E & M Qualifier

August 20, 2015

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

1. (a) [4 pts] Use δ-functions to give volume charge densities ρ_f for each of the following:

   i. Give $\rho_f(\rho, \phi, z)$ in cylindrical-polar coordinates for a cylindrical shell of charge of a radius $\rho = b$, centered on the z-axis, which has a surface charge density $\sigma_f(\phi, z)$.

   ii. Give $\rho_f(\rho, \phi, z)$ in cylindrical-polar coordinates for a line of charge located at $\rho = b, \phi = \beta$ which has a charge/length $= \lambda_f(z)$.

   iii. Give $\rho_f(r, \theta, \phi)$ in spherical-polar coordinates for a spherical shell of charge of radius $r = a$, centered on the origin, which has a surface density $\sigma_f(\theta, \phi)$.

   (b) [2 pts] Use Gauss's law to compute the electric field caused by the cylindrically symmetric charge density

$$\rho_f(\rho) = \frac{\lambda_0}{\pi b^2} e^{-\rho^2/b^2}.$$

   (c) [2 pts] What charge density produces an electrostatic potential

$$\Phi(z) = V_0 e^{-z^2/a^2}.$$

   (d) [2 pts] What charge density produces an electrostatic potential

$$\Phi(z) = -E_0 \left| z \right|.$$

1

(a) (i.) $\rho_f(\rho,\phi,z) = \boxed{\sigma_f(\phi,z)\,\delta(\rho-b)}$

(ii.) $\rho_f(\rho,\phi,z) = \boxed{\dfrac{\lambda_f(z)}{b}\,\delta(\phi-\beta)\,\delta(\rho-b)}$

(iii.) $\rho_f(r,\theta,\phi) = \boxed{\sigma_f(\theta,\phi)\,\delta(r-a)}$

(b) $Q(<\rho) = \lambda_0 b\left[1-e^{-\rho^2/b^2}\right]$ ← from $\int \rho\, d\,\mathrm{vol}$

$(\epsilon_0)\,2\pi\rho E b = Q(<\rho)$

$$\Rightarrow \boxed{E = \frac{\lambda_0}{\epsilon_0}\,\frac{\left[1-e^{-\rho^2/b^2}\right]}{2\pi\rho} \quad \hat{\rho} \text{ direction}}$$

SI →

for Gaussian $\frac{1}{\epsilon_0} \to 4\pi$!

(c) $\nabla^2\left(V_0 e^{-z^2/a^2}\right) = -\frac{1}{\epsilon_0}\rho$ ← SI (or $-4\pi\rho$ in Gauss)

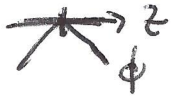
$\parallel$

$V_0\left[4z^2/a^4 - \frac{2}{a^2}\right]e^{-z^2/a^2}$

$\epsilon_0 \to \frac{1}{4\pi}$ for Gaussian

$$\boxed{\therefore\ \rho = \frac{2\epsilon_0 V_0}{a^2}\left[1-2z^2/a^2\right]e^{-z^2/a^2} \quad \text{SI}}$$

#1



(d) $\nabla^2(-E_0|z|) = -\dfrac{\rho}{\epsilon_0}$ or $-4\pi\rho$ in Gaussian

$= 0$ for $z \neq 0$ $\Rightarrow$ $\boxed{\rho = 0 \text{ for } z \neq 0}$

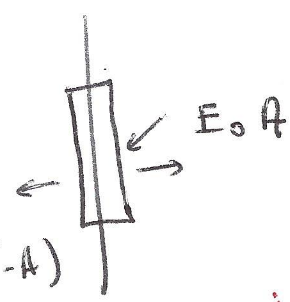
∴ only a surface charge

$\vec{E} = -\vec{\nabla}\phi$

$= E_0\hat{z}\ ,\ z > 0$

$= -E_0\hat{z}\ ,\ z < 0$

using Gaussian law



$(-E_0)(-A)$

$2E_0 A = \dfrac{\sigma A}{\epsilon_0}$

$\boxed{\rho = \sigma\,\delta(z)}$

∴ $\sigma = 2E_0\epsilon_0$ for SI

$= \dfrac{2E_0}{4\pi}$ for Gauss

2. The magnetic field of a plane wave in vacuum is

$$\mathbf{B} = B_0 \sin(kx - \omega t)\,\hat{\mathbf{y}},$$

where $\hat{\mathbf{y}}$ is a unit vector pointing in the positive y-direction.

