Relativistic

FALL 2013

5. An infinitely long, uniformly charged wire of radius a and total charge per unit length λ, is at rest on the z-axis of the lab frame.

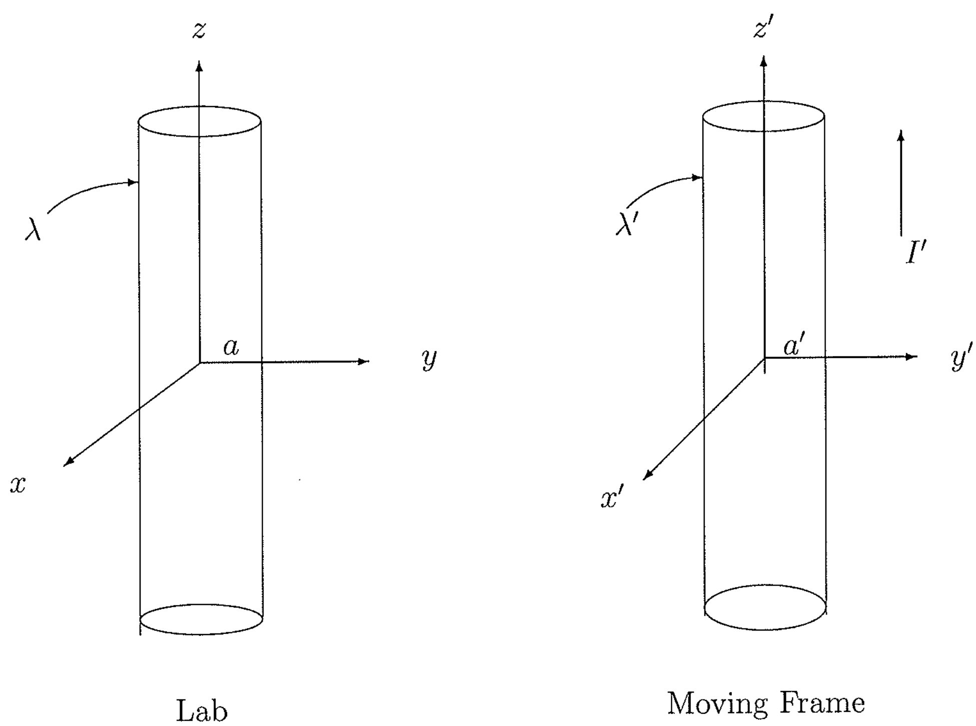
   (a) [2 pts] Compute the electric field $\mathbf{E}(x,y,z)$ exterior to the wire in the lab frame by solving Gauss's law in that frame. What is the magnetic induction $\mathbf{B}(x,y,z)$ in this frame?

   (b) [2 pts] If you are moving in the lab's negative z direction with speed v how are your spatial and time coordinates related to those of the lab's? To answer this question simply give the Lorentz boost $x'^{\sigma} = L^{\sigma}{}_{\mu} x^{\mu}$ that relates the two sets of coordinates.

   (c) [2 pts] In your frame what is the radius a' of the wire? What is the charge/length λ' of the wire and what is the current I' in the wire?

   (d) [1 pts] Combine the E and B fields in the lab into a single electromagnetic field tensor $F^{\alpha\beta}$ using $F^{\sigma\mu} = -F^{\mu\sigma}$ and $F^{0i} = -E^{i}$. In Gaussian units $F^{12} = -B^{z}$, $F^{23} = -B^{x}$ and $F^{13} = B^{y}$, and in SI units $F^{12} = -c\,B^{z}$, $F^{23} = -c\,B^{x}$ and $F^{13} = c\,B^{y}$, .

   (e) [3 pts] What electric field $\mathbf{E}\,'(x',y',z')$ and what magnetic induction $\mathbf{B}\,'(x',y',z')$ will you measure exterior to the wire in your frame? To answer this part you can use your answers for part (c) or you can compute $F' = LFL^{T}$.



