Wire / Cylinder

SPRING 2009

2 Rods

In this problem you will determine the magnetic field produced by three different infinitely long, cylindrical conducting rods. The figures on the next page are useful for vizualizing the differences in the rods.

a) State whether you are using MKS or cgs units (**0-points**).

b) **Rod 1**: Consider an infinitely long, solid, cylindrical conducting rod (known as Rod 1) with radius $2a$ that is concentric with the z-axis and carries a uniform current density $+J_0$ in the $+z$ direction (Fig. 1). Let $r = (x^2 + y^2)^{1/2}$ be the perpendicular distance to the z-axis and θ be the angle r makes with the positive x-axis. (See Fig. 1.) Find the magnitude and direction of the magnetic field $\vec{\mathbf{B}}_1$ produced by Rod 1 for all r. Give the direction in Cartesian coordinates using the unit vectors: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. **(3-points)**

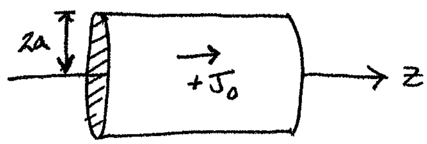
c) **Rod 2**: Consider a second, infinitely long, solid, cylindrical conducting rod (i.e. Rod 2) that is parallel to the z-axis and has radius a. The axis of Rod 2 is centered at $(x, y) = (+a, 0)$. Rod 2 carries a uniform current density $-J_0$ in the $-z$ direction. Let ρ be the radial distance from the axis of the Rod 2 and let ϕ be the angle that ρ makes with the positive x-axis. (See Fig. 2.) Find the magnitude and direction of the magnetic field $\vec{\mathbf{B}}_2$ produced by Rod 2 for all values of ρ and ϕ. Give the direction in Cartesian coordinates using the unit vectors: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. **(1-points)**

d) **Rod 3**: Consider an infinitely long, cylindrical conducting rod (i.e. Rod 3) with radius $2a$ that is concentric with the z-axis and carries a uniform current density J_0 in the $+z$ direction. However in this conductor, an (infinitely long) hole of radius a is drilled parallel to the z-axis at the position $(x, y) = (+a, 0)$. (See Fig 3). Find the magnitude and direction of the magnetic field $\vec{\mathbf{B}}_3$ produced by Rod 3 on the x-axis (at $y = 0$) for all values of $x > 0$. Give the direction in the Cartesian cordinates using the unit vectors: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. **(6-points)**

Part (a)

I will use SI, or MKS, units for this problem.

Part (b)



We want to find the magnetic field both inside and outside of the rod.

Inside

Using Ampère's law with a loop of radius $r < 2a$, we have

$$\oint \vec{B} \cdot d\vec{L} = \mu_0 I_{enc}.$$

In this case, since the current density is uniform, we have

$$\frac{I_{enc}}{A_{enc}} = \frac{I_{tot}}{A_{tot}}$$

$$\begin{aligned} I_{enc} &= A_{enc} \frac{I_{tot}}{A_{tot}} \\ &= A_{enc} J_0 \\ &= \pi r^2 J_0 \end{aligned}$$

and

$$\begin{aligned} B \oint dL &= \mu_0 \pi r^2 J_0 \\ B(2\pi r) &= \mu_0 \pi r^2 J_0 \\ \vec{B} &= \frac{\mu_0 r J_0}{2} \hat{\phi}. \end{aligned}$$

But $\hat{\phi} = -\sin\phi \,\hat{i} + \cos\phi \,\hat{j}$, so

$$\boxed{\vec{B}_{in} = \frac{\mu_0 r J_0}{2} (-\sin\phi \,\hat{i} + \cos\phi \,\hat{j})},$$

where $r = \sqrt{x^2+y^2}$.

Part (b), cont'd

Now using Ampère's law with a loop of radius $r > 2a$, we have

$$\oint \vec{B} \cdot d\vec{L} = \mu_0 I_{enc}$$

$$B(2\pi r) = \mu_0 (J_0 A_{enc})$$

$$B(2\pi r) = \mu_0 J_0 \pi (2a)^2$$

$$= \mu_0 J_0 4\pi a^2$$

$$\vec{B}_{out} = 2 \mu_0 J_0 \frac{a^2}{r} \hat{\phi}$$

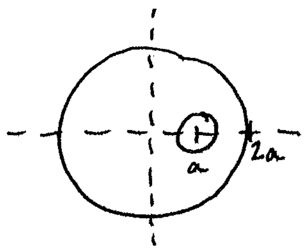
or

$$\boxed{\vec{B}_{out} = 2\mu_0 J_0 \frac{a^2}{r} \left(-\sin\phi \, \hat{i} + \cos\phi \, \hat{j}\right)},$$

where $r = \sqrt{x^2+y^2}$. This decreases as r increases, which is reasonable as we move away from the rod.

In both of these cases, I used ϕ instead of θ out of habit.

Part (d)



We can use superposition to determine the magnetic field of this rod.

Without the hole, we know the magnetic field is

$$\vec{B}_1 = \frac{1}{2}\mu_0 r J_0 \left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right)$$

from part (a), for inside the rod. If the hole were at the origin, the magnetic field within the hole would simply be the opposite in sign as the answer above, so

$$\vec{B}_2 = -\frac{1}{2}\mu_0 r J_0 \left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right).$$

We can move to cartesian coordinates,

$$\vec{B}_2 = -\frac{1}{2}\mu_0 J_0 \left(-y\,\hat{\imath} + x\,\hat{\jmath}\right)$$

But we want the field at $y=0$, so

$$\vec{B}_2 = -\frac{1}{2}\mu_0 J_0 x\,\hat{\jmath}.$$

Since the hole is at $x=a$, we simply shift our result, so

$$\vec{B}_2 = -\frac{1}{2}\mu_0 J_0 (x-a)\,\hat{\jmath}.$$

Then by superposition,

$$\vec{B}_{tot,in} = \vec{B}_1 + \vec{B}_2$$

$$= \frac{1}{2}\mu_0 J_0 x\,\hat{\jmath} + \left(-\frac{1}{2}\mu_0 J_0 (x-a)\,\hat{\jmath}\right)$$

$$\boxed{\vec{B}_{tot,in} = \frac{1}{2}\mu_0 J_0 a\,\hat{\jmath}}$$

Part (d), cont'd

We use the same principle for the field outside the cylinder. We still have

$$\vec{B}_2 = -\tfrac{1}{2}\mu_0 J_0 (x-a)\hat{j},$$

but

$$\vec{B}_1 = 2\mu_0 J_0 \frac{a^2}{r}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$
$$= 2\mu_0 J_0 \frac{1}{x^2}(-y\,\hat{i} + x\hat{j})$$
$$\vec{B}_1 = \frac{2\mu_0 J_0}{x}\hat{j}.$$

So the total field is

$$\vec{B}_{tot,out} = \vec{B}_1 + \vec{B}_2$$
$$= \frac{2\mu_0 J_0}{x}\hat{j} + \left(-\tfrac{1}{2}\mu_0 J_0 (x-a)\hat{j}\right)$$
$$= \mu_0 J\left(\frac{2}{x} - \frac{x-a}{2}\right)\hat{j}$$
$$= \mu_0 J\left(\frac{4 - x(x-a)}{2x}\right)\hat{j}$$
$$\boxed{\vec{B}_{tot,out} = \mu_0 J\left(\frac{4 + ax - x^2}{2x}\right)\hat{j}}$$

This doesn't seem right, but oh well. The result does "decrease," but it doesn't go to zero as I would expect for $x \to \infty$.