(a) [1 pts] Give the wavelength λ of this wave as a function of k, ω and/or c (the speed of light).

(b) [2 pts] Write an expression for the wave part of the electric field $\mathbf{E}$ associated with the above magnetic field.

(c) [1 pts] What is the direction and magnitude of the Poynting vector $\mathbf{S}$ associated with this wave?

(d) [1 pts] Assume this plane wave is totally reflected by a thin conducting sheet lying in the y-z plane at x=0. What is the resulting time averaged radiation pressure on the sheet? Recall that the momentum density $\mathbf{g}$ and the Poynting vector of the incoming and reflected waves are related by $\mathbf{g} = \mathbf{S}/c^2$

(e) [2 pts] The component of an electric field parallel to the surface of an ideal conductor must be zero on the surface. Using this fact, find expressions for the reflected electric and magnetic fields. Recall that the electric and magnetic fields vanish within an ideal conductor.

(f) [3 pts] An oscillating surface current $\mathbf{K}$ flows in the thin conducting sheet as a result of this reflection. Along which axis does $\mathbf{K}$ point and what is its amplitude? Hint: To find $\mathbf{K}$ use an Amperian loop with one side just inside the conducting sheet and one side just outside the sheet.

#2

(a) $c = \omega/k$, $k = 2\pi/\lambda$ $\Rightarrow$ $\boxed{\lambda = 2\pi/k = 2\pi c/\omega}$

(b) $\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$ or $\vec{\nabla} \times \vec{H} - \frac{1}{c}\frac{\partial}{\partial t}\vec{D} = \frac{4\pi}{c}\vec{J}$

SI ... Gaussian

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_0} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} = 0$$

$$k\frac{B_0}{\mu_0}\cos(kx - \omega t)\hat{z} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} = 0 \Rightarrow \vec{E} = -\frac{kB_0}{\omega \epsilon_0 \mu_0}\sin(kx-\omega t)\hat{z}$$

$\therefore$ $\boxed{\vec{E} = -cB_0 \sin(kx - \omega t)\hat{z} \quad \text{SI}}$

$= -B_0 \sin(\ )\hat{z}$ in Gaussian

(c) $\boxed{\underset{\text{SI}}{\vec{S}} = \vec{E} \times \vec{H} = \frac{cB_0^2}{\mu_0}\sin^2(kx - \omega t)\hat{x}}$

$\underset{\text{Gaussian}}{=} \frac{c}{4\pi}\vec{E} \times \vec{H} = \frac{c}{4\pi}B_0^2 \sin^2(kx-\omega t)\hat{x}$

(d) $\rightleftarrows|$ Rad Pressure $= 2\langle c\vec{g}_{in}\rangle = \frac{2}{c}\langle \vec{S}_{in}\rangle$

$$\boxed{\begin{aligned} &= \frac{B_0^2}{\mu_0}\hat{x} \quad \text{SI} \\ &= \frac{B_0^2}{4\pi}\hat{x} \quad \text{Gaussian} \end{aligned}}$$

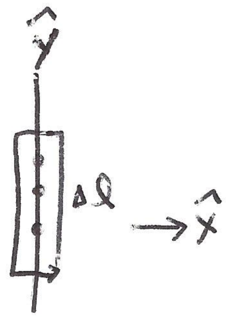
#2 $\qquad B_0 \leftarrow$ Gaussian

(e)

$$\vec{E}_{ref} = cB_0 \sin(-kx-\omega t)\,\hat{z} \qquad (SI)$$

$$\vec{B}_{ref} = B_0 \sin(-kx-\omega t)\,\hat{y} \qquad \hookrightarrow\ E\times B \propto -\hat{x}$$

