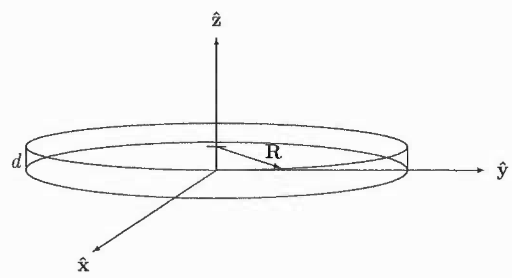
E & M Qualifier

August 15, 2014

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias **(NOT YOUR REAL NAME)** on every page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. number every page as follows
   (a) put the problem number on every page you hand in for that problem,
   (b) starting numbering each problem with page 1,
   (c) when you finish a problem put the total number of pages you used for that problem on every page you hand in for that problem.
6. **DO NOT** staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



1. A large flat thin disk of linear magnetic material of thickness d and radius $R \gg d$ which has magnetic permeability μ is placed in a uniform magnetic field $\mathbf{H} = H_0\hat{\mathbf{z}}$ as shown in the figure. The bottom of the slab is in the x-y plane at $z = 0$ and the top is at $z = d$. Assume the source of the uniform magnetic field is far away and assume the slab is infinite ($R \to \infty$) in the x-y directions. In addition to possessing a linear magnetic susceptibility χ_m related to the materials permeability, the slab also possesses a **uniform permanent magnetization** $M_0\hat{\mathbf{z}}$, producing a total magnetization density

$$\mathbf{M} = \chi_m\mathbf{H} + M_0\hat{\mathbf{z}} \quad \text{where} \quad \chi_m^{SI} = 4\pi\chi_m^G.$$

Recall that in SI (mks) and Gaussian (cgs) units

$$\mathbf{B}^{SI} = \mu_0(\mathbf{H}^{SI} + \mathbf{M}^{SI}), \qquad \mathbf{B}^G = \mathbf{H}^G + 4\pi\mathbf{M}^G.$$

(a) [1 pts] In this problem you are to write the magnetic field $\mathbf{H}$ as the gradient of a scalar potential

$$\mathbf{H} = -\boldsymbol{\nabla}\Phi_M.$$

Explain why you can do this.

(b) [3 pts] What is the form of the Poisson equation satisfied by Φ_M inside and outside the slab, i.e.,

$$\nabla^2\Phi_M = ?$$

Solve this equation for the 3 spatial regions separated by $z \neq 0$ and $z \neq d$. Observe that there is no x or y dependence in this problem. Make sure your Φ_M far above and below the slab produces the uniform magnetic field $\mathbf{H} = H_0\hat{\mathbf{z}}$.

(c) [2 pts] What general boundary conditions are satisfied by $\mathbf{H}$ and $\mathbf{B}$ at the two junctions $z = 0$ and $z = d$. What conditions are placed on Φ_M and its z-derivative by these junction conditions for this particular problem?

(d) [2 pts] Use your solutions from (b) and boundary conditions from (c) to find Φ_M inside and outside the slab.

(e) [2 pts] Calculate $\mathbf{H}$ and $\mathbf{B}$ inside and outside the slab.

1

(a) Ampère's law $\vec{\nabla}\times\vec{H} - 0 = 0$

$\Rightarrow \quad \vec{H} = -\vec{\nabla}\phi_m$

These are identically zero everywhere for this problem.

(b) $\vec{\nabla}\cdot\vec{B} = 0$

$\downarrow$

$\mu_0(\vec{H} + \chi_m\vec{H} + M_0\hat{z})$ inside

$\mu_0(\vec{H})$ outside

$\therefore \quad \mu_0\vec{\nabla}\cdot\left[-\vec{\nabla}\phi_m(1+\chi_m)\right] = 0$ inside

$\mu_0\vec{\nabla}\cdot\left[-\nabla\phi_m\right] = 0$ outside

$\therefore \quad \boxed{\nabla^2\Phi_m = 0 \text{ inside + out}}$ but not at $z = 0$ or d!

$\downarrow$

$\frac{d^2}{dz^2} \Rightarrow \Phi_m = \alpha + \beta z$ inside or out

$\therefore \quad \boxed{\Phi_m = \alpha - H_0 z}$

outside the slab!

α might be different above + below slab.