Part (a)

From Gauss's law, we have

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$$

$$E \oint da = \frac{\lambda L}{\varepsilon_0}$$

$$E(2\pi r L) = \frac{\lambda L}{\varepsilon_0}$$

$$\vec{E} = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{r} \hat{r} .$$

But we want this in Cartesian coordinates. We know

$$r = \sqrt{x^2+y^2}$$

since the axis of the cylinder is aligned with the z-axis. We also know

$$\hat{r} = \cos\phi \, \hat{x} + \sin\phi \, \hat{y} ,$$

so

$$\boxed{\vec{E} = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{\sqrt{x^2+y^2}} (\cos\phi \, \hat{x} + \sin\phi \, \hat{y})} .$$

Since there is no current flowing through the wire, then we must have

$$\boxed{\vec{B} = 0} .$$

Part (b)

To relate the spatial and time coordinates of the moving frame to those of the lab's frame, we use the Lorentz boost,

$$x'^{\sigma} = L^{\sigma}_{\mu} x^{\mu}.$$

Since we are moving in the lab's $-z$-direction with speed v, we can say

$$L_z = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix},$$

where the β's have picked up a negative sign due to the direction of motion, making them positive in our matrix. Using matrix notation to make things simpler,

$$x'^{\alpha} = L_z x^{\alpha}.$$

Then

$$x'^{\alpha} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$= \begin{pmatrix} \gamma ct + \beta\gamma z \\ x \\ y \\ ct\beta\gamma + \gamma z \end{pmatrix}.$$

So

$$ct' = \gamma ct + \beta\gamma z \qquad z' = ct\beta\gamma + \gamma z \qquad \boxed{x' = x}$$

$$\boxed{t' = \frac{\gamma}{c}(ct + \beta z)} \qquad \boxed{z' = \gamma(ct\beta + z)} \qquad \boxed{y' = y}$$

Part (c)

We can determine the charge density λ' and current I' in the wire by boosting the 4-current. So

$$
\begin{aligned}
J'^{\alpha} &= L_z J^{\alpha} \\
&= \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} c\lambda \\ J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} c\lambda' \\ J_x' \\ J_y' \\ J_z' \end{pmatrix} \\
&= \begin{pmatrix} \gamma c\lambda + \beta\gamma J_z \\ J_x \\ J_y \\ \beta\gamma c\lambda + \gamma J_z \end{pmatrix}.
\end{aligned}
$$

But $\vec{J} = 0$, so

$$
\begin{pmatrix} c\lambda' \\ J_x' \\ J_y' \\ J_z' \end{pmatrix} = \begin{pmatrix} \gamma c\lambda \\ 0 \\ 0 \\ \beta\gamma c\lambda \end{pmatrix}
$$

and

$$c\lambda' = \lambda c\gamma$$

$$\boxed{\lambda' = \gamma\lambda}$$

$$\boxed{J_z' = \gamma\beta c\lambda}.$$

This means in the moving frame, current flows in the z-direction. The radius is still 'a' since we are moving in the z-direction.

We can also say

$$
\begin{aligned}
I' &= \oint \vec{J} \cdot d\vec{a} \\
&= J_z' (\pi a^2) \\
I' &= \pi a^2 \gamma\beta c\lambda .
\end{aligned}
$$

Part (d)

Our electromagnetic field tensor is

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

In the lab, $\vec{B} = 0$ and $E_z = 0$, so

$$F^{\alpha\beta}_{lab} = \begin{pmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & 0 \\ E_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Part (e)

We know

$$F'^{\alpha\beta} = L_z F^{\alpha\beta} L_z^T ,$$

so

$$F'^{\alpha\beta} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & 0 \\ E_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\gamma E_x & -\gamma E_y & 0 \\ E_x & 0 & 0 & 0 \\ E_y & 0 & 0 & 0 \\ 0 & -\beta\gamma E_x & -\beta\gamma E_y & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\gamma E_x & -\gamma E_y & 0 \\ \gamma E_x & 0 & 0 & \beta\gamma E_x \\ \gamma E_y & 0 & 0 & \beta\gamma E_y \\ 0 & -\beta\gamma E_x & -\beta\gamma E_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & -cB_z' & cB_y' \\ E_y' & cB_z' & 0 & -cB_x' \\ E_z' & -cB_y' & cB_x' & 0 \end{pmatrix}$$

So

$$E_x' = \gamma E_x \qquad cB_x' = -\beta\gamma E_y$$

$$E_y' = \gamma E_y \qquad cB_y' = \beta\gamma E_x$$

$$E_z' = 0 \qquad cB_z' = 0$$

$$\vec{E}' = \gamma\left(E_x + E_y\right) \qquad \vec{B}' = \frac{\beta\gamma}{c}\left(-E_y + E_x\right)$$

$$\boxed{\vec{E}' = \frac{1}{2\pi\varepsilon_0}\frac{\gamma\lambda}{\sqrt{x^2+y^2}}\left(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right)} \qquad \boxed{\vec{B}' = \frac{1}{c2\pi\varepsilon_0}\frac{\beta\gamma\lambda}{\sqrt{x^2+y^2}}\left(\sin\phi\,\hat{x} - \cos\phi\,\hat{y}\right)}$$

It makes sense that we now have a magnetic field according to the moving frame since this frame also observes a current.