

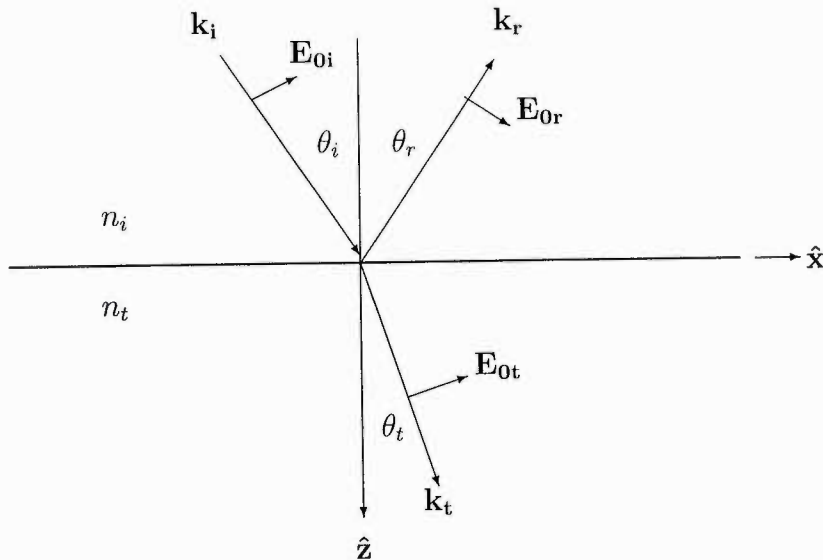
E & M Qualifier

August 15, 2013

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias (**NOT YOUR REAL NAME**) on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. **do not** staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



1. A plane monochromatic electromagnetic wave of wave vector \mathbf{k}_i , amplitude \mathbf{E}_{0i} , and angular frequency ω is incident at the planar interface of two dielectric, non-magnetic ($\mu = \mu_0$ in SI units), non-absorbing media (i.e., have real indices of refraction n_i and n_t). The angle of incidence is equal to θ_i . Part of the incident wave is reflected at an angle $\theta_r = \theta_i$ and part of it is transmitted into the second medium at a transmission angle θ_t . Assume the electric fields of the incident, reflected, and refracted waves lie in the plane of incidence as shown in the figure. Assume coordinates are chosen so that the dielectric interface is the $z = 0$ plane and the polarization is in the x - z plane.

- (a) [2 pts] Use Maxwell's equations to derive an expression for the magnetic induction \mathbf{B} associated with a plane monochromatic electromagnetic wave whose electric field is

$$\mathbf{E} = \mathbf{E}_0 \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

traveling in a homogeneous material described by a real index of refraction n . Give the relationship of $|\mathbf{k}|$ to ω .

- (b) [2 pts] From the above figure give \mathbf{k}_i , \mathbf{k}_r , and \mathbf{k}_t in terms of their $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components, and evaluate $\mathbf{k} \cdot \mathbf{r}$ in the $z = 0$ plane.
- (c) [1 pts] From the above figure give \mathbf{E}_{0i} , \mathbf{E}_{0r} , and \mathbf{E}_{0t} in terms of their $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components.
- (d) [1 pts] State the 4 boundary conditions satisfied by the fields \mathbf{E} , \mathbf{B} , \mathbf{H} , and \mathbf{D} at the above $z = 0$ junction.
- (e) [1 pts] Use one of these junction conditions to prove Snell's law, $n_t \sin \theta_t = n_i \sin \theta_i$ (only 2 of the 4 are independent).
- (f) [3 pts] Use two of the junction conditions to determine the ratio of the magnitude of the amplitudes of the reflected and transmitted to the incident electric fields, i.e., evaluate $|\mathbf{E}_{0r}|/|\mathbf{E}_{0i}|$ and $|\mathbf{E}_{0t}|/|\mathbf{E}_{0i}|$ as shown in the figure.

#1 $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$ $n = c/v = \sqrt{\epsilon_r \mu_r}$ 1/4

non-magnetic $\Rightarrow \mu_r = 1$

Max eqns $\vec{\nabla} \cdot \vec{D} = 0 \Rightarrow \vec{k} \cdot \vec{E}_0 = 0$
 $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{k} \cdot \vec{B}_0 = 0$, $\vec{B} = \vec{B}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

Faraday $\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow i \vec{k} \times \vec{E}_0 - \frac{i\omega}{c} \vec{B}_0 = 0$

$\therefore \vec{B}_0 = c \frac{\vec{k}}{\omega} \times \vec{E}_0$

Red = Gaussian

Ampere $\vec{\nabla} \times \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} = 0 \Rightarrow i \vec{k} \times \frac{\vec{B}_0}{\mu} + \frac{i\omega}{c} \epsilon \vec{E}_0 = 0$

