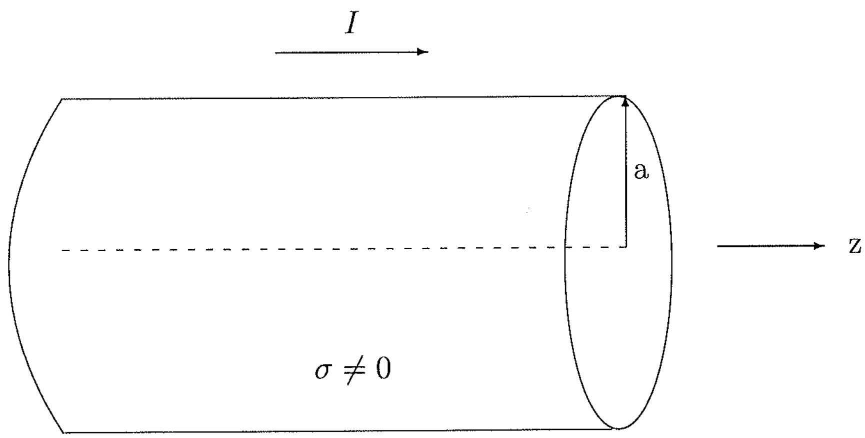
Wire / Cylinder

SPRING 2012

3. Poynting Vector



A straight metal wire of conductivity σ and cross-sectional area $A = \pi a^2$ carries a uniform, steady current I.

(a) (2 pts) Calculate $\mathbf{E}$ at the surface of the wire.

(b) (2 pts) Calculate $\mathbf{B}$ at the surface of the wire.

(c) (1 pts) Calculate the direction and magnitude of the Poynting vector at the surface of the wire.

(d) (3 pts) Integrate the normal component of the Poynting vector over the surface of the wire for a segment of length L.

(e) (2 pts) compare your result for (d) with the Joule heat produced in this segment.

The Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

Part (a)

From Ohm's law, we know

$$\vec{J} = \sigma \vec{E}.$$

The current is given by

$$I = \oint \vec{J} \cdot d\vec{a},$$

so

$$\begin{aligned} I &= \oint \sigma \vec{E} \cdot d\vec{a} \\ &= \sigma \oint \vec{E} \cdot d\vec{a} \\ &= \sigma E \oint da \\ &= \sigma E \left(\pi a^2\right) \end{aligned}$$

and so

$$\boxed{\vec{E} = \frac{I}{\sigma \pi a^2} \hat{z}}$$

Part (b)

Ampère's law says

$$\oint \vec{B} \cdot d\vec{L} = \mu_0 I_{enc}.$$

In this case,

$$I_{enc} = I$$

if our Ampèrian loop has radius 'a'. Then

$$B \oint dL = \mu_0 I$$

$$B(2\pi a) = \mu_0 I$$

and

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\phi}}.$$

Since $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$, we can write this as

$$\vec{B} = \frac{\mu_0 I}{2\pi a}(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}),$$

or

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\left(\sin\phi\,(-\hat{x}) + \cos\phi\,\hat{y}\right).$$

Part (c)

The Poynting vector is given by

$$\vec{S} = \vec{E} \times \vec{H}$$
$$= \vec{E} \times \left(\frac{1}{\mu_0} \vec{B}\right)$$
$$= \frac{1}{\mu_0} \vec{E} \times \vec{B} .$$

Then

$$\vec{E} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & E \\ b_x & b_y & 0 \end{vmatrix}$$

$$= \hat{x}\left(-E b_y\right) - \hat{y}\left(-E b_x\right)$$

$$= \hat{x}\left[-\left(\frac{I}{\sigma \pi a^2}\right)\left(\frac{\mu_0 I}{2\pi a}\cos\phi\right)\right] + \hat{y}\left[\left(\frac{I}{\sigma \pi a^2}\right)\left(-\frac{\mu_0 I}{2\pi a}\sin\phi\right)\right]$$

$$= \frac{\mu_0 I^2}{2\pi^2 \sigma a^3}\left[\cos\phi\,(-\hat{x}) + \sin\phi\,(-\hat{y})\right]$$

and

$$\vec{S} = \frac{1}{\mu_0}\frac{\mu_0 I^2}{2\pi^2 \sigma a^3}\left[\cos\phi\,(-\hat{x}) + \sin\phi\,(-\hat{y})\right]$$

$$\boxed{\vec{S} = \frac{I^2}{2\pi^2 \sigma a^3}\left(-\hat{r}\right)} .$$

Part (d)

We want to determine

$$\oint \vec{S} \cdot d\vec{a}$$

for a segment of length L. Since $\vec{S}$ points radially inward, it is parallel to $d\vec{a}$. So

$$\oint \vec{S} \cdot d\vec{a} = S \oint da$$

$$= \frac{I^2}{2\pi^2 \sigma a^3} \oint da$$

$$= \frac{I^2}{2\pi^2 \sigma a^3} (2\pi a L)$$

$$\boxed{\oint \vec{S} \cdot d\vec{a} = \frac{I^2 L}{\pi \sigma a^2}} .$$

Part (e)

We know that the conductivity is simply the inverse of the resistivity, and

$$\rho = \frac{RA}{L},$$

where R is the resistance, L is the length of the segment, and A is the cross-sectional area. Then since

$$\rho = \frac{1}{\sigma},$$

$$\begin{aligned}
\oint \vec{S} \cdot d\vec{a} &= \frac{I^2 L}{\pi \sigma a^2} \\
&= \frac{I^2 L}{\pi a^2} \cdot \rho \\
&= \frac{I^2 L}{\pi a^2} \frac{RA}{L} \\
&= \frac{I^2 L}{\pi a^2} \frac{R(\pi a^2)}{L} \\
&= I^2 R.
\end{aligned}$$

This is, in fact, the Joule heat produced in the segment, or the dissipated power.