E & M Qualifier

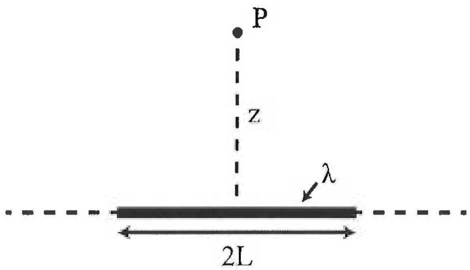
August 17, 2016

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

1. This problem asks you to compute electrostatic potentials and fields on the z-axis produced by two different continuous charge distributions located in the xy-plane. **The two calculations are remarkably similar even though the first is for a line of charge and the second is for a charged disc.** Choose the center of each distribution as your origin and the reference point for the potentials at infinity, e.g. $\lim_{r\to\infty} \Phi = 0$. You will need the indefinite integral

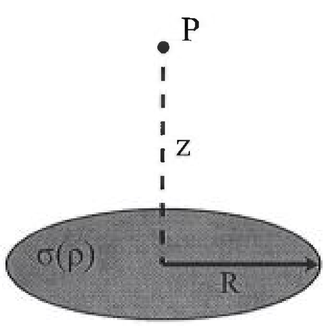
$$\int \frac{dx}{\sqrt{x^2+z^2}} = \ln[x + \sqrt{x^2+z^2}] + \text{constant}.$$



(a) [2 pt] For the line charge distribution shown above with a uniform linear charge density λ and length $2\,L$, calculate the potential $\Phi(z)$ at point P a distance z above the center of the line charge.

(b) [1 pt] Calculate the electric field $\mathbf{E}(z)$ at point P from your potential (use symmetries).

(c) [2 pt] Show that $\Phi(z)$ and $\mathbf{E}(z)$ reduce to the expected values when $z \gg L$.



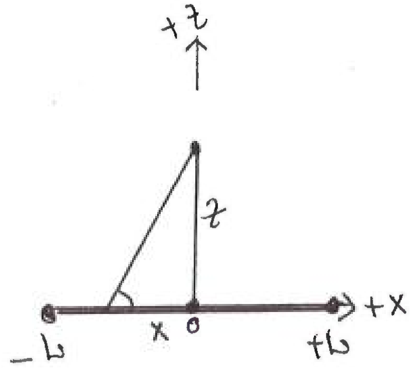
(d) [3 pt] For the thin disc, shown above, with radius R that has a *non-uniform* surface charge distribution $\sigma = \sigma_0\, R/\rho$, where $\rho = \sqrt{x^2+y^2}$ is the radial distance from the center of the disc, calculate the electric potential $\Phi(z)$ at a point P at a distance z above the center of the disc.

(e) [1 pt] Calculate the electric field $\mathbf{E}(z)$ at point P from your potential (use symmetries).

(f) [1 pt]) Write $\Phi(z)$ as a function of the total charge Q on the disc when $z \gg R$.

#1

(a)



$$\phi(z) = \frac{1}{4\pi\epsilon_0}\int_{-b}^{+b}\frac{\lambda\, dx}{\sqrt{x^2+z^2}}$$

SI

$$= \frac{2\lambda}{4\pi\epsilon_0}\int_0^b \frac{dx}{\sqrt{x^2+z^2}}$$

For Gaussian drop the $4\pi\epsilon_0$

$$\boxed{\phi(z) = \frac{2\lambda}{4\pi\epsilon_0}\ln\left[\frac{b}{z}+\sqrt{\left(\frac{b}{z}\right)^2+1}\right]}$$

(b) $$E_z = -\frac{\partial}{\partial z}\phi(z) = \frac{-2\lambda}{4\pi\epsilon_0}\left[\frac{-b/z^2 - \frac{b^2}{z^3}\frac{1}{\sqrt{(b/z)^2+1}}}{b/z+\sqrt{(b/z)^2+1}}\right]$$

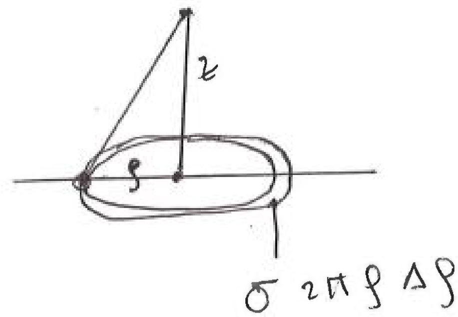
$$\boxed{E_z = \frac{2\lambda}{4\pi\epsilon_0}\frac{b}{z^2\sqrt{(b/z)^2+1}}}$$

(c) $$\sqrt{(b/z)^2+1} \cong 1+\frac{1}{2}(b/z)^2+\cdots$$

$$\ln\left[\frac{b}{z}+\sqrt{\quad}\right] \cong \ln\left[1+\frac{b}{z}+\cdots\right] \cong 0+\frac{b}{z}+\cdots$$

$$\therefore \boxed{\begin{aligned}\phi(z) &\cong \frac{2\lambda}{4\pi\epsilon_0}\frac{b}{z}\\ E_z &\cong \frac{2\lambda b}{4\pi\epsilon_0 z^2}\end{aligned}}$$

(d) $$\phi = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{\sigma(\rho) 2\pi\rho\, d\rho}{\sqrt{z^2+\rho^2}}$$



$$\sigma = \sigma_0 R/\rho$$

$$\phi = \frac{2\pi\sigma_0 R}{4\pi\epsilon_0}\int_0^R \frac{d\rho}{\sqrt{z^2+\rho^2}}$$

$$\boxed{\phi(z) = \frac{2\pi\sigma_0 R}{4\pi\epsilon_0}\ln\left[\frac{R+\sqrt{R^2+z^2}}{z}\right]}$$

$$E_z = -\frac{\partial\phi}{\partial z} = -\frac{2\pi\sigma_0 R}{4\pi\epsilon_0}\left[\frac{z/\sqrt{R^2+z^2}}{R+\sqrt{R^2+z^2}} - \frac{1}{z}\right]$$