#1 - cont

2/3

(c.) The tangential component of $\vec{H}$ + the normal component of $\vec{B}$ are continuous at $z=0$ + d!

$\Rightarrow$ Φ_m is continuous at $z=0+d$

$$B^z = \mu_0 (H^z + M^z) = \mu_0 [(1+\chi_m) H^z + M_0] \quad \text{in slab}$$

$$= \mu_0 [H^z] \quad \text{outside slab}$$

$$\therefore \boxed{(1+\chi_m)\left(-\frac{d\phi_m^{in}}{dz}\right) + M_0 = -\frac{d\phi_m^{out}}{dz}}$$

$\uparrow$ at $z=0+d$

(d)

$z=d$: $\alpha_{up} - H_0 z$

$\alpha_{in} + \beta z$ $\quad \phi_m$

$z=0$: $\alpha_{down} - H_0 z$

$\phi_{cont} \Rightarrow$ $\alpha_{in} = \alpha_{down} = \alpha$

$\alpha_{in} + \beta d = \alpha_{up} - H_0 d$

$\Rightarrow \boxed{\alpha_{up} = \alpha + (\beta + H_0) d}$

#1 - cont

B^z cont $\Rightarrow$

$$(1+\chi_m)(-\beta) + M_0 = H_0 \quad \text{at } z=d$$

$$(1+\chi_m)(-\beta) + M_0 = H_0 \quad \text{at } z=0$$

$$\Rightarrow \boxed{\beta = \frac{-(H_0 - M_0)}{1+\chi_m}} \Rightarrow \boxed{\alpha_{up} = \alpha + \left(\frac{M_0 + \chi_m H_0}{1+\chi_m}\right) d}$$

$$\phi_m = \alpha + \left(\frac{M_0 + \chi_m H_0}{1+\chi_m}\right) d - H_0 z \qquad z \geq d$$

$$= \alpha - \frac{(H_0 - M_0)}{1+\chi_m} z \qquad 0 \leq z \leq d$$

$$= \alpha - H_0 z \qquad z \leq 0$$

α is arbitrary!

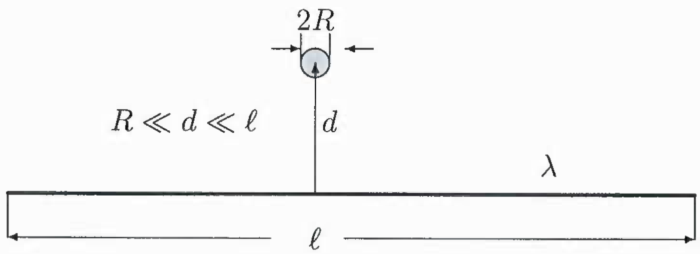
(e)

$$\vec{H} = H_0 \hat{z} \qquad z \geq d \qquad \vec{B} = \mu_0 H_0 \hat{z}$$

$$= \frac{(H_0 - M_0)}{1+\chi_m} \hat{z} \qquad 0 \leq z \leq d \qquad = \mu_0 H_0 \hat{z}$$

$$= H_0 \hat{z} \qquad z \leq 0 \qquad = \mu_0 H_0 \hat{z}$$

B^z is cont!



2. Consider a tiny sphere of radius R, composed of a linear dielectric material of susceptibility χ_e and permittivity ϵ which is a distance d from a thin but very long ($R << d << \ell$) wire possessing a uniform line charge per unit length λ. Recall that

$$\mathbf{D} = \epsilon\mathbf{E} \qquad \text{where} \qquad \epsilon^G = \epsilon^{SI}/\epsilon_0 = 1 + \chi_e^{SI} = 1 + 4\pi\chi_e^G$$

$$\mathbf{P} = \chi_e\mathbf{E} \qquad \text{where} \qquad \chi_e^{SI} = 4\pi\chi_e^G$$

$$\mathbf{D}^{SI} = \epsilon_0(\mathbf{E}^{SI} + \mathbf{P}^{SI}), \qquad \mathbf{D}^G = \mathbf{E}^G + 4\pi\mathbf{P}^G$$

The electrostatic potential for a point dipole at the origin is

$$\Phi^{SI} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3},$$

$$\Phi^G = \frac{\mathbf{p}\cdot\mathbf{r}}{r^3},$$

(a) [2 pts] Calculate the magnitude of the electric field E_{wire} **at the center of the sphere** caused by the charge on the wire.

