E & M Qualifier

January xx, 2015

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

1. (a) [2 pts] Write down all four of Maxwell's equations in differential form for the four fields $\mathbf{E}, \mathbf{H}, \mathbf{D}$, and $\mathbf{B}$. Which two are homogeneous equations and which two are non-homogeneous equations? Which one is called Faraday's law of induction, which one is called Ampère's law, and which one is equivalent to Gauss's law?

   (b) [1 pts] Maxwell's equations simplify for fields defined in static, nonconducting, homogeneous, isotropic, and linear materials. Eliminate $\mathbf{H}$ and $\mathbf{D}$ from two of the four equations and simplify these two equations by assuming $\mathbf{B} = \mu\mathbf{H}$, and $\mathbf{D} = \epsilon\mathbf{E}$, where the permittivity ϵ and permeability μ are both real positive constants. (Do not assume the free charge density or the free current density vanishes.)

   (c) [2 pts] Using your Maxwell equations from part (a) explain exactly why $\mathbf{E}$ and $\mathbf{B}$ can be replaced by potentials ϕ and $\mathbf{A}$. What freedom (non-uniqueness) exists in ϕ and $\mathbf{A}$ for a given pair of fields $\mathbf{E}$ and $\mathbf{B}$?

   (d) [2 pts] Replace $\mathbf{E}$ and $\mathbf{B}$ by ϕ and $\mathbf{A}$ in your four Maxwell equations of part (b) and explain how it is possible to solve them for ϕ and $\mathbf{A}$ if these quantities are not unique?

   (e) [3 pts]Given charge and current densities $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ bounded in space (i.e., contained entirely in $r < R$) and static before some early time $t = t_0$, simplify your Maxwell equations from part (d) by using the Coulomb ($\nabla \cdot \mathbf{A} = 0$) gauge constraint. Give the retarded solution for ϕ and $\mathbf{A}$ to your Maxwell equations as 3-d spatial integrals.

1. (a)

Homo

$\vec{\nabla}\cdot\vec{B} = 0$

Faraday's Law $\quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$

inHomo

$\vec{\nabla}\cdot\vec{D} = \rho$ Gauss's Law

$\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\rho$ Ampère's law

$\boxed{\text{red} = \text{Gaussian}}$

(b) $\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$

$\vec{\nabla}\cdot\vec{E} = \frac{4\pi}{\epsilon}\rho$

$\vec{\nabla}\times\vec{B} - \frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t} = \mu\frac{4\pi}{c}\vec{J}$

(c) When can you solve $\boxed{\vec{\nabla}\times\vec{A} = \vec{B}}$ for $\vec{A}$ given $\vec{B}$?

eliminate $\vec{B}$

$\frac{\partial A^y}{\partial x} - \frac{\partial A^x}{\partial y} = B^z$ etc

The answer is when $\vec{\nabla}\cdot\vec{B} = 0$!

$\vec{A}'$

But! The answer for $\vec{A}$ is not unique $(\vec{A} + \vec{\nabla}\Lambda)$ gives the same $\vec{B}$!

$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = 0 \Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{A}\right) = 0$

eliminate $\vec{E}$

When can you solve for ϕ $\boxed{\vec{\nabla}\phi = -\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)}$ given $E + \vec{A}$?

The answer is when $\vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0$! But ϕ is unique up only to a constant!

$\therefore$ if $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$

$\vec{\nabla}\phi' = \vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\Lambda \Rightarrow \phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} + \text{con}$

can be absorbed into Λ !

$\therefore (\vec{E},\vec{B}) \to (\phi,\vec{A}) \to (\phi',\vec{A}')$

when $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$ and $\phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$

(d) $\vec{\nabla}\cdot\vec{B} = 0$ + $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$ are identically satisfie[d] when $(\phi, \vec{A})$ are introduced!

$$\vec{\nabla}\cdot\vec{E} = \frac{4\pi}{\epsilon}\rho \Rightarrow \vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}\right) = \frac{4\pi}{\epsilon}\rho$$

Gauss's law ① $\therefore \boxed{\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\frac{4\pi}{\epsilon}\rho}$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\epsilon\mu\frac{\partial}{\partial t}\vec{E} = \frac{\mu 4\pi}{c}\vec{J}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \frac{\epsilon\mu}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = \frac{\mu 4\pi}{c}\vec{J}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} + \frac{\epsilon\mu}{c}\frac{\partial}{\partial t}\vec{\nabla}\phi + \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \frac{\mu 4\pi}{c}\vec{J}$$

② $\boxed{\underbrace{\Box_v \vec{A}} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{\epsilon\mu}{c}\frac{\partial\phi}{\partial t}\right) = \mu\frac{4\pi}{c}\vec{J}}$ Ampère's l[aw]

To solve for a unique $(\phi, \vec{A})$ you must impose some addit[io]nal constraints on $(\phi\ \vec{A})$ to eliminate the Λ freedom,

e.g., $\vec{\nabla}\cdot\vec{A} = 0$ ⊕ boundary conditions on $\vec{A}$ will do + gives the coulomb gauge!

or e.g., $\vec{\nabla}\cdot\vec{A} + \frac{\epsilon\mu}{c}\frac{\partial\phi}{\partial t} = 0$ + boundary conditions will do + gives the Lorentz gauge.