$\therefore \vec{E}_0 = - \frac{\vec{k}}{\omega \mu \epsilon} \times \vec{B}_0$

combine Faraday + Ampere

to get $\frac{|\vec{k}|^2}{\omega^2} \left(\frac{c^2}{\mu \epsilon} \right) = 1$

$\therefore |\vec{k}| = \frac{n \omega}{c}$

(a) 2 pts

\parallel
 v^2

$\vec{B}_0 = \frac{n}{c} \hat{k} \times \vec{E}_0$

$\hat{k} \equiv \frac{\vec{k}}{|\vec{k}|}$

Green = SI

$n = \sqrt{\epsilon_r \mu_r}$

(b) $\vec{R}_i = k_i (\sin \theta_i \hat{x} + \cos \theta_i \hat{z})$ 3/4

~~2 pt~~ $\vec{R}_r = R_r (\sin \theta_i \hat{x} - \cos \theta_i \hat{z})$ $R_r = k_i$

$\vec{R}_t = k_t (\sin \theta_t \hat{x} + \cos \theta_t \hat{z})$

For $\vec{r} = x \hat{x} + y \hat{y}$

$\vec{R}_i \cdot \vec{r} = x k_i \sin \theta_i$

From (a)

$\vec{R}_r \cdot \vec{r} = x R_r \sin \theta_i$

$k_r = k_i = \frac{m_i}{c} \omega$

$\vec{R}_t \cdot \vec{r} = x k_t \sin \theta_t$

$k_t = \frac{m_t}{c} \omega$

(c) $\vec{E}_i = E_{0i} [\cos \theta_i \hat{x} - \sin \theta_i \hat{z}]$

~~1 pt~~ $\vec{E}_{or} = E_{0r} [\cos \theta_i \hat{x} + \sin \theta_i \hat{z}]$

$\vec{E}_{ot} = E_{0t} [\cos \theta_t \hat{x} - \sin \theta_t \hat{z}]$

(d) $\hat{z} \cdot \vec{D}$, $\hat{z} \cdot \vec{B}$, $\hat{z} \times \vec{E}$, and $\hat{z} \times \vec{H}$ are
~~1 pt~~ all continuous across $z=0$ boundary

$\hat{z} \cdot \vec{B} = 0$, $\hat{z} \cdot \vec{D}$ and $\hat{z} \times \vec{H}$ are not independent

(e) From $\hat{z} \cdot \vec{D} = \text{const}$ we have

3/4

$$\epsilon_i \hat{z} \cdot (\vec{E}_i + \vec{E}_r) = \epsilon_r \hat{z} \cdot \vec{E}_t$$

$$\epsilon_i \left[-E_{oi} \sin \theta_i + E_{or} \sin \theta_i \right] e^{i(x k_i \sin \theta_i - \omega t)} \\ = \epsilon_r \left[-E_{ot} \sin \theta_t \right] e^{i(x k_t \sin \theta_t - \omega t)}$$

// functions must be the same

$\therefore \Rightarrow$ 1st $\left| \begin{array}{l} k_i \sin \theta_i = k_t \sin \theta_t = \text{Snell's law} \\ (n_i \omega/c) \quad (n_t \omega/c) \Rightarrow n_i \sin \theta_i = n_t \sin \theta_t \end{array} \right.$

[Eq 1] and (f) 3 pts

$$\epsilon_i (E_{oi} - E_{or}) \sin \theta_i = \epsilon_t E_{ot} \sin \theta_t$$

$\downarrow \quad \quad \quad \downarrow$
 $n_i^2 \epsilon_0 \quad \quad \quad n_t^2 \epsilon_0$

From $\hat{z} \times \vec{E} = \text{const}$ we have

$$(E_{oi} + E_{or}) \cos \theta_i \hat{y} e^{i(x k_i \sin \theta_i - \omega t)} \\ = E_{ot} \cos \theta_t \hat{y} e^{i(x k_t \sin \theta_t - \omega t)}$$

Same from above

[Eq 2]

\Rightarrow

$$(E_{oi} + E_{or}) \cos \theta_i = E_{ot} \cos \theta_t$$

[Eq 1] \Leftrightarrow

$$n_i (E_{oi} - E_{or}) = n_t E_{ot}$$

Solve: $E_{g2} + E_{g1}$ for E_{ot} and E_{or} !