(b) [2 pts] As an approximation, assume the **dielectric sphere** is centered at the origin in a uniform electric field of the form $E_{wire}\,\hat{\mathbf{x}}$. The polarization charge induced on the sphere's surface produces an electric dipole field $\mathbf{E}_{dipole}$ outside the sphere and makes a uniform contribution to the net uniform field $E_0\,\hat{\mathbf{x}}$ that exists inside the sphere. Give an expression for the electric dipole field $\mathbf{E}_{dipole}$ as a function of the sphere's uniform polarization density $\mathbf{P}$ if the dipole is oriented in the $\hat{\mathbf{x}}$ direction, i.e., if $\mathbf{p} = p_0\,\hat{\mathbf{x}} = 4/3\,\pi R^3\,\mathbf{P}$.

(c) [3 pts] What boundary conditions must $\mathbf{E}$ and $\mathbf{D}$ satisfy at the sphere's surface? Use these boundary conditions to calculate the net electric dipole moment $p_0\hat{\mathbf{x}}$ of the sphere?

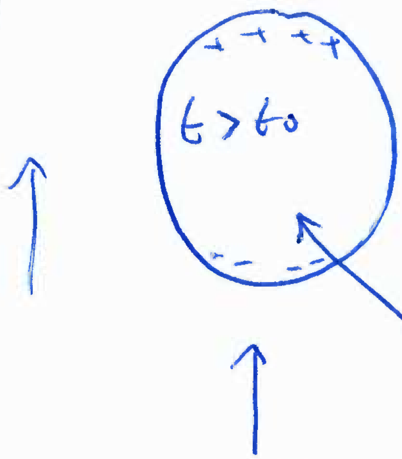
(d) [3 pts] Compute the force exerted on the sphere by the wire by computing the force on a point dipole in the non-uniform electric field caused by the wire. Is the sphere attracted or repelled by the charged wire?

#2 (a) use Gauss's law

$$2\pi \rho\, l\, E(\rho) = \frac{\lambda l}{\epsilon_0} \Rightarrow \boxed{E(\rho) = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{\rho}}$$

$$E_{wire} = E(\rho = d) = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{d}$$

(b)



$$\vec{E}_{out} = E_{wire}\,\hat{x} + \vec{E}_{dipol}$$

$$\vec{E}_{in} = E_{wire}\,\hat{x} + E_0\,\hat{x}$$

$$\vec{E}_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}}{r^3}$$

(c) $\vec{E}_{tang} = \text{cont}$ $\quad \vec{D}_{normal} = \text{cont} \Rightarrow \epsilon \vec{E}_{in}\cdot\hat{r} = \epsilon_0 \vec{E}_{out}\cdot\hat{r}$

$\downarrow$

$\vec{E}\cdot\hat{\theta}$ + $\vec{E}\cdot\hat{\phi}$ are cont at $r = R$

$$\vec{E}_{out}\cdot\hat{\theta} = E_{wire}\,\hat{x}\cdot\hat{\theta} + \frac{1}{4\pi\epsilon_0}\frac{(-p_0)(\hat{x}\cdot\hat{\theta})}{R^3}$$

$\parallel$

$$\vec{E}_{in}\cdot\hat{\theta} = E_{wire}\,\hat{x}\cdot\hat{\theta} + E_0\,\hat{x}\cdot\hat{\theta}$$

(same form for $\hat{\phi}$!)

$$\Rightarrow \boxed{E_0 = \frac{-p_0}{4\pi\epsilon_0 R^3}}$$ from $\vec{E}_{tang}$ = continuous

From $\vec{D}\cdot\hat{r}$ continuous

$$\epsilon\left(E_{wire}\,\hat{x}\cdot\hat{r} + E_0\,\hat{x}\cdot\hat{r}\right) = \epsilon_0\left(E_{wire}\,\hat{x}\cdot\hat{r} + \frac{p_0}{4\pi\epsilon_0}\frac{3\hat{r}\cdot\hat{x} - \hat{r}\cdot\hat{x}}{R^3}\right)$$

$$\therefore \boxed{\epsilon/\epsilon_0\left[E_{wire} + E_0\right] = E_{wire} + \frac{2p_0}{4\pi\epsilon_0 R^3}}$$

$$E_0 \to \frac{-p_0}{4\pi\epsilon_0 R^3}$$

$$\Rightarrow \left(\epsilon/\epsilon_0 - 1\right)E_{wire} = \frac{p_0}{4\pi\epsilon_0 R^3}\left[2 + \epsilon/\epsilon_0\right]$$

$$\Rightarrow \boxed{p_0 = \frac{\left(\epsilon/\epsilon_0 - 1\right)E_{wire}\,4\pi\epsilon_0 R^3}{\left(\epsilon/\epsilon_0 + 2\right)}}$$

