

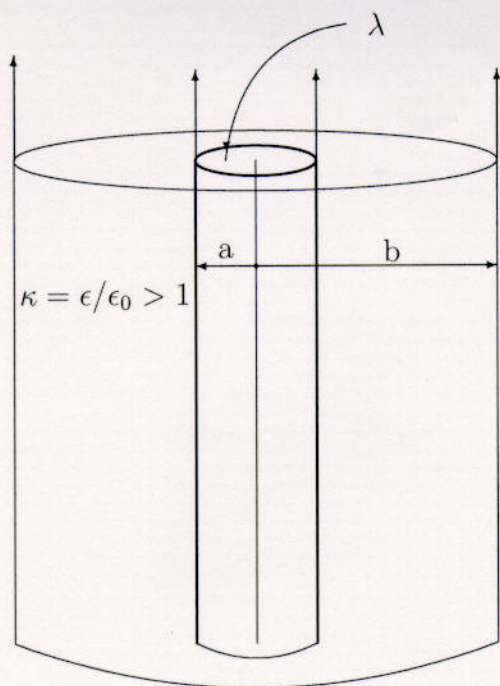
E & M Qualifier

August XX, 2010

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



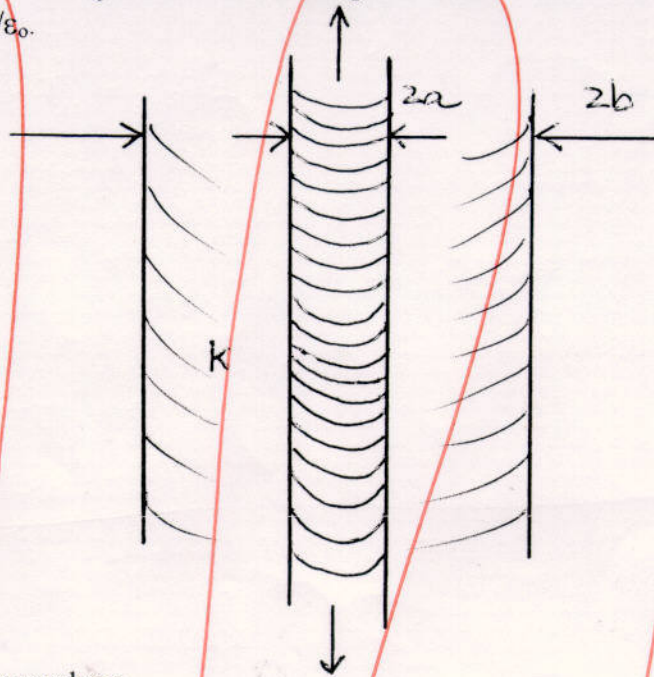
1. A very long conducting wire of radius a , carrying free positive charge per unit length λ , is surrounded by a dielectric coating of outside radius b and relative dielectric constant $\kappa = \epsilon/\epsilon_0$.
 - (a) {2 pts} Find the displacement vector \mathbf{D} everywhere.
 - (b) {2 pts} Find the electric field \mathbf{E} everywhere.
 - (c) {2 pts} Find the polarization density \mathbf{P} everywhere.
 - (d) {2 pts} Find the bound volume charge density ρ_b and the bound surface charge density σ_b everywhere.
 - (e) {2 pts} Show that the total charge densities, bound and free, produce the same \mathbf{E} found in (a).

#1

RD -1

#1

A very long cylindrical conducting wire of radius a , carrying free positive charge per unit length λ is surrounded by a dielectric coating of outside radius b . The relative dielectric constant is $\kappa = \epsilon/\epsilon_0$.



points



- 2 (a) Find \mathbf{D} everywhere.
- 2 (b) Now find from your result in (a) \mathbf{E} everywhere.
- 2 (c) From your results in (a) and (b) find the polarization \mathbf{P} .
- 2 (d) Now find the bound volume charge density ρ_b and the bound surface charge density σ_b everywhere.
- 2 (e) Show that the total charge densities, bound and free, now lead to the same \mathbf{E} found in part (a).

Solution

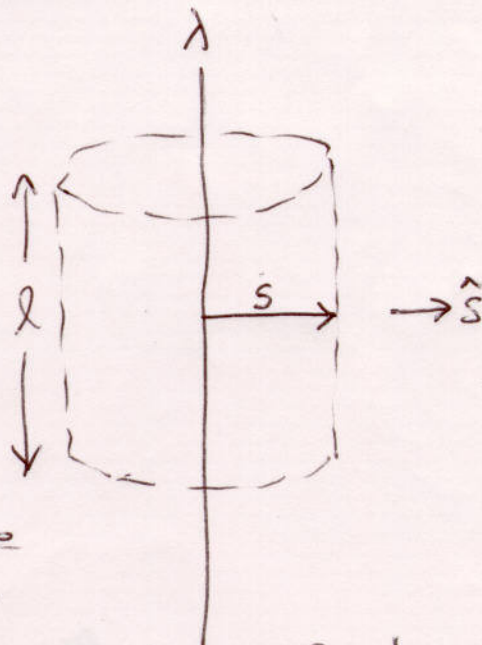
(a) Gauss: $\oint \vec{D} \cdot d\vec{a} = Q_f^{\text{enc}} = \lambda \ell$

$$\vec{D} = D(s) \hat{s}$$

$$\therefore \oint \vec{D} \cdot d\vec{a} = D \int da = D(2\pi s \ell)$$

$$\Rightarrow \vec{D} = \frac{\lambda \ell}{2\pi s \ell} \hat{s} = \frac{\lambda}{2\pi s} \hat{s} \text{ outside wire.}$$

inside conductor $\vec{D} = 0$ λ is on the surface!



