Wire / Cylinder

FALL 2009

3 Wire

An infinitely-long, thin wire (radius b) is coated with a dielectric (relative dielectric constant $k = \epsilon/\epsilon_0$ with radius $a > b$). The metal wire has charge per unit length λ

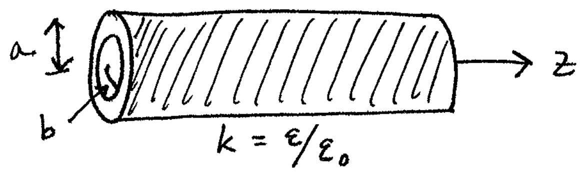
a) Find the electric displacement $\vec{\mathbf{D}}$ everywhere. (2 points)

b) Find the electric field $\vec{\mathbf{E}}$ everywhere. (2 points)

c) Find the polarization $\vec{\mathbf{P}}$ everywhere. (3 points)

d) Find **all** the bound charge everywhere. (3 points)

Part (a)



Charge per unit length $= \lambda$

We want to find the electric displacement in three regions: I: $r<b$, II: $b<r<a$, and III: $r>a$.

I: $r<b$

By Gauss's law,

$$\oint \vec{D} \cdot d\vec{a} = q_{enc}.$$

Should have multiplied q_{enc} by 4π

In this region, $q_{enc} = 0$ so

$$\boxed{\vec{D}_I = 0}.$$

II: $b<r<a$

$$\oint \vec{D} \cdot d\vec{a} = q_{enc}$$

$$D \oint d\vec{a} = \lambda l,$$

where $\oint da$ is the area of the cylinder minus the ends. So

$$D(2\pi r l) = \lambda l$$

$$\boxed{\vec{D}_{II} = \frac{\lambda}{2\pi r}\hat{r}},$$

where r is the radius of our Gaussian surface.

Part (a), cont'd

III: $r > a$

We get the same result as in region II, so

$$\boxed{\vec{D}_{III} = \frac{\lambda}{2\pi r}\hat{r}}.$$

Part (b)

Now we want to find the electric field in each of these regions. We know

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$
$$= \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E}$$
$$= \varepsilon_0 (1+\chi_e) \vec{E}$$
$$= \varepsilon_0 k \vec{E}$$
$$\vec{D} = \varepsilon_0 \left(\frac{\varepsilon}{\varepsilon_0}\right) \vec{E}$$
$$\vec{D} = \varepsilon \vec{E}.$$

So ...

I: $r<b$

$$\vec{E}_I = \frac{1}{\varepsilon} \vec{D}_I$$
$$\boxed{\vec{E}_I = 0}$$

II: $b<r<a$

$$\vec{E}_{II} = \frac{1}{\varepsilon} \vec{D}_{II}$$
$$\boxed{\vec{E}_{II} = \frac{\lambda}{2\pi\varepsilon r} \hat{r}}$$

III: $r>a$

$$\vec{E}_{III} = \frac{1}{\varepsilon_0} \vec{D}_{III}$$

since this is outside of the material and $\vec{P}=0$. So

$$\boxed{\vec{E}_{III} = \frac{\lambda}{2\pi\varepsilon_0 r} \hat{r}}.$$

Part (c)

We know

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P},$$

so...

I: $r<b$

We know $\vec{D}_I = 0$, $\vec{E}_I = 0$ so

$$\boxed{\vec{P}_I = 0}.$$

II: $b<r<a$

We know $\vec{D}_{II} = \frac{\lambda}{2\pi r}\hat{r}$ and $\vec{E}_{II} = \frac{\lambda}{2\pi\varepsilon r}\hat{r}$, so

$$\frac{\lambda}{2\pi r}\hat{r} = \varepsilon_0 \frac{\lambda}{2\pi\varepsilon r}\hat{r} + \vec{P}_{II}$$

$$\boxed{\vec{P} = \frac{\lambda}{2\pi r}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\hat{r}}.$$

III: $r>a$

We know $\vec{D}_{III} = \frac{\lambda}{2\pi r}\hat{r}$ and $\vec{E}_{III} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$, so

$$\frac{\lambda}{2\pi r}\hat{r} = \varepsilon_0 \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r} + \vec{P}_{III}$$

$$\boxed{\vec{P}_{II} = 0}$$

Part (d)

We want to find the bound charge in each region, both surface and volume. Surface bound charge is given by

$$\sigma_b = \vec{P} \cdot \hat{n} \,\Big|_{surface}$$

and volume bound charge is given by

$$\rho_b = -\vec{\nabla} \cdot \vec{P} \,.$$

I: $r<b$

$$\vec{P}_I = 0$$

So $\boxed{\rho_b = 0}$.

III: $r>a$

$$\vec{P}_{III} = 0$$

So $\boxed{\rho_b = 0}$.

II: $b<r<a$

$$\vec{P}_{II} = \frac{\lambda}{2\pi r}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\hat{r}$$

So

$$\sigma_{b,a} = \frac{\lambda}{2\pi r}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\hat{r} \cdot (+\hat{r}) \,\Big|_{r=a}$$

$$\boxed{\sigma_{b,a} = \frac{\lambda}{2\pi a}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)}$$

$$\sigma_{b,b} = \frac{\lambda}{2\pi r}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\hat{r} \cdot (-\hat{r}) \,\Big|_{r=b}$$

$$\boxed{\sigma_{b,b} = \frac{\lambda}{2\pi b}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)}$$

$$\rho_b = -\frac{1}{r}\frac{\partial}{\partial r}\left(r P_{II}\right)$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\lambda}{2\pi}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\right)$$

$\boxed{\rho_b = 0}$.

So there is only surface bound charge.