(d) $$\vec{F} = q\,\vec{E}(x+\delta) - q\,\vec{E}(x) = q\frac{\partial E^x}{\partial x}\delta\,\hat{x}$$

$$= \vec{p}\cdot\vec{\nabla}\vec{E} = p_0\frac{\partial}{\partial\rho}\left(\frac{\lambda}{2\pi\epsilon_0}\frac{1}{\rho}\right)_{\rho=d}\hat{x}$$

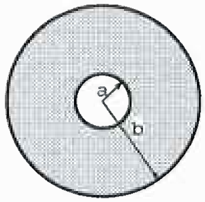
#2 cont

$$\therefore \vec{F} = \frac{\left(\frac{\epsilon}{\epsilon_0} - 1\right)}{\left(\frac{\epsilon}{\epsilon_0} + 2\right)} E_{\text{uno}} 4\pi\epsilon_0 R^3 \frac{\lambda}{2\pi\epsilon_0} \left(-\frac{1}{d^2}\right) \hat{x}$$

$$\boxed{\vec{F} = \frac{\left(\frac{\epsilon}{\epsilon_0} - 1\right)}{\left(\frac{\epsilon}{\epsilon_0} + 2\right)} \frac{\lambda^2}{\pi\epsilon_0} \left(\frac{R}{d}\right)^3 \left(-\hat{x}\right)}$$

attractive !

$$\left(\frac{\chi_e}{3 + \chi_e}\right)$$



3. Consider two concentric conducting spherical shells of radii a and b with $b > a$. The space between the two shells is a filled with Ohmic material of constant conductivity σ, permittivity ϵ_0, and permuability μ_0. The system is charged such that at time $t = 0$ the inner conductor has charge $+Q_0$ and the outer conductor has charge $-Q_0$. At times $t > 0$ the charge will flow from the inner shell to the outer shell.

   (a) [2 pts] Use Gauss's law to relate the electric field $\mathbf{E}(t, \mathbf{r})$ between the plates to the charge $Q(t)$ on the inner plate.

   (b) [4 pts] Use the conservation of charge and

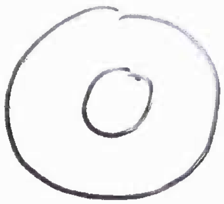
   $$\mathbf{J}(t, \mathbf{r}) = \sigma \, \mathbf{E}(t, \mathbf{r}),$$

   to find $Q(t)$.

   (c) [2 pts] Use Faraday's law and your electric field to show that $\mathbf{B}(t, \mathbf{r}) = 0$.

   (d) [2 pts] Confirm that Ampère's law is satisfied.

3.



(a)

$$\oint \vec{D}\cdot d\vec{a} = Q$$

$$\| $$

$$D(4\pi r^2) \Rightarrow D = \frac{Q(t)}{4\pi r^2}$$

$$\therefore \boxed{\vec{E} = \frac{Q(t)}{4\pi\epsilon_0 r^2}\hat{r}}$$

(b) $$\dot{Q} = -\int_{r=\text{const}} \vec{J}\cdot d\vec{a} = -J(r,t)\,4\pi r^2$$

$$\downarrow$$

$$\sigma E(r,t)$$

$$\dot{Q} = -\frac{\sigma\,Q(t)}{4\pi\epsilon_0 r^2}4\pi r^2 = -\frac{\sigma}{\epsilon_0}Q(t)$$

$$\text{so}\quad \boxed{Q(t) = Q(0)\,e^{-\sigma/\epsilon_0\, t}}$$

(c) $$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{B} = \vec{B}(\vec{r})$$

$\uparrow$ no t!