4/4

$$2n_i \cos \theta_i E_{oi} = (n_i \cos \theta_t + n_t \cos \theta_i) E_{ot}$$

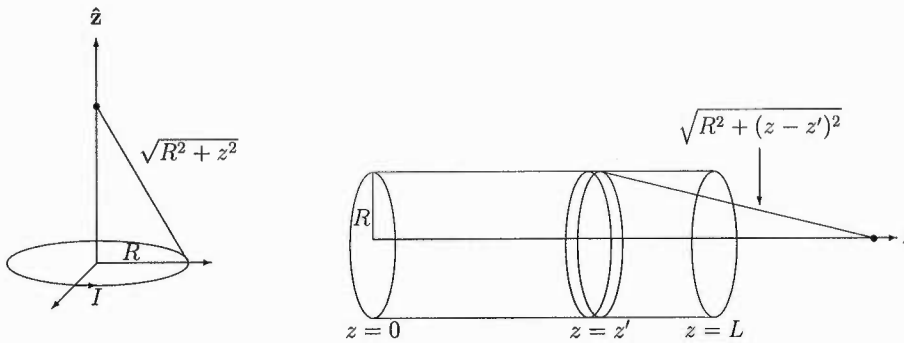
$$\Rightarrow \boxed{\frac{E_{ot}}{E_{oi}} = \frac{2n_i \cos \theta_i}{n_i \cos \theta_t + n_t \cos \theta_i}}$$

and $n_t (E_{oi} + E_{or}) \cos \theta_i - n_i (E_{oi} - E_{or}) \cos \theta_t = 0$

$$\Rightarrow \boxed{\frac{E_{or}}{E_{oi}} = \frac{-n_t \cos \theta_i + n_i \cos \theta_t}{n_t \cos \theta_i + n_i \cos \theta_t}}$$

where $\cos \theta_t = \sqrt{1 - \sin^2 \theta_t}$

$$= \sqrt{1 - \frac{n_i^2}{n_t^2} \sin^2 \theta_i}$$



2. (a) [3 pts] A circular loop of radius R , centered on the origin and $z = 0$ plane, carries a current I . Find the magnetic field B on the axis of the loop as a function of the distance z from the center of the loop.
- (b) [4 pts] Use the result of part (a) to find B along the axis of a solenoid of radius R and length L , uniformly wound with $n = N/L$ turns per unit length.
- (c) [3 pts] Assume that instead of a solenoid you had a cylinder of radius R and length L made out of a piece of uniformly magnetized iron with magnetization \mathbf{M} pointing along the axis of the solenoid. Use the solution of part (b) to calculate the magnetic field strength \mathbf{H} and the magnetic induction \mathbf{B} along the axis of the cylinder, both inside and outside.

HINTS:

$$\int \frac{dw}{[R^2 + w^2]^{3/2}} = \frac{w}{\sqrt{R^2 + w^2}} + \text{constant}.$$

The bound volume and surface current densities associated with a smooth magnetization density are respectively

$$\mathbf{J}_b|_{SI} = \nabla \times \mathbf{M},$$

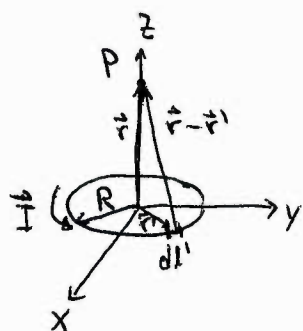
and

$$\mathbf{K}_b|_{SI} = \mathbf{M} \times \mathbf{n},$$

where \mathbf{n} is the outward unit normal at the magnet's boundary. The Gaussian expressions for \mathbf{J}_b and \mathbf{K}_b contain an additional factor of c in the numerators.

2) (a) Consider a circular loop of radius R

1/2



* From Biot-Savart law, the contribution from dl' to \vec{B} is

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{I} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dl' \quad \text{SI}$$

* Due to symmetry \vec{B} will be pointing in the z -direction.

$$\Rightarrow \vec{B} = \frac{\mu_0}{2\pi} \int \frac{[\vec{I} \times (\vec{r} - \vec{r}')] \cdot \hat{z}}{|\vec{r} - \vec{r}'|^3} dl'$$

$$\frac{\mu_0}{4\pi} \rightarrow \frac{1}{c} \text{ Gaussian}$$

For this problem

$$\vec{I} = I \hat{\phi}$$

$$\vec{r} = z \hat{z}$$

$$\vec{r}' = R \hat{s}$$

$$dl' = R d\phi'$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{[\hat{\phi} \times (z\hat{z} - R\hat{s})] \cdot \hat{z} R d\phi' \hat{z}}{(z^2 + R^2)^{3/2}}$$

