Green = SI , Red = Gaussian

1.

(a) $\phi = \dfrac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}$

(b) $\Phi = \dfrac{pz}{4\pi\epsilon_0 r^3} - \dfrac{pz}{4\pi\epsilon_0 a^3} = \boxed{\dfrac{p}{4\pi\epsilon_0}\left(\dfrac{1}{r^2} - \dfrac{r}{a^3}\right)\cos\theta}$

(c) $E_n = \dfrac{4\pi\sigma}{\epsilon_0}$ (normal component of E at surface)

$\Rightarrow \sigma = -\dfrac{\epsilon_0}{4\pi}\dfrac{\partial}{\partial n}\Phi\Big|_{r=a}$, $\dfrac{\partial}{\partial n} = -\dfrac{\partial}{\partial r}$

$$\boxed{\sigma = -\frac{3p\cos\theta}{4\pi a^3}}$$

Red = Gaussian

#2

(a)

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

$$\vec{\nabla}\cdot\left(\downarrow\right) \Rightarrow 0 - \frac{1}{c}\frac{\partial}{\partial t}\underbrace{\vec{\nabla}\cdot\vec{D}}_{4\pi\rho} = \frac{4\pi}{c}\vec{\nabla}\cdot\vec{J}$$

$$\therefore \quad \boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$

(b)

V_3 = fixed vol containing all ρ + $\vec{J}$!

no J on boundary

$$Q \equiv \int_{V_3} \rho \, d\mathrm{vol} \Rightarrow \boxed{\dot{Q}} = \int_{V_3} \frac{\partial}{\partial t}\rho \, d\mathrm{vol}$$

$$= \int_{V_3} (-\vec{\nabla}\cdot\vec{J})\, d\mathrm{vol}$$

↓ flux theorem

$$= -\oint_{\partial V_3} \vec{J}\cdot d\vec{a} = \boxed{0}$$

(c) $$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$$J^0 \equiv c\rho \quad , \quad J^i = (J^1, J^2, J^3)$$

$$\frac{\partial J^0}{\partial (ct)} + \frac{\partial J^i}{\partial x^i} = 0$$

$ct \to x^0$

$\uparrow$ Σ i implied

$$\therefore \boxed{\frac{\partial}{\partial x^\mu} J^\mu = 0} \qquad \mu = 0,1,2,3$$

$\uparrow$ $\sum_\mu$ implied

(d) $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{B} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0$ + part(a) eqns

$$\vec{E}\cdot\left(\nabla\times\vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \underset{\text{no } J}{\underset{\uparrow}{0}}\right) \Rightarrow \vec{E}\cdot\vec{\nabla}\times\vec{H} - \frac{\vec{E}}{c}\cdot\frac{\partial \vec{D}}{\partial t} = 0$$

$$\vec{E}\cdot\vec{\nabla}\times\vec{H} = -\vec{\nabla}\cdot(\vec{E}\times\vec{H}) - \vec{H}\cdot\underbrace{\vec{\nabla}\times\vec{E}}_{-\frac{1}{c}\frac{\partial \vec{B}}{\partial t}}$$

$$\therefore \quad -\vec{\nabla}\cdot(\vec{E}\times\vec{H}) - \frac{\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial}{\partial t}\vec{B}\right)}{c} = 0$$

$$\vec{S} \equiv c\frac{\vec{E}\times\vec{H}}{4\pi}, \quad u \equiv \frac{\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}}{4\pi}$$

For static isotropic and linear materials

$$\vec{D} = \epsilon\vec{E} \qquad \vec{H} = \frac{\vec{B}}{\mu}$$

no $\underline{\underline{t}}$ dependence

$$\vec{\nabla}\cdot\vec{S} + \frac{\partial}{\partial t}\left(\frac{\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}}{2}\right) = 0$$

$$\boxed{\vec{\nabla}\cdot\vec{S} + \frac{\partial}{\partial t}u = 0}$$

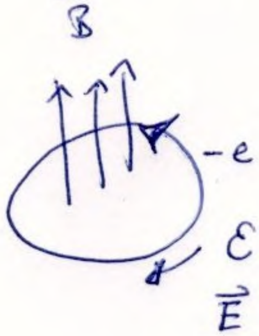
#3

$$\vec{m} = \frac{1}{2c}\int \vec{r} \times \vec{J}\, dvol = \frac{I}{2c}\oint \vec{r} \times d\vec{l} = \frac{I}{c}\vec{A}$$

(-e)

(a) $$\vec{m} = \frac{I}{c}\vec{A} = \frac{\frac{q\omega}{2\pi}\pi a^2 \hat{z}}{c} = \boxed{\frac{q\omega a^2}{2c}\hat{z}}$$

(b)



$$\mathcal{E} = \frac{\pi a^2}{c}\frac{\Delta B}{\Delta t} = E(2\pi a)$$

$$\therefore E = \frac{a\,\Delta B}{2c\,\Delta t}$$

$(-e)\vec{E}$ in same direction as $\vec{v}$

$$m\frac{\Delta v}{\Delta t} = eE = \frac{ea}{2c}\frac{\Delta B}{\Delta t}$$

$$\Rightarrow \Delta v = \frac{e}{m}\frac{a}{2c}\Delta B$$

$$\| $$

$$a\Delta\omega \qquad\qquad \Delta B = B - 0$$

$$\Rightarrow \boxed{\Delta\omega = \frac{e}{mc}\frac{1}{2}B}$$

(c) increase in the $-\hat{z}$ direction!

