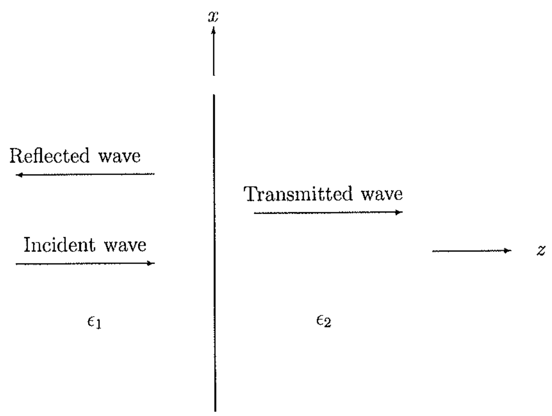
Waves

SPRING 2010



5. A plane-polarized harmonic ($e^{-i\omega t}$) plane electromagnetic wave traveling to the right in a homogeneous dielectric medium described by an dielectric constant ϵ_1, strikes a second homogeneous dielectric material described by dielectric constant $\epsilon_2 > \epsilon_1$ (see the figure). Assume that both materials have the same magnetic permeability μ_0 and that the incidence angle is 0^o (i.e., the wave is traveling perpendicular to the junction). Assume the incoming wave is polarized in the $\hat{x}$ direction and that its electric field amplitude is E_0, i.e., assume the incoming electric field is the real part of

$$\mathbf{E} = E_0\, e^{i(kz-wt)}\,\hat{x}.$$

(a) {3 pts} Give the magnetic induction $\mathbf{B}$ associated with the above incoming wave. Make sure your wave satisfies Maxwell's equations, e.g., give k as a function of ω, the direction of $\mathbf{B}$, and the amplitude of $\mathbf{B}$ as a function of E_0.

(b) {1 pts} Give similar expressions for the $\mathbf{E}$ and $\mathbf{B}$ components of the reflected and transmitted waves. Use E_0'' and E_0' for the respective amplitudes of reflected and transmitted waves.

(c) {2 pts} In general, what conditions must be satisfied at the junction between two materials by the electromagnetic fields $\mathbf{E}, \mathbf{B}, \mathbf{D}$, and $\mathbf{H}$, if Maxwell's equations are to be satisfied?

(d) {2 pts}Apply these junction conditions to the combined incoming, reflected, and transmitted wave to compute E_0'' and E_0' as functions of E_0 and the two dielectric constants ϵ_1 and ϵ_2.

(e) {2 pts} Evaluate the time averages of the Poynting vectors of the incident, reflected, and transmitted waves. Recall that

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} \equiv \frac{1}{4\pi}\mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

The sum of the magnitudes of the reflected and transmitted time averaged Poynting vectors should equal the magnitude of the incident wave's time averaged Poynting vector.

Part (a)

We want to determine the magnetic induction $\vec{B}$ associated with the electric field

$$\vec{E} = E_0 e^{i(kz-\omega t)} \hat{x}.$$

In general,

$$\vec{B} = n\hat{k} \times \vec{E},$$

where n is the index of refraction. Since we want the magnetic induction associated with the incoming wave,

$$n = \sqrt{\mu_0 \varepsilon_1} \ .$$

Working in Gaussian units, we have $\mu_0 = 1$, so

$$n = \sqrt{\varepsilon_1} \ .$$

Since the wave is propagating in the z-direction, we have

$$\hat{k} = \hat{z}$$

so

$$\begin{aligned} n\hat{k} \times \vec{E} &= n\hat{z} \times \vec{E} \\ &= \sqrt{\varepsilon_1}\, \hat{z} \times E_0 e^{i(kz-\omega t)} \hat{x} \\ &= E_0 \sqrt{\varepsilon_1}\, e^{i(kz-\omega t)} (\hat{z} \times \hat{x}) \end{aligned}$$

and using the right-hand rule,

$$\boxed{\vec{B} = E_0 \sqrt{\varepsilon_1}\, e^{i(kz-\omega t)} \hat{y}},$$

which is perpendicular to both the direction of propagation and the direction of the electric field, as expected. Now we want to verify that this wave satisfies Maxwell's equations.

