Sphere

SPRING 2012

1. Dielectric Sphere

   A dielectric sphere of radius $R$ is polarized so that $\mathbf{P} = (K/r)\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ is the unit radial vector. Assume the sphere is in an empty vacuum and that the sphere's dielectric material is linear and isotropic, calculate

   (a) (3 pts) the volume and the surface densities of bound charge,

   (b) (2 pts) the volume density of free charge,

   (c) (2 pts) the electric field inside the sphere,

   (d) (3 pts) the electric field outside the sphere.

   Your answers should be given in terms of $K, \chi_E, \epsilon_0, \epsilon$, and/or $\epsilon_r$. Recall that for linear isotropic materials:

   In SI units,

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \epsilon_0\chi_E\mathbf{E}$$

$$\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = 1 + \chi_E$$

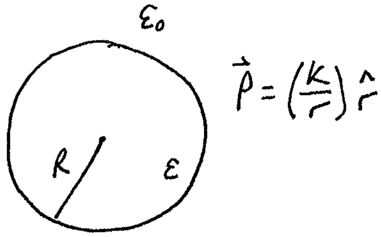
   In Gaussian units,

$$\mathbf{D} = \epsilon\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}$$

$$\mathbf{P} = \chi_E\mathbf{E}$$

$$\epsilon = 1 + 4\pi\chi_E = \epsilon_r$$

## Part (a)




The surface bound charge density is

$$\sigma_b = \vec{P}\cdot\hat{n}\big|_{surface}$$
$$= \vec{P}\cdot\hat{r}\big|_{r=R}$$
$$= \left(\frac{K}{r}\hat{r}\right)\cdot\hat{r}\,\Big|_{r=R}$$
$$\boxed{\sigma_b = \frac{K}{R}}.$$

The volume bound charge density is given by

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{K}{r}\right)$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(Kr)$$
$$\boxed{\rho_b = \frac{-K}{r^2}}$$

# Part (b) and Part (c)

The volume density of free charge is given by

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\varepsilon_0},$$

so we need the electric field inside the material. We have

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_0\vec{E} + \vec{P}$$

$$(\varepsilon-\varepsilon_0)\vec{E} = \vec{P}$$

$$\vec{E} = \frac{1}{\varepsilon-\varepsilon_0}\vec{P}$$

and so

$$\boxed{\vec{E} = \frac{1}{\varepsilon-\varepsilon_0}\frac{K}{r}\hat{r}},$$

where $r<R$. Then

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(\frac{1}{\varepsilon-\varepsilon_0}\frac{K}{r}\hat{r}\right) = \frac{\rho_f}{\varepsilon_0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\varepsilon-\varepsilon_0}\frac{K}{r}\right) = \frac{\rho_f}{\varepsilon_0}$$

$$\frac{K\varepsilon_0}{(\varepsilon-\varepsilon_0)r^2} = \rho_f$$

and

$$\rho_f = \frac{K}{\varepsilon/\varepsilon_0 - 1}\frac{1}{r^2}$$

$$\boxed{\rho_f = \frac{K}{\varepsilon_r - 1}\frac{1}{r^2}}.$$

## Part (d)

The electric field outside the sphere must be due to all of the free and bound charge. Using Gauss's law,

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0} .$$

But

$$q_{enc} = \int \sigma_b \, da + \int \rho_b \, dV + \int \rho_f \, dV$$

$$q_{enc} = \int_0^{2\pi}\int_0^{\pi} \frac{K}{R} R^2 \sin\theta \, d\theta \, d\phi + \int_0^{2\pi}\int_0^{\pi}\int_0^{R} -\frac{K}{r^2} r^2 \sin\theta \, d\theta \, d\phi \, dr + \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{K}{\varepsilon_r - 1} \frac{1}{r^2} r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 4\pi K R - 4\pi K R + \frac{4\pi K R}{\varepsilon_r - 1}$$

$$= \frac{4\pi K R}{\varepsilon_r - 1} .$$

So

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} \frac{4\pi K R}{\varepsilon_r - 1}$$

$$E(4\pi r^2) = \frac{1}{\varepsilon_0} \frac{4\pi K R}{\varepsilon_r - 1}$$

$$\boxed{\vec{E} = \frac{K}{\varepsilon_0} \frac{R}{r^2(\varepsilon_r - 1)} \hat{r}}$$

outside of the sphere.