#4

(a) $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$

$\downarrow$

$i k E_0 \underbrace{(\hat{n}\times\hat{j})}_{(-\hat{i})} e^{i(kz-\omega t)} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$

$\therefore \vec{B} = -\frac{ck}{\omega} E_0 \hat{i} e^{i(kz-\omega t)} = \boxed{-B_0 \hat{i} e^{i(kz-\omega t)}}$

where $\omega = kc$

assume E_0 is real! $\boxed{B_0 = \frac{E_0}{c}}$

(b) $\Phi_B \cong N\vec{B}\cdot\vec{A} = NBA\cos\measuredangle$ $\quad \hat{i}\cdot\hat{n}_{\text{normal}}$

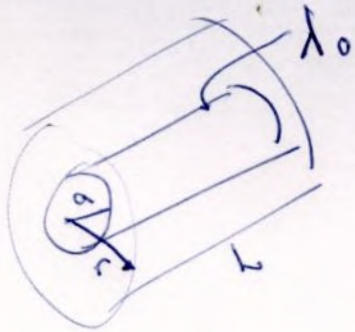
$= \underbrace{\frac{E_0}{c} N\pi\left(\frac{d}{2}\right)^2 \cos 60°}_{\cos 60° \to \sqrt{3}/2;\ \equiv \Phi_{0B}} \cos(kz-\omega t)$ $\quad kz \approx 0$

(c) $\mathcal{E} = -\frac{1}{c}\dot{\Phi}_B = \underbrace{N\frac{E_0\pi d^2}{c}\frac{\sqrt{3}\omega}{8}}\sin(\omega t)$

Peak Emf $\longrightarrow$ $\boxed{\mathcal{E}_0 = N E_0(\pi d^2)\frac{\sqrt{3}}{8}\left(\frac{\omega}{c}\right)}$

#5

(a)



Red = Gaussian

Green = SI

$$(2\pi r \cancel{L})D = 4\pi\lambda_0 \cancel{L}$$

$$\therefore\ \frac{D}{\epsilon_0} = \boxed{E} = \frac{4\pi\lambda_0}{2\pi\epsilon_0 r} = \boxed{\frac{2\lambda_0}{4\pi\epsilon_0 r}}$$

$\uparrow$

in $\hat{\rho}$ direction

4D eqn of motion

(b) $$\boxed{\frac{dp^\mu}{dy} = \frac{q}{c}F^{\mu\lambda}u_\lambda}$$

$u_\lambda = \gamma(c, -\vec{v})$

$u^\lambda = \gamma(c, \vec{v})$

$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

$\vec{\beta} \equiv \vec{v}/c$

$$p^\mu \equiv \Big(\frac{\mathcal{E}}{c}, \underbrace{m\gamma\vec{v}}_{\vec{p}}\Big)$$

$\mathcal{E} \equiv m\gamma c^2$

$$F^{\mu\lambda} = -F^{\lambda\mu} = \left(\begin{array}{c|ccc} 0 & -E^x & -E^y & -E^z \\ \hline E^x & 0 & -cB^z & cB^y \\ E^y & cB^z & 0 & -cB^x \\ E^z & -cB^y & cB^x & 0 \end{array}\right)$$

(b) cont

(3+1) D equations of motion

$$\left[\begin{array}{l} \frac{d}{dt} m\gamma c^2 = q\,\vec{E}\cdot\vec{v} \\ \\ \underbrace{\frac{d}{dt}\vec{p}}_{m\gamma\vec{v} \text{ as above}} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \end{array}\right] \leftarrow \text{3D Lorentz force}$$

(c) $$\frac{d\mathcal{E}}{dt} = q\vec{E}\cdot\vec{v} = q\frac{2\lambda_0}{4\pi\epsilon_0}\frac{dr}{dt}$$

$$\therefore\ \mathcal{E} - \mathcal{E}_0 = q\frac{2\lambda_0}{4\pi\epsilon_0}\int_a^b \frac{dr}{r}$$

$$mc^2(\gamma_b - 1) = \frac{q\,2\lambda_0}{4\pi\epsilon_0}\ln b/a$$

$$\therefore\ \boxed{\mathcal{E}_b - mc^2 = q\frac{2\lambda_0}{4\pi\epsilon_0}\ln b/a}$$