Part (a), continued

Maxwell's equations in Gaussian units are:

$$(1) \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$(2) \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$(3) \quad \vec{\nabla} \times \vec{\ } = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{\ }}{\partial t} \, .$$

These are the equations involving $\vec{B}$. So...

$$(1) \quad \vec{\nabla} \cdot \vec{B} \stackrel{?}{=} 0$$

$$\vec{\nabla} \cdot \vec{B} = \frac{\partial}{\partial y}\left(E_0 \sqrt{\varepsilon_1}\, e^{i(kz-\omega t)}\right)$$

$$\stackrel{\checkmark}{=} 0$$

$$(2) \quad \vec{\nabla} \times \vec{E} \stackrel{?}{=} -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E & 0 & 0 \end{vmatrix}$$

$$= (0)\hat{x} - \hat{y}\left(-\frac{\partial}{\partial z}E\right) + \hat{z}\left(-\frac{\partial}{\partial y}E\right)$$

$$= \frac{\partial}{\partial z}\left(E_0 e^{i(kz-\omega t)}\right)\hat{y}$$

$$= ikE_0 e^{i(kz-\omega t)}\,\hat{y}$$

$$-\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \frac{i\omega}{c}E_0\sqrt{\varepsilon_1}\, e^{i(kz-\omega t)}\,\hat{y}$$

So $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ as long as

$$\boxed{k = \frac{\omega\sqrt{\varepsilon_1}}{c}} \, .$$

Part (a), cont'd

③ $\vec{\nabla} \times \vec{H} \stackrel{?}{=} \frac{4\pi}{c} \vec{J}_f + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$

We know $\vec{J}_f = 0$, so we want to check that

$$\vec{\nabla} \times \vec{H} \stackrel{?}{=} \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \frac{1}{\mu_0}\vec{B} \stackrel{?}{=} \frac{1}{c}\frac{\partial}{\partial t}(\varepsilon_1 \vec{E})$$

$$\vec{\nabla} \times \vec{B} \stackrel{?}{=} \frac{\varepsilon_1}{c}\frac{\partial}{\partial t}\vec{E}$$

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & B & 0 \end{vmatrix}$$

$$= \hat{x}\left(-\frac{\partial}{\partial z}B\right) - \hat{y}(0) + \hat{z}\left(\frac{\partial}{\partial x}B\right)$$

$$= -\frac{\partial}{\partial z}\left(E_0\sqrt{\varepsilon_1}\, e^{i(kz-\omega t)}\right)\hat{x}$$

$$= -ikE_0\sqrt{\varepsilon_1}\, e^{i(kz-\omega t)}\,\hat{x}$$

$$\frac{\varepsilon_1}{c}\frac{\partial \vec{E}}{\partial t} = -\frac{i\omega\varepsilon_1}{c}E_0\, e^{i(kz-\omega t)}\,\hat{x}.$$

So again, if $k = \frac{\omega\sqrt{\varepsilon_1}}{c}$, then this equation holds. Therefore, Maxwell's equations hold.

The amplitude of $\vec{B}$ as a function of E_0 is

$$\vec{B}_0 = n\hat{k} \times \vec{E}_0$$

$$= \sqrt{\varepsilon_1}\,\hat{z} \times E_0\hat{x}$$

$$= \sqrt{\varepsilon_1}\,E_0\,\hat{y},$$

so

$$\boxed{B_0 = \sqrt{\varepsilon_1}\,E_0}.$$

Part (b)

Now we want to find expressions for $\vec{E}$ and $\vec{B}$ of the reflected and transmitted waves.