$$(b) \quad s > b: \vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s}$$

RD cont - 2

$$a < s < b: \vec{E} = \frac{\vec{D}}{\epsilon} = \frac{\vec{D}}{K\epsilon_0} = \frac{\lambda}{2\pi K\epsilon_0 s} \hat{s}$$

$$s < a: \vec{E} = 0$$

$$(c) \quad \vec{P} = \vec{D} - \epsilon_0 \vec{E}$$

$$\Rightarrow s > b: \vec{P} = \frac{\lambda \hat{s}}{2\pi s} - \epsilon_0 \frac{\lambda}{2\pi\epsilon_0 s} \hat{s} = 0$$

$$a < s < b: \vec{P} = \frac{\lambda}{2\pi s} \hat{s} - \frac{\epsilon_0 \lambda}{2\pi K\epsilon_0 s} \hat{s} = \frac{\lambda \hat{s}}{2\pi s} \left(1 - \frac{1}{K}\right)$$

$$s < a: \vec{P} = 0$$

$$(d) \quad P_b = -\nabla \cdot \vec{P} \Rightarrow s > b \quad P_b = 0$$

$$a < s < b \quad P_b = -\frac{1}{s} \frac{\partial}{\partial s} (s P_s) = -\frac{1}{s} \frac{\partial}{\partial s} \left(\frac{\lambda}{2\pi} \left(1 - \frac{1}{K}\right) \right) = 0$$

$$s < a \quad P_b = 0$$

So $P_b = 0$ everywhere.

$$\text{at } s=b \quad \sigma_b = \vec{P} \cdot \hat{n} = \vec{P} \cdot \hat{s} = \frac{\lambda \hat{s}}{2\pi s} \left(1 - \frac{1}{K}\right) \cdot \hat{s} \Big|_{s=b} = \frac{\lambda}{2\pi b} \left(1 - \frac{1}{K}\right)$$

$$\text{at } s=a \quad \sigma_b = \vec{P} \cdot \hat{n} = \vec{P} \cdot (-\hat{s}) = \frac{\lambda \hat{s}}{2\pi s} \left(1 - \frac{1}{K}\right) \cdot (-\hat{s}) \Big|_{s=a} = -\frac{\lambda}{2\pi a} \left(1 - \frac{1}{K}\right)$$

(e) use Gauss: $\int \vec{E} \cdot d\vec{a} = \frac{Q_{\text{encl.}}}{\epsilon_0}$ for Gaussian cylinder of (radius s): (length l)

$$s > b: Q_{\text{encl.}}^{\text{TOTAL}} = \lambda l - \frac{\lambda}{2\pi a} \left(1 - \frac{1}{K}\right) (2\pi a l) + \frac{\lambda}{2\pi b} \left(1 - \frac{1}{K}\right) (2\pi b l) = \lambda l$$

$$\Rightarrow \vec{E} = \frac{\lambda l \hat{s}}{\epsilon_0 (2\pi s l)} = \frac{\lambda \hat{s}}{2\pi\epsilon_0 s} \checkmark$$

$$a < s < b: Q_{\text{encl.}}^{\text{TOTAL}} = \lambda l - \frac{\lambda}{2\pi a} \left(1 - \frac{1}{K}\right) (2\pi a l) = \lambda l \left(1 - 1 + \frac{1}{K}\right) = \frac{\lambda l}{K}$$

$$\Rightarrow \vec{E} = \frac{(\lambda l / K) \hat{s}}{\epsilon_0 (2\pi s l)} = \frac{\lambda}{2\pi K\epsilon_0 s} \hat{s} \checkmark$$

$$s < a: \text{inside conductor } Q_{\text{encl.}}^{\text{TOTAL}} = 0 \Rightarrow \vec{E} = 0 \checkmark$$

2. (a) {2 pts} In a homogeneous, linear and isotropic conducting material whose electromagnetic properties (at low frequencies) are described by constant values of the permittivity, permeability, and conductivity respectively ϵ , μ , and σ , show that Maxwell's equations require that the electric field satisfy

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0, \quad (SI)$$

$$\nabla^2 \mathbf{E} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \mathbf{E}}{\partial t} = 0. \quad (Gaussian)$$

- (b) {2 pts} Given a plane polarized plane wave of angular frequency ω whose electric field is of the form

$$\mathbf{E}(z, t) = \text{Real} \{ \hat{\mathbf{i}} E_0 e^{i(kz - \omega t)} \},$$

evaluate k^2 as a function of ϵ , μ , σ , and ω .

- (c) {2 pts} Find the real and imaginary parts of k assuming $\sigma \gg \omega\epsilon$.
 (d) {2 pts} Using your results from (c) find the skin depth δ of the conductor. The skin depth is defined by the depth at which the wave's amplitude decreases by e^{-1} , i.e.,

$$\frac{|\mathbf{E}(z + \delta, t)|}{|\mathbf{E}(z, t)|} = \frac{1}{e}$$

- (e) {2 pts} Using Maxwell's equations, find the magnetic field $\mathbf{H}(x, t)$ associated with $\mathbf{E}(z, t)$ given in (b) and discuss their phase difference when $\sigma \gg \omega\epsilon$.

#2

Red = Gaussian

1/3

Faraday's $\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$

Ampere's $\vec{\nabla} \times \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c} \vec{J} = \frac{4\pi}{c} \sigma \vec{E}$

$$\frac{1}{c} \frac{\partial}{\partial t} (\downarrow) = \vec{\nabla} \times \frac{\partial \vec{H}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \vec{D}}{\partial t^2} = \frac{4\pi}{c^2} \sigma \frac{\partial \vec{E}}{\partial t}$$

↓

$$\vec{\nabla} \times \frac{\partial}{\partial t} \left(\frac{\vec{B}}{\mu} \right) - \frac{\epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{4\pi}{c^2} \sigma \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (-\vec{\nabla} \times \vec{E}) - \frac{\mu \epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{4\pi}{c^2} \sigma \mu \frac{\partial \vec{E}}{\partial t}$$

||

$$-\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) + \nabla^2 \vec{E}$$

↓

$$\overset{0}{\text{from}} \vec{\nabla} \cdot \vec{D} = 4\pi \rho = 0$$

(a) ∴ $\left[\nabla^2 \vec{E} - \frac{\mu \epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi \sigma \mu}{c^2} \frac{\partial \vec{E}}{\partial t} \right]$