(e)

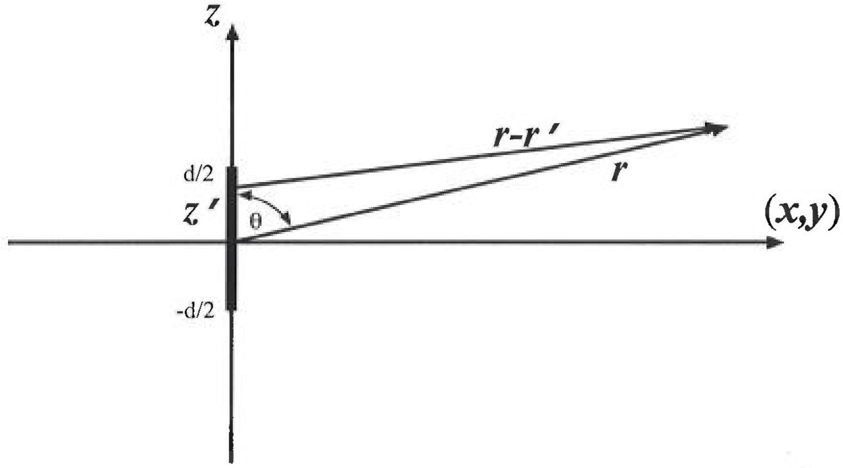
$$\boxed{E_z = \frac{2\pi\sigma_0 R^2}{4\pi\epsilon_0\, z\sqrt{R^2+z^2}}}$$

(f) $$Q = \int_0^R \sigma\, 2\pi\rho\, d\rho = \sigma_0 R\, 2\pi R = 2\pi\sigma_0 R^2$$

$$\therefore \phi(z) = \frac{Q}{(4\pi\epsilon_0)R}\ln\left[\frac{R+\sqrt{R^2+z^2}}{z}\right] \cong \frac{Q}{(4\pi\epsilon_0)R}\ln\left[1+R/z+\cdots\right]$$

$$\to 0 + R/z$$

$$\boxed{\phi(z) \cong \frac{Q}{4\pi\epsilon_0\, z}}$$



2. A thin linear (full wave) antenna of length $d = \lambda$ is centered on the origin and aligned along the z-axis as shown in the figure. An oscillating current density of the form

$$\mathbf{J}(\mathbf{r}, t) = I_0 \delta(x)\delta(y) \sin\left(\frac{2\pi z}{d}\right) e^{i\omega t}\, \hat{\mathbf{z}},$$

is produced in the antenna by applying an oscillating voltage of angular frequency $\omega = 2\pi c/d$.

The resulting oscillating sinusoidal current makes one full wavelength of oscillation within the antenna with nodes at $z = \pm d/2$.

(a) [4 pt] At any point $\mathbf{r}$ which is a large distance from the antenna ($d/r << 1$), calculate the **radiation part** of the retarded vector potential. Recall that the retarded vector potential at $\mathbf{r}$ caused by a current at $\mathbf{r}'$ (in SI units) is

$$\mathbf{A}(\mathbf{r}, t) = \left(\frac{\mu_0}{4\pi}\right) \int \frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'. \tag{1}$$

In Gaussian units the factor $(\mu_0/4\pi)$ is replaced by $(1/c)$. To obtain the radiation parts you will need to approximate $|\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \cdots = r - \cos\theta\, z' + \cdots$, and keep only the appropriate terms. You will also need the integral

$$\int_{-d/2}^{d/2} \sin\left(\frac{2\pi z'}{d}\right) e^{i\left(\frac{2\pi z'}{d}\right)\cos\theta} dz' = i\left(\frac{d}{\pi}\right) \frac{\sin(\pi\cos\theta)}{\sin^2\theta}.$$

(b) [3 pt] Calculate the **radiation part** of the magnetic induction using $\mathbf{B} = \nabla \times \mathbf{A} = (\nabla A^z) \times \hat{\mathbf{z}}$.

(c) [3 pt] Use the Poynting vector $\mathbf{S}$ to calculate the time averaged power radiated per unit solid angle in the θ direction. Recall that in Gaussian units $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{H}$) and in SI units the $(c/4\pi)$ factor is missing. Also recall that in SI units $\mathbf{E}_{rad} = c\mathbf{B}_{rad} \times \hat{\mathbf{r}}$ and in Gaussian units the factor c is missing.

Electrodynamics Qualifier — Solution

2. A thin linear anntenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in the figure (frequency $\omega = 2\pi c/d$). The current can be written in the following form

$$\vec{\mathbf{J}} = I_0 \delta(x)\delta(y) \sin\left(\frac{2\pi z}{d}\right) \hat{\mathbf{z}} e^{i\omega t'}.$$

(a) Calculate the retarded vector potential keeping only the radiation term.
The retarded vector potential is

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{J}}(\vec{\mathbf{r}}', t - r'/c)}{r'} dV' = \frac{\mu_0}{4\pi}\int \frac{I(\vec{\mathbf{r}}', t - r'/c)}{r'} dz' = \frac{\mu_0}{4\pi} I_0 e^{i\omega t} \hat{\mathbf{z}} \int_{-\lambda/2}^{\lambda/2} \frac{\sin(kz')}{r'} e^{-ikr}\, dz',$$

where $k = 2\pi/d$. The term r' is

$$r' = \sqrt{r^2 - 2rz'\cos\theta + z'^2} \approx r - z'\cos\theta$$
$$\frac{1}{r'} \approx \frac{1}{r}.$$

