Relativistic

SPRING 2010

6. Maxwell's equations in 4 dimensions

(a) {2 pts} Write the Maxwell equations in the absence of polarizable materials using 4-vector notation, making use of the field strength tensor $F_{\mu\nu}$.

(b) {4 pts} Show that the equations of part (a) reduce to the usual form of Maxwell's equations in 3-vector notation.

(c) {2 pts} The Lagrangian density of the EM field is given by

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}, \qquad (SI)$$

or

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}. \qquad (Gaussian)$$

Recall that all repeated Greek indices are summed over 4-dimensions (1 time and 3 space). Show that the Lagrangian density is invariant under a gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha(x)$, where α is an arbitrary function of spacetime $x \equiv (ct, \vec{x})$.

(d) {2 pts} If we add an interaction term $\mathcal{L} \to \mathcal{L} + \Delta\mathcal{L}$ where

$$\Delta\mathcal{L} = j^\mu A_\mu, \qquad (SI)$$

or

$$\Delta\mathcal{L} = \frac{1}{c} j^\mu A_\mu, \qquad (Gaussian)$$

to the Lagrangian– where j^μ is some spatially bounded and conserved 4-current density– how does the action $I \equiv \int \mathcal{L} d^4 r$ change under a gauge transformation and do the resulting equations of motion change?

Part (a)

In the absence of polarizable media, Maxwell's equations can be written as...

Inhomogeneous

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$
$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \quad \rightarrow \quad \boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$$

Homogeneous

$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \quad \rightarrow \quad \boxed{\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0}$$