(f)



$$\int\left(\vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{a} = \int \vec{J}\cdot d\vec{a}$$

Gaussian $\downarrow$

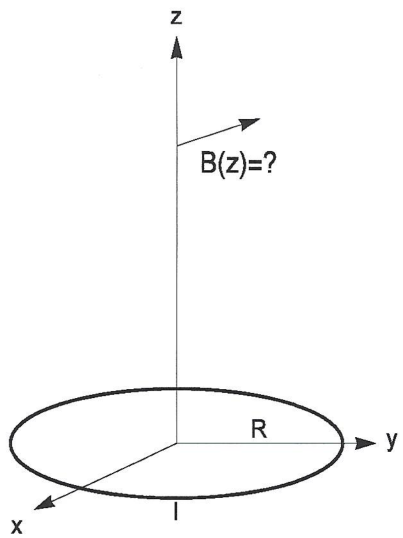
$$\underbrace{\oint \vec{H}\cdot d\vec{l}}_{= 2\frac{B_0}{\mu_0}\sin(-\omega t)\Delta l} - 0 = I = \underbrace{K}_{\text{surface current}}\Delta l \qquad \left(\frac{4\pi}{c} I\right)$$

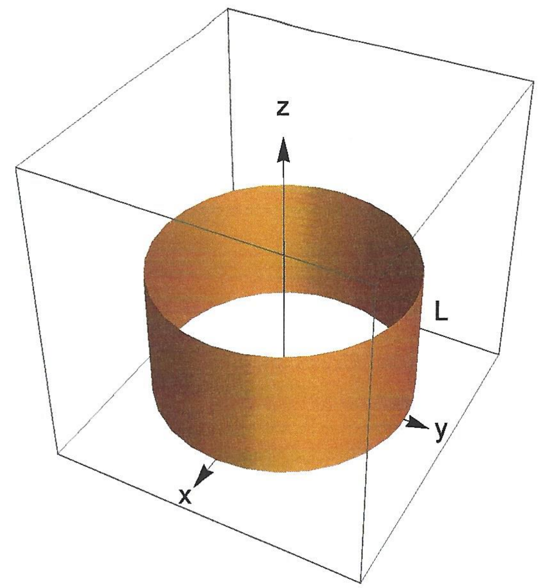
$$\vec{J} = \vec{K}\,\delta(x)$$

$$\therefore\quad \boxed{\vec{K} = \frac{2B_0}{\mu_0}\sin(-\omega t)\,\hat{z}} \qquad SI$$

$$= \frac{c}{4\pi}(2B_0)\sin(-\omega t)\,\hat{z} \qquad \text{Gaussian}$$

3.





(a) [5 pts] A circular loop of wire of radius R carries a current I as shown in the first figure. Find the magnitude and direction of the magnetic induction $\mathbf{B}(z)$ **on the axis** of the loop as a function of z.

(b) [5 pts] Use the result of part (a) to find $\mathbf{B}(z)$ along the axis of a solenoid of radius R and length L wound with n turns per unit length (total turns $N = n \times L$).

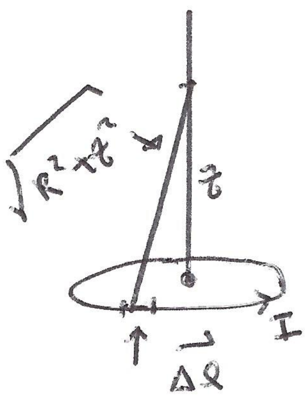
Hint:
$$\int \frac{dx}{[a^2 + x^2]^{3/2}} = \frac{x}{a^2[a^2 + x^2]^{1/2}} + \text{constant}.$$

#3

Gaussian: $\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}$

SI: $\vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$

(a)



$\vec{D}=0$ $\qquad \vec{H} = \frac{\vec{B}}{\mu_0} \leftarrow$ SI

$\vec{m}=0$

$\therefore$ Gaussian: $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$

SI: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

For static bound $\vec{J}$ (Biot-Savart law)

$$\vec{B}(\vec{r}) = (\cdot)\int \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d^3r'$$