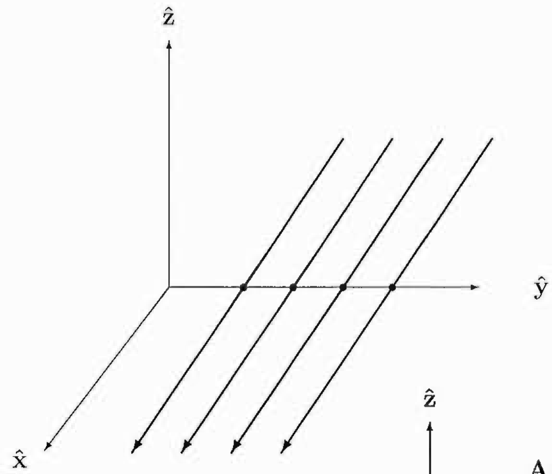
But as $t\to\infty$ $\vec{J}, Q$, etc $\to 0$

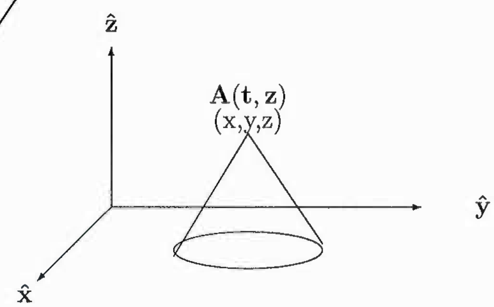
$$\therefore \boxed{\vec{B} = 0 \text{ at all time}}$$

(d) $$\vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t} = \vec{J}$$

$$\downarrow$$

$$0 \quad -\cancel{\epsilon_0}\frac{\partial}{\partial t}\left(\frac{Q(t)}{4\pi\cancel{\epsilon_0}r^2}\right) = \sigma E = -\frac{\dot{Q}}{4\pi r^2} = \text{Same}$$





4. A uniform sheet of current in the (x, y) plane at $z = 0$ suddenly turns on at $t = 0$ and has a surface current density

$$\begin{aligned} \mathbf{K}(t, \mathbf{r}) &= 0, \qquad t < 0, \\ \mathbf{K}(t, \mathbf{r}) &= K_0\, \hat{\mathbf{x}}, \qquad t \geq 0, \end{aligned} \tag{1}$$

where K_0 has units of current/length. The corresponding volume current density is

$$\mathbf{J}(t, \mathbf{r}) = \mathbf{K}(t, \mathbf{r})\, \delta(z).$$

The retarded vector potential in SI units and in the Lorentz gauge for an arbitrary current source can found by integrating

$$\mathbf{A}(t, \mathbf{r}) = \left(\frac{\mu_0}{4\pi}\right) \int \frac{\mathbf{J}(t - |\mathbf{r} - \mathbf{r}'|/c, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'.$$

In Gaussian units the factor $\mu_0/4\pi$ is replaced by $1/c$.

(a) [4 pts] In cylindrical polar coordinates evaluate 2 of the 3 integrals in the above expression for $\mathbf{A}(t, \mathbf{r})$, i.e., integrate over z' and ϕ' leaving $\mathbf{A}(t, \mathbf{r})$ as an integral over the single coordinate ρ'.

(b) [3 pts] Evaluate the ρ' integral giving $\mathbf{A}(t, \mathbf{r})$ as a function of t and z only.

(c) [3 pts] Compute the magnetic induction from your vector potential.

#4

$$\vec{J}(t,\vec{r}) = K_0 \hat{x}\, \Theta(t)\, \delta(z)$$

(a)

$$\vec{A}(t,\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{K_0 \hat{x}\, \Theta\left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \delta z'\, dx'\,dy'\,dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= \frac{\mu_0 K_0 \hat{x}}{4\pi} \iint \frac{\Theta\left(t - \frac{\sqrt{(x-x')^2+(y-y')^2+z^2}}{c}\right) dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}$$

change to $x'' = x' - x \quad y'' = y' - y$ coord

$$\vec{A}(t,\vec{r}) = \frac{\mu_0 K_0 \hat{x}}{4\pi} \iint \frac{\Theta\left(t - \frac{\sqrt{\rho''^2+z^2}}{c}\right)}{\sqrt{\rho''^2+z^2}} \begin{matrix} dx''\,dy'' \\ \rho''\,d\rho''\,d\phi'' \end{matrix}$$

