Waves

SPRING 2008

4. Consider a monochromatic plane electromagnetic wave of frequency ω propagating in vacuum in the z direction and polarized in the x direction, which impinges upon a perfect conductor at $z = 0$, as shown in the figure. The incident electric field is

$$\mathbf{E}_I(z,t) = \hat{\mathbf{x}} E_{0I} e^{i(kz-\omega t)}.$$

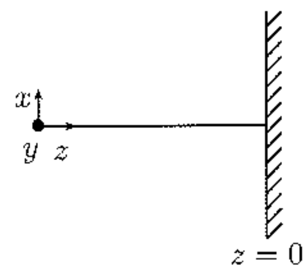

Figure 1: Plane wave normally incident on a perfectly conducting plane at $z = 0$.

a) 1 pt. Use Maxwell's equations to determine the relation between k and ω.

b) 1 pt. Use Maxwell's equations to determine the incident magnetic field, $\mathbf{B}_I(z,t)$.

c) 1 pt. What are the forms of the reflected wave $\mathbf{E}_R(z,t)$, $\mathbf{B}_R(z,t)$?

d) 2 pt. Apply the appropriate boundary conditions at the interface between the vacuum and the conductor to determine the reflected amplitudes E_{0R} and B_{0R} in terms of E_{0I}.

e) 1 pt. What is the phase of the incident and reflected electric fields? Are they in phase or out of phase at $z = 0$?

f) 2 pt. What is the force exerted on the conducting surface by the reflection of the plane wave? Answer this question by computing the momentum transferred from the field to the conductor.

g) 2 pt. Answer the same question by computing the discontinuity of the normal-normal component of the stress tensor across the interface, ΔT_{zz}.

Part (a)

This is a perfect conductor, so we know that $\sigma_f = 0$, $J_f = 0$, and $\rho_f = 0$. From Faraday's law,

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}.$$

Taking the curl, we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}).$$

Since $\rho_f = 0$, we know

$$\vec{\nabla} \cdot \vec{E} = 0,$$

so

$$-\vec{\nabla}^2 \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}).$$

From the Ampère-Maxwell law,

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J}_f + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \frac{1}{\mu}\vec{B} = \frac{\varepsilon}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{\varepsilon\mu}{c}\frac{\partial \vec{E}}{\partial t},$$

so

$$\vec{\nabla}^2 \vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\varepsilon\mu}{c}\frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla}^2 \vec{E} = \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

or

$$\vec{\nabla}^2 \vec{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0.$$

Part (a), cont'd

Since

$$\vec{E}_I = \hat{x}\, E_{0_I}\, e^{i(kz-\omega t)},$$

then

$$\vec{\nabla}^2 \vec{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0$$

$$\frac{\partial^2 \vec{E}}{\partial z^2} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0$$

$$-k^2 E_{0_I} e^{i(kz-\omega t)}\hat{x} + \frac{\varepsilon\mu}{c^2}\omega^2 E_{0_I} e^{i(kz-\omega t)}\hat{x} = 0.$$

So

$$k^2 = \frac{\varepsilon\mu}{c^2}\omega^2$$

and

$$\boxed{k = \sqrt{\varepsilon\mu}\,\frac{\omega}{c}}.$$

Part (b)

We can use Faraday's law to determine $\vec{B}_I(z,t)$.

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\hat{x}E_{0_I}e^{i(kz-\omega t)} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_I & 0 & 0\end{vmatrix} = \hat{x}(0) - \hat{y}\left(-\frac{\partial}{\partial z}E_I\right) + \hat{z}\left(-\frac{\partial}{\partial y}E_I\right)$$

$$\frac{\partial E_I}{\partial z}\hat{y} = -\frac{1}{c}\frac{\partial\vec{B}_I}{\partial t}$$

$$ikE_{0_I}e^{i(kz-\omega t)}\hat{y} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{B}_I = -ikcE_{0_I}\int e^{i(kz-\omega t)}dt\ \hat{y}$$

$$= -ikcE_{0_I}\cdot -\frac{1}{i\omega}e^{i(kz-\omega t)}\hat{y}$$

$$\vec{B}_I = \frac{kc}{\omega}E_{0_I}e^{i(kz-\omega t)}\hat{y}$$

$$= \sqrt{\varepsilon\mu}\frac{\omega}{c}\frac{c}{\omega}E_{0_I}e^{i(kz-\omega t)}\hat{y}$$

$$\boxed{\vec{B}_I = \sqrt{\varepsilon\mu}\,E_{0_I}e^{i(kz-\omega t)}\hat{y}}$$

But $\varepsilon = \varepsilon_0 = 1$, $\mu = \mu_0 = 1$ in this case. So

$$\vec{B}_I = E_{0_I}e^{i(kz-\omega t)}\hat{y}.$$

Part (c)

The reflected electric field is

$$\vec{E}_R(z,t) = \hat{x}\, E_{0_R}\, e^{i(k(-z)-\omega t)}$$

$$\boxed{\vec{E}_R(z,t) = \hat{x}\, E_{0_R}\, e^{-i(kz+\omega t)}}.$$

The reflected magnetic field is

$$\vec{B}_R(z,t) = n\,\hat{k}\times\vec{E}$$

$$= \sqrt{\varepsilon_0\mu_0}\,(-\hat{z})\times\hat{x}\,E_{0_R}\,e^{-i(kz+\omega t)}$$

$$\boxed{\vec{B}_R(z,t) = E_{0_R}\,e^{-i(kz+\omega t)}\,(-\hat{y})}.$$

There is no transmitted electric field or magnetic field since this is a perfect conductor.

Part (d)

The boundary conditions we want to apply are

$$\vec{E}_{1,\parallel} - \vec{E}_{2,\parallel} = 0$$

$$\vec{H}_{1,\parallel} - \vec{H}_{2,\parallel} = \frac{4\pi}{c} \vec{K}_f \hat{n} = 0 .$$

In this case, the first is all we need to use. So

$$\vec{E}_I + \vec{E}_R = 0$$

$$\hat{x} E_{0_I} e^{i(kz-\omega t)} + \hat{x} E_{0_R} e^{-i(kz+\omega t)} = 0$$

At the interface, $z=0$ so

$$E_{0_I} + E_{0R} = 0$$

$$\boxed{E_{0R} = -E_{0_I}} .$$

Then since

$$B_{0R} = E_{0R} ,$$

$$\boxed{B_{0R} = -E_{0_I}} .$$

Part (e)

We have

$$\vec{E}_I(z,t) = \hat{x}\, E_{0_I}\, e^{i(kz-\omega t)}$$
$$\vec{E}_R(z,t) = \hat{x}\left(-E_{0_I}\, e^{-i(kz+\omega t)}\right).$$

At $z=0$,

$$\vec{E}_I(0,t) = \hat{x}\, E_{0_I}\, e^{-i\omega t}$$
$$\vec{E}_R(0,t) = \hat{x}\left(-E_{0_I}\, e^{-i\omega t}\right).$$

We can write

$$\vec{E}_R(0,t) = \hat{x}\, E_{0_I}\, e^{-i\omega t}\, e^{i\pi},$$

or

$$\vec{E}_R(0,t) = \vec{E}_I(0,t)\, e^{i\pi},$$

so the incident and reflected electric fields are out of phase by $\boxed{\phi = \pi}$.