(b) put into (a) result 2/3

$$-k^2 \hat{E}_0 e^{i(\dots)} - \frac{\mu \epsilon}{c^2} (-\omega^2) \hat{E}_0 e^{i(\dots)} + \frac{4\pi \sigma \mu \omega}{c^2} \hat{E}_0 e^{i(\dots)} = 0$$

$$\therefore -k^2 + \omega^2 \frac{\mu \epsilon}{c^2} + \frac{4\pi \sigma \mu \omega}{c^2} i = 0$$

$$\therefore k^2 = \left(\frac{\omega}{c} \right)^2 \mu \epsilon \left[1 + i \frac{4\pi \sigma}{\omega \epsilon} \right]$$

(d) if $\sigma/\omega\epsilon \gg 1$

$$k^2 \approx \left(\frac{\omega}{c} \right)^2 \mu \epsilon i \frac{4\pi \sigma}{\omega \epsilon}$$

$$\therefore k \approx \frac{1}{c} \sqrt{4\pi \mu \omega \sigma} \sqrt{i} = \sqrt[4]{e^{i\pi/2}} = e^{i\pi/4} = \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$$

$$k = \frac{1}{c} \sqrt{\frac{4\pi \mu \omega \sigma}{2}} (1+i)$$

$$= \frac{1}{\delta} (1+i)$$

$$\delta \equiv c \sqrt{\frac{2}{4\pi \mu \sigma \omega}}$$

(e)
skin
depth

$$e^{i \left(\left(\frac{1+i}{\delta} \right) z - \omega t \right)} = e^{-z/\delta} e^{i \left(\frac{z}{\delta} - \omega t \right)}$$

$$E_0 e^{-z/\delta} = E_0 e^{-1} \text{ when } z = \delta !$$

(d)

3/3

From Faraday's law

$$\frac{\partial \vec{B}}{\partial t} = -c \vec{\nabla} \times \vec{E} = -c \vec{\nabla} \times \hat{x} E_0 e^{i(kz - \omega t)}$$

$$\frac{\partial \vec{B}}{\partial t} = -c \hat{y} k E_0 i e^{i(kz - \omega t)}$$

$$\Rightarrow \vec{B} = c \frac{\hat{y} k E_0}{\omega} e^{i(kz - \omega t)}$$

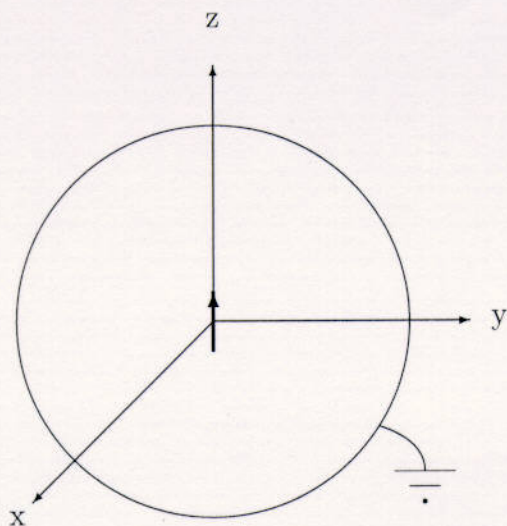
$$c \frac{k}{\omega} = \sqrt{\mu \epsilon} \sqrt{1 + \frac{i 4\pi \sigma}{\omega \epsilon}} \neq \text{real}$$

$\therefore \vec{B} + \vec{E}$ are not in phase

for $\sigma \gg \omega \epsilon$

$$c \frac{k}{\omega} = \sqrt{\mu \epsilon} \sqrt{\frac{4\pi \sigma}{\omega \epsilon}} \left(\frac{1+i}{\sqrt{2}} \right) = e^{i\pi/4} = 45^\circ \text{ phase difference}$$

3. A point electric dipole with dipole moment $\mathbf{p} = p_0 \hat{k}$ is located at the center of a hollow, grounded, conducting sphere.



- (a) {5 pts} Compute the electrostatic potential inside the sphere.
(b) {5 pts} Compute the charge density σ on the inside surface on the grounded sphere.

#3

y1

$$\phi = \frac{\vec{P} \cdot \vec{r}}{4\pi\epsilon_0 r^3} + \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\phi(r=a)=0 = \frac{P}{4\pi\epsilon_0 a^2} \cos\theta + \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta)$$

all $A_l = 0$ except
 $l=1$

$$\left(\frac{P}{a^2} + A_1 a \right) \cos\theta = 0$$

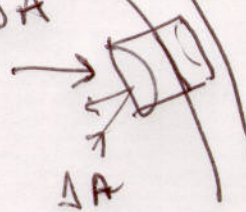
$$\Rightarrow A_1 = -\frac{P}{a^3}$$

(a)

$$\therefore \phi = \frac{P}{4\pi\epsilon_0} \left(\frac{1}{r^2} - \frac{r}{a^3} \right) \cos\theta$$

SI

Gaussian

$$\oint \vec{D} \cdot d\vec{a} = 4\pi \sigma \Delta A$$


ΔA

$$\therefore \sigma = -E_r(a) \frac{\epsilon_0}{4\pi}$$

SI

(b)

$$E_r = \frac{1}{4\pi\epsilon_0} P \left(\frac{2}{a^3} + \frac{1}{a^3} \right) \cos\theta = \frac{3P}{a^3} \cos\theta$$

\therefore

$$\sigma = -\frac{3P}{4\pi a^3} \cos\theta$$

4. (25 points) A 50 MeV electron ($m\gamma c^2 = 50$ MeV, $mc^2 = 0.5$ MeV) moving along the z-axis is decelerated and brought to a stop after traveling 10 cm in a uniform electric field $\mathbf{E} = E_0 \hat{\mathbf{k}}$.