(e) choose $\vec{A}$ that satisfies $\vec{\nabla}\cdot\vec{A}=0$

$\therefore$ your Maxwell eqn should be

① $$\nabla^2\phi = -\frac{4\pi}{\epsilon}\rho$$

and $$\Box_v \vec{A} = \mu\frac{4\pi}{c}\vec{J} - \frac{\epsilon\mu}{c}\frac{\partial}{\partial t}\vec{\nabla}\phi$$

$\downarrow$

② $$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\vec{A} = \mu\frac{4\pi}{c}\vec{J}_{tr}$$ (transverse part of $\vec{J}$)

$$v \equiv c/\sqrt{\epsilon\mu} = \text{constant} \qquad \vec{J}_{tr} \equiv \vec{J} - \frac{1}{4\pi}\frac{\partial}{\partial t}\vec{\nabla}\phi$$

$$\boxed{\begin{aligned} &① \Rightarrow \phi = \frac{1}{4\pi\epsilon}\int \frac{\rho(\vec{r}',t)\,d^3r'}{|\vec{r}-\vec{r}'|} \\ &② \Rightarrow \vec{A} = \frac{\mu}{4\pi c}\int \frac{\vec{J}_{tr}\left(t-\frac{|\vec{r}-\vec{r}'|}{v},\vec{r}'\right)d^3r'}{|\vec{r}-\vec{r}'|} \end{aligned}}$$

2. Consider a capacitor composed of two thin concentric spherical metal shells, the inner one with radius a and the outer one with radius b. The region between the spherical metal shells is filled with a linear dielectric with permittivity $\epsilon = k/r^2$. A charge $+Q$ exists on the inner metallic shell and $-Q$ on the outer metallic shell.

   (a) [2 pts] Find the electric displacement $\mathbf{D}$ everywhere in space.

   (b) [3 pts] Find the capacitance of the configuration.

   (c) [5 pts] Calculate the bound charge densities within the dielectric and on its surfaces, and verify that the total net bound charge is zero.

(a) #2

$$\vec{\nabla}\cdot\vec{D} = 4\pi\,\rho_{free} \Rightarrow \boxed{\vec{D} = \frac{Q}{4\pi r^2}\,\hat{r}}$$

$a<r<b$

$=0$ outside

(b) $$\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{Q}{4\pi k}\,\hat{r}$$ $a<r<b$

$=0$ outside

$$\Rightarrow \quad \Delta V = \frac{Q}{4\pi k}(b-a)$$

$$\Rightarrow \boxed{C = Q/\Delta V = \frac{4\pi k}{(b-a)}}$$

(c) $$\vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot\left(\epsilon_0\vec{E} + 4\pi\vec{P}\right) = 4\pi\rho_f$$

$$\epsilon_0\vec{\nabla}\cdot\vec{E} = 4\pi(\rho_f \underbrace{-\vec{\nabla}\cdot\vec{P}}_{\rho_b})$$

$$\rho_f = 0 \Rightarrow \rho_b = \frac{\epsilon_0}{4\pi}\vec{\nabla}\cdot E = \frac{\epsilon_0}{4\pi}\vec{\nabla}\cdot\left(\frac{Q}{4\pi k}\hat{r}\right)$$

$a<r<b$ $$\boxed{\rho_b} = \frac{\epsilon_0 Q}{4\pi k}\vec{\nabla}\cdot\hat{r} = \boxed{\frac{\epsilon_0\, 2Q}{4\pi k r}}$$

from $\rho_b = -\vec{\nabla}\cdot\vec{P}$ we have $\sigma_b = \vec{P}\cdot\hat{n}$

$$\vec{P} = \frac{\vec{D} - \epsilon_0 \vec{E}}{4\pi} = \frac{(\epsilon - \epsilon_0)\vec{E}}{4\pi}$$

$$\Rightarrow \boxed{\begin{aligned} &\sigma_b = \frac{\left(k/b^2 - \epsilon_0\right)}{\cancel{4\pi}} \frac{Q}{\cancel{4\pi}\, k} \quad \text{at } r = b \\ &\qquad\qquad 4\pi \\ &\sigma_b = -\left(\frac{k}{a^2} - \epsilon_0\right)\frac{Q}{4\pi \quad k} \quad \text{at } r = a \end{aligned}}$$

$$\int_a^b \rho_b \, 4\pi r^2 dr = \frac{\epsilon_0 Q}{k}\left(b^2 - a^2\right)$$

$$\sigma_b * 4\pi b^2 = \left(k - \epsilon_0 b^2\right)\frac{Q}{k} \quad \text{on } r = b$$

$$\sigma_b * 4\pi a^2 = -\left(k - \epsilon_0 a^2\right)\frac{Q}{k} \quad \text{on } r = a$$

$$\boxed{\Sigma_i = 0}$$

3. In this problem you will construct the 4-dimensional (4-d) electromagnetic stress-energy-momentum tensor from the 4-d electromagnetic field tensor $F^{\mu\nu}$ in **Gaussian** units. Recall that $F^{\mu\nu}$ is antisymmetric ($F^{\mu\nu} = -F^{\nu\mu}$) and is constructed from components of the electric and magnetic induction fields **E** and **B** by choosing

