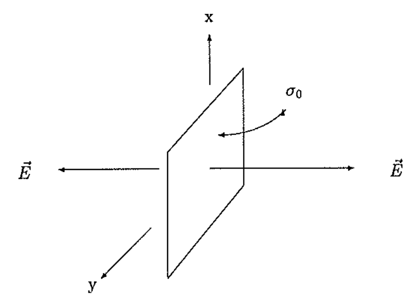
Relativistic

6. Consider a thin very large sheet of charge (as shown in the figure) such that you can ignore edge effects,



(a) [2 points] Compute the 4-current $J^\alpha(x^\beta) \equiv (c\rho, \mathbf{J})$ for the stationary sheet of charge located at $z = 0$ in the lab (see the figure). Assume the surface charge density is a constant σ_0.

(b) [2 points] Compute the E & M fields in the lab frame for the above 4-current source.

(c) [2 points] Now assume you move with speed $v < c$ in the $+x$-direction relative to the lab. Compute the 4-current $J'^\alpha(x'^\beta)$ in your frame. You can compute J'^α from the lab's J^α by a Lorentz boost, i.e., $J'^\alpha = L^\alpha{}_\beta J^\beta$ or in matrix notation $J' = LJ$. If you derived J' some other way than using the above Lorentz boost, check to see that your J' satisfies $J' = LJ$.
Recall

$$L^\alpha{}_\beta = \left\{ \begin{array}{cccc} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\}.$$

(d) [1 points] Combine the above E & M fields into a single E & M tensor $F^{\alpha\beta}$ defined (in Gaussian units) by

$$F^{\alpha\beta} = \left\{ \begin{array}{cccc} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{array} \right\}.$$

For SI units each $\mathbf{B}$ component in $F^{\alpha\beta}$ above is divided by c.

(e) [3 points] What will the $\mathbf{E}'$ & $\mathbf{B}'$ fields be in your frame?
You can compute them by using a Lorentz boost, i.e., $F'^{\alpha\beta} = L^\alpha{}_\mu L^\beta{}_\nu F^{\mu\nu}$ or in matrix notation $F' = LFL^T$. If you derived $\mathbf{E}'$ & $\mathbf{B}'$ some other way than boosting F show that the F' constructed with your $\mathbf{E}'$ & $\mathbf{B}'$ fields is related to the lab's $F^{\alpha\beta}$ by $F' = LFL^T$.

Part (a)

We can write the 4-current as

$$J^\alpha = \begin{pmatrix} c\sigma \\ J_x \\ J_y \\ J_z \end{pmatrix}$$

$$\boxed{J^\alpha = \begin{pmatrix} c\sigma_0\,\delta(z) \\ 0 \\ 0 \\ 0 \end{pmatrix}},$$

since the charge density is constant over the infinitely large, thin sheet.

Part (b)

Using Gauss's law...

$$\oint \vec{E}\cdot d\vec{a} = 4\pi q_{enc}$$

$$E \oint da = 4\pi(\sigma_0 A)$$

$$E\,2A = 4\pi\sigma_0 A$$

$$E = 2\pi\sigma_0$$

and

$$\boxed{\vec{E} = \begin{cases} 2\pi\sigma_0\hat{z}, & z>0 \\ -2\pi\sigma_0\hat{z}, & z<0 \end{cases}}$$

Since there are no moving charges in the lab frame, $\boxed{\vec{B}=0}$.

Part (c)

If the moving frame travels in the $+\hat{x}$-direction, our transformation matrix is

$$\mathcal{L}_x = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The 4-current in the moving frame is then

$$J'^{\alpha} = \mathcal{L}_x J^{\alpha}$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\sigma_0 \delta(z) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma c\sigma_0 \delta(z) \\ -\beta\gamma c\sigma_0 \delta(z) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c\sigma' \\ J_x' \\ J_y' \\ J_z' \end{pmatrix}$$

So

$$\boxed{J'^{\alpha} = \begin{pmatrix} \gamma c\sigma_0 \delta(z) \\ -\beta\gamma c\sigma_0 \delta(z) \\ 0 \\ 0 \end{pmatrix}},$$

implying that from the perspective of the moving frame, there is now a current in the $-\hat{x}$-direction.

Part (d)

The electromagnetic field tensor is

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\boxed{F^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix}}$$

in the lab frame.

Part (e)

In the moving frame...

$$F'^{\alpha\beta} = L_x F^{\alpha\beta} L_x^{T}$$

$$F'^{\alpha\beta} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ \gamma E_z & -\beta\gamma E_z & 0 & 0 \end{pmatrix}.$$

So

$$E_z' = \gamma E_z \quad \text{and} \quad B_y' = \beta\gamma E_z,$$

and thus,

$$\boxed{\vec{E}' = \begin{cases} \gamma 2\pi\sigma_0\, \hat{z}, & z>0 \\ -\gamma 2\pi\sigma_0\, \hat{z}, & z<0 \end{cases}} \qquad \vec{B} = \begin{cases} \beta\gamma 2\pi\sigma_0\, \hat{y}, & z>0 \\ -\beta\gamma 2\pi\sigma_0\, \hat{y}, & z<0 \end{cases}$$