Red = GAUSSIAN

Green = SI

#6

(a)

$$\oint \vec{D}\cdot d\vec{a} = 4\pi q$$

$$\Rightarrow \quad E_0 = \frac{4\pi\sigma}{\epsilon_0} \quad \leftarrow \text{on the plate at } z=0$$

$$= -\frac{4\pi}{\epsilon_0}\sigma \quad \leftarrow \text{on the plate at } z=d$$

(b)

$$F^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & & & \\ 0 & & 0 & \\ 1 & & & \end{pmatrix} E_0 \quad \leftarrow x \text{ frame}$$

$$L = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & 0 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix} \qquad \beta = \frac{1}{2} \qquad \gamma = 2/\sqrt{3}$$

$$x' = Lx \quad \leftarrow \text{moved in } +x \text{ direction}$$

$$F' = LFL^T = E\begin{pmatrix} 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & +\beta\gamma \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} L^T$$

#6

$$F' = E_0 \begin{pmatrix} 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & \beta\gamma \\ 0 & 0 & 0 & 0 \\ \gamma & -\beta\gamma & 0 & 0 \end{pmatrix} = E_0' \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 1 & -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

$z = z'$

$\therefore \quad E_0' = \gamma E_0 = \frac{2}{\sqrt{3}} E_0$

$$\boxed{\begin{aligned} \vec{E}_0' &= \left(0, 0, \tfrac{2}{\sqrt{3}} E_0\right) \\ \vec{B}_0' &= \left(0, \tfrac{1}{\sqrt{3}} \tfrac{E_0}{c}, 0\right) \end{aligned}} \quad \text{for } 0 \le z \le d$$

(c) $J^\sigma = (c\rho_0, \vec{J}) = [c(\sigma\,\delta(z) - \sigma\,\delta(z-d)), 0]$

$J'^\sigma = L^\sigma{}_\lambda J^\lambda = L^\sigma{}_0\, c\sigma\left(\delta(z) - \delta(z-d)\right)$

$= (c\rho', \vec{J}')$

$$\boxed{\begin{aligned} \rho' &= \overset{2/\sqrt{3}}{\gamma}\,\sigma\left(\delta(z') - \delta(z'-d)\right) \\ \vec{J}' &= \underbrace{-\beta\gamma}_{1/\sqrt{3}}\, c\sigma\left(\delta(z') - \delta(z'-d)\right)\hat{x} \end{aligned}}$$

(d) $\vec{\nabla}' \cdot \vec{D}' \overset{?}{=} 4\pi \rho'$

$\parallel$

$\epsilon_0 \vec{\nabla}' \cdot \vec{E}' \overset{?}{=} 4\pi \rho'$

$$\epsilon_0 \left(\frac{\partial}{\partial z'} E^{z'} \right) \overset{?}{=} 4\pi \frac{2\sigma}{\sqrt{3}} \left(\delta(z') - \delta(z'-d) \right)$$

vanishes outside [0,d]! Heaviside step functions

$$\epsilon_0 \frac{2}{\sqrt{3}} E_0 \left(\Theta(z') \Theta(d-z') \right) \overset{?}{=}$$

$\parallel$

$$\frac{2}{\sqrt{3}} \underset{\frac{4\pi\sigma}{\epsilon_0}}{E_0} \left(\delta(z') - \delta(d-z') \right) \overset{?}{=}$$

$\checkmark$ $=$

$$\vec{\nabla}' \times \vec{H}' - \frac{1}{c} \frac{\partial \vec{D}'}{\partial t'} = \frac{4\pi}{c} \vec{J}'$$

$\parallel$

0

$$\vec{\nabla}' \times \left[\hat{j} \frac{1}{\sqrt{3}} \frac{E_0}{\underset{\mu_0}{c}} \left(\Theta(z') \Theta(d-z') \right) \right] \overset{?}{=} \frac{4\pi}{c} \left(-\frac{1}{\sqrt{3}} c\sigma \right) \left(\delta(z') - \delta(z'-d) \right) \hat{x}$$

$$-\hat{1}\ \frac{1}{\sqrt{3}}\frac{E_0}{c\mu_0}\frac{\partial}{\partial z'}\left(\theta(z')\,\theta(d-z')\right) \overset{?}{=}$$

$$-\frac{\overset{\nearrow \frac{4\pi\sigma}{\epsilon_0}}{E_0}}{\sqrt{3}\,c\,\mu_0}\left(\delta(z')-\delta(d-z')\right) \overset{?}{=} \frac{4\pi}{c}\left(-\frac{c\sigma}{\sqrt{3}}\right)\left(\delta(z')-\delta(d-z')\right)$$

$$-\frac{E_0\,4\pi c\,\sigma}{\sqrt{3}}\left(\delta(z')-\delta(d-z')\right) \overset{\checkmark}{=} -\frac{4\pi c\sigma}{\sqrt{3}}\left(\delta(z')-\delta(d-z')\right)$$