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon^{ijk} B^k.$$

Here we use the Einstein convention of summing over repeated indices, where Greek letters run from 0 to 3, while Latin letters run from 1 to 3. The totally anti-symmetric 3-dimensional Levi-Civita symbol is defined by $\epsilon^{ijk} = \epsilon_{ijk}$ with $\epsilon^{123} = +1$. The time coordinate is given by $x^0 = ct$, where c is the speed of light and the 4-d metric is chosen as one of the two possibilities $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

(a) [3 pts] Define the 4-current J^μ and show that in a region containing no polarizable materials ($\epsilon = \mu = 1$) Maxwell equations are written in 4-d form as

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.$$

(b) [1 pts] From Maxwells equations prove that charge is conserved, i.e., show that

$$\partial_\mu J^\mu = 0.$$

(c) [3 pts] The 4-d stress-energy-momentum tensor is a traceless symmetric second-rank tensor, quadratic in the field strengths defined by

$$T^{\mu\nu} = \frac{1}{4\pi} \left[F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\beta\alpha} \right].$$

Show that the 4 parts of $T^{\mu\nu}$ can be identified with the electromagnetic energy density u by $T^{00} = u$, the momentum density $\mathbf{g}$ and the Poynting vector $\mathbf{S}$ by $T^{0i} = T^{i0} = cg^i = S^i/c$, and the 3-d Maxwell stress tensor $\overleftrightarrow{\mathbf{T}}_M$ by $T^{ij} = -T^{ij}_M$. Be sure to give u, $\mathbf{g} = \mathbf{S}/c^2$, and $\overleftrightarrow{\mathbf{T}}_M$ as functions of **E** and **B**.

(d) [3 pts] In a region where $J^\mu = 0$, show that $\partial_\mu T^{\mu\nu} = 0$ and this one equation in 4-d is equivalent to the local conservation of electromagnetic energy and momentum in 3-d, i.e., that

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = 0,$$

and

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \overleftrightarrow{\mathbf{T}}_M.$$

$\frac{1}{4}$

3

(a) $J^\mu \equiv \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$ $\qquad \partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu$

$\mu = 0 \qquad \partial_0 F^{00} + \partial_i F^{i0} = \frac{4\pi}{c} c\rho$

$\qquad \overset{\shortparallel}{0} \qquad \partial_i E^i = 4\pi\rho \quad \hookrightarrow \quad \boxed{\vec{\nabla}\cdot\vec{E} = 4\pi\rho}$

$\mu = j \qquad \partial_0 F^{0j} + \partial_i F^{ij} = \frac{4\pi}{c} J^j$

$\qquad \partial_0(-E^j) + \partial_i(-\epsilon^{ijk}B^k)$

$\qquad -\frac{1}{c}\frac{\partial E^j}{\partial t} + \epsilon^{jik}\frac{\partial}{\partial x^i}B^k = \frac{4\pi}{c}J^j \quad \hookrightarrow \quad \boxed{\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}}$

$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$

$\lambda, \mu, \nu = i, j, k$ (all different)

$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0 \qquad ijk = 1, 2, 3$

$\partial_1(F_{23}) + \partial_2(F_{31}) + \partial_3(F_{12}) = 0$

$\partial_1(-B^1) + \partial_2(-B^2) + \partial_3(-B^3) = 0 \Leftrightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$

$\lambda, \mu, \nu = 0, i, j \qquad (i \neq j)$

$\boxed{(\vec{\nabla}\times\vec{E}) + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0}$

$\uparrow$ cycle 123

$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0$

$\partial_0(-\epsilon^{ijk}B^k) + \partial_i(E^j) + \partial_j(E^i) = 0$

$i, j = 1, 2 \qquad \partial_0(-B^3) + \partial_1 E^2 + \partial_2 E^1 = 0 \Rightarrow \boxed{(\vec{\nabla}\times\vec{E})^3 + \frac{1}{c}\frac{\partial B^3}{\partial t} = 0}$

(b) $\partial_\mu J^\mu \stackrel{?}{=} 0$

$$\begin{aligned}
&\partial_\mu(\partial_\nu F^{\nu\mu}) = \underbrace{\partial_\mu\partial_\nu}_{\text{derivatives commute}} F^{\nu\mu} = \partial_\nu\partial_\mu F^{\overbrace{\nu\mu}^{\text{anti symmetric exchange}}} = -\partial_\nu\partial_\mu F^{\mu\nu} \\
&\quad \therefore -\partial_\nu\partial_\mu F^{\mu\nu} \quad \leftarrow \text{rename dummy indices} \\
&\quad = -\partial_\mu\partial_\nu F^{\nu\mu} \quad \boxed{= 0}
\end{aligned}$$

is the only real # equal to its negative!