- (a) {3 pts} Compute $\gamma(t)$ assuming the electron starts its deceleration at $t=0$.
- (b) {3 pts} How long does it take the electron to stop?
- (c) {3 pts} Compute the total energy radiated by the electron during the 10 cm stopping process.
- (d) {1 pts} What fraction of the electrons initial energy was lost to radiation?

Hint: The general Larmor formula for power radiated by an accelerating point charge is

$$P(t) = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 c} \gamma^6 [(\dot{\vec{\beta}})^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2], \quad (SI)$$

$$P(t) = \frac{2}{3} \frac{q^2}{c} \gamma^6 [(\dot{\vec{\beta}})^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2]. \quad (Gaussian)$$

It might be useful to use $(\dot{\gamma}\beta) = \gamma^3 \dot{\beta}$.

$$\begin{aligned} 1e &= 4.8 \times 10^{-10} \text{ statcoul} = 1.6 \times 10^{-19} \text{ coul}, \\ 1eV &= 1.6 \times 10^{-12} \text{ ergs} = 1.6 \times 10^{-19} \text{ J}. \end{aligned}$$

Solution
#14

stop with $\beta = 0$ here!

$$E_0^x > 0, v < 0 \quad \frac{1}{3}$$

(a) $m \dot{\gamma} v^x = q E_0^x = -e E_0^x = e v / d$

$$\Rightarrow (\dot{\gamma} \beta) = \frac{e v}{m c d} \Rightarrow \gamma \beta - \gamma_0 \beta_0 = \frac{e v}{m c d} (t - t_0)$$

\downarrow
0

(b) $\Delta t_{\text{stop}} = \gamma_0 \beta_0 \left(\frac{m c d}{-e v} \right)$

need

$$\gamma_0 = \frac{50 \text{ MeV}}{0.5 \text{ MeV}} = 100$$

$$\Rightarrow \beta_0 = \sqrt{1 - 10^{-4}} \approx 1$$

$$\gamma \beta = \gamma_0 \beta_0 + \frac{e v}{m c d} t = \gamma_0 \beta_0 \left(1 - \frac{t}{\Delta t_s} \right)$$

||

$$(\gamma \sqrt{1 - \beta^2})^2$$

$$\gamma \sqrt{1 - \beta^2}$$

||
 $\gamma^2 - 1$

(a) $\gamma(t) = \sqrt{1 + \gamma_0^2 \beta_0^2 \left(1 - t / \Delta t_s \right)^2}$

need $V = \text{potential difference to stop!}$

from $(m \gamma c^2) = q \vec{E} \cdot \vec{v} = -e E_0 \dot{x}$

$$\gamma - \gamma_0 = \frac{-e E_0}{m c^2} (x - x_0)$$

\downarrow
0

$$\therefore 1 - \gamma_0 = \frac{(-e)(-v/d)}{m c^2} d$$

$$(v < 0)$$

$$1 - 100 = \frac{e v}{m c^2}$$

||
-99

from above

$$\therefore \Delta t_{stop} = \gamma_0 \beta_0 \left(\frac{d/c}{100-1} \right) \approx \frac{100}{99} \times \frac{10}{3 \times 10^{10}} \text{ sec} \quad (w)$$

$$\approx 3.4 \times 10^{-10} \text{ sec}$$

$$(c) \quad (\gamma \dot{\beta}) = \left(\frac{\dot{\beta}}{1-\beta^2} \right) = \dot{\gamma} \beta + \frac{\beta \dot{\beta} \dot{\beta}}{(\gamma)^3} = \dot{\gamma} \dot{\beta} \left(1 + \frac{\beta^2}{1-\beta^2} \right)$$

$$= \gamma^3 \dot{\beta}$$

\therefore if $\gamma \beta = \text{constant as above, so is } \gamma^3 \dot{\beta} !$

$$\therefore P(t) = \frac{2}{3} \frac{e^2}{c} (\gamma^3 \dot{\beta})^2 = \frac{2}{3} \frac{e^2}{c} (\gamma_0-1)^2 \left(\frac{c}{d} \right)^2 = \text{const}$$

$$[\gamma \dot{\beta}]^2$$

||

$$\left(\frac{e v}{m c d} \right)^2$$

||

$$\left(-99 \frac{c}{d} \right)^2$$

$$\Delta E_{rad} = P \times \Delta t_s = \frac{2}{3} \frac{e^2}{c} (\gamma_0-1)^2 \frac{c^2}{d^2} \frac{\gamma_0 \beta_0}{\gamma_0-1} \frac{d}{c}$$

$$= \frac{2}{3} \frac{e^2}{d} (\gamma_0-1) \gamma_0 \beta_0 \approx \frac{2}{3} 99 \times 100 \times \frac{(4.8 \times 10^{-10})^2}{10} \uparrow$$

ergs

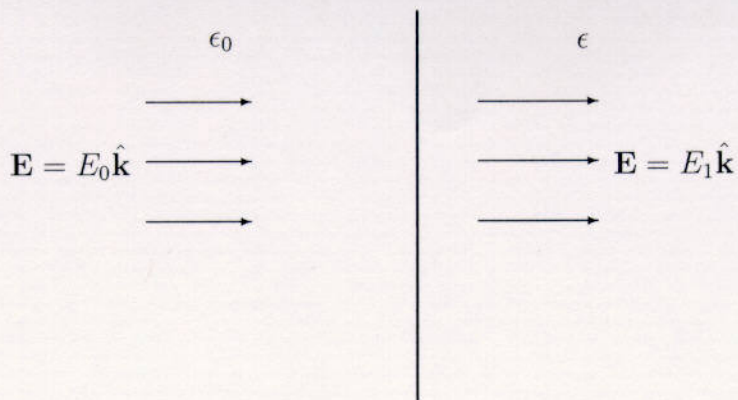
$$\frac{\Delta E_{\text{rad}}}{m \delta_0 c^2} = \frac{\frac{2}{3} (4.8)^2 \times 10^{-17} \text{ ergs}}{50 \times 10^6 \times 1.6 \times 10^{-12} \text{ ergs}} \approx \underline{1.9 \times 10^{-12}}$$

↖

↙

50 MeV

$$1 \text{ eV} = 1.6 \times 10^{-12} \text{ ergs}$$



5. A uniform electric field is orthogonal to a large sheet of dielectric material with constant $\epsilon > \epsilon_0$. Electrostatic forces on the surface of the sheet will rip it apart unless it can withstand these forces.

