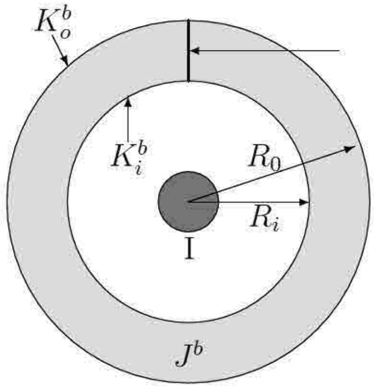
E & M Qualifier

August 16, 2012

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. number every page starting with 1 for each problem,
6. put the total # of pages you use for that problem on every page,

Use only the reference material supplied (Schaum's Guides).



1. A long wire of radius R_{wire} carries a current I and is surrounded by a long hollow iron cylinder. The inner radius of the cylinder is R_i and the outer radius is R_o ($R_{wire} < R_i < R_o$, see the figure, assume the current flows out of the page).

   (a) (2 pts) Compute the flux of $\mathbf{B}$ through a rectangular section of the iron cylinder L meters long and $R_o - R_i$ wide.

   (b) (3 pts) Find the <u>bound</u> surface current densities flowing along the inner and outer iron surfaces, respectively K_i^b and K_o^b, and find the direction of these currents relative to the current in the wire.

   (c) (2 pts) Find the <u>bound</u> volume current density J^b inside the iron.

   (d) (3 pts) Find $\mathbf{B}$ at distances $r > R_0$ from the wire. Would this value of $\mathbf{B}$ be affected if the iron cylinder were removed?

   Recall that the magnetization $\mathbf{M}$ is related to the magnetic field strength $\mathbf{H}$ and the susceptibility χ_m by

$$\begin{aligned}
\mathbf{M} &= \chi_m^{SI}\mathbf{H} && \text{in SI units} \\
&= \chi_m^{G}\mathbf{H} && \text{in Gaussian units} \\
\mathbf{B} &= \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0(1+\chi_m^{SI})\mathbf{H} && \text{in SI units} \\
&= (\mathbf{H}+4\pi\mathbf{M}) = (1+4\pi\chi_m^{G})\mathbf{H} && \text{in Gaussian units}
\end{aligned}$$

   For all substances $4\pi\chi_m^G = \chi_m^{SI}$. For iron χ_m is in the range 10 to 1000.

#1

Red = Gaussian

(a) $\vec{\nabla}\times\vec{H} - \frac{1}{c}\underbrace{\frac{\partial\vec{D}}{\partial t}}_{0} = \frac{4\pi}{c}\vec{J}$ Green = SI

(ρ,ϕ,z) coords

+z out of page

$\Rightarrow$

$$\oint \vec{H}\cdot d\vec{l} = \frac{4\pi}{c} I_{in}$$

$$\Rightarrow H = \frac{4\pi}{c}\frac{I}{2\pi\rho} \quad \rho > R_{wire} \quad (\text{in } \hat{\phi} \text{ direction})$$

$$\mu_0(H + 4\pi M) = B = \mu H \Rightarrow \boxed{B = \frac{4\pi}{c}\frac{\mu}{2\pi}\frac{I}{\rho}}$$

$$\phi_B = \int \vec{B}\cdot d\vec{a} = \frac{4\pi}{c}\frac{\mu}{2\pi} I L \int_{R_i}^{R_o} \frac{d\rho}{\rho}$$

$$\boxed{\phi_B = \frac{4\pi}{c}\frac{\mu}{2\pi} I L \ln R_o/R_i}$$

(b) $\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}_{Free}$

$$\vec{H} \downarrow \left(\frac{B}{\mu_0} - 4\pi M\right), \qquad \frac{\partial\vec{D}}{\partial t} \downarrow 0$$

$$\therefore \vec{\nabla}\times\vec{B} = \mu_0\frac{4\pi}{c}\left(\vec{J}_{free} + \underbrace{c\,\vec{\nabla}\times\vec{M}}_{\vec{J}_{bound}}\right)$$

