4. A source located at the origin of our coordinate system emits a sinusoidal plane polarized electromagnetic wave with a linearly increasing amplitude. Assume the amplitude of the wave at time t at the source is given by $E_m = E_0(1 + \alpha t)$ with $\alpha \ll \omega$, therefore the amplitude E_m varies slightly over one period.

   (a) [2 points] Write down expressions for the electric field intensity $\vec{\mathbf{E}}$ and the magnetic field intensity $\vec{\mathbf{B}}$ at the source.

   (b) [2 points] Write down expressions for the electric field intensity $\vec{\mathbf{E}}$ and the magnetic field intensity $\vec{\mathbf{B}}$ at a distance z from the source at time t.

   Now consider a imaginary cylinder a distance z_0 from the source, with length L and radius R centered on an axis along the direction of propagation that passes through the origin.

   (c) [2 points] Calculate the time average Poynting vector ($\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$) over one period at an arbitrary point z. Again assume that E_0 varies slightly over one period, so that averages can be calculated as usual.

   (d) [2 points] Evaluate the time average Poynting vector at the entrance and exit of the cylindrical volume.

   (e) [2 points] Integrate the average Poynting vector over the surface area of the cylindrical volume to calculate the net average outward energy flow.

Part (a)

In general, we know

$$\vec{E}(z,t) = A e^{i(kz-\omega t)} \hat{x},$$

assuming the wave propagates in the $\hat{k} = \hat{z}$ direction and the wave is polarized in the $\hat{x}$ direction. Then if the amplitude is

$$A = E_m = E_0(1+\alpha t),$$

we have

$$\vec{E}(z,t) = E_0(1+\alpha t) e^{i(kz-\omega t)} \hat{x} .$$

The magnetic field is then

$$\vec{B} = n \hat{k} \times \vec{E}$$

$$= n \hat{z} \times E_0(1+\alpha t) e^{i(kz-\omega t)} \hat{x}$$

$$\vec{B} = n E_0(1+\alpha t) e^{i(kz-\omega t)} \hat{y} .$$

At the source, $z=0$, so the intensity of these fields at the source is

$$\boxed{\begin{aligned} \vec{E} &= E_0(1+\alpha t) e^{-i\omega t} \hat{x} \\ \vec{B} &= n E_0(1+\alpha t) e^{-i\omega t} \hat{y} \end{aligned}}$$

Part (b)

As we found in part (a)...

$$\boxed{\begin{aligned}\vec{E}(z,t) &= E_0(1+\alpha t)e^{i(kz-\omega t)}\,\hat{x}\\ \vec{B}(z,t) &= nE_0(1+\alpha t)e^{i(kz-\omega t)}\,\hat{y}\end{aligned}}\,.$$

Part (c)

We know that in general

$$\langle\vec{S}\rangle = \frac{c}{8\pi}\,Re\left[\frac{n}{\mu}\right](\vec{E}\cdot\vec{E}^*)\,\hat{k}$$

in Gaussian units. Since $n=\sqrt{\varepsilon\mu}$ and $\hat{k}=\hat{z}$, we have

$$\langle\vec{S}\rangle = \frac{c}{8\pi}\,Re\left[\sqrt{\frac{\varepsilon}{\mu}}\right](\vec{E}\cdot\vec{E}^*)\,\hat{k}$$

$$= \frac{c}{8\pi}\left[\frac{1}{2}\left(\sqrt{\frac{\varepsilon}{\mu}}+\sqrt{\frac{\varepsilon}{\mu}}^{\,*}\right)\right](\vec{E}\cdot\vec{E}^*)\,\hat{z}$$

$$\boxed{\langle\vec{S}\rangle = \frac{c}{16\pi}\left(\sqrt{\frac{\varepsilon}{\mu}}+\sqrt{\frac{\varepsilon}{\mu}}^{\,*}\right)E_0^{\,2}(1+\alpha t)^2\,\hat{z}}\,.$$

Part (d)

We assume that the wave enters the cylinder at $z = z_0$. If the cylinder has length L, then the wave exits at $z = z_0 + L$. The time at which the wave enters is $t = t_0$ and when the wave leaves, $t = t_f$.

Then ...

$$z = z_0 : \quad \langle \vec{S} \rangle = \frac{c}{16\pi}\left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon}{\mu}}^{\,*}\right) E_0^{\,2}(1+\kappa t_0)^2 \,\hat{z}$$

$$z = z_0 + L : \quad \langle \vec{S} \rangle = \frac{c}{16\pi}\left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon}{\mu}}^{\,*}\right) E_0^{\,2}(1+\kappa t_f)^2 \,\hat{z} \,.$$

In this case, we can say

$$t_0 = \frac{z_0}{c}$$

and

$$t_f = \frac{z_0 + L}{c} \,.$$

Part (e)

We want to determine the net average outward energy flow. So

$$\oint \langle \vec{S} \rangle \cdot d\vec{a}\,.$$

We want to integrate using the top and bottom of the cylinder as our surfaces of integration. Then

$$\oint \langle \vec{S} \rangle \cdot d\vec{a} = \left(\langle \vec{S} \rangle_{z=z_0+L} - \langle \vec{S} \rangle_{z=z_0} \right)\left(\pi R^2\right)$$

$$= \left(\frac{c}{16\pi}\left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon}{\mu}}^{\,*}\right) E_0^2 \right)\left[\left(1+\alpha t_f\right)^2 - \left(1+\alpha t_0\right)^2\right]\left(\pi R^2\right)$$

$$= \frac{cR^2}{16} E_0^2 \left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon}{\mu}}^{\,*}\right)\left[1 + 2\alpha t_f + \alpha^2 t_f^2 - 1 - 2\alpha t_0 - \alpha^2 t_0^2\right]$$

$$= \frac{cR^2}{16} E_0^2 \left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon}{\mu}}^{\,*}\right)\left[\frac{2\alpha}{c}\left(z_0 + L - z_0\right) + \frac{\alpha^2}{c^2}\left(z_0^2 + 2z_0 L + L^2 - z_0^2\right)\right]$$

$$= \frac{cR^2}{16c^2} E_0^2 \left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon}{\mu}}^{\,*}\right)\left[2\alpha Lc + \alpha^2 L^2 + 2\alpha^2 z_0 L\right]$$

$$\boxed{\oint \langle \vec{S} \rangle \cdot d\vec{a} = \frac{R^2}{16c} E_0^2 \left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon}{\mu}}^{\,*}\right)\left[2c\alpha L + \alpha^2 L^2 + 2\alpha^2 z_0 L\right]}\,.$$