(c)

$$T^{\mu\nu} \equiv \frac{1}{4\pi}\left[F^{\mu\lambda}F_\lambda{}^\nu - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\beta\alpha}\right]$$

$$F^{0\lambda}F_\lambda{}^0 = F^{0i}F_i{}^0 = (-E^i)(-E^i) = |\vec{E}|^2$$

$$F^{0\lambda}F_\lambda{}^i = F^{i\lambda}F_\lambda{}^0 = (-E^j)(F_j{}^i) = -E^j\epsilon^{jik}B^k = (\vec{E}\times\vec{B})^i$$

$$\begin{aligned}
F^{i\lambda}F_\lambda{}^j &= F^{i0}F_0{}^j + F^{ik}F_k{}^j \\
&= E^i(-E^j) + \underbrace{(-\epsilon^{ikm}B^m)(\epsilon^{kjp}}_{-(\delta^j_m\delta^p_i - \delta^j_i\delta^p_m)B^mB^p} B^p) \\
&= -E^iE^j - B^iB^j + \delta^{ij}|\vec{B}|^2
\end{aligned}$$

$$\therefore F^{\alpha\beta}F_{\beta\alpha} = |\vec{E}|^2 + |\vec{E}|^2 + |\vec{B}|^2 - 3|\vec{B}|^2 = 2(|\vec{E}|^2 - |\vec{B}|^2)$$

$$\therefore\ T^{00} = \frac{1}{4\pi}\left[|\vec{E}|^2 - \frac{1}{4}\left(2|\vec{E}|^2 - 2|\vec{B}|^2\right)\right] = \frac{1}{8\pi}\left(|\vec{E}|^2 + |\vec{B}|^2\right) \equiv \boxed{u}$$

Energy density $\mu=\epsilon=1$

$$T^{0i} = T^{i0} = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)^i = \frac{S^i}{c} = c g^i \qquad \vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\vec{H}\right)$$

Poynting vector ↑ in Gaussian

$$T^{ij} = \frac{1}{4\pi}\left[-E^iE^j - B^iB^j + \delta^{ij}|\vec{B}|^2 + \frac{\delta^{ij}}{4}\left(2|\vec{E}|^2 - 2|\vec{B}|^2\right)\right]$$

$$= -\frac{1}{4\pi}\left[E^iE^j + B^iB^j - \frac{\delta^{ij}}{2}\left(|\vec{E}|^2 + |\vec{B}|^2\right)\right] \equiv -T_m^{ij}$$

Maxwell's stress tensor

(d)
$$\partial_\mu T^{\mu\nu} = \frac{1}{4\pi}\Big[\underbrace{(\partial_\mu F^{\mu\lambda})}_{= -J^\lambda} F^\nu{}_\lambda + F^{\mu\lambda}\partial_\mu F^\nu{}_\lambda$$
$$- \frac{1}{4} g^{\mu\nu}\left((\partial_\mu F^{\alpha\beta})F_{\beta\alpha} + F^{\alpha\beta}\partial_\mu F_{\beta\alpha}\right)\Big]$$

same

$$= \frac{1}{4\pi}\left[F^\nu{}_\lambda J^\lambda + g^{\nu\mu} F^{\alpha\beta}\left(\partial_\alpha F_{\beta\mu} - \frac{1}{2}\partial_\mu F_{\beta\alpha}\right)\right]$$

$$= \frac{1}{4\pi}\left[F^\nu{}_\lambda J^\lambda + \frac{g^{\nu\mu}F^{\alpha\beta}}{2}\underbrace{\left[\partial_\alpha F_{\beta\mu} - \underbrace{\partial_\beta F_{\alpha\mu}}_{= -\partial_\alpha F_{\mu\beta}} - \partial_\mu F_{\beta\alpha}\right]}_{0 \text{ from homogeneous Max eqns}}\right]$$

$$\boxed{\partial_\mu T^{\mu\nu} = \frac{1}{4\pi} F^\nu{}_\lambda J^\lambda}$$

$= 0$ when $J^\lambda = 0$!

$$\partial_\mu T^{\mu\gamma} = 0 \Rightarrow \partial_0 T^{0\gamma} + \partial_i T^{i\gamma} = 0$$

$$\parallel$$

$$\frac{1}{c}\frac{\partial}{\partial t} T^{0\gamma} + \partial_i T^{i\gamma} = 0$$

$$\gamma = 0 \Rightarrow \quad \frac{1}{c}\frac{\partial}{\partial t} T^{00} + \partial_i T^{i0} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t} u + \partial_i \left(\frac{1}{c} S^i\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0}$$

$$\gamma = j \Rightarrow \quad \frac{1}{c}\frac{\partial}{\partial t} T^{0j} + \partial_i T^{ij} = 0$$

$$\parallel$$

$$\frac{\partial}{\partial t} g^j - \partial_i T_m^{ij} = 0$$

$$\therefore \boxed{\frac{\partial}{\partial t} \vec{g} = \vec{\nabla} \cdot \overleftrightarrow{T}_m}$$

4. A solution of dextrose, which is optically active, is characterized by a polarization vector $\mathbf{P} = \gamma \nabla \times \mathbf{E}$ where γ is a real constant that depends on the concentration of dextrose. The solution is non-conducting ($\mathbf{J}_{\text{free}} = 0$) and non-magnetic ($\mathbf{M} = 0$). Consider a plane electromagnetic wave of angular frequency ω propagating along the $+z$-axis in such a solution.