1 cont

$$\therefore \oint \vec{B}\cdot d\vec{l} = \mu_0 \frac{4\pi}{c}\left(I_{Free} + I_{bound}\right)_{\text{inside loop}}$$

$$I_{bound} = c\oint \vec{M}\cdot d\vec{l} = c\,\underline{M(\rho)}\,2\pi\rho$$

$$\boxed{I_{bound} = 0 \quad \boxed{\text{for } \rho < R_i}}$$

$$= c\,\underline{\chi H(\rho)}\,2\pi\rho \,,\; R_i < \rho < R_o$$

$$= c\,\underline{\chi \frac{4\pi}{c}\frac{I_F}{2\pi\rho}}\,2\pi\rho$$

a constant

$$\boxed{I_{bound} = 4\pi\chi\, I_{Free} \quad \boxed{R_i < \rho < R_o}}$$

$$\boxed{I_{bound} = 0 \quad \text{for} \quad \boxed{R_o < \rho}}$$

$$\therefore \boxed{\begin{array}{ll} K_i^b = \dfrac{4\pi\chi\, I_F}{2\pi R_i} & \text{same direction as } I \\[2ex] K_o^b = -\dfrac{4\pi\chi\, I_F}{2\pi R_o} & \text{opposite direction to } I \end{array}}$$

$$4\pi\chi_m^{G} = \chi_m^{SI}$$

(c) #1 cont

From (b) $\vec{J}_{bound} = c\vec{\nabla} \times \vec{M}$

$\downarrow$

$\chi \vec{H}$

$= c\chi \underbrace{\vec{\nabla} \times \vec{H}}_{= 0}$ in iron

(no $\vec{J}_{Free}$ there)

$\therefore$ $\boxed{\vec{J}_{bound} = 0 \text{ inside pipe}}$

(d) $\vec{B} = \mu\vec{H} = \mu_0 \vec{H}$ for $\rho > R_0$

$\downarrow$

$$\boxed{\vec{B} = \mu_0 \frac{4\pi}{c} \frac{I}{2\pi\rho}}$$

Same with or without iron pipe

2. (a) (3 pts) From Maxwell's Equations, derive the wave equation for $\mathbf{E}$ with no sources ($\rho = 0, \mathbf{J} = 0$) in a homogeneous, isotropic, linear medium of permittivity ϵ and permeability μ.

   (b) (1 pts) Show that if $\mathbf{E} = E(t, z)\,\hat{\mathbf{y}}$, the wave equation reduces to

$$\begin{aligned} \frac{\partial^2 E}{\partial z^2} &= \epsilon\mu \frac{\partial^2 E}{\partial t^2}, && \text{in SI units} \\ \frac{\partial^2 E}{\partial z^2} &= \frac{\epsilon\mu}{c^2} \frac{\partial^2 E}{\partial t^2}. && \text{in Gaussian units} \end{aligned}$$

   (c) (4 pts) By introducing the change of variables

$$\begin{aligned} \xi &= t + \sqrt{\epsilon\mu}\, z, && \text{in SI units} \\ \xi &= ct + \sqrt{\epsilon\mu}\, z, && \text{in Gaussian units} \\ \eta &= t - \sqrt{\epsilon\mu}\, z, && \text{in SI units} \\ \eta &= ct - \sqrt{\epsilon\mu}\, z, && \text{in Gaussian units} \end{aligned}$$

   show that the wave equation assumes a form that is easily integrated.

   (d) (2 pts) Integrate the equation to obtain

$$E(z, t) = E_1(\xi) + E_2(\eta),$$

   where E_l and E_2 are arbitrary functions.

