsilon)} - \frac{1}{\varepsilon_0}\right)\frac{KR}{r^2}\,\hat{r}}\ .$$

Part (e)

Now we must determine the electric potential both inside and outside the sphere. Outside the sphere, taking the zero point at infinity ...

$$\vec{E} = -\vec{\nabla} V$$
$$= -\frac{\partial}{\partial r} V$$

$$V_{out} = -\int_{\infty}^{r} E_{out}\, dr$$
$$= -\int_{\infty}^{r} \left(\frac{\varepsilon}{\varepsilon_0^2 (1-\varepsilon)} - \frac{1}{\varepsilon_0} \right) \frac{KR}{r^2}\, dr$$
$$= \left(\frac{\varepsilon}{\varepsilon_0^2 (1-\varepsilon)} - \frac{1}{\varepsilon_0} \right) KR \left(\frac{1}{r} \right) \Big|_{\infty}^{r}$$
$$\boxed{V_{out} = \left(\frac{\varepsilon}{\varepsilon_0^2 (1-\varepsilon)} - \frac{1}{\varepsilon_0} \right) \frac{KR}{r}} .$$

Inside the sphere...

$$V_{in} = -\int_{\infty}^{r} E\, dr$$
$$= -\int_{\infty}^{R} E_{out}\, dr - \int_{R}^{r} E_{in}\, dr$$
$$= \left(\frac{1}{\varepsilon_0} - \frac{\varepsilon}{\varepsilon_0^2 (1-\varepsilon)} \right) K - \int_{R}^{r} \frac{K}{\varepsilon_0 (1-\varepsilon) r}\, dr$$
$$V_{in} = \left(\frac{1}{\varepsilon_0} - \frac{\varepsilon}{\varepsilon_0^2 (1-\varepsilon)} K \right) - \frac{K}{\varepsilon_0 (1-\varepsilon)} \ln \left(\frac{r}{R} \right)$$
$$\boxed{V_{in} = \frac{K}{\varepsilon_0 (1-\varepsilon)} \left[\frac{1-\varepsilon}{K} - \frac{\varepsilon}{\varepsilon_0 (1-\varepsilon)} - \ln \left(\frac{r}{R} \right) \right]} .$$