Wire / Cylinder

FALL 2008

5. Consider an infinitely long, solid, nonmagnetic conducting rod of radius a centered on the z axis. An infinitely long, hollow, conducting cylinder with inner radius $b > a$ and outer radius d is coaxial with the rod. Let r be the radial distance perpendicular to the axis of the rod and the cylinder. The region between the conducting rod and the conducting cylinder (that is, $a < r < b$) is filled with a nonconducting, linear, isotropic magnetic material with a constant relative permeability $K = \mu/\mu_0$, where μ is the permeability of the material, and μ_0 is the permeability of free space ($\mu_0 = 1$ in Gaussian units).

   The rod carries a current I in the $+z$ direction while the concentric cylinder carries a current I in the $-z$ direction. We assume that the current density $\mathbf{j}$ is uniform and of the same magnitude in both the rod and the cylinder,

$$j = \frac{I}{\pi a^2} = \frac{I}{\pi(d^2 - b^2)}.$$

a) 3 pts. Calculate the magnetic field $H(r)$ for the four regions

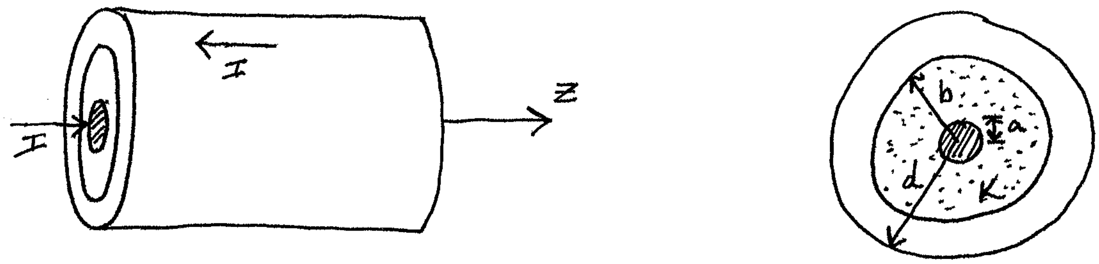
$$\text{I: } r \leq a, \quad \text{II: } a \leq r \leq b, \quad \text{III: } b \leq r \leq d, \quad \text{IV: } d \leq r.$$

b) 3 pts. Calculate the magnetic flux (per unit length in the z direction) crossing a half-plane extending from the axis of the coaxial system and extending to infinity, that is, the surface defined by $x > 0$, $y = 0$, $-\infty < z < \infty$. Use this result to find the self-inductance L per unit length of the coaxial conductor.

c) 2 pts. Compute the magnetic energy U per unit length along the z axis stored in the region filled with the linear magnetic material, that is for region II, $a < r < b$.

d) 2 pts. Using the result from part c), show that the contribution to L coming from the region $a \leq r \leq b$, L_{II}, is consistent with the contribution from the same region that you calculated in part b) above. That is, compute $\frac{1}{2}L_{\mathrm{II}}I^2$ and compare with the result of part c).

Part (a)



$K = \mu/\mu_0$

Current density is uniform and given by

$$j = \frac{I}{\pi a^2} = \frac{I}{\pi(d^2 - b^2)}$$

for both the rod and cylinder. We want to calculate the magnetic field in the various regions, as follows.

I: $r \leq a$

Using Ampère's law with a loop of radius r ...

$$\oint \vec{B} \cdot d\vec{L} = \mu_0 I_{enc}$$

$$B \oint dL = \mu_0 \oint \vec{j} \cdot d\vec{a}$$

$$B(2\pi r) = \mu_0 j \oint da$$

$$B(2\pi r) = \mu_0 j (\pi r^2)$$

$$B = \frac{\mu_0}{2} j r$$

$$= \frac{\mu_0}{2} \left(\frac{I}{\pi a^2}\right) r$$

and

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{I r}{a^2} \hat{\phi}.$$

but

$$\vec{H} = \frac{1}{\mu_0} \vec{B} + \vec{M}$$

$$= \frac{1}{\mu_0} \vec{B} + 0$$

$$\boxed{\vec{H} = \frac{I r}{2\pi a^2} \hat{\phi}}, \text{ for } r \leq a.$$

Part (a), cont'd

II: $a \le r \le b$

In this case, we have current contributions from solely the rod, but the magnetic material between $a < r < b$ provides a magnetization. Using Ampère's law with a loop of radius r...