#2

$$\vec{\nabla}\cdot\vec{D}=0 \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{H}-\frac{1}{c}\frac{\partial\vec{D}}{\partial t}=0 \qquad \vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})+\frac{1}{c}\vec{\nabla}\times\frac{\partial B}{\partial t}=0$$

$$\underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})}_{=\vec{\nabla}\left(\vec{\nabla}\cdot\frac{D}{\epsilon}\right)=0}-\nabla^2\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}\underbrace{(\nabla\times\mu H)}_{=\left(\frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}\right)}=0$$

(a) $\therefore$ $$\boxed{-\nabla^2\vec{E}+\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}=0}$$

$$\Rightarrow \quad \Box\,\vec{E}=0$$

(b) In rectangular coords - if $\vec{E}=E(t,z)\,\hat{y}$

$$-\frac{\partial^2}{\partial z^2}E(t,z)+\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}E(t,z)=0$$

$\therefore$ $$\boxed{\frac{\partial^2}{\partial z^2}E=\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}E}$$

(c) $\xi = ct + \sqrt{\epsilon\mu}\, z$

$\eta = ct - \sqrt{\epsilon\mu}\, z$

$$\frac{\partial}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} = \sqrt{\epsilon\mu}\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)$$

$$\frac{1}{c}\frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{1}{c}\frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)$$

$$\Rightarrow -\left(\frac{\partial}{\partial z}\right)^2 + \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}$$

$$= 4\epsilon\mu \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}$$

$$\therefore \boxed{\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} E(t,z) = 0}$$

(d) $E(t,z) = \underbrace{E_1(\xi) + E_2(\eta)}$

arbitrary function

$$\boxed{E(t,z) = E_1\left(ct + \sqrt{\epsilon\mu}\, z\right) + E_2\left(ct - \sqrt{\epsilon\mu}\, z\right)}$$

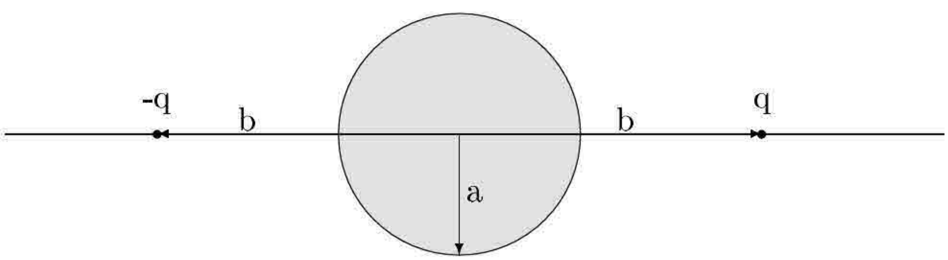
3. Two charges $\pm q$ are on opposite sides of a dielectric sphere (ϵ = constant) as shown in the figure. The three objects are on a common axis, the sphere is of radius a and the two charges are a distance $b > a$ from the sphere's center.

    (a) (2 pts) Give the form of potential $\Phi(r, \theta)$ inside the sphere ($r < a$) as a series of Legendre polynomials, $P_\ell(\cos\theta)$, with coefficients A_ℓ. Give the correct r dependence of each term and do not include ℓ values that vanish from symmetry.

    (b) (2 pts) Give the form of the potential $\Phi(r, \theta)$ outside the sphere ($r > a$) as the sum of two terms; one a series of Legendre polynomials with coefficients B_ℓ caused by the polarization charges on the dielectric, and the other term caused by the two point charges. In the series part keep only non-vanishing ℓ values and give the correct r dependence of each term.

    (c) (3 pts) In the outside region where $r > a$, expand the part of the part of the potential caused by the point charges as a single series in P_ℓ. Give two explicit forms of this series, one good for $a < r < b$ and one good for $r > b$.

    (d) (3 pts) You do not have to evaluate the constants A_ℓ and B_ℓ but write down the two sets of equations from which you can determine them (the boundary matching conditions).



#3 (choose coords $r\theta\phi$ with $r=0$ at center of sphere + $+q$ on the $+z$ axis!)