G: $\frac{1}{c}$ $\qquad$ SI: $\frac{\mu_0}{4\pi}$

$\therefore$ For a current confined to a wire

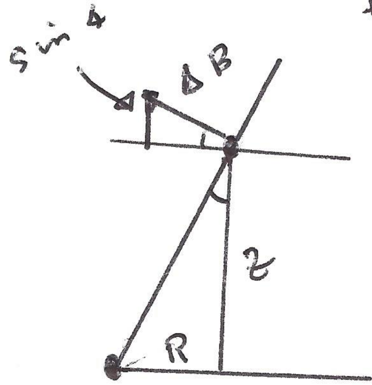
$$\vec{J}\, d^3r' = I\, d\vec{l}'$$

$\therefore$

$$\boxed{\vec{B}(\vec{r}) = (\cdot)\, I \int \frac{d\vec{l}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \qquad \text{G: } \frac{1}{c} \text{ or } \frac{\mu_0}{4\pi} \text{ for SI}}$$

use this to get $\vec{B}(\vec{r})$!!

#3



$$\Delta B = \left(\overset{\frac{1}{c} \text{ or } \frac{\mu_0}{4\pi}}{\downarrow \atop \cdot}\right) I \frac{\Delta l}{[R^2+z^2]}$$

$$\sin\phi = \frac{R}{\sqrt{R^2+z^2}}$$

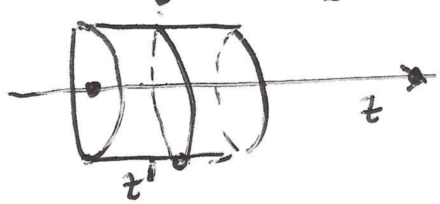
(a)

$$\therefore \boxed{B = (\cdot)\frac{I(2\pi R)R}{[R^2+z^2]^{3/2}}}$$ in the $\hat{z}$ direction

G: $\frac{1}{c}$ or SI: $\frac{\mu_0}{4\pi}$

(b)

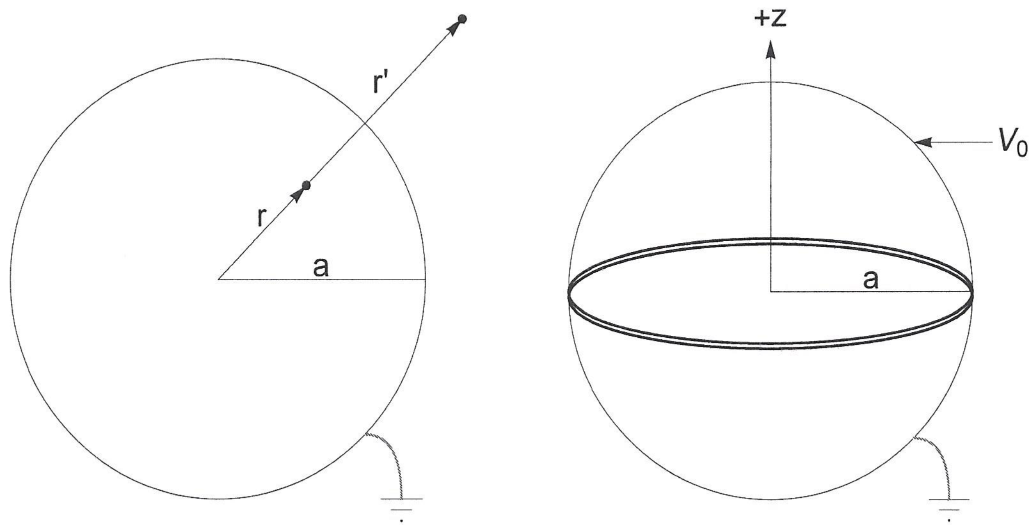
$I' = \frac{\Delta z'}{L}(N I) = n I \Delta z'$



$$B = (\cdot)\, 2\pi R^2 n I \int_0^L \frac{\Delta z'}{[R^2+(z-z')^2]^{3/2}}$$

$$\boxed{B = (\cdot)\, 2\pi \; n I \left[\frac{z-L}{[R^2+(z-L)^2]^{3/2}} - \frac{z}{[R^2+z^2]^{3/2}}\right]}$$

G: $\frac{1}{c}$ or SI: $\frac{\mu_0}{4\pi}$



4. (a) [3 pts] What is the Dirichlet Green's function $G^D(\mathbf{r},\mathbf{r}')$ for the Laplace operator for the 3-dimensional volume interior to a sphere of radius $r = a$?

