Relativistic

FALL 2011

5. A charged particle with charge $q$ and mass $m$ starts from rest on the inner plate (radius $a$) of a cylindrical capacitor and is accelerated towards the outer plate (radius $b$). Orient the coordinates so that $q$ starts at $(x, y) = (a, 0)$ and is accelerated along the +x direction until it reaches $(x, y) = (b, 0)$.

   (a) {2 pts} If the charge/length on the capacitor $\lambda_0$ is constant find the electric field causing the acceleration by using Gauss's law. Assume the capacitor is very long compared to the radii $a$ and $b$ and assume the electric field can is cylindrically symmetric.

   (b) {4 pts} Write down the 4-D [or (3+1) D] special relativistic version of Newton's equations for the motion of a point charge experiencing the Lorentz force (the force due to an arbitrary external **E** and **B** field).

$$\frac{dp^\mu}{d\tau} = ?$$

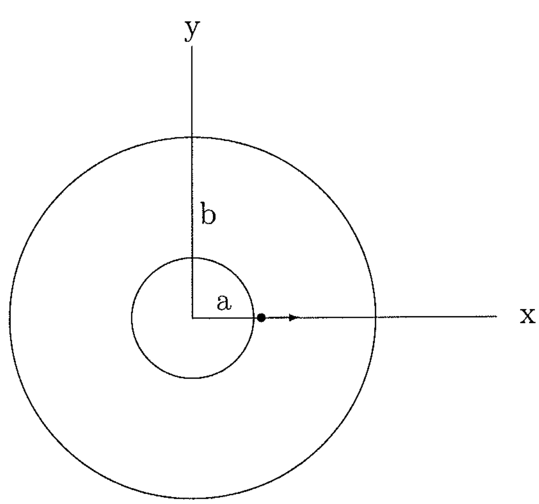
   or

$$\frac{dm\gamma c^2}{dt} = ? \qquad \text{and} \qquad \frac{d\vec{p}}{dt} = ?$$

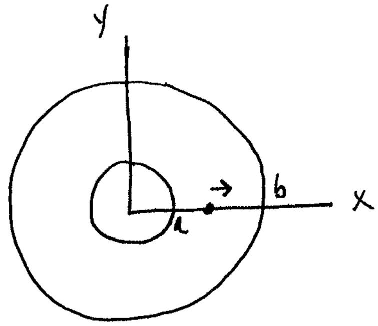
   Be sure to define $p^\mu$ and $\vec{p}$ as well as the Lorentz force terms that appear on the right hand sides of the above equations.

   (c) {4 pts} Apply your answer to part (b) to the field you found in part (a) and integrate your dynamical equations to obtain the particle's energy when it reaches the outer plate. You can easily compute $\gamma(x)$ from Newton's equations even though the acceleration is not constant.

   You are **not** asked to compute $x(t)$ or $x(\tau)$ nor how long it took to reach $x = b$.



## Part (a)




Charge density : $\lambda_0$

The field must be directed toward the $+\hat{r}$-direction since the charge accelerates toward the outer cylinder.

By Gauss's Law...

$$\oint \vec{E} \cdot d\vec{a} = 4\pi q_{enc}$$

$$E(2\pi r l) = 4\pi(\lambda_0 l) \qquad (\text{for } a < r < b)$$

$$\boxed{\vec{E} = \frac{2\lambda_0}{r}\hat{r}}.$$

## Part (b)

Newton's 2nd law is given by

$$\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta .$$

So

$$\begin{aligned} p^\alpha &= mU^\alpha \\ &= m\frac{dx^\alpha}{d\tau} \\ &= m\frac{d}{d\tau}(ct, \vec{x}) . \end{aligned}$$