Substitute the approximation and perform the integral

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi}\frac{I_0 e^{i(\omega t - kr)}}{r}\hat{\mathbf{z}} \int_{-\lambda/2}^{\lambda/2} \sin(kz') e^{ikz'\cos\theta}\, dz' = i\frac{\mu_0}{2\pi}\frac{I_0}{kr}\frac{\sin(\pi\cos\theta)}{\sin^2\theta} e^{i(\omega t - kr)}\hat{\mathbf{z}}.$$

(b) Calculate the magnetic field $\vec{\mathbf{B}}$ and electric field $\vec{\mathbf{E}}$ at the field point.
From $\vec{\nabla}\cdot\vec{\mathbf{B}} = 0$ we have $\vec{\mathbf{B}} = \vec{\nabla}\times\vec{\mathbf{A}}$, which for this specific vector potential gives

$$\vec{\mathbf{B}} = -i\vec{\mathbf{k}}\times\vec{\mathbf{A}} = -\frac{\mu_0}{2\pi}\frac{I_0}{r}\frac{\sin(\pi\cos\theta)}{\sin\theta} e^{i(\omega t - kr)}\hat{\boldsymbol{\phi}},$$

where $\vec{\mathbf{k}} = k\hat{\mathbf{r}}$, $\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$, and $\hat{\mathbf{r}}\times\hat{\mathbf{z}} = -\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}\sin\theta = -\hat{\boldsymbol{\phi}}\sin\theta$ were used. Next use the Maxwell equation

$$\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\epsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t}$$
$$-i\vec{\mathbf{k}}\times\vec{\mathbf{B}} = i\mu_0\epsilon_0\omega\vec{\mathbf{E}} = i\frac{k}{c}\vec{\mathbf{E}},$$

which leads to

$$\vec{\mathbf{E}} = -\frac{c\vec{\mathbf{k}}}{k}\times\vec{\mathbf{B}} = c\vec{\mathbf{B}}\times\hat{\mathbf{r}}.$$

(c) Finally, calculate the average radiated power per unit solid angle.
Start by calculating the time average Poynting vector

$$\left\langle\vec{\mathbf{S}}\right\rangle = \frac{1}{2\mu_0}\mathrm{Re}(\vec{\mathbf{E}}^*\times\vec{\mathbf{B}}) = \frac{c}{2\mu_0}|\vec{\mathbf{B}}|^2\hat{\mathbf{r}} = \frac{\mu_0\, cI_0^2}{8\pi^2 r^2}\left[\frac{\sin(\pi\cos\theta)}{\sin\theta}\right]^2\hat{\mathbf{r}}.$$

Finally, the radiated power per unit solid angle is

$$\frac{dP}{d\Omega} = r^2\left\langle\vec{\mathbf{S}}\right\rangle\cdot\hat{\mathbf{r}} = \frac{I_0^2}{8\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\left[\frac{\sin(\pi\cos\theta)}{\sin\theta}\right]^2.$$

2

(a) $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{I_0 \delta(x')\delta(y')\sin\left(\frac{2\pi z'}{d}\right) e^{i\omega\left[t - \frac{|\vec{r}-\vec{r}'|}{c}\right]}}{|\vec{r}-\vec{r}'|} \hat{z}\, d^3r'$ $\quad 1/2$

SI

$d = \lambda$

radiation terms are $\propto 1/r$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} I_0 \hat{z} \int_{-d/2}^{+d/2} \frac{\sin\left(\frac{2\pi z'}{d}\right) e^{i\omega\left[t - \frac{|\vec{r}-z'\hat{z}|}{c}\right]}}{|\vec{r}-z'\hat{z}|} dz'$$

$$|\vec{r}-z'\hat{z}| = \sqrt{r^2 + z'^2 - 2zz'} \qquad \underline{z = r\cos\theta}$$

$$= r\sqrt{1 - 2\cos\theta \frac{z'}{r} + \frac{z'^2}{r^2}} \cong r - \cos\theta z' + \cdots$$

keep

$$\frac{1}{|\vec{r}-z'\hat{z}|} \cong \frac{1}{r}\left[1 + \cos\theta\, z'/r + \cdots\right]$$

keep

$$\therefore \vec{A}(\vec{r},t) \cong \frac{\mu_0}{4\pi} I_0 \frac{\hat{z}}{r} e^{i\omega[t - r/c]} \underbrace{\int_{-d/2}^{+d/2} \sin\left(\frac{2\pi z'}{d}\right) e^{\frac{i\omega\cos\theta z'}{c}} dz'}$$

$\omega = c2\pi/\lambda = \frac{c2\pi}{d}$

from hint $\longrightarrow$ $i\frac{d}{\pi}\frac{\sin(\pi\cos\theta)}{\sin^2\theta}$

$$\boxed{\therefore \vec{A}(\vec{r},t) \cong i\frac{\mu_0}{4\pi} I_0 \frac{\hat{z}}{r} e^{i\omega[t - r/c]} \frac{d}{\pi}\frac{\sin(\pi\cos\theta)}{\sin^2\theta}}$$

2

$$\vec{B}_{rad} = \frac{i\mu_0}{4\pi} I_0 \frac{\left(\vec{\nabla} e^{i\omega(t-r/c)}\right)\times\hat{z}}{r} \frac{d}{\pi} \frac{\sin(\pi\cos\theta)}{\sin^2\theta}$$

$$\vec{\nabla} e^{i\omega(t-r/c)} = -i\frac{\omega}{c} e^{i\omega(t-r/c)}\hat{r}$$

