Relativistic

FALL 2008

3. A *relativistic* particle of rest mass m and charge e is moving in a uniform (constant and static) magnetic field $\mathbf{B}$. The equations of motion for the particle momentum $\mathbf{p}$ and its energy E are (Gaussian units)

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c}\mathbf{v} \times \mathbf{B}, \quad \frac{dE}{dt} = 0.$$

a) 1 pt. Why is the particle energy conserved?

b) 1 pt. Express $\mathbf{p}$ in terms of m and the particle velocity $\mathbf{v}$, and E in terms of m and $\mathbf{v}$.

c) 3pts. Show that these equations of motion can be written as

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\omega} \times \mathbf{v},$$

and express $\boldsymbol{\omega}$ in terms of e, E, and $\mathbf{B}$. This says that the velocity vector precesses with angular velocity $\boldsymbol{\omega}$.

d) 3pts. Now suppose the motion is confined to the plane perpendicular to $\mathbf{B}$, that is, $\mathbf{B} \perp \mathbf{v}$. Then show that the particle moves with angular speed ω in a circle of radius R. Give an equation for R in terms of v, E, e, and B.

e) 2pts. Now give an equation relating the magnitude of the particle momentum p to the radius R found in part d). Thus show that a measurement of the radius of the orbit determines the particle momentum. If the velocity of the particle is independently known, we can then determine the mass m of the particle, according to the relation given in part b).

Part (a)

The particle energy is conserved because magnetic fields do not do any work, and we only have a force due to the magnetic field since

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v}\times\vec{B} = \vec{F}.$$

Since $W = \Delta K$, we must have $\Delta K = 0$ and therefore, energy is conserved.

Part (b)

We know

$$\boxed{\vec{p} = \gamma m \vec{v}}$$

and

$$E^2 = (pc)^2 + (mc^2)^2$$

$$E = \sqrt{(\gamma m v c)^2 + (mc^2)^2}$$

$$= \sqrt{m^2c^4\left(\gamma^2\frac{v^2}{c^2} + 1\right)}$$

$$E = mc^2\sqrt{\gamma^2\beta^2 + 1}$$

$$= mc^2\sqrt{\beta^2\frac{1}{1-\beta^2} + 1}$$

$$= mc^2\sqrt{\frac{\beta^2}{1-\beta^2} + \frac{1-\beta^2}{1-\beta^2}}$$

$$= mc^2\sqrt{\frac{1}{1-\beta^2}}$$

$$\boxed{E = \gamma mc^2}.$$

Part (c)

We want to show

$$\frac{d\vec{v}}{dt} = \vec{\omega} \times \vec{v}.$$

We know

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v} \times \vec{B},$$

so

$$\frac{d}{dt}(\gamma m \vec{v}) = \frac{e}{c}\vec{v} \times \vec{B}$$

$$\gamma m \frac{d\vec{v}}{dt} = -\frac{e}{c}\vec{B} \times \vec{v}$$

$$\frac{d\vec{v}}{dt} = -\frac{e}{\gamma m c}\vec{B} \times \vec{v}$$

and thus,

$$\vec{\omega} = -\frac{e\vec{B}}{\gamma m c}.$$

but

$$E = \gamma m c^2$$

and so

$$\frac{E}{c} = \gamma m c.$$

Therefore,

$$\vec{\omega} = -\frac{e\vec{B}}{E/c}$$

$$\boxed{\vec{\omega} = -\frac{ec\vec{B}}{E}}.$$

Part (d)

If $\vec{B} \perp \vec{v}$, then since

$$\vec{\omega} = -\frac{ec\vec{B}}{E},$$

$$\frac{d\vec{v}}{dt} = \vec{\omega} \times \vec{v}$$

$$= |\vec{\omega}||\vec{v}|.$$

But

$$\omega = \frac{v}{R},$$

so

$$\frac{dv}{dt} = \frac{v^2}{R},$$

which corresponds to centripetal acceleration, implying that the particle moves in a circle of radius R. Then

$$\omega = \frac{v}{R} = \frac{ecB}{E}$$

$$R = \frac{vE}{ecB}$$

$$\boxed{R = \frac{\beta E}{eB}}.$$

Part (e)

We know

$$p = \gamma m v$$

and

$$v = \frac{ecBR}{E},$$

so

$$p = \gamma m \left(\frac{ecBR}{E}\right)$$

$$= \gamma m c \, \frac{eBR}{E}$$

$$= \left(\frac{E}{c}\right) \frac{eBR}{E}$$

and

$$\boxed{p = \frac{eBR}{E}}.$$