$$\oint \vec{B} \cdot d\vec{L} = \mu_0 I_{enc}$$

$$B(2\pi r) = \mu_0 j (\pi a^2)$$

$$B = \frac{\mu_0}{2} j \frac{a^2}{r}$$

$$= \frac{\mu_0}{2} \left(\frac{I}{\pi a^2}\right) \frac{a^2}{r}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi},$$

which is just the magnetic field outside of a wire, as we would expect. We know

$$\vec{H} = \frac{1}{\mu_0} \vec{B} + \vec{M}$$

$$= \frac{1}{\mu_0} \vec{B} + \chi_m \vec{H},$$

where χ_m = magnetic susceptibility. Then

$$(1 - \chi_m) \vec{H} = \frac{1}{\mu_0} \vec{B},$$

but $(1 - \chi_m)$ is simply the relative permeability, $K = \mu / \mu_0$, so

$$\frac{\mu}{\mu_0} \vec{H} = \frac{1}{\mu_0} \vec{B}$$

$$\boxed{\vec{H} = \frac{\mu_0 I}{\mu 2\pi r} \hat{\phi}}, \text{ for } a \le r \le b.$$

Part (a), cont'd

III : $b \le r \le d$

In this case, we have current contributions from both the rod and the cylindrical shell, as well as magnetization from the magnetic material. Using Ampère's law with a loop of radius r...

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$B(2\pi r) = \mu_0 \left[j_{rod} \oint da_{rod} - j_{shell} \oint da_{shell} \right]$$

$$= \mu_0 \left[\frac{I}{\pi a^2} (\pi a^2) - \frac{I}{\pi (d^2 - b^2)} (\pi (r^2 - b^2)) \right]$$

$$B(2\pi r) = \mu_0 I - \mu_0 I \frac{(r^2 - b^2)}{(d^2 - b^2)}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \left(1 - \frac{r^2 - b^2}{d^2 - b^2} \right) \hat{\phi} .$$

Then

$$\vec{H} = \frac{\vec{B}}{\mu_0} + \vec{M}$$

$$= \frac{1}{\mu_0} \vec{B} + \chi_m \vec{H}$$

$$(1 - \chi_m) \vec{H} = \frac{1}{\mu_0} \vec{B}$$

$$\frac{\mu}{\mu_0} \vec{H} = \frac{1}{\mu_0} \vec{B}$$

and

$$\boxed{\vec{H} = \frac{\mu_0 I}{\mu 2\pi r} \left(1 - \frac{r^2 - b^2}{d^2 - b^2} \right) \hat{\phi}} \text{, for } b \le r \le d$$

Part (a), cont'd

IV : $d \le r$

Using Ampère's law with a loop of radius r...

$$\oint \vec{B} \cdot d\vec{L} = \mu_0 I_{enc}$$

$$B(2\pi r) = \mu_0 \left[\frac{I}{\pi a^2}(\pi a^2) - \frac{I}{\pi(d^2-b^2)}(\pi(d^2-b^2)) \right]$$

$$B(2\pi r) = \mu_0 [I - I]$$
$$= 0$$

So

$$\vec{B} = 0.$$

This means

$$\vec{H} = \frac{\vec{B}}{\mu_0} + \vec{M}$$
$$(1-\chi_m)\vec{H} = 0$$

and

$$\boxed{\vec{H} = 0}.$$

This makes sense because the currents run opposite one another.

Part (b)

We can calculate the magnetic flux per unit length in each of the regions, then sum them at the end. In general,

$$\frac{\Phi}{l} = \frac{1}{l}\oint \vec{B}\cdot d\vec{a}$$

$$= \frac{1}{l}\iint B\,dx\,dz.$$

<u>I: $r \leq a$</u>

$$\frac{\Phi}{l} = \frac{1}{l}\int_0^l\int_0^a B\,dx\,dz$$

$$= \frac{1}{l}\int_0^l\int_0^a \frac{\mu_0}{2\pi}\frac{Ir}{a^2}\,dx\,dz$$

$$= \frac{1}{l}\,z\Big|_0^l \frac{\mu_0 I}{2\pi a^2}\int_0^a \sqrt{x^2+y^2}\,dx$$

but $y=0$, so

$$\frac{\Phi}{l} = \frac{\mu_0 I}{2\pi a^2}\left(\frac{1}{2}x^2\right)\Big|_0^a$$

$$\frac{\Phi}{l} = \frac{\mu_0 I}{4\pi}$$

Part (b), cont'd

II: $a \le r \le b$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^{\ell}\int_a^b B\,dx\,dz$$