- (a) {3 pts} Use the junction conditions at the interface of the dielectric to relate E_1 to E_0 .
- (b) {5 pts} Use the Maxwell stress tensor to compute the force/area on the dielectric's surface.
- (c) {2 pts} If $\epsilon/\epsilon_0 = 2$ and the electric field strength $E_0 = 900$ Volts/cm, evaluate the net force on a $10\text{cm} \times 10\text{cm}$ square dielectric surface.

Recall that

$$T_M^{ij} = \left[\epsilon E^i E^j + \mu H^i H^j - \frac{1}{2} \delta^{ij} (\epsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2) \right], \quad (SI)$$

$$T_M^{ij} = \frac{1}{4\pi} \left[\epsilon E^i E^j + \mu H^i H^j - \frac{1}{2} \delta^{ij} (\epsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2) \right], \quad (Gaussian)$$

for linear and isotropic materials that can be described by constant permittivities ϵ and permeabilities μ . In SI units $\mu_0 = 4\pi \times 10^{-7}$ H/m, and $\epsilon_0 \mu_0 = 1/c^2$ where $c = 3.0 \times 10^8$ m/s. (In Gaussian units $\epsilon_0 = \mu_0 = 1$, and 1 Volt = 1/300 statvolts.)

#5 ~~#3~~ ~~#10~~ Solution

3-1/2

$$\hat{n} = \hat{n}_{\text{left}} \leftarrow \begin{array}{c} T_{\text{left}}^{is} \\ T_{\text{right}}^{is} \end{array} \rightarrow \hat{n}_{\text{right}} = \hat{k}$$

$$T_R^{is} \hat{n}_R + T_b^{is} \hat{n}_b = \frac{F^i}{\text{area}}$$

Junction condition

$$D_L^3 = D_R^3$$

$$\Rightarrow \epsilon_0 F_b^3 = \epsilon E_R^3 + 5$$

$$\frac{F^i}{\text{area}} = \left(T_R^{is} - T_b^{is} \right) + 5$$

$$= \epsilon \left(E_R^i E_R^3 - \frac{1}{2} (E_R^3)^2 \right)$$

$$- \epsilon_0 \left(E_b^i E_b^3 - \frac{1}{2} E_b^{3^2} \right)$$

$$\frac{F^i}{\text{area}} = \left[\frac{\epsilon}{2} (E_R^3)^2 - \frac{\epsilon_0}{2} (E_b^3)^2 \right] \delta_j^i + 5$$

$$= \left[\frac{\epsilon}{2} \left(\frac{\epsilon_0}{\epsilon} \right)^2 - \frac{\epsilon_0}{2} \right] (E_b^3)^2 \delta_j^i$$

$$\left(\frac{F^i}{\text{area}} = \frac{1}{2} \epsilon_0 \left(\frac{\epsilon_0}{\epsilon} - 1 \right) (E_b^3)^2 \delta_j^i \right) + 5$$

↑
neg. \therefore direction is left!

3-12

$$E = 1000 \text{ V/m} \\ = 10^5 \text{ V/m}$$

$$\mathcal{E} = - \frac{d\Phi}{dt} \\ \text{Volt} = \text{Henry} \frac{\text{Coul}}{\text{s}^2} \\ \downarrow \\ \text{H}$$

$$\frac{F^i}{\text{area}} = \frac{1}{2} \frac{1}{(5.0 \times 10^8 \text{ m/s})^2 4\pi \times 10^{-7} \text{ H/m}} \left(\frac{1}{2} - 1\right) (10^5)^2 \frac{1}{\text{m}} \delta_g^i$$

$$= \frac{-(10)^{10-16+7}}{2^2 \cdot 3^2 \cdot 4\pi} \left(\frac{\text{S}^2}{\text{m H}} \frac{\text{V}^2}{\text{m}^2} \right) = \text{Nt}$$

$$\left(\frac{F^i}{\text{area}} = \left(\frac{-5}{2^3 3^2 \pi} \right) \frac{\text{Nt}}{\text{m}^2} \delta_g^i \right. \\ \left. = \left(\frac{-5}{72 \pi} \right) \quad \quad \quad \right) + \checkmark$$

6. A plane polarized electromagnetic plane wave traveling in the lab in vacuum is the of the form:

$$\mathbf{E} = \text{Real} \left\{ E_0 \hat{\mathbf{i}} e^{i(kz - \omega t)} \right\},$$

$$\mathbf{B} = \text{Real} \left\{ B_0 \hat{\mathbf{j}} e^{i(kz - \omega t)} \right\},$$

where $k = \omega/c$ and

$$\begin{aligned} B_0 &= E_0/c, & (SI) \\ B_0 &= E_0. & (Gaussian) \end{aligned}$$

A relativistic particle moving in the z-direction ($\mathbf{v} = v_0 \hat{\mathbf{k}}$, where $\gamma \gg 1$) encounters this wave. For this problem you are to find the form of the wave in the rest frame of the particle at the instant the wave is first encountered (before the particle's velocity is changed because of an interaction with the wave).