   (a) [5 pts] Using Maxwell's equations show that left and right circularly polarized waves travel at 2 distinct speeds ($v_{\pm}$) in this medium. Calculate the indices of refraction $n_{\pm} = (ck_{\pm})/\omega = c/v_{\pm}$ as a function of ω and γ for left and right circularly polarized waves. Recall that left ($+$) and right ($-$) circularly polarized waves are of the form

$$\mathbf{E} = E_0\,(\hat{\mathbf{x}} \pm i\,\hat{\mathbf{y}})\,e^{i(kz-\omega t)}$$

   (b) [5 pts] Suppose linearly polarized light is incident on the dextrose solution. After traveling a distance L through the solution, the light is still linearly polarized but its direction of polarization rotated by an angle $\Delta\phi$. Calculate $\Delta\phi$ in terms of L, γ, and ω. Hint: Write $k_{\pm} = \overline{k} \pm \Delta k$ where

$$\overline{k} \equiv \frac{k_{+} + k_{-}}{2} \quad \text{and} \quad \Delta k \equiv \frac{k_{+} - k_{-}}{2}.$$

   Also recall that the amplitude of a wave linearly polarized at an angle ϕ relative to the x-direction can be written as a combination of circularly polarized amplitudes as

$$(\cos\phi\;\hat{\mathbf{x}} + \sin\phi\;\hat{\mathbf{y}}) = e^{i\phi}\left(\frac{\hat{\mathbf{x}} - i\,\hat{\mathbf{y}}}{2}\right) + e^{-i\phi}\left(\frac{\hat{\mathbf{x}} + i\,\hat{\mathbf{y}}}{2}\right).$$

#4 $\vec{P} = \gamma \vec{\nabla}\times\vec{E}$ $\qquad \vec{J}_{Free} = 0\ ,\ \vec{M} = 0$

$$\therefore\ \vec{D} = \epsilon_0 \vec{E} + 4\pi\vec{P} = \epsilon_0\vec{E} + 4\pi\gamma\vec{\nabla}\times\vec{E}$$

$\gamma \downarrow$ constant

$$\vec{H} = \frac{\vec{B}}{\mu_0}$$

$(\vec{\nabla}\cdot\vec{\nabla}\times\vec{E} = 0)$

4 Max eqns!

① $\vec{\nabla}\cdot\vec{D} = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = 0}$

② $\vec{\nabla}\cdot\vec{B} = 0$

③ $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$

④ $\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = 0 \Rightarrow \frac{\vec{\nabla}\times\vec{B}}{\mu_0} - \frac{1}{c}\frac{\partial}{\partial t}\left(\epsilon_0\vec{E} + 4\pi\gamma\vec{\nabla}\times\vec{E}\right) = 0$

$$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)} \qquad \vec{B} = \vec{B}_0 e^{i(kz-\omega t)}$$

① $\Rightarrow \hat{z}\cdot\vec{E}_0 = 0$ $\qquad$ ② $\Rightarrow \hat{z}\cdot\vec{B}_0 = 0$

③ $\Rightarrow ik\hat{z}\times\vec{E}_0 - \frac{i\omega}{c}\vec{B}_0 = 0 \Rightarrow \boxed{\vec{B}_0 = \frac{ck}{\omega}\hat{z}\times\vec{E}_0}$

substitute

④ $\Rightarrow \frac{\cancel{i}k\hat{z}\times\vec{B}_0}{\mu_0} + \frac{\cancel{i}\omega}{c}\left(\epsilon_0\vec{E}_0 + i4\pi\gamma k\hat{z}\times\vec{E}_0\right) = 0$

$$\frac{ck^2}{\mu_0\omega}(-\vec{E}_0) + \frac{\omega\epsilon_0}{c}\vec{E}_0 + \frac{i\omega 4\pi\gamma}{c}k(\hat{z}\times\vec{E}_0) = 0$$

df $\vec{E}_0 = E_0(\hat{x} \pm i\hat{y})$ $\qquad$ + = left circularly pol

$\qquad$ − = right "

$$\Rightarrow \hat{z}\times\vec{E}_0 = E_0(\hat{z}\times\hat{x} \pm i\hat{z}\times\hat{y}) = E_0(\hat{y} \mp i\hat{x})$$

$$= \mp i E_0(\hat{x} \pm i\hat{y}) = \boxed{\mp i\vec{E}_0}$$

substitute

$\sqrt{\epsilon_0 \mu_0} = c^{-1}$

$$\therefore \quad -\frac{ck^2}{\mu_0 \omega} + \frac{\omega \epsilon_0}{c} \pm \frac{\omega \gamma k 4\pi}{c} = 0$$

$$\therefore \quad \left(\frac{ck}{\omega}\right)^2 \mp \frac{4\pi\gamma\omega}{\epsilon_0 c}\left(\frac{ck}{\omega}\right) - 1 = 0$$

quadratic = 2 roots for each sign

$$\frac{ck_\pm}{\omega} = \frac{\pm\left(\frac{4\pi\gamma\omega}{\epsilon_0 c}\right) (\pm) \sqrt{\left(\frac{4\pi\gamma\omega}{\epsilon_0 c}\right)^2 + 4}}{2}$$

only 1 root is physical for each sign!