$$= \int_a^b \frac{\mu_0 I}{2\pi r}\,dx$$

$$= \int_a^b \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}}\,dx.$$

But $y=0$, so

$$\frac{\Phi}{\ell} = \frac{\mu_0 I}{2\pi}\int_a^b \frac{1}{x}\,dx$$

$$= \frac{\mu_0 I}{2\pi}\ln x\Big|_a^b$$

$$\frac{\Phi}{\ell} = \frac{\mu_0 I}{2\pi}\ln\left(\frac{b}{a}\right).$$

III: $b \le r \le d$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^{\ell}\int_b^d B\,dx\,dz$$

$$= \int_b^d \frac{\mu_0 I}{2\pi r}\left(1 - \frac{r^2-b^2}{d^2-b^2}\right)dx$$

$$= \int_b^d \frac{\mu_0 I}{2\pi x}\left(1 - \frac{x^2-b^2}{d^2-b^2}\right)dx$$

$$= \frac{\mu_0 I}{2\pi}\int_b^d \left(\frac{1}{x} - \frac{x}{d^2-b^2} + \frac{b^2}{x(d^2-b^2)}\right)dx$$

$$= \frac{\mu_0 I}{2\pi}\left[\ln x - \frac{1}{2}\frac{x^2}{d^2-b^2} + \frac{b^2}{d^2-b^2}\ln x\right]\Big|_b^d$$

Part (b), cont'd

Then

$$\frac{\Phi}{\ell} = \frac{\mu_0 I}{2\pi}\left[\ln\left(\frac{d}{b}\right) - \frac{1}{2} + \frac{b^2}{d^2-b^2}\ln\left(\frac{d}{b}\right)\right]$$

$$= -\frac{\mu_0 I}{4\pi} + \frac{\mu_0 I}{2\pi}\left[\left(1+\frac{b^2}{d^2-b^2}\right)\ln\left(\frac{d}{b}\right)\right].$$

IV : $d \le r$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^{\ell}\int_d^{\infty} B\,dx\,dz$$

$$\frac{\Phi}{\ell} = 0$$

since $B = 0$ in this region. Then overall,

$$\frac{\Phi}{\ell} = \frac{\mu_0 I}{4\pi} + \frac{\mu_0 I}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_0 I}{2\pi}\left(1+\frac{b^2}{d^2-b^2}\right)\ln\left(\frac{d}{b}\right) - \frac{\mu_0 I}{4\pi}$$

$$\boxed{\frac{\Phi}{\ell} = \frac{\mu_0 I}{2\pi}\left[\ln\left(\frac{b}{a}\right) + \left(1+\frac{b^2}{d^2-b^2}\right)\ln\left(\frac{d}{b}\right)\right]}.$$

Now we want to find the self-inductance per unit length, L. So since

$$\Phi = LI,$$

we have

$$\frac{L}{\ell} = \frac{\Phi}{\ell I}$$

$$\boxed{\frac{L}{\ell} = \frac{\mu_0}{2\pi}\left[\ln\left(\frac{b}{a}\right) + \left(1+\frac{b^2}{d^2-b^2}\right)\ln\left(\frac{d}{b}\right)\right]}$$

Part (c)

We want to determine the magnetic energy per unit length, U/ℓ, in the region $a < r < b$. In general,

$$U = \frac{1}{2\mu_0} \int B^2 d^3x ,$$

where $d^3x = r\,dr\,d\phi\,dz$ in cylindrical coordinates. Then

$$U = \frac{1}{2\mu_0} \int_a^b \int_0^{2\pi} \int_0^{\ell} \left(\frac{\mu_0 I}{2\pi r} \right)^2 r\,dr\,d\phi\,dz$$

$$= \frac{\ell}{2\mu_0} (2\pi) \int_a^b \left(\frac{\mu_0 I}{2\pi} \right)^2 \frac{1}{r}\,dr$$

$$= \frac{\pi \ell}{\mu_0} \left(\frac{\mu_0 I}{2\pi} \right)^2 \ln\left(\frac{b}{a}\right)$$

$$U = \frac{\mu_0 I^2 \ell}{4\pi} \ln\left(\frac{b}{a}\right) ,$$

and

$$\boxed{\frac{U}{\ell} = \frac{\mu_0 I^2}{4\pi} \ln\left(\frac{b}{a}\right)}$$

Part (d)

We know

$$\Phi = \mathcal{L} I,$$

so

$$\frac{\mathcal{L}}{l} = \frac{\Phi}{l I}.$$

In region II,

$$\frac{\mathcal{L}_{II}}{l} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right),$$

so

$$\mathcal{L}_{II} = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right),$$

and

$$\frac{1}{2} \mathcal{L}_{II} I^2 = \frac{\mu_0 l I^2}{4\pi} \ln\left(\frac{b}{a}\right).$$

This is indeed equivalent to U from part (c).