$\hat{z}$ θ $\hat{r}$

$$\hat{r}\times\hat{z} = -\sin\theta\,\hat{\phi}$$

$\frac{\omega d}{c} = \frac{2\pi f\lambda}{c} = 2\pi$

(b)

$$\Rightarrow \boxed{\vec{B}_{rad} = \left(\frac{\mu_0}{4\pi}\right) I_0\, 2\frac{e^{i\omega(t-r/c)}}{r} \frac{\sin(\pi\cos\theta)}{\sin\theta}(-\hat{\phi})}$$

$*\left(\frac{1}{c}\right)$ in G

$$\vec{S}_G = \left(\frac{c}{4\pi}\right)\vec{E}\times\vec{H} \qquad \left(\frac{c}{4\pi}\right)\to 1 \text{ in SI}$$

real part $\vec{B}$ only

$$\vec{S}_{SI} = (c\vec{B}_{rad}\times\hat{r})\times\frac{\vec{B}_{rad}}{\mu_0} = \frac{c}{\mu_0}(\vec{B}_{rad}\cdot\vec{B}_{rad})\hat{r}$$

$$\langle\vec{S}_{SI}\rangle = \frac{c}{\mu_0}\frac{1}{2}(\vec{B}_{rad}\cdot\vec{B}^*_{rad})\hat{r}$$

time ave

$$= \frac{c}{\mu_0}\frac{1}{2}\left(\frac{\mu_0}{2\pi}\right)^2 I_0^2 \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta\, r^2}\hat{r}$$

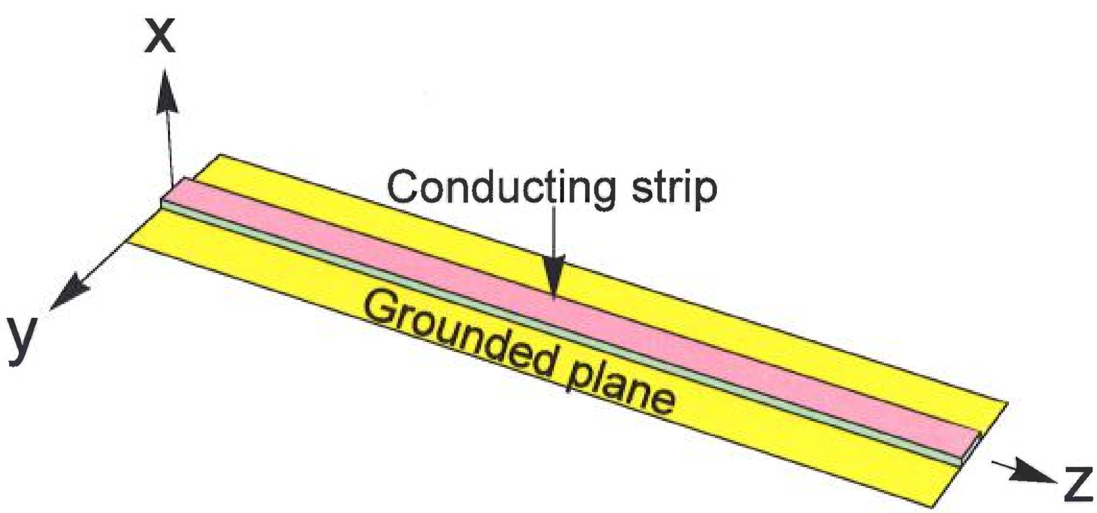
(c)

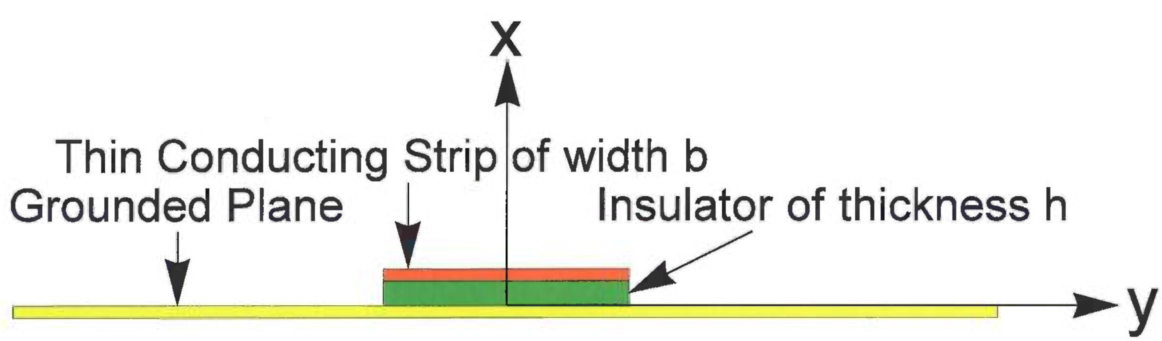
$c\mu_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$

$$\therefore \boxed{\frac{dP}{d\Omega} = \frac{c\mu_0 I_0^2}{2\,(2\pi)^2}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}}$$

SI

$$= \frac{I_0^2}{2\pi c}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} \quad \text{Gaussian}$$





Enlarged end view of Strip

3. A transverse electromagnetic (TEM) wave is transmitted using a conducting micro-strip built into a printed circuit board as illustrated above. The circuit consists of a large grounded conducting plane and a thin flat conducting ribbon of width b kept at a fixed distance h ($h << b$) above the grounding plane by an insulating dielectric material. The conductors are perfect, i.e., $\sigma \sim \infty$, and the region between the conductors is filled with a polarizable material of real permittivity and permeability (ϵ, μ), indicated by the green $b \times h$ rectangle in the bottom figure. The TEM wave propagates in the $\hat{\mathbf{z}}$ direction (into the page in the bottom figure) and is confined to the volume between the conductors defined by the green rectangle $b \times h$ (i.e., **neglect edge effects**).