$$\vec{A}(t,\vec{r}) = \boxed{\frac{\mu_0 K_0 \hat{x}}{2} \int_0^{\infty} \frac{\Theta\left(t - \frac{\sqrt{\rho''^2+z^2}}{c}\right)}{\sqrt{\rho''^2+z^2}}\, \rho''\,d\rho''}$$

$$\int_0^{\sqrt{c^2t^2-z^2}} + d\sqrt{\rho''^2+z^2}$$

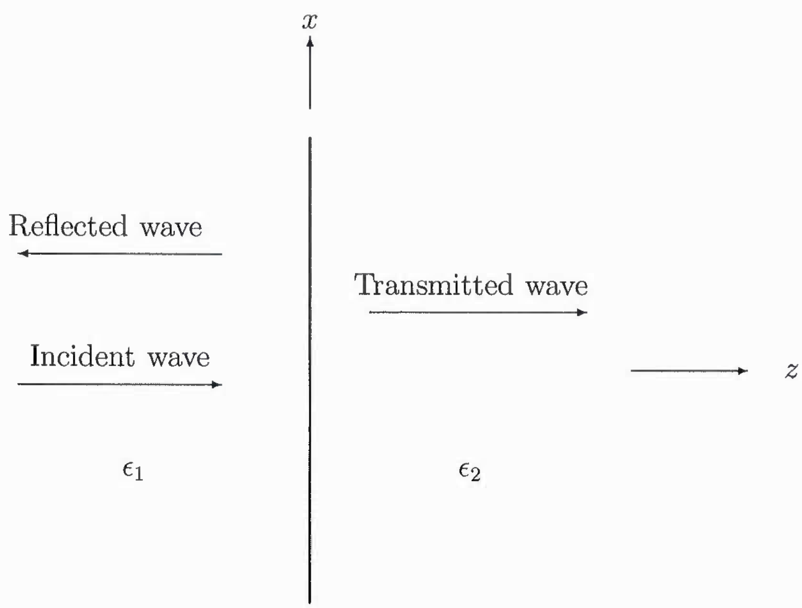
#4 cont

(b) $$\boxed{\vec{A}(t,\vec{r}) = \frac{\mu_0 K \hat{x}}{2}\left[ct - z\right]\Theta(ct-z)}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{y}\frac{\partial}{\partial z}A^{x}$$

$$= \frac{\mu_0 K}{2}\hat{y}\left[\underbrace{(ct-z)\delta(ct-z)}_{=0} - \Theta(ct-z)\right]$$

(c) $$\boxed{\vec{B} = -\frac{\mu_0 K}{2}\Theta(ct-z)\,\hat{y}}$$



5. A linearly-polarized harmonic ($e^{-i\omega t}$) plane electromagnetic wave traveling to the right in a homogeneous dielectric medium described by a real dielectric constant ϵ_1, strikes a second homogeneous dielectric material described by another real dielectric constant $\epsilon_2 > \epsilon_1$ (see the figure). Assume that both materials have no magnetic susceptibility, $\chi_m = 0$, and that the incidence angle is 0^o (i.e., the wave is traveling perpendicular to the junction). Assume the incoming wave is polarized in the $\hat{x}$ direction and that its electric field amplitude is E_0, i.e., assume the incoming electric field is the real part of

$$\mathbf{E} = E_0 \, e^{i(kz-wt)} \, \hat{x}.$$

   (a) [2 pts] Give the direction of the magnetic induction $\mathbf{B}$ associated with the above incoming wave and give its amplitude B_0 as a function of E_0. Also give k as a function of ω.

   (b) [2 pts] Give similar expressions for $\mathbf{E}$ and $\mathbf{B}$ of the reflected and transmitted waves. Use E_0'' and E_0' for the respective electric field amplitudes of the reflected and transmitted waves.

   (c) [3 pts] Apply the boundary conditions at the junction/interface between the dielectrics to the incoming, reflected, and transmitted wave to compute E_0'' and E_0' as functions of E_0 and the two dielectric constants ϵ_1 and ϵ_2.