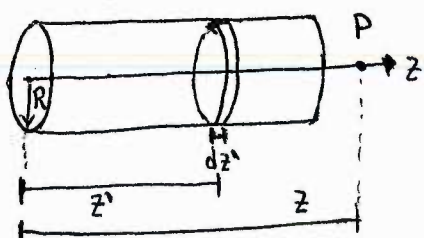
where

$$[\hat{\phi} \times (z\hat{z} - R\hat{s})] \cdot \hat{z} = (z\hat{s} + R\hat{z}) \cdot \hat{z} = R$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{4\pi} \frac{R^2}{(z^2 + R^2)^{3/2}} \int_0^{2\pi} d\phi' \hat{z} = \frac{\mu_0 I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}} \hat{z}$$

$$\therefore \vec{B}(z) = \frac{\mu_0 I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}} \hat{z}$$

(b) We can now use the result of part (a) to find the magnetic field due to a solenoid of length L .



* The solenoid can be seen to be composed of circular loops with width dz and current

$$I \rightarrow n I dz'$$

• The contribution from one of these circular loops to \vec{B} can be obtained from the result of part (a) after adopting the position of the loop

$$z \rightarrow z - z'$$

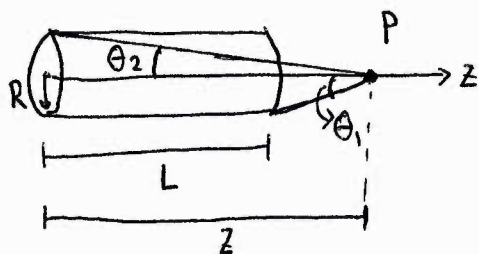
$$\Rightarrow d\vec{B} = \frac{\mu_0 n I dz'}{2} \frac{R^2}{[R^2 + (z - z')^2]^{3/2}} \hat{z}$$

$$\vec{B} = \frac{\mu_0 n I}{2} \int_0^L \frac{R^2}{[R^2 + (z-z')^2]^{3/2}} dz' \hat{z} = \frac{\mu_0 n I}{2} \left[\frac{z}{\sqrt{R^2 + z^2}} + \frac{L-z}{\sqrt{R^2 + (L-z)^2}} \right] \hat{z}$$

2/2

$$\Rightarrow \vec{B}(z) = \frac{\mu_0 n I}{2} \left[\frac{z}{\sqrt{R^2 + z^2}} + \frac{L-z}{\sqrt{R^2 + (L-z)^2}} \right] \hat{z}$$

We can rewrite this result in terms of the angles from the edge of the solenoid to point z



$$\cos \theta_1 = \frac{z-L}{\sqrt{R^2 + (z-L)^2}}$$

$$\cos \theta_2 = \frac{z}{\sqrt{R^2 + z^2}}$$

$$\Rightarrow \vec{B}(z) = \frac{\mu_0 n I}{2} (\cos \theta_2 - \cos \theta_1) \hat{z}$$

(C) The magnetization in the iron will lead to bound currents.

For this problem

$$\vec{M} = M_0 \hat{z}$$

so the bound current are $-\hat{\phi}$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$$

$$\vec{K}_b = \vec{M} \times \hat{n} = M_0 \hat{z} \times \hat{s} = M_0 \hat{\phi} \text{ si} = \mu_0 M_0 (-\hat{\phi}) \text{ in Gaussian}$$

\Rightarrow The piece of iron can be treated as a solenoid with

$$nI = |\vec{K}_b| = M_0$$

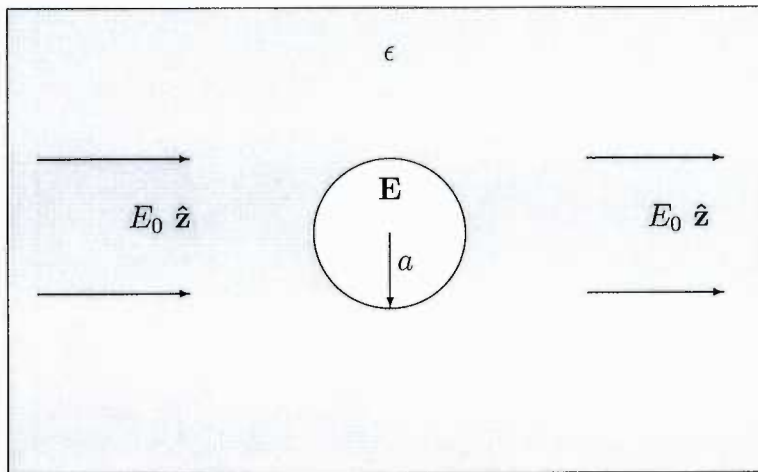
$$\Rightarrow \vec{B}(z) = \frac{\mu_0 M_0}{2} (\cos \theta_2 - \cos \theta_1) \hat{z} \quad \text{everywhere along the axis}$$