$$\boxed{\begin{aligned} \frac{ck_\pm}{\omega} &= \sqrt{\left(\frac{4\pi\gamma\omega}{2\epsilon_0 c}\right)^2 + 1} \pm \left(\frac{4\pi\gamma\omega}{2\epsilon_0 c}\right) \\ v_\pm &= \omega / k_\pm, \quad n_\pm = c/v_\pm = \frac{ck_\pm}{\omega} \end{aligned}}$$

(b) Consider a linearly polarized wave at $z=0$

$$\vec{E}(z=0) = |E_0|(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y})e^{-i\omega t}$$

$$= |E_0|\left[e^{i\varphi}\frac{(\hat{x}-i\hat{y})}{2} + e^{-i\varphi}\frac{(\hat{x}+i\hat{y})}{2}\right]e^{-i\omega t}$$

↑ See Hint

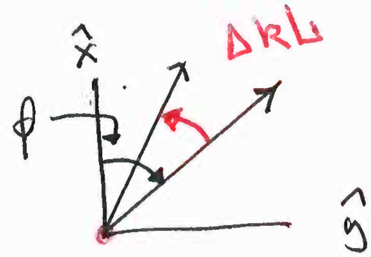
at $z = b$

$$\vec{E}(z=b) = |E_0|\left[e^{i\varphi}\frac{(\hat{x}-i\hat{y})}{2} e^{+ik_- b} + e^{-i\varphi}\frac{(\hat{x}+i\hat{y})}{2} e^{ik_+ b}\right] * e^{-i\omega t}$$

$$= |E_0|\left[e^{i\varphi}\frac{(\hat{x}-i\hat{y})}{2} e^{-i\Delta k b} + e^{-i\varphi}\frac{(\hat{x}+i\hat{y})}{2} e^{i\Delta k b}\right] e^{i(\bar{k} b)}$$

$$= |E_0|\left[e^{i(\varphi-\Delta k b)}\frac{(\hat{y}-i\hat{y})}{2} + e^{-i(\varphi-\Delta k b)}\frac{(\hat{x}+i\hat{y})}{2}\right] e^{i(\bar{k} b-\omega t)}$$

$$= |E_0|\left[\cos(\varphi-\Delta k b)\,\hat{x} + \sin(\varphi-\Delta k b)\,\hat{y}\right] e^{i(\bar{k} b-\omega t)}$$

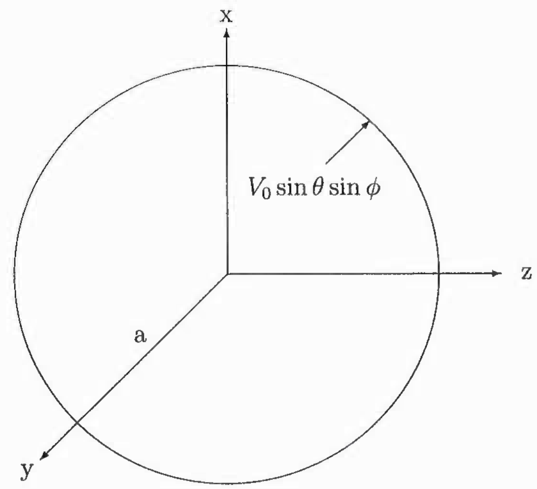


$$\bar{k} = \frac{\omega}{c}\sqrt{1+\left(\frac{4\pi\gamma\omega}{2\epsilon_0 c}\right)^2}$$

$$\Delta k = \left(\frac{\omega}{c}\right)^2 \frac{\gamma\, 4\pi}{2\epsilon_0}$$

$$\boxed{\Delta\varphi = -\Delta k b}$$

$$\boxed{\Delta k = \left(\frac{\omega}{c}\right)^2 \frac{\gamma\, 4\pi}{2\epsilon_0}}$$



5. Assume that in spherical polar coordinates (r, θ, ϕ), the potential on the surface of a sphere of radius a, centered on the origin, is known to be $V(\theta, \phi)$.

   (a) [2 pts] If the space inside the sphere is empty give an expression for the potential $\Phi(r, \theta, \phi)$ everywhere inside as an expansion in spherical harmonics with arbitrary constants. If you knew the potential $V(\theta, \phi)$ on the surface how would you evaluate the constants in your expansion?