   { Hint: The Dirichlet Green's function vanishes on $r = a$ and the method of images gives the Green's function as the sum of potentials of two point charges, a positive unit charge located inside the sphere and a larger negative point charge located outside. }

   (b) [2 pts] If a grounded conducting sphere of radius $r = a$ contains a positive charge q at $z = a/2$ and a negative charge $-q$ at $z = -a/2$, what will the electrostatic potential be inside the sphere?

   (c) [5 pts] If a conducting sphere of radius $r = a$, centered at the origin, is cut in two halves along the $z = 0$ plane and the top half ($z > 0$) is held at potential V_0 and the bottom half ($z < 0$) is grounded, what is the potential **on the $+z$-axis, inside** the sphere $r \le a$.

   { Hint: Use spherical polar coordinates and recall that the Dirichlet Green's function can be used to calculate the electrostatic potential via:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{r}')G^D(\mathbf{r},\mathbf{r}')d^3r' - \frac{1}{4\pi}\int_{\partial V} \Phi(\mathbf{r}')\,\frac{\partial G^D(\mathbf{r},\mathbf{r}')}{\partial n'}da'.$$

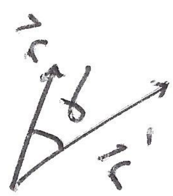
   In this case $\frac{\partial}{\partial n'} = \frac{\partial}{\partial r'}$ and $da' = a^2 \sin\theta' d\theta' d\phi'$.}

#4

(a) $$G^D(\vec r,\vec r') = \frac{1}{|\vec r-\vec r'|} - \frac{a/r'}{|\vec r - \vec r'\, a^2/r'^2|} \qquad r, r' < a$$

(b) $$\Phi = q\left[\frac{1}{|\vec r - \frac{a}{2}\hat z|} - \frac{2}{|\vec r - 2a\hat z|}\right] - q\left[\frac{1}{|\vec r + \frac{a}{2}\hat z|} - \frac{2}{|\vec r + 2a\hat z|}\right]$$

(c) $$G^D = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{a/r'}{\sqrt{r^2+\frac{a^4}{r'^2}-2\frac{a^2 r}{r'}\cos\gamma}}$$



$$G^D = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2r'^2}{a^2}+a^2-2rr'\cos\gamma}}$$

#4

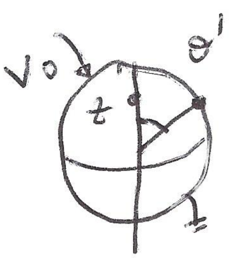
$$\frac{\partial G^D}{\partial n'}\bigg|_{\partial V} = \frac{\partial}{\partial r'} G^D \bigg|_{r'=a}$$

$$= \frac{-a + r\cos\gamma}{[r^2+a^2-2ra\cos\gamma]^{3/2}} + \frac{\frac{r^2}{a} - r\cos\gamma}{[r^2+a^2-2ra\cos\gamma]^{3/2}}$$

$$= \frac{(r^2-a^2)}{a\,[r^2+a^2-2ra\cos\gamma]^{3/2}}$$

If $\vec{r}$ is on the $+z$ axis inside the sphere then

$r = z$ and $\cos\gamma = \cos\theta'$



$$\Phi(z) = -\frac{V_0}{4\pi}\int_0^{\pi/2}\int_0^{2\pi} \frac{(z^2-a^2)\; a^2 \sin\theta' d\theta' d\phi'}{a[z^2+a^2-2za\cos\theta']^{3/2}}$$

