Waves

SPRING 2007

3. Light with wavenumber **k** and frequency ω is incident from vacuum normally along the z axis on a crystalline-plane parallel plate of thickness d. The crystal is cut so that a plane wave polarized along the x axis is refracted with index of refraction n_0 and one polarized along the y axis is refracted with index of refraction n_E.

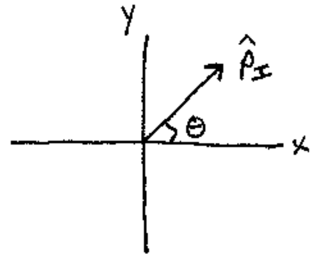
   (a) [2 pts.] If the incident light is polarized at angle θ with respect to the x axis, write the form of the incident wave (using ω, θ, and space and time coordinates), assuming an incident amplitude of E_I.

   (b) [2 pts.] Now write the form of the refracted wave inside the crystal (using ω, θ, n_0, n_E, and space and time coordinates) assuming that the refracted wave amplitude is E_T.

   (c) [6 pts.] Find the conditions of θ and d such that the refracted wave at d (the opposite side of the plate) is circularly polarized (either right or left).

Part (a)

The incident light is polarized at an angle θ with respect to $\hat{x}$. So we have a setup like



where $\hat{\rho}_I$ is the direction of polarization of our incident wave. We can write this as

$$\hat{\rho}_I = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}.$$

Then

$$\vec{E}_i = E_I e^{i(kz-\omega t)}\hat{\rho}_I$$

$$\boxed{\vec{E}_i = E_I e^{i(kz-\omega t)}\left(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right)}.$$

Part (b)

We know that along the x-axis within the material, we have $k = k_o$. Along the y-axis, we have $k = k_E$. If $k = \frac{n\omega}{c}$, then

$$k_o = \frac{n_o \omega}{c}$$

$$k_E = \frac{n_E \omega}{c}.$$

The refracted wave is then given by

$$\boxed{\vec{E}_t = E_T\left[e^{i(k_o z - \omega t)}\cos\theta\,\hat{x} + e^{i(k_E z - \omega t)}\sin\theta\,\hat{y}\right]},$$

where E_T is the refracted wave amplitude.

Part (c)

The wave is circularly polarized when the amplitudes in the x- and y-directions are equal, but offset by a phase difference of $\frac{\pi}{2}$.

Setting the amplitudes equal, we can put a restraint on θ.

$$E_{T_x}\cos\theta = E_{T_y}\sin\theta$$

$$\tan\theta = \frac{E_{T_x}}{E_{T_y}}.$$

To determine E_{T_x} and E_{T_y}, we want to apply the boundary conditions at the interface. Specifically, we want to apply the tangential boundary conditions

$$\vec{E}_{1,\parallel} - \vec{E}_{2,\parallel} = 0$$

$$\vec{H}_{1,\parallel} - \vec{H}_{2,\parallel} = 0,$$

which take into account the fact that there is no free current or free surface charge density. Then

$$\vec{E}_{1,\parallel} - \vec{E}_{2,\parallel} = 0$$

$$(\vec{E}_i + \vec{E}_r - \vec{E}_t)\times\hat{n} = 0,$$

which ends up giving us

$$E_I + E_R - E_T = 0.$$

Applying the second boundary condition...

$$\vec{H}_{1,\parallel} - \vec{H}_{2,\parallel} = 0$$

$$\frac{1}{\mu_0}\vec{B}_{1,\parallel} - \frac{1}{\mu_0}\vec{B}_{2,\parallel} = 0$$

$$\vec{B}_{1,\parallel} - \vec{B}_{2,\parallel} = 0$$

$$(\vec{B}_i + \vec{B}_r - \vec{B}_t)\times\hat{n} = 0$$

$$B_I - B_R - B_T = 0$$

$$B_I - B_R = B_T$$

Part (c), cont'd

But

$$\vec{B} = n\hat{k} \times \vec{E},$$

so since $n = \sqrt{\varepsilon\mu}$, and $\varepsilon = \varepsilon_0 = 1$ and $\mu = \mu_0 = 1$ in vacuum, we have

$$E_I - E_R = nE_T .$$

Plugging in E_R from our first condition...

$$E_I - (E_T - E_I) = nE_T$$

$$2E_I = (n+1)E_T$$

$$E_T = \frac{2}{n+1} E_I .$$

Then

$$E_{T_x} = \frac{2}{n_0+1} E_I$$

$$E_{T_y} = \frac{2}{n_E+1} E_I$$

and

$$\tan\theta = \frac{E_{T_x}}{E_{T_y}}$$

$$= \frac{\frac{2}{n_0+1} E_I}{\frac{2}{n_E+1} E_I}$$

$$= \frac{n_E+1}{n_0+1}$$

$$\boxed{\theta = \arctan\left(\frac{n_E+1}{n_0+1}\right)} .$$

Part (c), cont'd

We want to determine the condition on d. At the end of the plate, $(z+d)$, we know that the phase difference between $\vec{E}_i$ and $\vec{E}_t$ is $\pi/2$. So...

$$[k_o(z+d)-\omega t] - [k_E(z+d)-\omega t] = \frac{\pi}{2}$$

$$(k_o-k_E)z + (k_o-k_E)d = \frac{\pi}{2}$$

$$(k_o-k_E)d = \frac{\pi}{2} - (k_o-k_E)z$$

$$d = \frac{\frac{\pi}{2}-(k_o-k_E)z}{k_o-k_E}$$

$$\boxed{d = \frac{\pi - 2(k_o-k_E)z}{2(k_o-k_E)}}.$$