- (a) {2 pts} Start by combining \mathbf{E} and \mathbf{B} into a single 4-tensor $F^{\alpha\beta}(x)$. This includes writing $(kz - \omega t) = \pm k_\alpha x^\alpha$. The sign \pm depends on your choice of Lorentz metrics $(-1, +, +, +)$ or $(1, -1, -1, -1)$. State which you are using.
- (b) {2 pts} Give the Lorentz transformation L^α_β that transforms the lab frame into the particle's rest frame $x'^\alpha = L^\alpha_\beta x^\beta$.
- (c) {2 pts} Apply your Lorentz transformation to $F^{\alpha\beta}(x)$ to find $F'^{\alpha\beta}(x')$, the electromagnetic 4-tensor in the particles rest frame.
- (d) {2 pts} From your results in (c) give the 3-dimensional propagation direction of the wave in the particle's frame and the $\mathbf{E}'(x')$ and $\mathbf{B}'(x')$ fields.
- (e) {2 pts} Compare the amplitudes and frequency of the wave as seen by the particle in its rest frame with those seen by a lab observer?

#6

1/4

(a)

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{matrix} E_0 e^{i(kz - \omega t)} \\ E_0 e^{\pm i k_2 x^2} \end{matrix}$$

(---) $k_2 x^2$

$$\begin{aligned} k_2 - \omega t &= + (k_0 ct + k_1 x + k_2 y + k_3 z) \\ &= (-k_0 ct + k_1 x + k_2 y + k_3 z) \\ &= \left(-\frac{\omega}{c} ct + 0 + 0 + k_3 z \right) \end{aligned}$$

$$\begin{aligned} k_2 - \omega t &= \text{(---)} - k_2 x^2 = - (k_0 ct + k_1 x + k_2 y + k_3 z) \\ &= - (k_0 ct - k_1 x - k_2 y - k_3 z) \\ &= -\frac{\omega}{c} ct + k_1 x + k_2 y + k_3 z \\ &= -\omega t + k_3 z \end{aligned}$$

(b)

$$L^2_F = \left(\begin{array}{cc|cc} \gamma - \beta\gamma & 0 & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

$$X'^2 = L^2_F X^B$$

\uparrow particle \uparrow lab

 $\frac{2}{4}$

$$F' = L F L^T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma - \beta\gamma & 0 \\ -\beta\gamma & \gamma \\ \hline 0 & 1 \\ 0 & 1 \end{pmatrix} E_0 e^{\pm i k_2 x^2}$$

$$= L \begin{pmatrix} \beta\gamma - \gamma & 0 & 0 & 0 \\ \gamma - \beta\gamma & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \beta\gamma - \gamma & 0 & 0 & 0 \end{pmatrix} E_0 e^{\pm i k_2 x^2}$$

$$= \begin{pmatrix} 0 & -\gamma^2 + \beta^2\gamma^2 & 0 & -\beta\gamma \\ -\beta^2\gamma^2 + \gamma^2 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ \beta\gamma & -\gamma & 0 & 0 \end{pmatrix} E_0 e^{\pm i k_2 x^2}$$

$$(c) \quad F' = \begin{pmatrix} 0 & -1 & 0 & -\beta\gamma \\ 1 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ \beta\gamma & -\gamma & 0 & 0 \end{pmatrix} E_0 e^{\pm i k_2 x^2}$$

$$k_\alpha x'^\alpha = k'_\alpha x'^\alpha = k'^\alpha x'_\alpha$$

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$$L^\alpha_\beta k^\beta$$

$$k = \omega/c$$

$$\left(\begin{array}{cc|cc} \gamma - \beta\gamma & 0 & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \end{array} \right) \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}$$

$$= \begin{pmatrix} \gamma k \\ -\beta\gamma k \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} \gamma \\ -\beta\gamma \\ 0 \\ 1 \end{pmatrix} k$$

$$\frac{\omega'}{c} = \gamma \frac{\omega}{c} \quad \hookrightarrow \quad k'^0 = \gamma k = k' = \omega'/c$$

$$k'^x = -\beta\gamma k \Rightarrow k'^x = k'(-\sin\theta') = \gamma k(-\sin\theta')$$

$$k'^y = 0$$

$$\therefore \sin\theta' = \beta$$

$$k'^z = k = k' \cos\theta'$$

$$= \gamma k \cos\theta' \quad \therefore \cos\theta' = \gamma^{-1}$$

(e) frequency

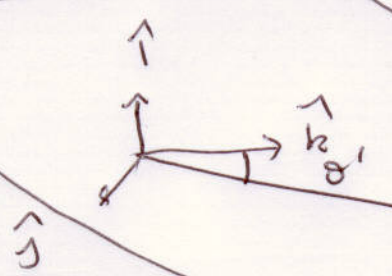
$$\therefore \omega' = \gamma \omega \quad \left| k'^\alpha = \gamma \frac{\omega}{c} (1, -\beta\hat{e}_x, \gamma^{-1}) \right|$$

$$\hat{j} \text{ prop dir} = (-\beta\hat{i} + \gamma^{-1}\hat{j})$$

$$e^{-i k'_{\alpha} x'_{\alpha}} = e^{-i \gamma \frac{\omega}{c} (ct' + \beta x' - \gamma^{-1} z)} \quad (+---) \quad \frac{4}{4}$$

$$e^{+i k'_{\alpha} x'_{\alpha}} = e^{i \gamma \frac{\omega}{c} (-ct' - \beta x' + \gamma^{-1} z)} \quad (-+++)$$