(a) $r < a$

$$\Phi(r,\theta) = \sum_{l=1}^{\infty} A_l r^l P_l(\cos\theta)$$

↑ odd only

because $\Phi(r, \pi-\theta) = -\Phi(r,\theta)$

$$P_l(\cos(\pi-\theta)) = (-1)^l P_l(\cos\theta)$$

(b) For $r>a$

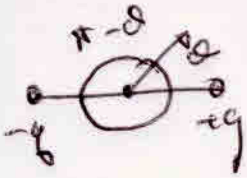
$$\Phi(r,\theta) = \sum_{\substack{l=\text{odd}\\ \text{only}}} \frac{B_l}{r^{l+1}} P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-b\hat{z}|} - \frac{1}{|\vec{r}+b\hat{z}|}\right)$$

(c) $$\frac{1}{|\vec{r}-b\hat{z}|} = \sum_{l=0}^{\infty} \frac{b^l}{r^{l+1}} P_l(\cos\theta) \quad b<r$$

$$= \sum_{l=0}^{\infty} \frac{r^l}{b^{l+1}} P_l(\cos\theta) \quad r<b$$

#3 cont

$$\frac{1}{|\vec r + b\hat z|} = \sum_{l=0}^{\infty} \frac{b^l}{r^{l+1}} P_l(\cos(\pi-\theta)) \qquad b<r$$



$$= \sum_{l=0}^{\infty} \frac{r^l}{b^{l+1}} P_l(\cos(\pi-\theta)) \qquad r<b$$

$$P_l(\cos(\pi-\theta)) = P_l(-\cos\theta) = (-1)^l P_l(\cos\theta)$$

$$\left(\frac{1}{|\vec r - b\hat z|} - \frac{1}{|\vec r + b\hat z|}\right) = 2\sum_l \frac{b^l}{r^{l+1}} P_l(\cos\theta) \qquad b<r$$

only odd l's

$$= 2\sum_{l\ \text{odd}} \frac{r^l}{b^{l+1}} P_l(\cos\theta) \qquad r<b$$

$$\frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec r - b\hat z|} - \frac{1}{|\vec r + b\hat z|}\right)$$

$$= \frac{2q}{4\pi\epsilon_0} \sum_{l\ \text{odd}} \frac{r^l}{b^{l+1}} P_l(\cos\theta) \qquad r<b$$

(c)

$$\left(\frac{b^l}{r^{l+1}}\right) \qquad \text{for } r>b$$

(d) Φ is continuous at $r = a$ (same as tangential E is continuous)

$\Phi_{in} = \Phi_{out}$ $P_\ell(\cos\theta)$ component

$$\Rightarrow \boxed{A_\ell a^\ell = \frac{B_\ell}{a^{\ell+1}} + \frac{2q}{4\pi\epsilon_0}\frac{a^\ell}{b^{\ell+1}} \quad (\ell = \text{odd})}$$

$\epsilon \frac{\partial \Phi}{\partial r}$ is continuous at $r = a$ (same as normal $\vec{D}$ is continuous)

$\epsilon \frac{\partial \Phi_{in}}{\partial r} = \epsilon \frac{\partial \Phi_{out}}{\partial r}$ $P_\ell(\cos\theta)$ component

$$\Rightarrow \boxed{\epsilon \ell A_\ell a^{\ell-1} = \epsilon_0\left(-\frac{(\ell+1)B_\ell}{a^{\ell+2}} + \frac{2q}{4\pi\epsilon_0}\frac{\ell(a)^{\ell-1}}{b^{\ell+1}}\right) \quad (\ell = \text{odd})}$$

4. The reflection of a circularly polarized plane wave at a metallic boundary.

   (a) (2 pts) Give expressions for the **E** and **B** fields of a monochromatic, right circularly polarized plane wave traveling in vacuum. Use rectangular Cartesian coordinates, assume the angular frequency is ω, assume the polarization plane is the x–y plane, and assume the propagation direction is in the positive z direction.