\vec{H} is related to \vec{B} by

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

$$\vec{H} = \vec{B} - 4\pi \vec{M} \text{ in Gaussian}$$

$$\Rightarrow \vec{H} = \begin{cases} \frac{M_0}{2} (\cos \theta_2 - \cos \theta_1) \hat{z} & \text{outside} \\ \left[\frac{M_0}{2} (\cos \theta_2 - \cos \theta_1) - M_0 \right] \hat{z} & \text{inside} \end{cases}$$



3. A small spherical vacuum bubble of radius a exists inside an otherwise homogeneous dielectric material whose electric polarization properties are described by a constant permittivity ϵ . Assume the bubble is centered on the origin and that the electric field far from vacuum bubble but still in the homogeneous material is of the form $\mathbf{E} = E_0 \hat{z}$. Because of the axial symmetry the electrostatic potential $\Phi(r, \theta)$ for this problem can be written as a linear combination of constants times Legendre polynomials.
 - (a) [2 pts] What are the boundary conditions satisfied by the fields \mathbf{E} and \mathbf{D} fields, and the potential Φ at the junction $r = a$?
 - (b) [2 pts] Give the electrostatic potential inside the bubble as a combination of constants and Legendre polynomials (keep only non-vanishing ℓ terms).
 - (c) [2 pts] Give the electrostatic potential outside the bubble as a combination of constants and Legendre polynomials (keep only non-vanishing ℓ terms).
 - (d) [2 pts] Use the boundary conditions at the $r = a$ junction from part (a) to evaluate the non-vanishing constants in parts (b) and (c).
 - (e) [2 pts] Express the electric field outside the bubble as an electric dipole field plus the uniform field $E_0 \hat{z}$ and give the value of the dipole moment.

#3

(a) $\vec{\nabla} \times \vec{E} = 0 \Rightarrow \underline{\vec{E}_{\text{tangential}} \text{ is continuous at } r=a}$
 $\vec{\nabla} \cdot \vec{D} = 0 \Rightarrow \underline{\vec{D}_{\text{normal}} \text{ is continuous at } r=0}$

$\vec{E}_{\text{tang}} \text{ is cont} \Leftrightarrow \frac{\partial \Phi}{\partial \theta} \text{ is continuous} \Leftrightarrow \underline{\Phi \text{ is cont.}}$

$\vec{D}_{\text{normal}} \text{ is cont} \Leftrightarrow \underline{\epsilon \frac{\partial \Phi}{\partial r} \text{ is continuous}}$

(b)
 $r \leq a$

$\Phi_{\text{in}} = B_l r^l \cos \theta$

($l=1$ only)

could have $l=0$ but
 chose $\phi=0$ at $r=0$!

$\frac{B_l}{r^{l+1}} \rightarrow \infty \text{ at } r \rightarrow 0$

+ there are no sources for $B_l r^l P_l$ $l \neq 1$
 terms

The $l=1$ source is $-E_0 z$!

(c) $r > a$

$$\Phi_{\text{out}} = \frac{B_1}{r^2} \cos \theta - E_0 r \cos \theta$$

↑

only $l=1$ at large r !no sources for

$$\frac{B_l}{r^{l+1}} P_l \quad l \neq 1 \text{ term!}$$

$$(d) \quad \phi_{\text{in}} = \phi_{\text{out}} \text{ at } r=a$$

$$A_1 a \cos \theta = \frac{B_1}{a^2} \cos \theta - E_0 a \cos \theta$$

$$\epsilon_0 \frac{\partial \phi}{\partial r} \text{ in} = \epsilon \frac{\partial \phi}{\partial r} \text{ out}$$

$$\epsilon_0 A_1 \cos \theta = -2\epsilon \frac{B_1}{a^3} \cos \theta - \epsilon E_0 \cos \theta$$

$$\Rightarrow A_1 + E_0 = B_1/a^3 \quad \Rightarrow \left(\frac{\epsilon}{\epsilon_0} - 1 \right) E_0 = - \left(\frac{2\epsilon}{\epsilon_0} + 1 \right) \frac{B_1}{a^3}$$

$$A_1 + \frac{\epsilon}{\epsilon_0} E_0 = -2\epsilon/\epsilon_0 \frac{B_1}{a^3}$$

$$B_1 = \frac{- \left(\frac{\epsilon}{\epsilon_0} - 1 \right) E_0 a^3}{\left(\frac{2\epsilon}{\epsilon_0} + 1 \right)}$$

$$\therefore A_1 = - \frac{3 \frac{\epsilon}{\epsilon_0}}{\left(\frac{2\epsilon}{\epsilon_0} + 1 \right)} E_0$$