$$= -\frac{V_0}{4\pi}\, 2\pi\, (z^2-a^2)\, a \int_0^{\pi/2} \frac{\sin\theta' d\theta'}{[z^2+a^2-2za\cos\theta']^{3/2}}$$

$$= \frac{V_0}{2}\,\frac{(z^2-a^2)\cancel{a}}{z\cancel{a}}\;\frac{1}{\sqrt{z^2+a^2-2za\cos\theta'}}\bigg|_0^{\pi/2}$$

$$\boxed{\Phi(z) = \frac{V_0}{2z}(z^2-a^2)\left[\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{(a-z)}\right]} \qquad |z| \le a$$

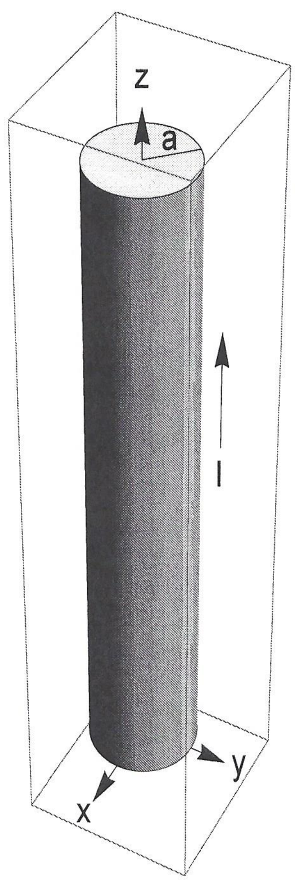
5. A very-very long (∞-length) wire of radius a is centered along the z-axis. The wire, at rest in the lab, is uncharged and carries a **uniform** current I in the +z direction.

   (a) [2 pts] Compute the magnetic induction $\mathbf{B}$ as a function of (x, y) for all $\sqrt{x^2 + y^2} \geq a$ (**outside the wire**).

   (b) [2 pts] Give the electro-magnetic field tensor $F^{\alpha\beta}(x, y)$ in the lab frame for $\sqrt{x^2 + y^2} \geq a$(**outside the wire**).

   (c) [3 pts]
   An observer is moving with constant velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$ in the lab's +z direction. Compute $F'^{\alpha\beta}(x', y')$, **outside the wire** in the frame of the moving observer. Assume the moving frame coordinates are simply a boost of the lab coordinate in the z direction. Don't forget to write your answer as functions of the moving frame's coordinates.

   (d) [3 pts]
   Give the electric and magnetic fields $\mathbf{E}'$ and $\mathbf{B}'$, **outside the wire** in the moving frame.



#5

(a) $\vec{\nabla} \times \vec{H} = \vec{J}$ in SI $\left(= \frac{4\pi}{c} \vec{J} \text{ in Gaussian} \right)$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \left(\text{or } \frac{4\pi}{c} \vec{J} \right)$$

$$\oint \vec{B} \cdot d\hat{l} = \mu_0 I_{in}$$

$$B \, 2\pi\rho = \mu_0 I \qquad (\rho \geq a)$$

$\hat{\phi}$

$$\vec{B} = \frac{\mu_0 I}{2\pi\rho} \left[-\sin\phi \, \hat{x} + \cos\phi \, \hat{y} \right]$$

$\mu_0 \to \frac{4\pi}{c}$ for Gaussian

$$\therefore \boxed{\vec{B} = \frac{\mu_0 I}{2\pi [x^2 + y^2]} \left[-y\hat{x} + x\hat{y} \right] \quad \text{for } \sqrt{x^2+y^2} \geq a}$$

$4\pi/c$

(b)

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{pmatrix}$$

common convention, some change $\vec{E} \to -\vec{E}$.

