Sphere

Electrodynamics Qualifier Examination

January 10, 2007

1. This problem deals with magnetostatics, described by a time-independent magnetic field, produced by a current density which is divergenceless,

$$\boldsymbol{\nabla} \cdot \mathbf{J} = 0.$$

Further, we are describing a nonmagnetic medium, so the relative permeability $\mu = 1$.

(a) [2 pts.] Show from Maxwell's equations that in the radiation gauge, where

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0,$$

the vector potential satisfies

$$-\nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J},$$

in Gaussian units.

(b) [2 pts.] Show this equation may be solved as

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int d^3 r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.$$

(c) [3 pts.] Now consider a sphere of radius a centered on the origin, which carries a total charge e distributed uniformly on its surface, and which is rotating with angular velocity $\boldsymbol{\omega}$, so that the velocity of a point $\mathbf{r}'$ on its surface is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}'$. Use the result of part 1b as well as the Legendre expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r^l}{a^{l+1}} P_l(\cos\gamma),$$

Sphere

SPRING 2007

for $|\mathbf{r}| < a = |\mathbf{r}'|$, with γ being the angle between $\mathbf{r}$ and $\mathbf{r}'$, to compute both the vector potential, and the magnetic field *inside* the sphere.

(d) [3 pts.] Proceed in the same way to compute the vector potential and the magnetic field outside the sphere. What is the magnetic dipole moment of a rotating charged spherical shell?

Part (a)

In the radiation gauge,

$$\vec{\nabla}\cdot\vec{A} = 0.$$

From Maxwell's equations,

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\left(4\pi\vec{J} + \frac{\partial\vec{D}}{\partial t}\right).$$

We know

$$\vec{H} = \vec{\nabla}\times\vec{A},$$

so

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{1}{c}\left(4\pi\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} = \frac{1}{c}\left(4\pi\vec{J} + \frac{\partial\vec{D}}{\partial t}\right).$$

But $\vec{\nabla}\cdot\vec{A} = 0$, so

$$-\vec{\nabla}^2\vec{A} = \frac{1}{c}\left(4\pi\vec{J} + \frac{\partial\vec{D}}{\partial t}\right).$$

Since this is a magnetostatics problem, we must have

$$\vec{D} = 0,$$

so indeed,

$$\boxed{-\vec{\nabla}^2\vec{A} = \frac{4\pi}{c}\vec{J}}.$$

Part (b)

We assume

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \, .$$

Plugging into the equation from part (a) ...

$$-\vec{\nabla}^2\vec{A} = -\vec{\nabla}^2\left[\frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right]$$

$$= -\frac{1}{c}\left[\int \frac{\vec{\nabla}^2\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r' + \int \vec{\nabla}^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\vec{J}(\vec{r}')d^3r'\right].$$

but $\vec{\nabla}^2\vec{J} = 0$ and $\vec{\nabla}^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\delta^3(\vec{r}-\vec{r}')$, so

$$-\vec{\nabla}^2\vec{A} = \frac{4\pi}{c}\int \vec{J}(\vec{r}')\,\delta^3(\vec{r}-\vec{r}')d^3r'$$

and

$$-\vec{\nabla}^2\vec{A} = \frac{4\pi}{c}\vec{J},$$

as expected. Thus,

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

is a solution.