(d)



$$\vec{E}' = E_0 (\hat{j} + \beta \gamma \hat{k}) e^{\pm i k'_{\alpha} x'_{\alpha}}$$

$$\vec{E}' = E_0 \gamma (\gamma^{-1} \hat{j} + \beta \hat{k}) e^{\pm i k'_{\alpha} x'_{\alpha}}$$

unit vector
" polarization direction

$$\vec{B}' = \gamma \hat{j} E_0 e^{\pm i k'_{\alpha} x'_{\alpha}}$$

(SI) $\rightarrow c$

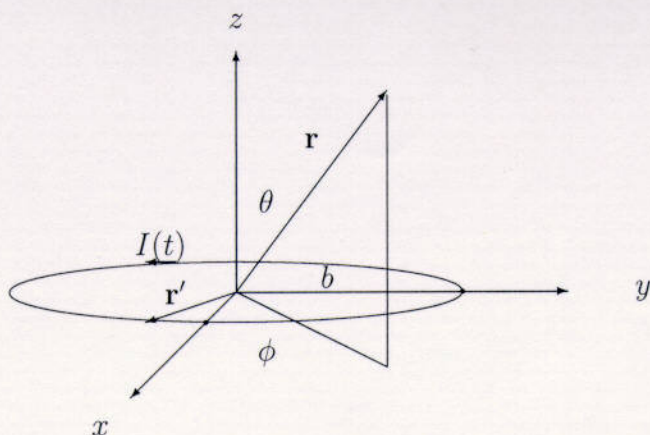
(e) amplitude

$$E'_0 = E_0 \gamma$$

"

$$B'_0 = \frac{E_0 \gamma}{c} = B_0 \gamma$$

(SI) $\rightarrow c$



7. A circular current loop of radius b lies in the x - y plane and is centered on the origin. If the current varies harmonically with time $I(t) = I_0 \cos(\omega t)$, use the following to carry out steps (a) through (e):

$$\begin{aligned}
 \mathbf{r}' &= b \{ \sin \theta' (\cos \phi' \hat{\mathbf{i}} + \sin \phi' \hat{\mathbf{j}}) + \cos \theta' \hat{\mathbf{k}} \} \\
 &= b (\cos \phi' \hat{\mathbf{i}} + \sin \phi' \hat{\mathbf{j}}), \\
 \mathbf{r} \cdot \mathbf{r}' &= b r \cos \gamma \\
 &= b r \{ \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \} \\
 &= b r \sin \theta \cos(\phi - \phi').
 \end{aligned}$$

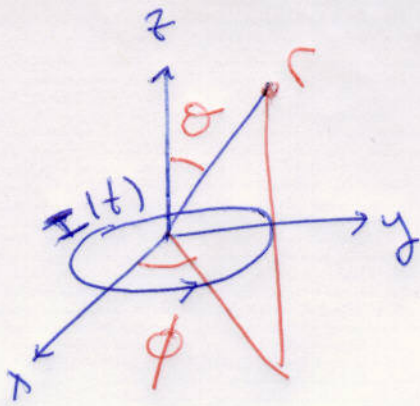
- (a) {2 pts} Compute time dependent magnetic dipole moment $\mathbf{m}(t)$ of the current loop. Recall that $\mathbf{m}_{\text{Gaussian}} = \mathbf{m}_{\text{SI}}/c$.
- (b) {2 pts} Give an integral expression for the retarded vector potential $\mathbf{A}(t, \mathbf{r})$.
- (c) {2 pts} Approximate the integral found in (b) for \mathbf{A} assuming $b \ll r$ and $b \ll c/\omega$. If you have made no mistakes your answer should agree with the potential for a point magnetic dipole, i.e., with:

$$\begin{aligned}
 \mathbf{A} &= \frac{\mu_0}{4\pi} \nabla \times \left\{ \frac{\mathbf{m}(t - r/c)}{r} \right\}, & \text{SI} \\
 \mathbf{A} &= \nabla \times \left\{ \frac{\mathbf{m}(t - r/c)}{r} \right\}. & \text{Gaussian}
 \end{aligned}$$

- (d) {2 pts} From your results for (c) or from the point magnetic dipole result, compute the radiation (far field) part of \mathbf{E} by assuming $b \ll c/\omega \ll r$.
- (e) {2 pts} Using only the radiation part, i.e., the part $\propto 1/r$, of \mathbf{E} and \mathbf{B} , and the Poynting vector, compute the time averaged electromagnetic energy flux radiated away by the dipole as a function of the spherical polar coordinates (r, θ, ϕ) . Recall that

$$\begin{aligned}
 \mathbf{B} &= \frac{1}{c} \hat{\mathbf{r}} \times \mathbf{E}, & (\text{SI}) \\
 \mathbf{B} &= \hat{\mathbf{r}} \times \mathbf{E}, & (\text{Gaussian})
 \end{aligned}$$

for radiation coming from a source at the origin.



$$\begin{aligned}
 \text{(a)} \quad \vec{m}(t) &\equiv I(t) \oint_{|\vec{r}'|=b} \frac{\vec{r}' \times d\vec{r}'}{2} && (\text{SI}) \\
 &= \frac{I(t)}{c} \oint_{|\vec{r}'|=b} \frac{\vec{r}' \times d\vec{r}'}{2} && (\text{Gaussian})
 \end{aligned}$$

$$\vec{r}' = b(\cos\phi' \hat{i} + \sin\phi' \hat{j})$$

$$d\vec{r}' = b(-\sin\phi' \hat{i} + \cos\phi' \hat{j}) d\phi'$$

$$\vec{r}' \times d\vec{r}' = \frac{b^2}{2} (\cos^2\phi' + \sin^2\phi') \hat{k} d\phi'$$

$$= \frac{2\pi b^2}{2} \hat{k}$$

$$\therefore \vec{m}(t) = I(t) \pi b^2 \hat{k} \quad (\text{SI})$$

$$= \frac{I(t)}{c} \pi b^2 \hat{k} \quad (\text{Gaussian})$$

$$\vec{m}(t) = m_0 \cos\omega t \hat{k}$$

$$m_0 = \frac{\pi b^2 I_0}{c}$$

$$I(t) \equiv I_0 \cos(\omega t)$$

(b)

$\frac{\mu_0}{4\pi}$ in SI

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int \frac{\vec{J}(t - |\vec{r} - \vec{r}'|/c, \vec{r}') d^3 r'}{|\vec{r} - \vec{r}'|}$$