   (a) [1 pt] For a wave traveling in the $\hat{\mathbf{z}}$ direction (into the page in the bottom figure)

#3

(a) [1 pt] For a wave traveling in the $\hat{\mathbf{z}}$ direction (into the page in the bottom figure) redraw an enlarged picture of the insulator (the green rectangle) and sketch the $\mathbf{E}$ and $\mathbf{B}$ field lines within it for some fixed value of t and z.

(b) [2 pt] Give expressions for $\mathbf{E}(t,z)$ and $\mathbf{B}(t,z)$ inside the volume defined by the green rectangle $b \times h$ that satisfy Maxwell's equations.

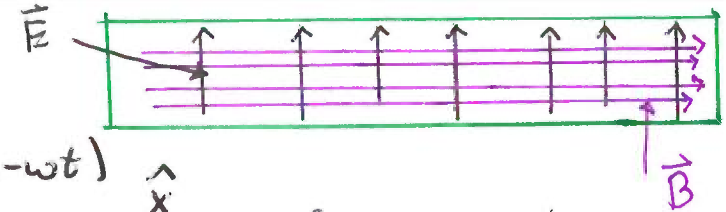
(c) [3 pt] Show that the instantaneous value of $I(t) * V(t)$ at a given value of z is equal to the Poynting vector integrated over the rectangular area $(b \times h)$. $V(t)$ is the potential of the strip at z relative to the grounded plane and $I(t)$ is the current flowing in the strip at that same value of z.

(d) [3 pt] Derive an expression for the characteristic impedance, $Z = V(t)/I(t)$, of the microstrip.

(e) [1 pt] What would the characteristic impedance be if a second grounded plane was added symmetrically (including the insulating material) above the strip?

#3 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 1/2

(a) see figure →



(b)
$$\boxed{\begin{aligned}\vec{E}(t,z) &= E_0\, e^{i(kz-\omega t)}\,\hat{x}\\ \vec{B}(t,z) &= B_0\, e^{i(kz-\omega t)}\,\hat{y}\end{aligned}}$$
inside $b \times h$

Maxwell's eqns require $\boxed{\dfrac{\omega}{k} = \dfrac{c}{\sqrt{\epsilon_r \mu_r}}}$ $\qquad \epsilon_r = \epsilon/\epsilon_0$, $\quad \mu_r = \mu/\mu_0$

$$= c/n = v$$

and $\boxed{B_0 = n E_0/c}$ $\qquad n \equiv \sqrt{\epsilon_r \mu_r}$

c is missing in Gaussian units

$\epsilon_0 = \mu_0 = 1$ in Gaussian

(c) $V(t) = E_0\, h\, e^{i(kz-\omega t)}$ ← real part of

$$\oint H \cdot dl = I \Rightarrow \frac{B}{\mu}\, b = {*}\, I$$

$\left(\frac{4\pi}{c}\right)$ in Gaussian

$\Rightarrow$ $I(t) = \dfrac{B_0 b}{\mu}\, e^{i(kz-\omega t)}$ ← real part of

$$\Rightarrow I(t)\,V(t) = \underbrace{\frac{E_0 B_0\, h b}{\mu}}\cos^2(kz-\omega t)$$

$$\boxed{I(t)\,V(t) = \frac{|E_0|^2\, n}{\mu c}\, hb\, \cos^2(kz-\omega t)} \quad \text{SI}$$

$$= \frac{c}{4\pi}\,\frac{|E_0|^2\, n}{\mu}\, hb\, \cos^2(kz-\omega t) \quad \text{Gaussian}$$

#3

$$\vec{S} = \vec{E}\times\vec{H} = \frac{\vec{E}\times\vec{B}}{\mu} = \frac{E_0 B_0}{\mu}\cos^2(kz-\omega t)$$

$$\boxed{\vec{S} = \frac{n|E_0|^2}{c\mu}\cos^2(kz-\omega t)} \quad \underline{\text{in SI}}$$

$$\therefore \quad \underline{\vec{S}\cdot\hat{z}\,(b*h)} \underset{\text{in SI}}{=} \frac{n|E_0|^2}{c\mu}(b\cdot h)\cos^2(kz-\omega t) = \underline{I*V}$$

In Gaussian $\vec{S} = \left(\frac{c}{4\pi}\right)\vec{E}\times\vec{H} = \frac{c}{4\pi}\frac{nE_0^2}{\mu}\cos^2(kz-\omega t)$

$$\therefore \quad \vec{S}\cdot\hat{z}\,(b*h) = \frac{c}{4\pi}\frac{nE_0^2}{\mu}(b\cdot h)\cos^2(kz-\omega t) \underset{\text{in Gaussian}}{=} I*V$$

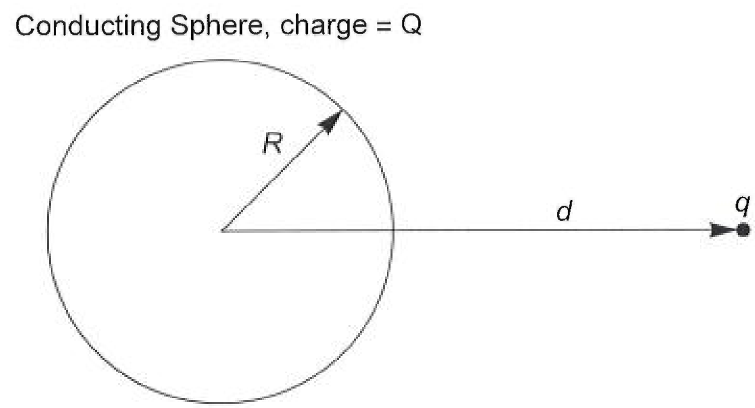
(d) $Z = V(t)/I(t) = \dfrac{E_0 h}{B_0 b/\mu} = \dfrac{E_0 h}{n E_0/c \; b/\mu} = \boxed{\dfrac{h}{b}\sqrt{\dfrac{\mu}{\epsilon}}}$

in Gaussian

$$= \frac{4\pi}{c}\frac{h}{b}\sqrt{\frac{\mu_r}{\epsilon_r}}$$

(e) Same as $b \to 2b$

$$\therefore \quad Z \to Z/2$$



4. Consider a point charge q located a distance d from the center of an isolated (i.e., not grounded) conducting sphere of total charge Q and radius R where $R < d$. Assume the sphere is centered on the coordinate origin and that the point charge q is on the positive z-axis at $\mathbf{r} = d\,\hat{\mathbf{z}}$.

