Relativistic

FALL 2012

5. An infinitely long, uniformly charged wire of radius a and total charge per unit length λ, is at rest on the z-axis of the lab frame.

   (a) (2 pts) Compute the electric field $\mathbf{E}(x, y, z)$ interior and exterior to the wire in the lab frame by solving Gauss's law in that frame.

   (b) Complete the next 4 steps to compute $\mathbf{E}'(x', y', z')$ and $\mathbf{B}'(x', y', z')$ in a frame moving in the positive z-direction with speed v.

      i. (2 pts) Give the Lorentz boost $x'^{\sigma} = L^{\sigma}_{\mu} x^{\mu}$ ($\mathbf{x}' = \mathbf{L}\mathbf{x}$) from the Lab to the moving frame (take $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$).

      ii. (2 pts) Construct the electromagnetic field tensor $F^{\alpha\beta}$ from the electric field you found in part (a).

      iii. (2 pts) Use your lorentz boost to compute the electromagnetic field tensor $F'^{\alpha\beta} = L^{\alpha}_{\mu} L^{\beta}_{\nu} F^{\mu\nu}$ ($\mathbf{F}' = \mathbf{L}\mathbf{F}\mathbf{L}^{\mathbf{T}}$) in the moving frame.

      iv. (2 pts) From your $F'^{\alpha\beta}$ give the answer to (b).

Hint: Recall that in both SI and Gaussian units $F^{\sigma\mu} = -F^{\mu\sigma}$ and $F^{0i} = -E^i$. In Gaussian units $F^{12} = -B^z$, $F^{23} = -B^x$ and $F^{13} = B^y$, but in SI units $F^{12} = -c\,B^z$, $F^{23} = -c\,B^x$ and $F^{13} = c\,B^y$

Part (a)

We can use Gauss's law to find the field exterior to the wire.

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$$

$$E(2\pi r L) = \frac{\lambda L}{\varepsilon_0}$$

$$\vec{E}_{out} = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{r} \hat{r}$$

$$\boxed{\vec{E}_{out} = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{\sqrt{x^2+y^2}} (\cos\phi\, \hat{x} + \sin\phi\, \hat{y})}$$

Since the wire is uniformly charged, then the charge within a Gaussian cylinder of radius $r < a$ is

$$\frac{q_{enc}}{A_{enc}} = \frac{q_{tot}}{A_{tot}}$$

$$q_{enc} = q_{tot} \frac{A_{enc}}{A_{tot}}$$

$$= \lambda L \frac{\pi r^2}{\pi a^2}$$

$$q_{enc} = \frac{\lambda L r^2}{a^2}.$$

Then

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$$

$$E(2\pi r L) = \frac{\lambda L r^2}{\varepsilon_0 a^2}$$

$$\vec{E} = \frac{1}{2\pi\varepsilon_0} \frac{\lambda r}{a^2} \hat{r}$$

$$\boxed{\vec{E} = \frac{1}{2\pi\varepsilon_0} \frac{\lambda\sqrt{x^2+y^2}}{a^2} (\cos\phi\, \hat{y} + \sin\phi\, \hat{y})}.$$

Part (b)

(i) If this frame moves along the $+z$-axis, the transformation matrix is

$$L_z = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}.$$

Then

$$x'^{\alpha} = L_z x^{\alpha}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \gamma ct - \beta\gamma z \\ x \\ y \\ \gamma z - \beta\gamma ct \end{pmatrix}.$$

So

$$ct' = \gamma ct - \beta\gamma z$$

$$\boxed{t' = \frac{\gamma}{c}(ct - \beta z)}$$

$$\boxed{\begin{aligned} x' &= x \\ y' &= y \end{aligned}}$$

$$z' = \gamma z - \beta\gamma ct$$

$$\boxed{z' = \gamma(z - \beta ct)}.$$

Part (b), cont'd

(ii) In general,

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

Then given our result in part (a) and noting that $\vec{B}=0$ since no current flows in the lab frame,

$$\boxed{F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & 0 \\ E_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}.$$

Part (b), cont'd

(iii) In the moving frame, the electromagnetic field tensor is

$$F'^{\alpha\beta} = L_z F^{\alpha\beta} L_z^T$$

$$F'^{\alpha\beta} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & 0 \\ E_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\gamma E_x & -\gamma E_y & 0 \\ E_x & 0 & 0 & 0 \\ E_y & 0 & 0 & 0 \\ 0 & \beta\gamma E_x & \beta\gamma E_y & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\gamma E_x & -\gamma E_y & 0 \\ \gamma E_x & 0 & 0 & -\beta\gamma E_x \\ \gamma E_y & 0 & 0 & -\beta\gamma E_y \\ 0 & \beta\gamma E_x & \beta\gamma E_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & -cB_z' & cB_y' \\ E_y' & cB_z' & 0 & -cB_x' \\ E_z' & -cB_y' & cB_x' & 0 \end{pmatrix}$$

$$\boxed{F'^{\alpha\beta} = \begin{pmatrix} 0 & -\gamma E_x & -\gamma E_y & 0 \\ \gamma E_x & 0 & 0 & -\beta\gamma E_x \\ \gamma E_y & 0 & 0 & -\beta\gamma E_y \\ 0 & \beta\gamma E_x & \beta\gamma E_y & 0 \end{pmatrix}}.$$

$E_x' = \gamma E_x$ $\qquad$ $cB_x' = \beta\gamma E_y$

$E_y' = \gamma E_y$ $\qquad$ $cB_y' = -\beta\gamma E_x$

$E_z' = 0$ $\qquad$ $cB_z' = 0$

$$\vec{E}' = \gamma(E_x + E_y) \qquad \vec{B}' = \frac{\beta\gamma}{c}(-E_x + E_y)$$

$$\boxed{\vec{E}' = \frac{\gamma\lambda}{2\pi\varepsilon_0}\begin{cases} \dfrac{\sqrt{x^2+y^2}}{a^2}(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}), & r < a \\ \dfrac{1}{\sqrt{x^2+y^2}}(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}), & r > a \end{cases}}$$

$$\boxed{\vec{B}' = \frac{\beta\gamma\lambda}{2\pi\varepsilon_0 c}\begin{cases} \dfrac{\sqrt{x^2+y^2}}{a^2}(\sin\phi\,\hat{x} - \cos\phi\,\hat{y}), & r < a \\ \dfrac{1}{\sqrt{x^2+y^2}}(\sin\phi\,\hat{x} - \cos\phi\,\hat{y}), & r > a \end{cases}}$$