   (b) [2 pts] If the space outside the sphere is empty give an expression for the potential $\Phi(r, \theta, \phi)$ everywhere outside as an expansion in spherical harmonics with arbitrary constants. If you knew the potential $V(\theta, \phi)$ on the surface how would you evaluate the constants in your expansion?

   (c) [6 pts] If $V(\theta, \phi) = V_0 \sin\theta \sin\phi$ give exact expressions for $\Phi(r, \theta, \phi)$ inside and out of the sphere.

   The spherical harmonics are ortho-normal on the sphere and for $\ell = 1$

$$
\begin{aligned}
Y_1^{-1}(\theta, \phi) &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}, \\
Y_1^{0}(\theta, \phi) &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\
Y_1^{1}(\theta, \phi) &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}.
\end{aligned}
$$

Y_2

#5

(a) $$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm}\, r^l\, Y_l^m(\theta,\phi) \qquad r \le a$$

Require a_{lm} are constrained to make

$$V(\theta\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm}(a)^l\, Y_l^m(\theta,\phi)$$

$$\Rightarrow \quad a_{lm} = \frac{1}{(a)^l} \int_0^{\pi}\int_0^{2\pi} Y_l^{m*}(\theta,\phi)\, V(\theta\phi)\, \sin\theta\, d\theta\, d\phi$$

(b) $$\Phi(r,\theta\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{b_{lm}}{r^{l+1}}\, Y_l^m(\theta\phi)$$

Require $$V(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{b_{lm}}{(a)^{l+1}}\, Y_l^m(\theta\phi)$$

$$\Rightarrow \quad b_{lm} = (a)^{l+1} \int_0^{\pi}\int_0^{2\pi} Y_l^{m*}(\theta\phi)\, V(\theta\phi)\, \sin\theta\, d\theta\, d\phi$$

(c) $$\sin\theta \sin\phi = \sin\theta\left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right)$$

$$= \sqrt{\frac{8\pi}{3}}\left(\frac{-Y_1^1 - Y_1^{-1}}{2i}\right) = \sqrt{\frac{2\pi}{3}}\; i\left(Y_1^1(\theta\phi) + Y_1^{-1}(\theta\phi)\right)$$

$\therefore$

$$a_{\ell m} = \frac{V_0}{(a)^\ell} \int_0^\pi \int_0^{2\pi} Y_\ell^{m*}(\theta\phi) \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1(\theta\phi) + Y_1^{-1}(\theta\phi) \right) \sin\theta\, d\theta\, d\phi$$

$$= \frac{V_0}{(a)^1} \sqrt{\frac{2\pi}{3}}\, i \left[\delta_\ell^1 \left(\delta_m^1 + \delta_m^{-1} \right) \right]$$

$$\Rightarrow \phi(r,\theta\phi) = \frac{V_0}{(a)^1} (r)^1 \sqrt{\frac{2\pi}{3}}\, i \left[Y_1^1(\theta\phi) + Y_1^{-1}(\theta\phi) \right]$$

$$\boxed{\phi(r,\theta\phi) = V_0 \left(\frac{r}{a} \right) \sin\theta \sin\phi \;, \quad r \leq a}$$

For $r > a$

$$b_{\ell m} = V_0 (a)^{\ell+1} \sqrt{\frac{2\pi}{3}}\, i \left[\delta_\ell^1 \left(\delta_m^1 + \delta_m^{-1} \right) \right]$$

$$\Rightarrow \phi(r,\theta\phi) = V_0 \left(\frac{a}{r} \right)^2 \sqrt{\frac{2\pi}{3}}\, i \left[Y_1^1(\theta\phi) + Y_1^{-1}(\theta\phi) \right]$$

$$\boxed{\phi(r,\theta\phi) = V_0 \left(\frac{a}{r} \right)^2 \sin\theta \sin\phi \;, \quad r \geq a}$$

6. In this problem you are to describe properties of waves penetrating into conductors. If the conductor is static, homogeneous, isotropic, linear, and ohmic, you can replace $\mathbf{D}, \mathbf{H}$, and $\mathbf{J}$ in Maxwell's equations using

$$\mathbf{D} = \epsilon \mathbf{E}, \ \mathbf{B} = \mu \mathbf{H}, \ \mathbf{J} = \sigma \mathbf{E},$$

where ϵ, μ, and σ are real positive constants. For simplicity you can also assume the wave is a harmonic plane wave propagating in the z-direction, e.g., its electric field and magnetic induction are of the form

$$\mathbf{E} = E_0 e^{i(kz-\omega t)}\hat{\mathbf{x}},$$

$$\mathbf{B} = B_0 e^{i(kz-\omega t)}\hat{\mathbf{y}}.$$

(a) [4 pts] Use Maxwell's equations to relate k to ω. Explain what the imaginary part of $k = k_{Re} + ik_{Im}$ does to the amplitude of the wave.

(b) [3 pts] Use Maxwell's equations to relate B_0 to E_0. Explain what the phase of $k = |k|e^{i\phi}$ does to the phase of $\mathbf{B}$ as compared to the phase of $\mathbf{E}$.