(e)

$$\vec{E}_{out} = -\vec{\nabla} \phi_{at}$$

$$= -\vec{\nabla} \left\{ -\frac{\left(\frac{\epsilon}{\epsilon_0} - 1\right)}{\left(2\frac{\epsilon}{\epsilon_0} + 1\right)} E_0 a^3 \frac{\cos \theta}{r^2} - E_0 r \cos \theta \right\}$$

$$= \frac{\left(\frac{\epsilon}{\epsilon_0} - 1\right)}{\left(2\frac{\epsilon}{\epsilon_0} + 1\right)} E_0 a^3 \vec{\nabla} \left(\frac{z}{r^3} \right) + E_0 \hat{z}$$

||

$$\left(\frac{\hat{z}}{r^3} - \frac{3z\vec{r}}{r^5} \right)$$

$$\vec{E}_{out} = \frac{1}{4\pi\epsilon_0} \left[\frac{\vec{p} - (\vec{p} \cdot \hat{r}) \hat{r}}{r^3} \right] + E_0 \hat{z}$$

$$\frac{\vec{p}}{4\pi\epsilon_0} \equiv \frac{\left(\frac{\epsilon}{\epsilon_0} - 1\right)}{\left(2\frac{\epsilon}{\epsilon_0} + 1\right)} E_0 a^3 \hat{z}$$

↑
direction of \vec{p}

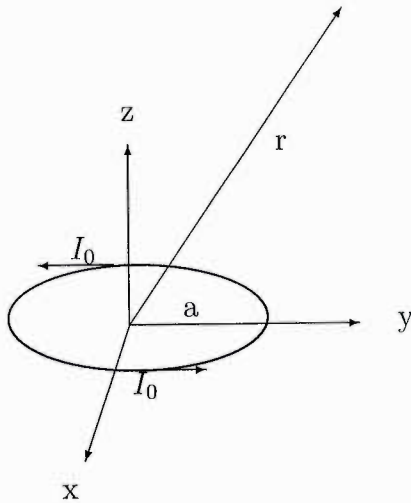
↑
 \vec{p} = dipole
moment

4. (a) [2 pts] Write down any vector potential that produces the uniform magnetic induction

$$\mathbf{B} = B_0 \hat{\mathbf{z}}.$$

- (b) [4 pts] What is the magnetic induction \mathbf{B} and an associated vector potential \mathbf{A} ($\mathbf{B} = \nabla \times \mathbf{A}$) produced by a very long wire located on the z -axis and carrying a current I_0 in the $+z$ direction?
- (c) [4 pts] A small circular loop of wire of radius a , centered at the origin and lying in the $z = 0$ plane, carries a current I_0 as shown in the figure. Derive an approximate expression for the vector potential at large distances ($r \gg a$) from the loop. Recall that

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r} \left\{ 1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \mathcal{O}\left(\frac{r'}{r}\right)^2 \right\}$$



#4

1/4

$$(a) \quad B_0 \hat{z} = \vec{\nabla} \times \vec{A} = \left(\frac{\partial}{\partial x} A^y - \frac{\partial}{\partial y} A^x \right) \hat{z}$$

can use $A^z = 0$, $A^x = -y/2 B_0$, $A^y = x/2 B_0$

$$\vec{A} = \frac{B_0}{2} (-y \hat{x} + x \hat{y}) = \frac{B_0}{2} \hat{z} \times \vec{r}$$

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla} \Lambda$$

$$(b) \quad \vec{\nabla} \times \vec{H} = \left(\frac{4\pi}{c} \right) \vec{J}$$

Gaussian

$$\Rightarrow \oint \vec{H} \cdot d\vec{l} = \frac{4\pi}{c} I_0$$

$$H 2\pi r$$

$$\Rightarrow H = \frac{4\pi}{c} \frac{I_0}{2\pi r}$$

$$\Rightarrow \vec{B} = \left(\frac{\mu_0}{4\pi} \right) \frac{2 I_0}{r} \hat{\phi} = \vec{\nabla} \times \vec{A}$$

SI

Gaussian

$$\hat{\phi} = \hat{z} \times \hat{r} \quad , \quad \frac{\hat{\phi}}{r} = \vec{\nabla} \ln r$$

$$\frac{\hat{\phi}}{r} = \hat{z} \times \vec{\nabla} \ln r = -\vec{\nabla} \times (\ln r \hat{z})$$

$$\therefore \vec{B} = \vec{\nabla} \times \left\{ -\frac{\mu_0}{4\pi} \frac{2 I_0}{c} \ln \rho \hat{z} \right\}$$

||

$$\left\{ \vec{A} + \vec{\nabla} \Lambda \right\}$$

eg. $\vec{A} = \frac{\mu_0}{4\pi} \frac{I_0}{c} \ln \left(\frac{\rho_0}{\rho} \right)^2 \hat{z}$ $\rho_0 = \text{arbitrary}$