We know $t = \gamma\tau$, so then $dt = \gamma d\tau$ and

$$\begin{aligned} \frac{d}{d\tau}(ct, \vec{x}) &= \frac{d}{dt}(ct, \vec{x}) \cdot \frac{dt}{d\tau} \\ &= \gamma\frac{d}{dt}(ct, \vec{x}) \end{aligned}$$

$$\begin{aligned} p^\alpha &= m\gamma\frac{d}{dt}(ct, \vec{x}) \\ &= m\gamma\left(\frac{d}{dt}(ct), \frac{d}{dt}\vec{x}\right) \end{aligned}$$

$$\boxed{p^\alpha = m\gamma(c, \vec{v})} .$$

The 3-momentum is simply

$$\boxed{\vec{p} = m\gamma\vec{v}} .$$

# Part (b), cont'd

We can say

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta = \frac{q}{c} F^{\alpha\beta} U^\alpha g_{\alpha\beta}$$

where

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$U_\beta = \gamma(c, \vec{v}) \cdot \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \gamma(-c, \vec{v})$$

and

$$\frac{dp^\alpha}{d\tau} = \frac{q\gamma}{c} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} -c \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

$$= \frac{q\gamma}{c} \begin{pmatrix} -(v_x E_x + v_y E_y + v_z E_z) \\ -cE_x - v_y B_z + v_z B_y \\ -cE_y + v_x B_z - v_z B_x \\ -cE_z - v_x B_y + v_y B_x \end{pmatrix}$$

$$= -\frac{q\gamma}{c} \begin{pmatrix} v_x E_x + v_y E_y + v_z E_z \\ cE_x + v_y B_z - v_z B_y \\ cE_y + v_z B_x - v_x B_z \\ cE_z + v_x B_y - v_y B_x \end{pmatrix}$$

## Part (b), cont'd

But

$$\vec{V} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ V_x & V_y & V_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{x}\left(V_y B_z - V_z B_y\right) - \hat{y}\left(V_x B_z - V_z B_x\right) + \hat{z}\left(V_x B_y - V_y B_x\right)$$

and

$$\vec{E} \cdot \vec{V} = V_x E_x + V_y E_y + V_z E_z$$

so

$$\frac{dp^\alpha}{d\tau} = -\frac{q\gamma}{c}\begin{pmatrix} \vec{E} \cdot \vec{V} \\ cE_x + [\vec{V} \times \vec{B}]_x \\ cE_y + [\vec{V} \times \vec{B}]_y \\ cE_z + [\vec{V} \times \vec{B}]_z \end{pmatrix}$$

$$\boxed{\frac{dp^\alpha}{d\tau} = -\frac{q\gamma}{c}\left(\vec{E} \cdot \vec{V},\ c\vec{E} + \vec{V} \times \vec{B}\right)}$$

## Part (c)

From the previous parts...

$$\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta$$

Considering $\alpha = 0$, we have

$$m\frac{dU^0}{d\tau} = \frac{q}{c}F^{0\beta}U_\beta$$

$$m\gamma\frac{dU^0}{dt} = \frac{q}{c}F^{0\beta}U_\beta$$

$$-m\gamma\frac{d}{dt}(\gamma c) = \frac{q}{c}\left[(-E^\beta)\cdot(\gamma v_\beta)\right]$$

$$mc^2\frac{d\gamma}{dt} = qE^\beta v_\beta$$

$$mc^2\frac{d\gamma}{dt} = qE^\beta\frac{dx}{dt}$$

and so

$$mc^2\,d\gamma = qE^\beta dx$$

$$\int_1^\gamma mc^2\,d\gamma = \int_a^b q\frac{2\lambda_0}{x}\,dx$$

$$mc^2\int_1^\gamma d\gamma = 2q\lambda_0\int_a^b\frac{1}{x}\,dx$$

$$mc^2\gamma - mc^2 = 2q\lambda_0 \ln x\Big|_a^b$$

$$\boxed{\gamma mc^2 = 2q\lambda_0\ln\left(\frac{b}{a}\right) + mc^2}.$$

This is the particle's energy when it reaches the outer plate.