Gaussian

$$\boxed{F^{\alpha\beta} = \frac{2 I}{c [x^2+y^2]} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & -x & -y & 0 \end{pmatrix}}$$

$\mu_0 \to \frac{4\pi}{c}$

in Gaussian

(c.) #5

$\vec{v} = v\hat{z}$

$$x'^{\alpha} = L^{\alpha}{}_{\beta} x^{\beta}$$

x' ↑ moving in lab; x ↑ lab

+β direction

$$L^{\alpha}{}_{\beta} = \left(\begin{array}{c|cc|c} \gamma & 0 & 0 & -\beta\gamma \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline -\beta\gamma & 0 & 0 & \gamma \end{array}\right) \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}} \qquad \beta = v/c$$

inverse trans $\quad X = L^{-1} X' \quad (\beta \to -\beta)$

$$F' = L F L^{T} \quad \Rightarrow \quad F'^{\alpha\beta} = L^{\alpha}{}_{\sigma} L^{\beta}{}_{\lambda} F^{\sigma\lambda}$$

$$L \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & -x & -y & 0 \end{array}\right) = \begin{pmatrix} 0 & \beta\gamma x & \beta\gamma y & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & -\gamma x & -\gamma y & 0 \end{pmatrix}$$

$$L \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & -x & -y & 0 \end{array}\right) L^{T} = \begin{pmatrix} 0 & \beta\gamma x & \beta\gamma y & 0 \\ -\beta\gamma x & 0 & 0 & \gamma x \\ -\beta\gamma y & 0 & 0 & \gamma y \\ 0 & -\gamma x & -\gamma y & 0 \end{pmatrix}$$

#5

$$\therefore F' = \frac{2I\gamma}{c[x^2+y^2]} \begin{pmatrix} 0 & \beta x & \beta y & 0 \\ -\beta x & 0 & 0 & x \\ -\beta y & 0 & 0 & y \\ 0 & -x & -y & 0 \end{pmatrix}$$

From $x = L^{-1} x'$

$$ct = \gamma ct' + \beta\gamma z'$$
$$x = x'$$
$$y = y'$$
$$z = \gamma z' + \beta\gamma ct'$$

$$\therefore \boxed{F' = \frac{2I\gamma}{c[x'^2+y'^2]} \begin{pmatrix} 0 & \beta x' & \beta y' & 0 \\ -\beta x' & 0 & 0 & x' \\ -\beta y' & 0 & 0 & y' \\ 0 & -x' & -y' & 0 \end{pmatrix}}$$

(d)

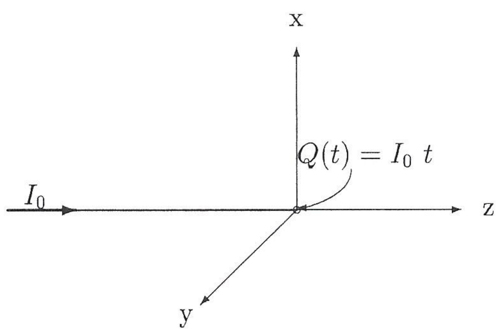
$$\therefore \vec{E}' = -\frac{2I\beta\gamma}{c}\frac{\hat{\rho}'}{\rho'} = -\frac{2I\beta\gamma}{c}\left[\frac{x'\hat{x}' + y'\hat{y}'}{(x'^2+y'^2)}\right]$$

$$\vec{B}' = \frac{2I\gamma}{c}\frac{\hat{\phi}'}{\rho'} = \frac{2I\gamma}{c}\left[\frac{-y'\hat{x} + x'\hat{y}}{x'^2+y'^2}\right]$$

6. In the absence of polarizable and/or magnetizable material (i.e., only free charges and currents present) Maxwell's equations, in the Lorenz gauge, reduce to the inhomogeneous wave equation:

$$\Box \left\{ \begin{array}{c} \Phi \\ A^x \\ A^y \\ A^z \end{array} \right\} = \frac{4\pi}{c} \left\{ \begin{array}{c} c\rho \\ J^x \\ J^y \\ J^z \end{array} \right\}, \text{ where } \Box \equiv \left(\frac{\partial}{c\partial t}\right)^2 - \nabla^2.$$

A time dependent charge $Q(t) = I_0\ t,\ t \geq 0$ is fixed at the origin



of a cylindrical polar coordinate system (ρ, ϕ, z) The charge increases linearly with time because a constant current I_0 flows in along a thin wire attached to the charge on its left, see the figure. Assume the wire carries no current for $t < 0$, however, at $t = 0$ a current I_0 abruptly starts flowing in the +z direction and remains constant for $t \geq 0$. Assume the wire remains neutral except for the charge that grows at the origin. Find the following quantities at time t for points (ρ, ϕ, z):

(a) [4 pts] The retarded scalar potential $\Phi(t, \rho, \phi, z)$, for all t and at all points in space.

