Wire / Cylinder

FALL 2008

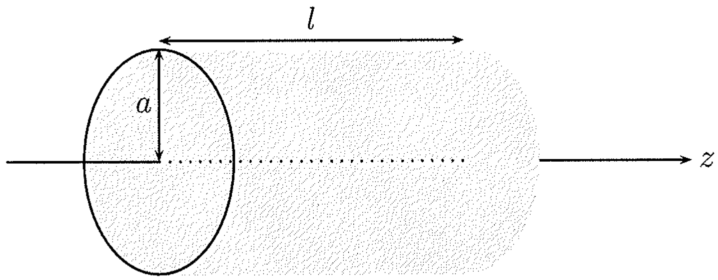

Figure 2: Hollow cylinder (radius a and length l) containing uniform gas flowing along the axis, the z direction, with velocity v. Protons are injected into the cylinder with velocity V parallel to the axis. As a result of magnetic forces, they are brought to a focus at a point on the z axis a distance p far from the cylinder, $p \gg l$.

4. Consider a hollow cylinder of radius a and length l filled with a completely ionized gas of uniform charge density ρ which is moving parallel to the axis of the cylinder with velocity v.

a) 3pts. Find the magnetic field (magnitude and direction) at a distance r from the axis of the cylinder, for $r < a$; assume that we are well inside the cylinder and that $l \gg r$ so that we can neglect edge effects. Assume that the gas is nonmagnetic.

b) 3pts. Suppose a beam of nonrelativistic protons of mass m and velocity V are sent into this cylinder with their initial velocities parallel to the z axis. Neglect electrostatic, edge effects, and collisions between protons and the gas. Show that while in the gas-filled cylinder, the protons experience a force pushing them toward the axis of the cylinder. Calculate the radial velocity V_r acquired by the protons when they exit the cylinder. Assume that the distance moved toward the axis while in the cylinder is negligible.

c) 2pts. After the protons leave the cylinder, they continue to move toward the z axis with constant radial velocity V_r. Calculate the time T required for the protons to reach the axis.

d) 2pts. As a result, the protons will travel through the cylinder and be focused at a point p on the z axis beyond the cylinder where $p \gg l$. Find p and show that it is independent of the initial distance of the protons from the axis when they enter the cylinder.

Part (a)

By Ampère's law,

$$\oint \vec{B} \cdot d\vec{L} = \mu_0 I_{enc}.$$

In this case, we can say that the ratio of enclosed current to the area of the Ampèrian loop is equal to the ratio of total current to total area. If the radius of our Ampèrian loop is r, where $r < a$, then

$$\frac{I_{enc}}{A_{enc}} = \frac{I_{tot}}{A_{tot}}$$

$$I_{enc} = A_{enc} \frac{I_{tot}}{A_{tot}}$$

$$= (\pi r^2) \frac{\rho v}{\pi a^2}$$

$$I_{enc} = \frac{\rho v r^2}{a^2}.$$

By the right hand rule, we know the magnetic field curls around the z-axis. This means $\vec{B} \parallel d\vec{L}$, so since $|\vec{B}|$ is constant,

$$\oint \vec{B} \cdot d\vec{L} = \mu_0 I_{enc}$$

$$B \oint dL = \mu_0 \left(\frac{\rho v r^2}{a^2} \right).$$

Since our loop is circular,

$$B (2\pi r) = \mu_0 \left(\frac{\rho v r^2}{a^2} \right)$$

and in the region $r < a$,

$$\boxed{\vec{B} = \frac{\mu_0}{2\pi} \frac{\rho v r}{a^2} \hat{\phi}}.$$

Part (b)

The force exerted on the protons by the magnetic field is

$$\vec{F} = q\vec{V} \times \vec{B}.$$

Evaluating the cross product...

$$\vec{V} \times \vec{B} = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ 0 & 0 & V \\ 0 & B & 0 \end{vmatrix}$$

$$= \hat{r}(-VB) - \hat{\phi}(0) + \hat{z}(0)$$

$$= -VB\hat{r}.$$

Thus,

$$\vec{F} = -qVB\,\hat{r}$$

$$= qVB(-\hat{r}),$$

and the protons experience a force pushing them toward the cylinder's axis.