   (a) [3 pt] Using spherical polar coordinates (r, θ, ϕ) give the electrostatic potential Φ inside and outside the sphere assuming the boundary condition $\lim \Phi \to 0$ as $r \to \infty$. Hint: Use image charges.

   (b) [2 pt] Show that your potential is constant on the surface of the sphere.

   (c) [2 pt] Show that the electrostatic force on the point charge q is

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0}\left\{\frac{Q + qR/d}{d^2} - \frac{qR}{d(d - R^2/d)^2}\right\}\hat{\mathbf{z}}.$$

   In Gaussian units $4\pi\epsilon_0 \to 1$.

   (d) [3 pt] Find the configuration energy of the system, i.e., find the total amount of external work required move the static point charge q from $r = \infty$ to $r = d$.

   Hint:

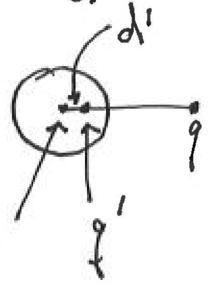
$$\int \frac{dx}{x^3 - a^2 x} = \frac{1}{2(a^2 - x^2)} + \text{constant}.$$

#4.

image at d' image at $r=0$

(a) $$\boxed{\Phi_{out} = \frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} + \frac{q'}{\sqrt{r^2+d'^2-2rd'\cos\theta}} + \frac{Q-q'}{r}}$$

$$q' = -q\frac{R}{d}, \quad d' = R^2/d, \quad \boxed{\Phi_{in} = \frac{Q-q'}{R}}$$

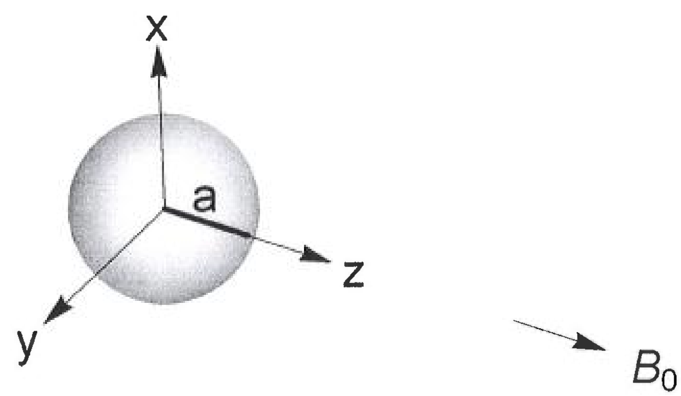


(b) $$\frac{q'}{\sqrt{R^2+d'^2-2Rd'\cos\theta}} \stackrel{?}{=} \frac{-q}{\sqrt{R^2+d^2-2Rd\cos\theta}}$$

plug in d' + q' values to check!

(c) $$\vec{F} = q\left[\frac{q'}{(d-d')^2} + \frac{Q-q'}{d^2}\right]\hat{z}$$ ← plug in q' + d' values

(d) $$U = -\int_{\infty}^{d} \vec{F}\cdot\delta\vec{d} = \boxed{\frac{qQ}{d} - \frac{q^2R^3}{2d^2(d^2-R^2)}}$$



5. A uniform sphere of radius a, made of linear magnetic material of permeability $\mu \neq \mu_0$, is placed in a region of empty space that contains an initially uniform magnetic induction $\mathbf{B} = B_0\,\hat{\mathbf{z}}$. When answering the following use spherical polar coordinates and assume the sphere is centered on the origin.

   (a) [5 pt] Use Legendre polynomials to find the magnetic scalar potential Φ_M both inside and outside the sphere. Recall that the magnetic field $\mathbf{H}$ is related to the magnetic scalar potential by $\mathbf{H} = -\nabla\Phi_M$.

   (b) [3 pt] Find the magnetic induction ($\mathbf{B} = \mu\mathbf{H}$) both inside and outside the sphere.

   (c) [2 pt] Give the magnetization $\mathbf{M}$ inside the sphere.

#5

$r>a$ $r<a$

(a) $\Phi_{1M} = A_1 r\cos\theta + \frac{C_1}{r^2}\cos\theta, \Phi_{2M} = A_2 r\cos\theta + \frac{C_2}{r^2}\cos\theta$

For $r \to \infty$ the magnetic field $\vec{B}_1$ is equal to $B_0\hat{z}$. Thus $\Phi_{1M} = -\frac{B_0 z}{\mu_0} = -\frac{B_0 r\cos\theta}{\mu_0}$

and $A_1 = -\frac{B_0}{\mu_0}$. The potential Φ_{2M} must be finite at the origin. Thus $C_2 = 0$

$$\Phi_{1M} = -\frac{B_0}{\mu_0} r\cos\theta + \frac{C_1}{r^2}\cos\theta \quad \text{and} \quad \Phi_{2M} = A_2 r\cos\theta$$

