Waves

FALL 2010

2. (a) {2 pts} In a homogeneous, linear and isotropic conducting material whose electromagnetic properties (at low frequencies) are described by constant values of the permittivity, pearmeability, and conductivity respectively ϵ, μ, and σ, show that Maxwell's equations require that the electric field satisfy

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0, \qquad (SI)$$

$$\nabla^2 \mathbf{E} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \mathbf{E}}{\partial t} = 0. \qquad (Gaussian)$$

(b) {2 pts} Given a plane polarized plane wave of angular frequency ω whose electric field is of the form

$$\mathbf{E}(z,t) = \text{Real}\left\{\hat{\imath} E_0 e^{i(kz-\omega t)}\right\},$$

evaluate k^2 as a function of ϵ, μ, σ, and ω.

(c) {2 pts} Find the real and imaginary parts of k assuming $\sigma >> \omega\epsilon$.

(d) {2 pts} Using your results from (c) find the skin depth δ of the conductor. The skin depth is defined by the depth at which the wave's amplitude decreases by e^{-1}, i.e.,

$$\frac{|\mathbf{E}(z+\delta,t)|}{|\mathbf{E}(z,t)|} = \frac{1}{e}$$

(e) {2 pts} Using Maxwell's equations, find the magnetic field $\mathbf{H}(x,t)$ associated with $\mathbf{E}(z,t)$ given in (b) and discuss their phase difference when $\sigma >> \omega\epsilon$.

Part (a) (Gaussian)

From the law of induction,

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}.$$

The Ampère-Maxwell law tells us

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\left(4\pi \vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right)$$

$$\vec{\nabla} \times \frac{1}{\mu}\vec{B} = \frac{4\pi}{c}\vec{J}_f + \frac{\varepsilon}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \frac{1}{\mu}\vec{B} = \frac{4\pi}{c}(\sigma \vec{E}) + \frac{\varepsilon}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi\mu\sigma}{c}\vec{E} + \frac{\varepsilon\mu}{c}\frac{\partial \vec{E}}{\partial t}.$$

Taking the curl of Faraday's law,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{\nabla}^2\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{4\pi\mu\sigma}{c}\vec{E} + \frac{\varepsilon\mu}{c}\frac{\partial \vec{E}}{\partial t}\right).$$

But

$$\vec{\nabla}\cdot\vec{E} = 0$$

since this is a conductor, so

$$-\vec{\nabla}^2\vec{E} = -\frac{\varepsilon\mu}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \vec{E}}{\partial t}$$

or

$$\boxed{\vec{\nabla}^2\vec{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \vec{E}}{\partial t} = 0},$$

as expected.

Part (b)

Plugging in

$$E = E_0 e^{i(kz-\omega t)}$$

into our expression from part (a)...

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E - \frac{\varepsilon\mu}{c^2} \frac{\partial^2}{\partial t^2} E - \frac{4\pi\sigma\mu}{c^2} \frac{\partial E}{\partial t} = 0$$

$$\frac{\partial^2}{\partial z^2} E - \frac{\varepsilon\mu}{c^2} \frac{\partial^2}{\partial t^2} E - \frac{4\pi\sigma\mu}{c^2} \frac{\partial E}{\partial t} = 0$$

$$-k^2 E_0 e^{i(kz-\omega t)} + \frac{\varepsilon\mu}{c^2} \omega^2 E_0 e^{i(kz-\omega t)} + \frac{4\pi\sigma\mu\omega}{c^2} i E_0 e^{i(kz-\omega t)} = 0$$

$$-k^2 + \frac{\varepsilon\mu}{c^2}\omega^2 + i\frac{4\pi\sigma\mu\omega}{c^2} = 0$$

and

$$\boxed{k^2 = \frac{1}{c^2}\left(\varepsilon\mu\omega^2 + i4\pi\sigma\mu\omega\right)}.$$

Part (c)

If $\sigma >> \omega\varepsilon$, then $\varepsilon << \frac{\sigma}{\omega}$. So

$$k^2 = \frac{1}{c^2}\left(\varepsilon\mu\omega^2 + i\,4\pi\sigma\mu\omega\right)$$
$$= \frac{\omega^2}{c^2}\left(\varepsilon\mu + i\,\frac{4\pi\sigma\mu}{\omega}\right).$$

But $\varepsilon << \sigma/\omega$, so

$$k^2 \approx \frac{\omega^2}{c^2}\left(i\,\frac{4\pi\sigma\mu}{\omega}\right)$$
$$= i\,\frac{4\pi\sigma\mu\omega}{c^2}$$

and

$$k = \sqrt{i\,\frac{4\pi\sigma\mu\omega}{c^2}}$$
$$= \sqrt{i}\,\sqrt{\frac{4\pi\sigma\mu\omega}{c^2}}.$$

Since $i = e^{\frac{i\pi}{2}}$, then $\sqrt{i} = e^{\frac{i\pi}{4}}$. We have

$$e^{\frac{i\pi}{4}} = i\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)$$
$$= i\,\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2},$$

so

$$k = \left(i\,\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\sqrt{\frac{4\pi\sigma\mu\omega}{c^2}}$$
$$= \sqrt{\frac{2\pi\sigma\mu\omega}{c^2}}\,(i+1).$$

So

$$\boxed{\begin{aligned} Re(k) &= \sqrt{\frac{2\pi\sigma\mu\omega}{c^2}} \\ Im(k) &= i\sqrt{\frac{2\pi\sigma\mu\omega}{c^2}} \end{aligned}}.$$

Part (d)

We can write the electric field as

$$\vec{E} = E_0 e^{i\left(Im(k)z + Re(k)z - \omega t\right)} \hat{\imath}$$
$$= E_0 e^{i\left(\sqrt{\frac{2\pi\sigma\mu\omega}{c^2}} - \omega t\right)z} e^{-\sqrt{\frac{2\pi\sigma\mu\omega}{c^2}} z} \hat{\imath}$$

In order for the amplitude to decrease by e^{-1}, we must have

$$z = \sqrt{\frac{c^2}{2\pi\sigma\mu\omega}},$$

which implies the skin depth δ must be

$$\boxed{\delta = \sqrt{\frac{c^2}{2\pi\sigma\mu\omega}}}.$$

Part (e)

From the law of induction,

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}.$$

So ...

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E & 0 & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}\left(-\frac{\partial}{\partial z}E\right) + \hat{k}\left(-\frac{\partial}{\partial y}E\right)$$

$$\frac{\partial \vec{E}}{\partial z} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$ikE_0 e^{i(kz-\omega t)}\hat{j} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\frac{\partial \vec{B}}{\partial t} = -ickE_0 e^{i(kz-\omega t)}\hat{j}$$

$$\vec{B} = -ickE_0 \int e^{i(kz-\omega t)}dt\,\hat{j}$$

$$= \frac{ck}{\omega}E_0 e^{i(kz-\omega t)}\hat{j}$$

but $\vec{H} = \frac{1}{\mu}\vec{B}$, so

$$\boxed{\vec{H} = \frac{kc}{\mu\omega}E_0 e^{i(kz-\omega t)}}.$$