(c) For a dipole field we expect

3/4

$$\vec{A}(\vec{r}) = \underbrace{\left(\frac{\mu_0}{4\pi}\right)}_{\substack{\uparrow \\ \text{SI}}} \frac{\vec{m} \times \vec{r}}{r^3} \quad , \quad \vec{m} = \frac{\pi a^2 I \hat{z}}{\underbrace{c}_{\substack{\uparrow \\ \text{Gaussian}}}}$$

mag

To derive this solve

$$\nabla^2 \vec{A} = -\mu_0 \frac{I}{c} \vec{J}(\vec{r}) \Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi c} \int \frac{\vec{J}(\vec{r}') d^3r'}{|\vec{r} - \vec{r}'|}$$

which for a wire is

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi c} \oint \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$= \frac{\mu_0 I}{4\pi c} \oint \frac{1}{r} \left[1 + \frac{\vec{r} \cdot \vec{r}'}{r^2} + \dots \right] d\vec{r}'$$

$$\approx \frac{\mu_0 I}{4\pi c} \frac{1}{r^3} \vec{r} \cdot \oint \vec{r}' d\vec{r}' + \mathcal{O}\left(\frac{1}{r^3}\right)$$

$$\vec{r}' d\vec{r}' = a^2 (\cos\phi' \hat{i} + \sin\phi' \hat{j}) (-\sin\phi' \hat{i} + \cos\phi' \hat{j}) d\phi'$$

$$\oint \vec{r}' d\vec{r}' = a^2 \left[\hat{i} \hat{j} \int_0^{2\pi} \cos^2 - \hat{j} \hat{i} \int_0^{2\pi} \sin^2 \right] d\phi'$$

$$= a^2 \pi [\hat{i} \hat{j} - \hat{j} \hat{i}]$$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0 I \pi a^2}{4\pi c} \frac{x \hat{j} - y \hat{i}}{r^3}$$

$$= \frac{\mu_0}{4\pi} \frac{m}{r^3} (\hat{\phi}) = \frac{\mu_0}{4\pi} \left(\frac{\vec{m} \hat{z} \times \vec{r}}{r^3} \right)$$

\downarrow
 $\hat{z} \times \vec{r}$

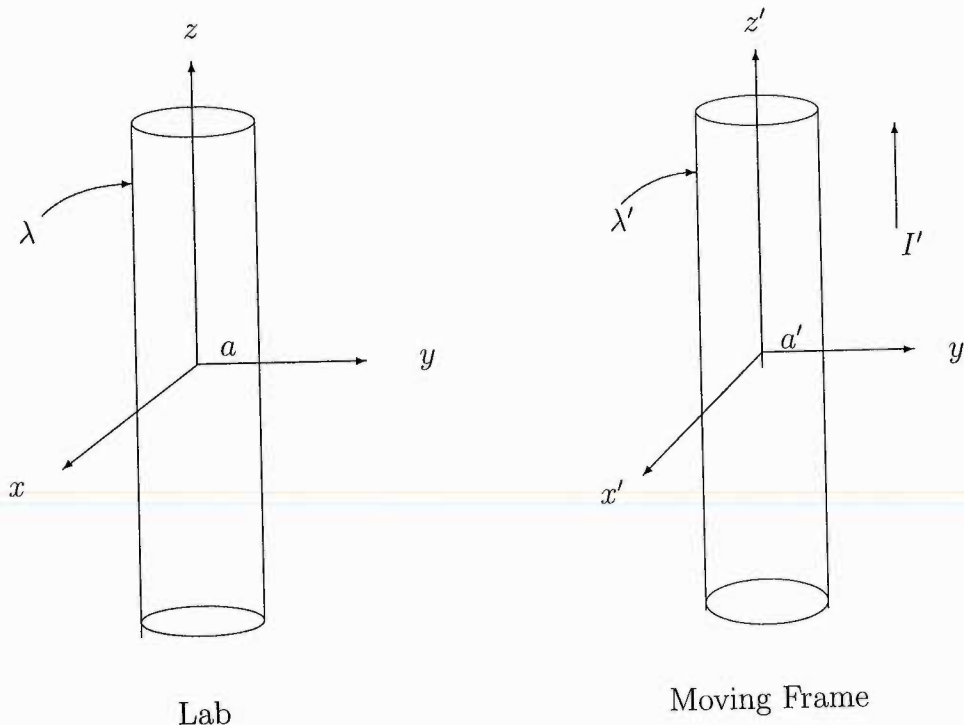
Dipole form!

$$\vec{m} \equiv m \hat{z} = \frac{I \pi a^2}{c} \hat{z}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} \text{ as expected}$$

5. An infinitely long, uniformly charged wire of radius a and total charge per unit length λ , is at rest on the z -axis of the lab frame.