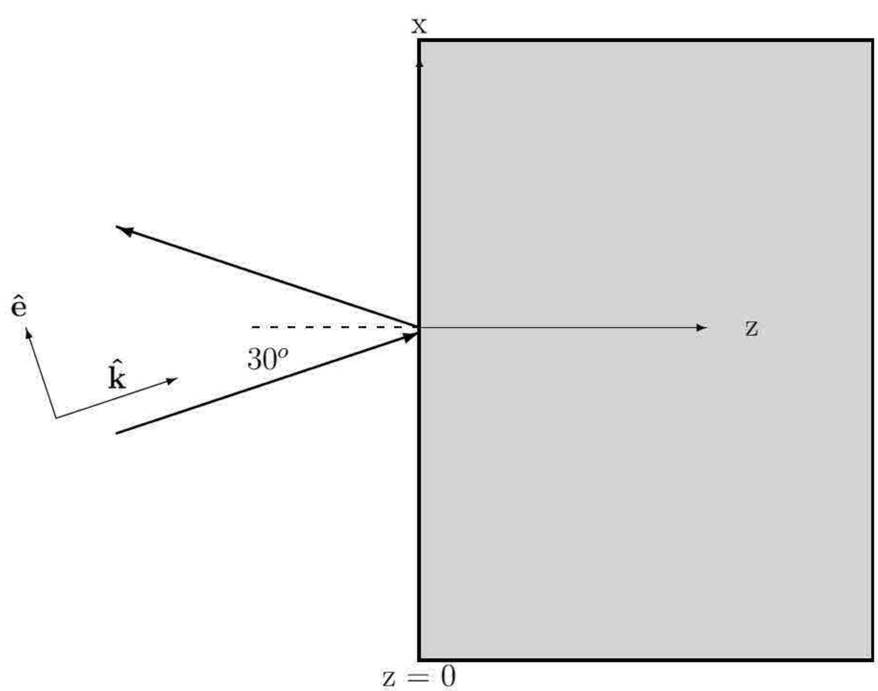
   (b) (1 pt) Explain in words what is meant by a monochromatic right circularly polarized wave.

   (c) (2 pts) Rewrite your **E** and **B** fields of part (a) assuming the propagation direction is 30^o above the $\hat{\mathbf{z}}$ direction as shown in the figure. You can use unit vectors $\hat{\mathbf{e}}$ and $\hat{\mathbf{k}}$ in your expressions but be sure to define what they are in terms of the coordinate directions $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

   (d) (2 pts) If the wave of part (c) strikes a flat perfectly conducting surface at $z = 0$ it will be reflected. What boundary conditions are satisfied by the combined **E** and **B** fields of the incoming and reflected waves at the $z = 0$ junction?

   (e) (2 pts) Give expressions for the reflected **E** and **B** fields. Make sure they satisfy your junction conditions of part (d).

   (f) (1 pt) Is the reflected wave right or left circularly polarized?



4

Real Part of !

Red = Gaussian Green = SI

(a) $\vec{E} = (\hat{x} - i\hat{y}) E_0 e^{i(\frac{z}{c} - t)\omega}$ RCP wave

$\vec{B} = i(\hat{x} - i\hat{y}) \frac{E_0}{c} e^{i(\frac{z}{c} - t)\omega}$

$\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$ checks phases of $\vec{B}$

(b) When facing the on-coming wave, the electric field rotates clockwise with constant magnitude and constant angular frequency.

(c) $\vec{E} = (\hat{e} - i\hat{y}) E_0 e^{i(\frac{\vec{r}\cdot\hat{k}}{c} - t)\omega}$

$\vec{B} = i(\hat{e} - i\hat{y}) \frac{E_0}{c} e^{i(\frac{\vec{r}\cdot\hat{k}}{c} - t)\omega}$

where $\hat{e} = \cos 30^\circ \hat{x} - \sin 30^\circ \hat{z} = \frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{z}$

$\hat{k} = \sin 30^\circ \hat{x} + \cos 30^\circ \hat{z} = \frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{z}$

(d) The tangential component of the total $\vec{E}$ field must vanish and the normal component of the total $\vec{B}$ field must vanish. ($\vec{E}+\vec{B}$ in the conductor vanishes) but surface charges + currents can exist)