Boundary conditions at $r = a$:

i. $H_{1\theta} = H_{2\theta} \to \frac{\partial\Phi_{1M}}{\partial\theta} = \frac{\partial\Phi_{2M}}{\partial\theta}$ at $r = a \to -\frac{B_0}{\mu_0} a\sin\theta + \frac{C_1}{a^2}\sin\theta = A_2 a\sin\theta \to$

$-\frac{B_0}{\mu_0} + \frac{C_1}{a^3} = A_2$ (eqs.1)

ii. $B_{1r} = B_{2r} \to \mu_0 \frac{\partial\Phi_{1M}}{\partial r} = \mu \frac{\partial\Phi_{2M}}{\partial r}$ at $r = a \to$

$-B_0\cos\theta - \frac{2\mu_0 C_1}{a^3}\cos\theta = \mu A_2\cos\theta \to$

$B_0 + \frac{2\mu_0 C_1}{a^3} = -\mu A_2$ (eqs.2)

We solve the system of equations 1 and 2 with A_2 and C_1 as unknowns and get:

$C_1 = \frac{B_0 a^3(\mu - \mu_0)}{\mu_0(\mu + 2\mu_0)}$ and $A_2 = -\frac{3B_0}{(\mu + 2\mu_0)}$ Thus we have:

$$\Phi_{1M} = -\frac{B_0}{\mu_0} r\cos\theta + \frac{B_0 a^3(\mu - \mu_0)}{r^2\mu_0(\mu + 2\mu_0)}\cos\theta = -\frac{B_0 z}{\mu_0} + \frac{B_0 a^3(\mu - \mu_0)}{r^2\mu_0(\mu + 2\mu_0)}\cos\theta \qquad r>a$$

$$\Phi_{2M} = -\frac{3B_0}{(\mu + 2\mu_0)} r\cos\theta = -\frac{3B_0 z}{(\mu + 2\mu_0)} \qquad r<a$$

b. $\vec{B}_1 = -\mu_0\vec{\nabla}\Phi_{1M} = B_0\hat{z} - \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{B_0 a^3(\mu - \mu_0)}{r^2(\mu + 2\mu_0)}\cos\theta\right]\hat{\theta} - \frac{\partial}{\partial r}\left[\frac{B_0 a^3(\mu - \mu_0)}{r^2(\mu + 2\mu_0)}\cos\theta\right]\hat{r}$

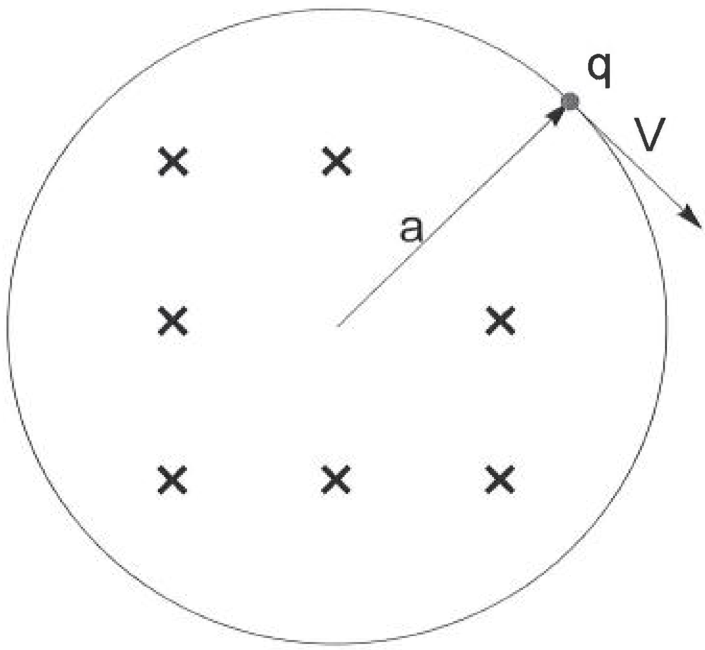
$\vec{B}_1 = B_0\hat{z} + \frac{B_0 a^3(\mu - \mu_0)}{r^3(\mu + 2\mu_0)}\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right] \qquad \vec{B}_2 = -\mu\vec{\nabla}\Phi_{1M} = \frac{3\mu B_0\hat{z}}{(\mu + 2\mu_0)}$

c. In general: $\vec{B} = \mu_0(\vec{H} + \vec{M}) \to \vec{M} = \frac{\vec{B}}{\mu_0} - \vec{H} = \frac{\vec{B}}{\mu_0} - \frac{\vec{B}}{\mu} = \vec{B}\left(\frac{\mu - \mu_0}{\mu\mu_0}\right)$

$$\vec{M} = \vec{B}_2\left(\frac{\mu - \mu_0}{\mu\mu_0}\right) = \frac{3B_0(\mu - \mu_0)\hat{z}}{\mu_0(\mu + 2\mu_0)}$$

6. (a) [3 pt] Write Newton's second law **in 4-dimensional form** for a point charges of mass m and charge q moving with a 4-velocity u^α in an external E&M field described by the Maxwell tensor $F^{\alpha\beta}$.

For the remainder of this problem assume the particle is a fast moving electron with total energy 10 MeV that enters a uniform magnetic induction $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ in the lab and that it moves in a circular orbit of radius $a = 10$ cm, orthogonal to the magnetic induction.



(b) [1 pt] What are the electron's γ and β values?

(c) [3 pt] What is the value of B_0?

(d) [1 pt] Find the time the electron takes to move around a complete circle as seen by a lab observer.