(b) [6 pts] The retarded vector potential $\mathbf{A}(t, \rho, \phi, z)$ for all t at all points $z > 0$.

Recall that the retarded solution to $\Box F(t, \mathbf{r}) = S(t, \mathbf{r})$ is

$$F(t, \mathbf{r}) = \frac{1}{4\pi} \int \frac{S(t - |\mathbf{r} - \mathbf{r}'|/c, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'.$$

You might need the indefinite integral

$$\int \frac{dX}{\sqrt{X^2 + a^2}} = \ln(\sqrt{X^2 + a^2} + X).$$

#6

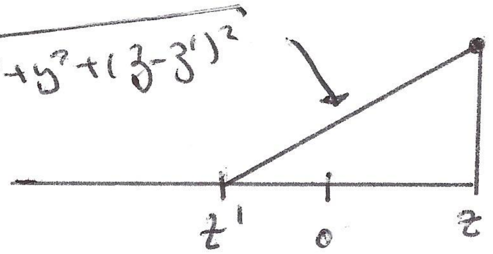
(a) $\Phi(t,\vec r) = \frac{1}{\cancel{4\pi}} \int \frac{\cancel{4\pi}\,\rho(t-|\vec r-\vec r\,'|/c, \vec r\,')\,d^3r'}{|\vec r-\vec r\,'|}$

$$= \frac{q(t-r/c)}{r} \qquad t \geq r/c$$

$$= \boxed{\frac{I_0[t-r/c]}{r}} \qquad t \geq r/c$$

(b) $\vec A(t,\vec r) = \frac{1}{4\pi}\int_{z'_{min}}^{0} \frac{4\pi}{c}\,\frac{I_0\,\hat z\,dz'}{\sqrt{x^2+y^2+(z-z')^2}}$

$ct \geq \sqrt{x^2+y^2+(z-z')^2}$



$\leftarrow \sqrt{x^2+y^2}$

$z > 0$!

$\boxed{\therefore\ ct \geq r}$

if $A \neq 0$!

z_{min} is defined by $ct = \sqrt{x^2+y^2+(z-z'_{min})^2}$

$$\Rightarrow \quad z'_{min} = z - \sqrt{(ct)^2-x^2-y^2} < 0$$

$$\vec A(t,\vec r) = \frac{I_0}{c}\,\hat z \int_{z'_{min}}^{0} \frac{dz'}{\sqrt{x^2+y^2+(z-z')^2}}$$

#6

$$\bar{X} \equiv z - z' \Rightarrow d\bar{X} = -dz'$$

$$z'_{min} = z - \sqrt{(ct)^2 - x^2 - y^2}$$

$$\Rightarrow \bar{X}_{min} = \sqrt{(ct)^2 - x^2 - y^2}$$

$$\vec{A}(t,\vec{r}) = \frac{I_0}{c}\hat{z} \int_{\sqrt{(ct)^2 - x^2 - y^2}}^{z} \frac{-d\bar{X}}{\sqrt{(x^2)+(y^2)+\bar{X}^2}}$$

$$= -\frac{I_0}{c}\hat{z} \ln\left[\frac{(r+z)}{(ct+\sqrt{(ct)^2 - x^2 - y^2}}\right]$$

$$\vec{A}(t,r) = \frac{I_0}{c}\hat{z} \ln\left[\frac{ct+\sqrt{(ct)^2 - x^2 - y^2}}{r+z}\right], \quad \boxed{ct > r}$$

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