Sphere

SPRING 2010

2. Consider a linear, homogeneous, isotropic, and non-dissipative dielectric (i.e., a dielectric where $\mathbf{D} = \epsilon\mathbf{E}$ and ϵ is a constant) in the shape of a sphere of radius R with a point charge Q embedded at its center.

   (a) {2 pts} Find the electric displacement vector $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization density $\mathbf{P}$ inside the dielectric.

   (b) {2 pts} Find the bound charge volume density ρ_D inside the dielectric.

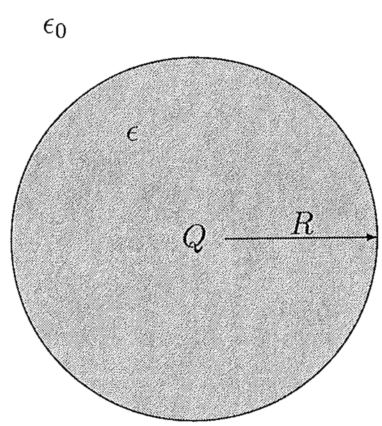
   (c) {1 pts} Find the total bound charge Q_D on the $r = R$ boundary of the dielectric.

   (d) {2 pts} Find the net charge (free plus bound) at the center of the dielectric.

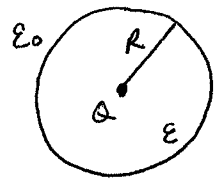
   (e) {1 pts} Find the electric displacement vector $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization density $\mathbf{P}$, outside the dielectric sphere.

   (f) {2 pts} Are $\mathbf{D}$ and $\mathbf{E}$ continuous at $r = R$? If not explain why.

   (If you use Gaussian units you can put $\epsilon_0 = 1$.)



Part (a)



We can use Gauss's law to determine the electric field inside the dielectric. So

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon}$$

$$E \oint da = \frac{Q}{\varepsilon}$$

$$E(4\pi r^2) = \frac{Q}{\varepsilon}$$

and so

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon} \frac{Q}{r^2} \hat{r}},$$

where r is the radius of our Gaussian sphere, and $r < R$. The displacement vector is

$$\vec{D} = \varepsilon \vec{E}$$

$$= \varepsilon \left(\frac{1}{4\pi\varepsilon} \frac{Q}{r^2} \hat{r} \right)$$

$$\boxed{\vec{D} = \frac{1}{4\pi} \frac{Q}{r^2} \hat{r}}.$$

We know

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P},$$

so

$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E}$$

$$= \frac{1}{4\pi} \frac{Q}{r^2} \hat{r} - \varepsilon_0 \frac{1}{4\pi\varepsilon} \frac{Q}{r^2} \hat{r}$$

$$\boxed{\vec{P} = \frac{Q}{4\pi r^2} \left(1 - \frac{\varepsilon_0}{\varepsilon}\right) \hat{r}}$$

Part (b)

We want to find the bound charge volume density within the dielectric. We know

$$\rho_b = -\vec{\nabla}\cdot\vec{P},$$

so

$$\rho_b = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P\right)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi}\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\right)$$

and

$$\boxed{\rho_b = 0}.$$

Part (c)

To find the bound charge on the surface, $r = R$, we first need to find the surface bound charge density, σ_b. We have

$$\sigma_b = \vec{P} \cdot \hat{n} \Big|_{surface}$$
$$= \vec{P} \cdot \hat{r} \Big|_{r=R}$$
$$= \frac{Q}{4\pi r^2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\hat{r} \cdot \hat{r} \Big|_{r=R}$$
$$= \frac{Q}{4\pi R^2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right).$$

Then the bound charge on the surface is

$$Q_b = \int \sigma_b \, da$$
$$= \int \frac{Q}{4\pi R^2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) da$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{Q}{4\pi R^2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) R^2 \sin\theta \, d\theta \, d\phi$$

and

$$\boxed{Q_b = Q\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)}.$$

Part (d)

Now we want to find the net charge at the center of the sphere. If we have

$$Q_b = Q\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)$$

on the surface, then at the center we have

$$Q_b = -Q_b$$
$$= -Q\left(1 - \frac{\varepsilon_0}{\varepsilon}\right).$$

Then since we have free charge

$$Q_f = Q$$

at the center, the total charge at the center is

$$Q_{tot} = Q_b + Q_f$$
$$= Q - Q\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)$$
$$\boxed{Q_{tot} = \frac{Q\varepsilon_0}{\varepsilon}}.$$

Part (e)

Outside of the dielectric,

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$$

$$E \oint da = \frac{q_{enc}}{\varepsilon_0}$$

$$E(4\pi r^2) = \frac{q_{enc}}{\varepsilon_0}$$

$$\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}},$$

where $r > R$. Then

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2}\hat{r}}.$$

Since we are outside of the material, we know

$$\boxed{\vec{P} = 0}.$$

Part (f)

At the surface, we have no free surface charge. Since the boundary condition on $\vec{D}$ at the surface is

$$D_\perp^{above} - D_\perp^{below} = \sigma_{free},$$

then

$$D_\perp^{above} - D_\perp^{below} = 0$$
$$D_\perp^{above} = D_\perp^{below}$$

and the normal component of $\vec{D}$ is continuous across the surface. However,

$$E_\perp^{above} - E_\perp^{below} = \frac{\sigma_{tot}}{\varepsilon_0}$$
$$E_\perp^{above} - E_\perp^{below} = \frac{\sigma_b}{\varepsilon_0},$$

so the electric field's normal component is discontinuous by σ_b/ε_0 across the surface boundary.