(c) [3 pts] If the conductor is a "good" conductor, i.e., if for low frequencies, $\epsilon \ll \sigma/\omega$, what does k simplify to, what is the attenuation distance (skin depth) of the wave, and what is the phase delay of the magnetic induction $\mathbf{B}$ relative to the electric field $\mathbf{E}$?

#6

(a) $\vec{\nabla}\cdot\vec{D}=0 \Rightarrow \vec{\nabla}\cdot\vec{E}=0$ satisfied by given $\vec{E}$

$\vec{\nabla}\cdot\vec{B}=0$ is satisfied by given $\vec{B}$

$$\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0$$

$$i\vec{k}\,\hat{z}\times(E_0\hat{x})-\frac{i\omega}{c}B_0\hat{y}=0$$

$$\therefore \quad \boxed{B_0=\frac{ck}{\omega}E_0} \qquad (1)$$

$$\vec{\nabla}\times\vec{H}-\frac{1}{c}\frac{\partial\vec{D}}{\partial t}=\frac{4\pi}{c}\vec{J}=\frac{4\pi}{c}\sigma\vec{E}$$

$$\downarrow$$

$$\frac{\vec{\nabla}\times\vec{B}}{\mu}+\frac{i\omega\epsilon}{c}E_0\hat{x}=\frac{4\pi}{c}\sigma E_0\hat{x}$$

$$\frac{ik\,\hat{z}\times\hat{y}B_0}{\mu}+\frac{i\omega\epsilon}{c}E_0\hat{x}=\frac{4\pi}{c}\sigma E_0\hat{x}$$

$$\Rightarrow \quad -\frac{ikB_0}{\mu}+\frac{i\omega\epsilon}{c}E_0=\frac{4\pi}{c}\sigma E_0$$

$$\therefore \quad \boxed{B_0\frac{k}{\mu}=\frac{1}{c}\left(\omega\epsilon+i4\pi\sigma\right)E_0} \qquad (2)$$

Combine (1) + (2) to get

$$\left(\frac{ck}{\omega\sqrt{\mu\epsilon}}\right)^2 = 1 + \frac{i4\pi\sigma}{\epsilon\omega}$$

$$\|$$

$$\left(\frac{vk}{\omega}\right)^2 = 1 + \frac{i4\pi\sigma}{\epsilon\omega} \qquad v \equiv c/\sqrt{\mu_r\epsilon_r}$$

$$= \frac{1}{\sqrt{\mu\epsilon}}$$

μ_r, ϵ_r = Gaussian $= \left(\frac{\mu}{\mu_0}, \frac{\epsilon}{\epsilon_0}\right)$ in SI $\qquad \uparrow$ SI

$$\Rightarrow \boxed{k = \frac{\omega}{v}\sqrt{1 + \frac{i4\pi\sigma}{\epsilon\omega}} = K_{Re} + iK_{Im}}$$

$$E_0\hat{x}e^{i(kz-\omega t)} = \underbrace{E_0 e^{-k_{Im}z}}\hat{x}\,e^{i(k_{Re}z-\omega t)}$$

exponentially reduce the amplitude inside the conductor!

(b)

$$B_0 = \frac{ck}{\omega}E_0 = \frac{c|k|}{\omega}E_0 e^{i\phi_R}$$

$$\therefore \quad \vec{B} = \frac{c|k|}{\omega}E_0\hat{y}e^{i(kz-\omega t+\phi_R)}$$

$$e^{-k_{Im}z}e^{i(k_{Re}z-\omega t+\phi_R)}$$

ϕ_R shifts the $\vec{B}$ wave in space $\Delta z = -\phi_R / k_{Re}$ behind the E wave, or in time it delays it $\Delta t = \frac{\phi_R}{\omega}$!

(c) $k = \frac{\omega}{v}\sqrt{1 + \frac{i 4\pi\sigma}{\epsilon\omega}} \underset{\text{Good conductor}}{\simeq} \frac{\omega}{v}\sqrt{\frac{4\pi\sigma}{\epsilon\omega}}\,\sqrt{i} = |k| e^{i\pi/4}$

$\sqrt{i} \rightarrow \frac{1+i}{\sqrt{2}}$

$\therefore \quad k_{Re} = k_{Im} = \sqrt{\frac{4\pi\sigma\omega}{2\epsilon}}\; v^{-1}$

$$v = c/\sqrt{\epsilon\mu}$$

$k_{Im} = \frac{1}{\delta} \Rightarrow \boxed{v\sqrt{\frac{2\epsilon}{4\pi\sigma\omega}} = \delta \text{ skin depth}}$

$$\vec{E} = E_0 \hat{x}\, e^{-z/\delta}\, e^{i(k_{Re} z - \omega t)}$$

$B_0 = \frac{c\,k}{\omega} E_0 = \frac{c}{v}\sqrt{\frac{4\pi\sigma}{\epsilon\omega}}\; e^{i\pi/4} E_0$ $\quad \Delta t \quad \Delta z$

$\boxed{B \text{ is delayed } 45° = \frac{\text{Period}}{8} = \frac{\lambda}{8}}$