Transmitted

$$\vec{E}' = E_0' e^{i(k'z - \omega t)} \hat{x}$$

$$\vec{B}' = E_0' \sqrt{\varepsilon_2}\, e^{i(k'z - \omega t)} \hat{y}$$

where

$$k' = \frac{\omega \sqrt{\varepsilon_2}}{c}.$$

Reflected

$$\vec{E}'' = E_0'' e^{i(k(-z) - \omega t)} \hat{x}$$

$$\vec{B}'' = E_0'' \sqrt{\varepsilon_1}\, e^{i(k(-z) - \omega t)} (-\hat{y}) \quad \text{(right-hand rule)}$$

where

$$k = \frac{\omega \sqrt{\varepsilon_1}}{c}.$$

Part (c)

We want to write down the boundary conditions.

$$B_{1,\perp} - B_{2,\perp} = 0$$

$$E_{1,\parallel} - E_{2,\parallel} = 0 \; .$$

So the normal component of B and the tangential component of E are both continuous. We also have

$$D_{1,\perp} - D_{2,\perp} = 4\pi\sigma_f \hat{n} = 0$$

if there is no free surface charge, which we can also write as

$$(\varepsilon_1 E_{1,\perp}) - (\varepsilon_2 E_{2,\perp}) = 0$$

And, lastly,

$$H_{1,\parallel} - H_{2,\parallel} = 4\pi K_f \times \hat{n} = 0,$$

or

$$\left(\frac{1}{\mu_0} B_{1,\parallel}\right) - \left(\frac{1}{\mu_0} B_{2,\parallel}\right) = 0$$

$$B_{1,\parallel} - B_{2,\parallel} = 0.$$

Part (d)

We want to determine E_0'' and E_0' from our boundary conditions, specifically our tangential conditions. So...

$$E_{1,\parallel} - E_{2,\parallel} = 0$$

$$\left[(\vec{E} + \vec{E}'') - \vec{E}'\right] \times \hat{n} = 0$$

$$\left[E\hat{x} + E''\hat{x} - E'\hat{x}\right] \times \hat{z} = 0$$

$$E\hat{y} + E''\hat{y} = E'\hat{y}$$

$$E_0 e^{i(kz-\omega t)}\hat{y} + E_0'' e^{i(k(-z)-\omega t)}\hat{y} = E_0' e^{i(k'z-\omega t)}\hat{y}$$

At the interface, $z=0$ so we end up with

$$E_0 + E_0'' = E_0'$$

as our first equation. Now

$$B_{1,\parallel} - B_{2,\parallel} = 0$$

$$\left[(\vec{B} + \vec{B}'') - \vec{B}'\right] \times \hat{n} = 0$$

$$\left[B\hat{y} + B''(-\hat{y}) - B'\hat{y}\right] \times \hat{z} = 0$$

$$\left[B\hat{y} - B''\hat{y} - B'\hat{y}\right] \times \hat{z} = 0$$

$$B\hat{x} - B''\hat{x} - B'\hat{x} = 0$$

$$E_0\sqrt{\varepsilon_1}\, e^{i(kz-\omega t)}\hat{x} - E_0''\sqrt{\varepsilon_1}\, e^{i(k(-z)-\omega t)}\hat{x} - E_0'\sqrt{\varepsilon_2}\, e^{i(k'z-\omega t)}\hat{x} = 0.$$

Again, $z=0$ so

$$E_0\sqrt{\varepsilon_1} - E_0''\sqrt{\varepsilon_1} = E_0'\sqrt{\varepsilon_2}$$

$$E_0 - E_0'' = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\, E_0' .$$

Part (d), cont'd

So we know

$$① \; E_0 + E_0'' = E_0'$$

$$② \; E_0 - E_0'' = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\, E_0' .$$

Rearranging ② for E_0' and plugging into ①...

$$E_0 + E_0'' = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\left(E_0 - E_0''\right)$$

$$E_0'' = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\left(E_0 - E_0''\right) - E_0$$

$$E_0''\left(1 + \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\right) = E_0\left(\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} - 1\right)$$

$$E_0''\left(\frac{\sqrt{\varepsilon_2} + \sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}}\right) = E_0\left(\frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_2}}\right)$$

$$\boxed{E_0'' = E_0\left(\frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}\right)} .$$

Rearranging ② for E_0'' and plugging into ①...

$$E_0 + \left(E_0 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\, E_0'\right) = E_0'$$

$$2E_0 = E_0'\left(1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

$$E_0'\left(\frac{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right) = 2E_0$$

$$\boxed{E_0' = \frac{2\sqrt{\varepsilon_1}\, E_0}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}} .$$