(e) $$\vec{E}_{Ref} = (\hat{e}_R + i\hat{y}) E_0 e^{i\left(\frac{\vec{r}\cdot\hat{k}_R}{c} - t\right)\omega}$$

$$\vec{B}_{ref} = -i(\hat{e}_R + i\hat{y}) \frac{E_0}{c} e^{i\left(\frac{\vec{r}\cdot\hat{k}_R}{c} - t\right)\omega}$$

where $\hat{e}_R = -\cos 30^\circ \hat{x} - \sin 30^\circ \hat{z} = -\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{z}$

$\hat{k}_R = \sin 30^\circ \hat{x} - \cos 30^\circ \hat{z} = \frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{z}$

(f) Left circularly Polarized

$$\vec{E}_{tan} = (\cos 30\,\hat{x} - i\hat{y})\,E_0\, e^{i\left(\frac{x}{2c} - t\right)\omega}$$

$$\vec{B}_{nomal} = i\,(-\sin 30^\circ\,\hat{z})\,\frac{E_0}{c}\, e^{i\left(\frac{x}{2c} - t\right)\omega}$$

$$\vec{E}_{R\,tan} = (-\cos 30^\circ\,\hat{x} + i\hat{y})\,E_0\, e^{i\left(\frac{x}{2c} - t\right)\omega}$$

$$\vec{B}_{R\,nomal} = -i\,(-\sin 30^\circ\,\hat{z})\,\frac{E_0}{c}\, e^{i\left(\frac{x}{2c} - t\right)\omega}$$

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above cancel to satisfy boundary conditions

5. An infinitely long, uniformly charged wire of radius a and total charge per unit length λ, is at rest on the z-axis of the lab frame.

   (a) (2 pts) Compute the electric field $\mathbf{E}(x, y, z)$ interior and exterior to the wire in the lab frame by solving Gauss's law in that frame.

   (b) Complete the next 4 steps to compute $\mathbf{E}'(x', y', z')$ and $\mathbf{B}'(x', y', z')$ in a frame moving in the positive z-direction with speed v.

      i. (2 pts) Give the Lorentz boost $x'^{\sigma} = L^{\sigma}_{\mu} x^{\mu}$ ($\mathbf{x}' = \mathbf{L}\mathbf{x}$) from the Lab to the moving frame (take $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$).

      ii. (2 pts) Construct the electromagnetic field tensor $F^{\alpha\beta}$ from the electric field you found in part (a).

      iii. (2 pts) Use your lorentz boost to compute the electromagnetic field tensor $F'^{\alpha\beta} = L^{\alpha}_{\mu} L^{\beta}_{\nu} F^{\mu\nu}$ ($\mathbf{F}' = \mathbf{L}\mathbf{F}\mathbf{L}^{\mathbf{T}}$) in the moving frame.

      iv. (2 pts) From your $F'^{\alpha\beta}$ give the answer to (b).

   Hint: Recall that in both SI and Gaussian units $F^{\sigma\mu} = -F^{\mu\sigma}$ and $F^{0i} = -E^{i}$. In Gaussian units $F^{12} = -B^{z}$, $F^{23} = -B^{x}$ and $F^{13} = B^{y}$, but in SI units $F^{12} = -c\,B^{z}$, $F^{23} = -c\,B^{x}$ and $F^{13} = c\,B^{y}$

#5 Green = SI Red = Gaussian

$$\boxed{\vec{E}(\rho) = \frac{2\lambda(\rho)}{4\pi\epsilon_0\, \rho}\hat{\rho}} \qquad \rho \equiv \sqrt{x^2+y^2}$$

λ inside ρ from Gauss's Law

(a)

$$\lambda(\rho) = \lambda \frac{\pi\rho^2}{\pi a^2} = \lambda \rho^2/a^2 \qquad \rho \le a$$

$$= \lambda \qquad \rho \geq a$$

(b)

(i) $$L^\alpha{}_\beta = L = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = v/c$$

$x' = Lx$ $\qquad x^{\alpha\prime} = L^\alpha{}_\beta x^\beta$

moving coord $\qquad$ lab coord

$$ct' = \gamma(ct - \beta z)$$
$$x' = x$$
$$y' = y$$
$$z' = \gamma(-\beta ct + z)$$