(e) [2 pt] How much total energy does the electron loose as radiation during one complete revolution? Hint: The total power radiated by an accelerated point particle is

$$P(t) = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c}\gamma^4\left[|\dot{\boldsymbol{\beta}}|^2 + \gamma^2(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\right], \qquad (SI)$$

$$P(t) = \frac{2}{3}\frac{q^2}{c}\gamma^4\left[|\dot{\boldsymbol{\beta}}|^2 + \gamma^2(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\right]. \qquad (Gaussian)$$

$m_e c^2 = 0.5 MeV$, 1 eV=1.6×10^{-12} ergs, e=4.8×10^{-10} statcoul= 1.6×10^{-19} Coulombs, 1 erg = 10^{-7} Joules.

6

(a) $\boxed{\frac{d}{d\tau}(m u^\alpha) = \frac{q}{c} F^{\alpha\beta} u_\beta} \Rightarrow \frac{d}{dt}(m u^\alpha) = \frac{q}{c}\gamma^{-1} F^{\alpha\beta} u_\beta$

where $u^\alpha = \gamma(c, \vec{v}) \leftarrow$ 4-velocity

$u_\beta = \gamma(c, -\vec{v})$

$$F^{\alpha\beta} = \left(\begin{array}{c|ccc} 0 & -E^x & -E^y & -E^z \\ \hline E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{array}\right) \quad \text{Gaussian}$$

$$= \left(\begin{array}{c|ccc} 0 & -E^x & -E^y & -E^z \\ \hline E^x & 0 & -cB^z & cB^y \\ E^y & cB^z & 0 & -cB^x \\ E^z & -cB^y & cB^x & 0 \end{array}\right) \quad \text{SI}$$

$$\frac{d}{d\tau} m\gamma c = \gamma \frac{q}{c} \vec{E}\cdot\vec{v} \quad = \frac{d}{dt}(m\gamma c) = \frac{q}{c}\vec{E}\cdot\vec{v}$$

$$\frac{d}{d\tau} m\gamma\vec{v} = \gamma q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \Rightarrow \frac{d}{dt}(m\gamma\vec{v}) = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

$\frac{d}{d\tau} = \gamma\frac{d}{dt}$

Gaussian

no c in SI !

#6 (b) From (a) with no $\vec{E}$

$$\frac{d}{dz}(m\gamma c) = 0 \Rightarrow \gamma + \beta \text{ are constants}$$

$$\boxed{\gamma} = \frac{m\gamma c^2}{mc^2} = \frac{10\,\text{MeV}}{0.5\,\text{MeV}} = \boxed{20}$$

$$\boxed{\beta} = \sqrt{1-\gamma^{-2}} \cong \boxed{1 - \frac{1}{800}}$$

(c) $\frac{d}{dz}(m\gamma\vec{v}) = \gamma q \frac{\vec{v}}{c} \times \vec{B} \;\hookrightarrow\; \frac{d}{dt}(m\gamma\vec{v}) = q\frac{\vec{v}}{c}\times\vec{B}$

$$\therefore \dot{\vec{v}} = \frac{q}{m\gamma}\frac{\vec{v}\times\vec{B}}{c} \qquad (\text{where } \vec{v}\cdot\vec{B}) = 0$$

c is ABSENT in SI !!

$\Rightarrow$ circular motion

$$\Rightarrow |\dot{\vec{v}}| = \frac{q v}{m\gamma}\frac{B_0}{c} \Rightarrow \frac{B_0}{c} = \frac{|\dot{\vec{v}}|}{qv/m\gamma}$$

For circular motion $v = \omega a$, $|\dot{\vec{v}}| = v^2/a$

$$\therefore \frac{B_0}{c} = \frac{v^2/a}{qv/m\gamma} = \frac{m\gamma v}{aq} = \frac{m\gamma c^2\beta}{caq} \qquad (q=e)$$

$$\frac{B_0}{c} = \frac{(10\,\text{MeV})(1-\frac{1}{800})}{c(1.6\times10^{-19}\,C)(0.10\,m)} = \frac{(10^7\,\text{Volts})(1-\frac{1}{800})}{c\,(0.10\,m)}$$

$$\therefore \boxed{B_0} \cong \frac{(10^7\,\text{Volts})}{3\times10^8\,m/s\,(0.10)\,m} = \boxed{\frac{1}{3}\text{ Tesla}} \quad \text{SI}$$

(d) $T = 2\pi/\omega = \frac{2\pi}{v/a} = \frac{2\pi(0.1\,m)}{3\times10^8\,m/s} = \boxed{2.09\times10^{-9}\,s}$

#6 3/3

(e) $P(t) * T = \frac{2 e^2}{4\pi e_0} \frac{\gamma^4}{3c} |\dot{\vec{\beta}}|^2 * (2.09 \times 10^{-9} s)$

$\dot{\vec{\beta}} = \dot{\vec{v}}/c = \beta v/a \cong c/a$

SI

$\frac{1}{4\pi e_0} = (3\times 10^8)^2 \, 10^{-7} \frac{J \cdot m}{C^2}$

$\Rightarrow P(t) * T \cong (3\times 10^8)^2 \, 10^{-7} \frac{J \cdot m}{\cancel{C}^2} \frac{2(1.6\times 10^{-19} \cancel{C})^2 (20)^4 c^{\cancel{2}\,1} * 2.09\times 10^{-9} s}{3 \cancel{(3\times 10^8 m/s)} (0.1 m)^2}$

$= \frac{3^{\cancel{2}\,1} 2 (1.6)^2 \, 2^4 (3)(2.09)}{\cancel{3}} 10^{+16-7-38+4+8-9+2} J$

$\boxed{P(t) * T} \cong 1541 * 10^{-24} J = \boxed{1.54 \times 10^{-21} J}$

$= \boxed{1.54 \times 10^{-14} ergs}$

$1 \text{ erg} = 6.24 \times 10^5 \text{ MeV}$

$= \boxed{9.62 \times 10^{-9} MeV}$