   (d) [3 pts] Evaluate the reflection and transmission coefficients, R and T, for above waves. Recall that R and T are computed from ratios of time averaged Poynting vectors which are defined by

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} \equiv \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

(#5) $\vec{E} = E_0 e^{i(kz-\omega t)}\hat{x}$

red = Gaussian
Green = SI

Has to satisfy Maxwell's eqns

(a) $\vec{\nabla}\cdot\vec{D} = 0$ is satisfied

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} \Rightarrow \boxed{\vec{B} = c\frac{k}{\omega}E_0 e^{i(kz-\omega t)}\hat{y}}$$

$$\parallel$$

$$ikE_0 e^{i(kz-\omega t)}\hat{y}$$

$\vec{\nabla}\cdot\vec{B} = 0$ is satisfied

$$e^{i(kz-\omega t)}\hat{x}$$

$$\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = 0 \Rightarrow \mu_0^{-1} c\frac{k}{\omega}E_0(ik)(-\hat{x}) - \frac{1}{c}\epsilon(-i\omega)E_0$$

$$\therefore \quad \frac{k^2}{\omega^2} - \frac{\epsilon\mu_0}{c^2} = 0$$

$$\boxed{k = \frac{\sqrt{\epsilon\mu_0}}{c}\,\omega}$$

(b) $\vec{E}'' = E_0'' e^{i(-k''z-\omega t)}\hat{x}$ $\qquad k \to -k''$

$$\Rightarrow \boxed{\vec{B}'' = c\frac{-k''}{\omega}E_0'' e^{i(-k''z-\omega t)}\hat{y}}$$

$$\left(\frac{k''}{\omega}\right)^2 = \frac{\epsilon_1\mu_0}{c^2} \Rightarrow \boxed{k'' = \frac{\sqrt{\epsilon_1\mu_0}}{c}\,\omega}$$

$$\boxed{= k}$$

#5 cont

$$\vec{E}' = E_0' e^{i(k'z-\omega t)}\hat{x}$$

$$\boxed{\vec{B}' = c\frac{k'}{\omega}E_0' e^{i(k'z-\omega t)}\hat{y}}$$

$$\left(\frac{k'}{\omega}\right)^2 = \frac{\epsilon_2\epsilon_0}{c^2} \Rightarrow \boxed{k' = \frac{\sqrt{\epsilon_2\epsilon_0}}{c}\omega}$$

(c) $\vec{E}_{tang} = \text{cont} \qquad \vec{H}_{tang} = \text{cont}$

① $\therefore \quad E_0 + E_0'' = E_0'$

$$\mu_0^{-1} c\frac{k}{\omega}E_0 + c\mu_0^{-1}\left(-\frac{k}{\omega}\right)E_0'' = \mu_0^{-1} c\frac{k'}{\omega}E_0'$$

② $\quad kE_0 - kE_0'' = k'E_0'$

$$\therefore \quad 2E_0 = \left(1+\frac{k'}{k}\right)E_0' \Rightarrow \boxed{E_0' = \frac{2E_0}{1+\sqrt{\frac{\epsilon_2}{\epsilon_1}}}}$$

$$E_0'' = E_0' - E_0$$

$$\Rightarrow \boxed{E_0'' = E_0\left(1-\sqrt{\epsilon_2/\epsilon_1}\right)}$$

#5 cont

$$R = \frac{\langle |\vec{S}''| \rangle}{\langle |\vec{S}_{in}| \rangle} = \frac{|\langle \vec{E}'' \times \vec{H}'' \rangle|}{|\langle \vec{E} \times \vec{H} \rangle|}$$

(d)

$$R = \frac{E_0'' \, H_0''}{E_0 \, H_0} = \boxed{\left(1 - \sqrt{\epsilon_2/\epsilon_1}\right)^2}$$

$$T = \frac{\langle |\vec{S}'| \rangle}{\langle |\vec{S}_{in}| \rangle} = \frac{E_0' \, H_0'}{E_0 \, H_0} = \frac{E_0'}{E_0} \frac{k' E_0'}{k E_0}$$

$$T = \boxed{\sqrt{\frac{\epsilon_2}{\epsilon_1}} \left(\frac{2}{1+\sqrt{\frac{\epsilon_2}{\epsilon_1}}}\right)^2}$$

$$\Rightarrow R + T = 1$$

6. In the lab you measure a uniform electric field and a uniform magnetic induction

$$\mathbf{E} = E_0(\cos 45^\circ\, \hat{\mathbf{x}} + \sin 45^\circ\, \hat{\mathbf{y}}),$$

$$\mathbf{B} = B_0\, \hat{\mathbf{x}},$$

where $B_0 = E_0$ in Gaussian units or $B_0 = E_0/c$ in SI units. The goal of this problem is to compute the $\mathbf{E}'$ and $\mathbf{B}'$ fields an observer sees if moving relative to the lab with a velocity $\mathbf{v} = v_0\, \hat{\mathbf{z}}$.