(ii)

$$F^{\alpha\beta} = \left(\begin{array}{c|c} 0 & -\vec{E} \\ \hline +\vec{E} & -\epsilon^{isk} c B^k \end{array}\right) \qquad \vec{E} = E(\rho)[\cos\phi\,\hat{x} + \sin\phi\,\hat{y}] \qquad \vec{B} = 0$$

$$= E(\rho)\left(\begin{array}{c|ccc} 0 & -\cos\phi & -\sin\phi & 0 \\ \hline \cos\phi & & & \\ \sin\phi & & 0 & \\ 0 & & & \end{array}\right)$$

$$E(\rho) \equiv \frac{2\lambda(\rho)}{4\pi\epsilon_0\,\rho}$$

(iii) $F' = \underline{L} F L^T$

$$= E(\rho)\begin{pmatrix} 0 & -\gamma\cos\phi & -\gamma\sin\phi & 0 \\ \cos\phi & 0 & 0 & 0 \\ \sin\phi & 0 & 0 & 0 \\ 0 & \beta\gamma\cos\phi & \beta\gamma\sin\phi & 0 \end{pmatrix} L^T$$

$$F' = E(\rho)\begin{pmatrix} 0 & -\gamma\cos\phi & -\gamma\sin\phi & 0 \\ \gamma\cos\phi & 0 & 0 & -\beta\gamma\cos\phi \\ \gamma\sin\phi & 0 & 0 & -\beta\gamma\sin\phi \\ 0 & \beta\gamma\cos\phi & \beta\gamma\sin\phi & 0 \end{pmatrix}$$

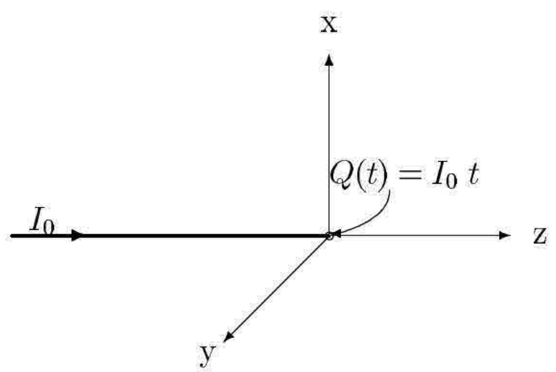
(iv) $\therefore$ $\vec{E}' = \gamma E(\rho)[\cos\phi\,\hat{x}' + \sin\phi\,\hat{y}'] = \gamma E(\rho)\,\hat{\rho}'$

$c\vec{B}' = \beta\gamma E(\rho)[\sin\phi\,\hat{x}' - \cos\phi\,\hat{y}'] = \beta\gamma E(\rho)(-\hat{\phi}')$

6. In the absence of polarizable and/or magnetizable material (i.e., only free charges and currents present) Maxwell's equations, in Gaussian units and in the Lorentz gauge, reduce to the inhomogeneous wave equation:

$$\Box \left\{ \begin{array}{c} \Phi \\ A^x \\ A^y \\ A^z \end{array} \right\} = \frac{4\pi}{c} \left\{ \begin{array}{c} c\rho \\ J^x \\ J^y \\ J^z \end{array} \right\}, \text{ where } \Box \equiv \left(\frac{\partial}{c\partial t}\right)^2 - \nabla^2.$$

A time dependent charge $Q(t) = I_0\ t,\ t \geq 0$ is fixed at the origin



of a cylindrical polar coordinate system (ρ, ϕ, z) The charge increases linearly with time because a constant current I_0 flows in along a thin wire attached to the charge on its left, see the figure. Assume the wire carries no current for $t < 0$, however, at $t = 0$ a current I_0 abruptly starts flowing in the +z direction and remains constant for $t \geq 0$. Assume the wire remains neutral as the charge at the origin grows. Find the following quantities at time t for points (ρ, ϕ, z):

(a) (2 pts) The charge density $\rho(t, \rho, \phi, z)$,

(b) (2 pts) The current density $\mathbf{J}(t, \rho, \phi, z)$,

(c) (2 pts) The retarded scalar potential $\Phi(t, \rho, \phi, z)$,

(d) (4 pts) The retarded vector potential $\mathbf{A}(t, \rho, \phi, z)$.

