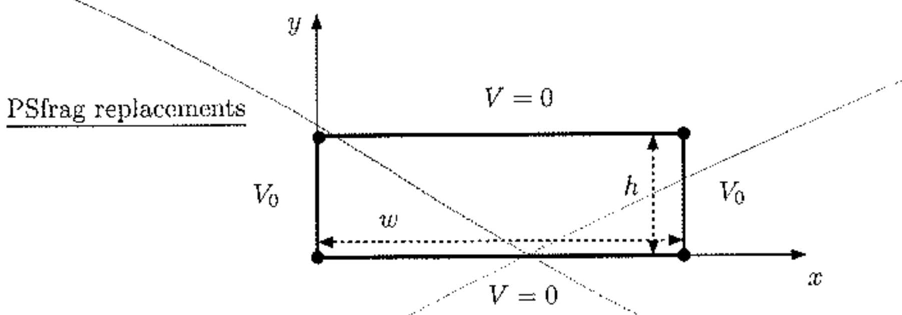
3. A very long channel (aligned with the z-axis) is made of four conducting plates insulated from each other. The top and bottom plates are grounded while the two side plates are held at a potential V_0 (see figure). Find the electric potential, $V(x, y)$, at any point inside the channel.



(a) [2 points] Using Laplace's equation and the method of separation of variables, find the differential equations for $X(x)$ and $Y(y)$ assuming $V(x, y) = X(x)Y(y)$.

(b) [2 points] Determine the general solutions of the differential equations in part (a).

(c) [3 points] Find the particular solutions in part (b) that satisfy the boundary conditions. Express your answer as a Fourier series with arbitrary coefficients.

(d) [3 points] Use Fourier's method to evaluate the coefficients in part (c).

4. Electromagnetic plane waves

(a) [1 point] Write down real expressions for the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields of a plane wave linearly polarized in the $\hat{\mathbf{x}}$ direction. (Take E_0 and B_0 as the amplitudes.)

(b) [2 points] For the wave of part a, determine the instantaneous and time-averaged energy density, u and $< u >$. Also determine the instantaneous Poynting vector and its time average $\vec{\mathbf{S}}$ and $< \vec{\mathbf{S}} >$.

(c) [1 point] A circular loop of wire can be used to detect electromagnetic waves. Suppose a 100 MHz FM station radiates 50 kW uniformly in all directions. What is the wavelength of the radiation.

(d) [6 points] What is the maximum rms voltage induced in a loop of radius 0.3 m at a distance of 10^5 m from the station in part c.

Part (a)

If the wave propagates in the $\hat{z}$ direction...

$$\vec{E} = E_0 e^{i(kz - \omega t)} \hat{x} .$$

We know

$$\vec{B} = n \hat{k} \times \vec{E}$$

where $n = \sqrt{\varepsilon\mu}$ and $\hat{k} = \hat{z}$. So

$$\vec{B} = \sqrt{\varepsilon\mu}\; \hat{z} \times \hat{x}\; E_0 e^{i(kz-\omega t)}$$

$$= \sqrt{\varepsilon\mu}\, E_0\, e^{i(kz-\omega t)} \hat{y}$$

$$\vec{B} = B_0 e^{i(kz-\omega t)} \hat{y} ,$$

where $B_0 = \sqrt{\varepsilon\mu}\, E_0$. Taking the real part of both of these expressions, we have

$$\boxed{\begin{aligned} \vec{E}_{Re} &= E_0 \cos(kz - \omega t)\, \hat{x} \\ \vec{B}_{Re} &= B_0 \cos(kz - \omega t)\, \hat{y} \end{aligned}}$$

Part (b)

We have...

$$u = \frac{1}{8\pi}\left(E^2 + B^2\right)$$

$$= \frac{1}{8\pi}\left(E_0^{\,2}\cos^2(kz-\omega t) + B_0^{\,2}\cos^2(kz-\omega t)\right)$$

$$\boxed{u = \frac{1}{8\pi}\left(E_0^{\,2}+B_0^{\,2}\right)\cos^2(kz-\omega t)}.$$

The time-averaged energy density is

$$\langle u \rangle = \frac{1}{16\pi}\left(\vec{E}\cdot\vec{E}^* + \vec{B}\cdot\vec{B}^*\right)$$

$$= \frac{1}{16\pi}\left(E_0^{\,2} e^{i(kz-\omega t)} e^{-i(kz-\omega t)} + B_0^{\,2} e^{i(kz-\omega t)} e^{-i(kz-\omega t)}\right)$$

$$\boxed{\langle u \rangle = \frac{1}{16\pi}\left(E_0^{\,2}+B_0^{\,2}\right)}.$$

The Poynting vector is

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\frac{1}{\mu}\vec{B}\right)$$

$$= \frac{c}{4\pi}\left(E_0\cos(kz-\omega t)\hat{x} \times \frac{1}{\mu}B_0\cos(kz-\omega t)\hat{y}\right)$$

$$\boxed{\vec{S} = \frac{c}{4\pi\mu}E_0 B_0\cos^2(kz-\omega t)\,\hat{z}}.$$

The time-averaged Poynting vector is

$$\langle\vec{S}\rangle = \frac{c}{8\pi}\,\mathrm{Re}\left[\frac{n}{\mu}\right]\left(\vec{E}\cdot\vec{E}^*\right)\hat{z}$$

$$= \frac{c}{8\pi\mu}\left(\frac{1}{2}(n+n^*)\right)\left(E_0^{\,2}\right)\hat{z}$$

$$= \frac{nc}{8\pi\mu}E_0^{\,2}\,\hat{z}$$

$$= \frac{nc}{8\pi\mu}E_0\frac{B_0}{\sqrt{\epsilon\mu}}\,\hat{z}$$

$$\boxed{\langle\vec{S}\rangle = \frac{c}{8\pi\mu}E_0 B_0\,\hat{z}}.$$

Part (c)

We know

$$\lambda = \frac{c}{f},$$

so

$$\lambda = \frac{3 \times 10^8 \text{ m/s}}{100 \times 10^6 \text{ 1/s}}$$

$$= 0.03 \times 10^2 \text{ m}$$

$$\boxed{\lambda = 3 \text{ m}}.$$