(a) [2 pts] Combine $\mathbf{E}$ and $\mathbf{B}$ into a single 4×4 anti-symmetric electromagnetic field tensor $F^{\alpha\beta}$.

(b) [2 pts] Give the 4×4 Lorentz boost L^{α}_{β} that transforms the lab coordinates (ct, x, y, z) into the moving frame's coordinates (ct', x', y', z') i.e., $x'^{\alpha} = L^{\alpha}_{\beta} x^{\beta}$ where $x^{\beta} = (ct, x, y, z)$. In matrix notation $x' = L\, x$.

(c) [3 pts] Find $\mathbf{E}'$ and $\mathbf{B}'$ by by boosting the F tensor, i.e., compute $F'^{\alpha\beta} = L^{\alpha}_{\sigma} L^{\beta}_{\lambda} F^{\sigma\lambda}$ which in matrix notation is $F' = LFL^{\top}$

(d) [3 pts] For what value of v_0 will $\mathbf{E}'$ and $\mathbf{B}'$ be parallel?

(the signs can differ from Jackson)

6

$$F^{0i} = -F^{i0} = -E^i$$

$$F^{ij} = -F^{ji} = -\epsilon^{ijk} B^k \leftarrow \text{Gaussian}$$

$$= -\epsilon^{ijk} c B^k \leftarrow \text{SI}$$

(a) $$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_0/\sqrt{2} & -E_0/\sqrt{2} & 0 \\ E_0/\sqrt{2} & 0 & 0 & 0 \\ E_0/\sqrt{2} & 0 & 0 & -E_0 \\ 0 & 0 & E_0 & 0 \end{pmatrix}$$

$E_0 = B_0$ in Gaussian

$E_0 = cB_0$ in SI

$$= \frac{E_0}{\sqrt{2}} \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$\beta = v_0/c$

$\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

$$\boxed{x'^\alpha = L^\alpha{}_\beta x^\beta} \quad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = (x') = (L)(x) = (L) \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Tensor notation

matrix notation

#6 cont

2/2

$$F' = \underbrace{L F}_{\to} L^T = L \frac{E_0}{\sqrt{2}} \begin{pmatrix} 0 & -1 & -1 & 0 \\ \gamma & 0 & 0 & -\beta\gamma \\ \gamma + \sqrt{2}\beta\gamma & 0 & 0 & -\beta\gamma - \sqrt{2}\gamma \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$F' = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 0 & -\gamma & -\gamma - \beta\gamma\sqrt{2} & 0 \\ \gamma & 0 & 0 & -\beta\gamma \\ \gamma + \sqrt{2}\beta\gamma & 0 & 0 & -\beta\gamma - \sqrt{2}\gamma \\ 0 & \beta\gamma & \beta\gamma + \sqrt{2}\gamma & 0 \end{pmatrix}$$

(c)

$$\Rightarrow \quad \vec{E}' = \frac{E_0}{\sqrt{2}} \gamma \hat{x} + \frac{E_0}{\sqrt{2}} \gamma (1 + \beta\sqrt{2}) \hat{y}$$

$$\vec{B}' = \frac{E_0}{\sqrt{2}} \gamma (\beta + \sqrt{2}) \hat{x} - \frac{E_0}{\sqrt{2}} \gamma \beta \hat{y}$$

(d) $\vec{E}' \propto \vec{B}'$ if $\frac{E'^y}{E'^x} = 1 + \sqrt{2}\beta = \frac{-\beta}{\beta + \sqrt{2}} = \frac{B'^y}{B'^x}$

$\therefore \quad \beta^2 + \frac{4}{\sqrt{2}}\beta + 1 = 0 \Rightarrow \beta = -\sqrt{2} \pm 1$

choose $\boxed{\beta = 1 - \sqrt{2} < 0} \Rightarrow \gamma = \frac{1}{\sqrt{1 - 3 + 2\sqrt{2}}} = \boxed{\sqrt{\frac{\sqrt{2}+1}{2}}}$