Hints: Parts (a) and (b) require the use of $\delta(x)$-functions and Heaviside step functions $\Theta(x) \equiv 1, 0$ respectively for $x > 0$ or $x < 0$. The retarded Green's function for the $\Box$ operator is:

$$G^{ret}(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta\left(t - t' - |\mathbf{r} - \mathbf{r}'|/c\right)}{4\pi\,|\mathbf{r} - \mathbf{r}'|},$$

which gives retarded potentials

$$\Big(\Phi(t, \mathbf{r}), \mathbf{A}(t, \mathbf{r})\Big)^{ret} = \frac{1}{c}\int \frac{\Big(c\rho(t - |\mathbf{r} - \mathbf{r}'|/c, \mathbf{r}'), \mathbf{J}(t - |\mathbf{r} - \mathbf{r}'|/c, \mathbf{r}')\Big)}{|\mathbf{r} - \mathbf{r}'|} d^3r'.$$

For part (d) you might need the integral

$$\int \frac{dX}{\sqrt{X^2 + a^2}} = \ln(\sqrt{X^2 + a^2} + X).$$

(#6)

(a) Done in Gaussian

$$\rho(t,\vec{r}) = I_0 t \ \Theta(t)\,\delta^3(\vec{r})$$

$$= I_0 t \ \Theta(t) \frac{\delta(\rho)\,\delta(z)}{2\pi\rho}$$

(b) $$\vec{J}(t,\vec{r}) = I_0 \Theta(t)\,\Theta(-z)\frac{\delta(\rho)}{2\pi\rho}\,\hat{z}$$

(c) $$\Phi^{ret}(t,\vec{r}) = \int \frac{I_0\left(t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\Theta\left(t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\delta^3(\vec{r}')\,d^3r'}{|\vec{r}-\vec{r}'|}$$

$$\boxed{\Phi^{ret}(t,\vec{r}) = \frac{I_0\,(t-r/c)\,\Theta(t-r/c)}{r}}$$

(d) $$\vec{A}(t,\vec{r}) = \frac{\hat{z}}{c}\int \frac{I_0\,\Theta\left(t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}\Theta(-z')\frac{\delta(\rho')}{2\pi\cancel{\rho'}}\,\cancel{\rho'}\,d\rho'\,d\phi'\,dz'$$

$$|\vec{r}-\vec{r}'| = \sqrt{(z-z')^2+\rho^2} \quad \text{(when } \rho'=0\text{)}$$

$$\therefore \vec{A}(t,\vec{r}) = \frac{\hat{z}\,I_0}{c}\int_{-\infty}^{0}\frac{\Theta\left(t-\frac{\sqrt{(z-z')^2+\rho^2}}{c}\right)}{\sqrt{(z-z')^2+\rho^2}}\,dz'$$

#6 cont

$$\vec{A}(t,\vec{r}) = \frac{\hat{z}\, I_0}{c} \int_{z_{min}}^{z_{max}} \frac{dz'}{\sqrt{(z-z')^2+\rho^2}}$$

$$= \frac{\hat{z}\, I_0}{c} \ln\left(\sqrt{(z-z')^2+\rho^2} + (z'-z) \right) \Bigg|_{z_{min}}^{z_{max}}$$

if $ct>\rho$ and $z-\sqrt{(ct)^2-\rho^2}<0$

$= 0$ if $ct<\rho$ or $z-\sqrt{(ct)^2-\rho^2}>0$

$$z_{min} = z - \sqrt{(ct)^2-\rho^2}$$

$$z_{max} = \begin{cases} z+\sqrt{(ct)^2-\rho^2} & \text{if } z+\Gamma<0 \\ 0 & \text{if } z+\Gamma>0 \end{cases}$$