Gaussian contains (c)

$$= \frac{\mu_0}{4\pi} \frac{1}{c} \oint \frac{I(t - |\vec{r} - \vec{r}'|/c) d\vec{\ell}'}{|\vec{r} - \vec{r}'|}$$

$$= \frac{\mu_0}{4\pi} \frac{I_0 b}{c} \int_0^{2\pi} \frac{\cos \omega(t - |\vec{r} - \vec{r}'|/c) (-R \sin \phi' \hat{i} + R \cos \phi' \hat{j}) d\phi'}{|\vec{r} - \vec{r}'|}$$

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + b^2 - 2rb \cos \phi}$$

$$\cos \phi = \underbrace{\cos \theta \cos \theta'}_0 + \underbrace{\sin \theta \sin \theta'}_1 \cos(\phi - \phi')$$

$$\therefore \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{I_0 b}{c} \int_0^{2\pi} \frac{\cos \omega(t - \sqrt{r^2 + b^2 - 2rb \cos(\phi - \phi')})}{\sqrt{r^2 + b^2 - 2rb \cos(\phi - \phi')}} (-R \sin \phi' \hat{i} + R \cos \phi' \hat{j}) d\phi'$$

hence

$$\Rightarrow \sqrt{r^2 + b^2 - 2rb \sin \theta \cos(\phi - \phi')}$$

$$\approx r \left\{ 1 - \frac{b}{r} \sin \theta \cos(\phi - \phi') \right\}$$

$$b \ll r \ll \frac{c}{\omega}$$

$$\Rightarrow \cos \omega \left(t - \frac{r}{c} \right) \approx \cos \omega \left(t - \frac{r}{c} \left\{ 1 - \frac{b}{r} \sin \theta \cos(\phi - \phi') \right\} \right)$$

$$= \cos \left[\omega \left(t - \frac{r}{c} \right) + \frac{\omega b}{c} \sin \theta \cos(\phi - \phi') \right] \quad \text{expand}$$

$$\approx \left[\cos \omega \left(t - \frac{r}{c} \right) - \sin \omega \left(t - \frac{r}{c} \right) \frac{\omega b}{c} \sin \theta \cos(\phi - \phi') \right] + \mathcal{O} \left(\frac{\omega b}{c} \right)^2$$

$$\vec{A}(\vec{r}, t) \approx \frac{\mu_0 I_0 b}{4\pi c r} \int \left[\cos \omega \left(t - \frac{r}{c} \right) - \sin \omega \left(t - \frac{r}{c} \right) \frac{\omega b}{c} \sin \theta \cos(\phi - \phi') \right]$$

$$\times \left\{ 1 + \frac{b}{r} \sin \theta \cos(\phi - \phi') \right\} d\phi'$$

$$(-\sin \phi' \hat{i} + \cos \phi' \hat{j}) d\phi'$$

$$\vec{A}(\vec{r}, t) \approx \frac{\mu_0}{4\pi} \frac{I_0 b}{c r} \int_0^{2\pi} \left[\cos \omega(t-r/c) - \frac{\omega b}{c} \sin \omega(t-r/c) \sin \theta \cos(\phi-\phi') + \frac{b}{r} \cos \omega(t-r/c) \sin \theta \cos(\phi-\phi') \right] (-\sin \phi' \hat{i} + \cos \phi' \hat{j}) d\phi'$$

gives '0'!

$$\vec{A} \approx \frac{\mu_0}{4\pi} \frac{I_0 b^2}{c r} \int_0^{2\pi} \left[-\frac{\omega}{c} \sin \omega(t-r/c) + \frac{1}{r} \cos \omega(t-r/c) \right] \sin \theta \cos(\phi-\phi') (-\sin \phi' \hat{i} + \cos \phi' \hat{j}) d\phi'$$

requires some work here!

point dipole result! $\star \pi \sin \theta \hat{\phi}(\phi) = \pi \hat{k} \times \hat{r}$

should be same as point dipole

$$\vec{A} = \frac{\mu_0}{4\pi} \nabla \times \left(\frac{\vec{m}(t-r/c)}{r} \right) = \frac{\mu_0}{4\pi} \left[-\frac{\vec{r} \times \vec{m}}{r^3} - \frac{\vec{r} \times \dot{\vec{m}}}{c r^2} \right]$$

For the "far" field $\frac{1}{r} \ll \omega/c$ the above reduces to

$$\vec{A} \approx -\frac{\mu_0}{4\pi} \frac{I_0 b^2}{c r} \frac{\omega}{c} \sin \omega(t-r/c) \sin \theta \int_0^{2\pi} \cos(\phi-\phi') (-\sin \phi' \hat{i} + \cos \phi' \hat{j}) d\phi'$$

requires some work here.

$$\pi (-\sin \phi \hat{i} + \cos \phi \hat{j}) = \pi \hat{\phi}(\phi)$$

$$\vec{A} = -\frac{\mu_0}{4\pi} \frac{\omega I_0 \pi b^2}{c r c} \sin \omega(t-r/c) \sin \theta \hat{\phi}$$

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0}{4\pi} \frac{\vec{r}}{c r^2} \times \frac{\dot{\vec{m}}}{c r^2} \quad \text{for point dipole}$$

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$$= -\frac{\mu_0}{4\pi} \frac{\vec{r}}{c r^2} \times \left(-\frac{\omega^2 \cos \omega t}{c} \frac{I_0 \pi b^2 \hat{k}}{c} \right) \quad \text{from above result (c)}$$

$$= -\frac{\mu_0}{4\pi} \frac{m_0 \omega^2 \cos \omega t}{c r} \sin \theta \hat{\phi}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{m_0 \omega^2}{c^2} \frac{1}{r} \cos \omega t \sin \theta \hat{\phi}$$

$$\langle \sin^2 \theta \rangle = \frac{1}{2}$$

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} = \frac{\mu_0 m_0^2 \omega^4}{4\pi^2 c^2} \frac{\sin^2 \theta \cos^2 \omega t}{r^2} \hat{r}$$