- (a) [2 pts] Compute the electric field $\mathbf{E}(x, y, z)$ exterior to the wire in the lab frame by solving Gauss's law in that frame. What is the magnetic induction $\mathbf{B}(x, y, z)$ in this frame?
- (b) [2 pts] If you are moving in the lab's negative z direction with speed v how are your spatial and time coordinates related to those of the lab's? To answer this question simply give the Lorentz boost $x'^{\sigma} = L^{\sigma}_{\mu} x^{\mu}$ that relates the two sets of coordinates.
- (c) [2 pts] In your frame what is the radius a' of the wire? What is the charge/length λ' of the wire and what is the current I' in the wire?
- (d) [1 pts] Combine the E and B fields in the lab into a single electromagnetic field tensor $F^{\alpha\beta}$ using $F^{\sigma\mu} = -F^{\mu\sigma}$ and $F^{0i} = -E^i$. In Gaussian units $F^{12} = -B^z$, $F^{23} = -B^x$ and $F^{13} = B^y$, and in SI units $F^{12} = -c B^z$, $F^{23} = -c B^x$ and $F^{13} = c B^y$.
- (e) [3 pts] What electric field $\mathbf{E}'(x', y', z')$ and what magnetic induction $\mathbf{B}'(x', y', z')$ will you measure exterior to the wire in your frame? To answer this part you can use your answers for part (c) or you can compute $F' = L F L^T$.



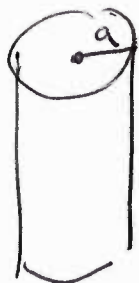
5
(a)

Red = Gaussian

Green = SI

1/3

2 pts



$$\epsilon_0 E 2\pi r b = \lambda L$$

$$\Rightarrow \vec{E} = \frac{2}{4\pi\epsilon_0} \frac{\lambda}{r} \hat{r}$$

$$\vec{B} = 0$$

(b) $x'^\sigma = L^\sigma{}_\mu x^\mu$

2 pts

$$L = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$ct' = \gamma ct + \beta\gamma z$$

$$x' = x$$

$$y' = y$$

$$z' = -\beta\gamma ct + \gamma z$$

$$\beta = v/c, \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

(c) $\boxed{a' = \sqrt{x'^2 + y'^2} = \sqrt{x^2 + y^2} = a}$ Same

2 pts

$$\Delta g = \lambda \Delta z = \lambda' \Delta z'$$

$\Delta z/\gamma$ Lorentz contracted

$$\therefore \boxed{\lambda' = \gamma \lambda}$$

$$I' = \frac{\Delta g}{\Delta t'} = \frac{\lambda' \Delta z'}{\Delta t'} = \lambda' v$$

$$\boxed{I' = \gamma \lambda v}$$

If preferred (c) can be worked using

$$\begin{pmatrix} c g \\ \vec{J} \end{pmatrix}' = L \begin{pmatrix} c p \\ \vec{J} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta \gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} c \lambda \delta(x) \delta(y) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow c g' = c \gamma \lambda \delta(x) \delta(y) = c \lambda' \delta(x') \delta(y')$$

$$\Rightarrow \underline{\lambda' = \gamma \lambda}$$

$$\vec{J}' = \beta \gamma c \lambda \delta(x) \delta(y) = \underbrace{\gamma \lambda}_{I'} \delta(x') \delta(y')$$

$g = g'$
 $\phi = \phi'$

(d) $F = \begin{pmatrix} 0 & -\omega \phi & -\sin \phi & 0 \\ \omega \phi & 0 & 0 & 0 \\ \sin \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \left(\frac{2}{4\pi\epsilon_0} \frac{1}{g'} \right)$

1 pt

(e) $F' = L \begin{pmatrix} 0 & -\omega \phi & -\sin \phi & 0 \\ \omega \phi & 0 & 0 & 0 \\ \sin \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \beta \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta \gamma & 0 & 0 & \gamma \end{pmatrix} = L \begin{pmatrix} 0 & -\omega \phi & -\sin \phi & 0 \\ \gamma \omega \phi & 0 & 0 & \beta \gamma \omega \phi \\ \gamma \sin \phi & 0 & 0 & \beta \gamma \sin \phi \\ 0 & 0 & 0 & 0 \end{pmatrix}$

3 pts

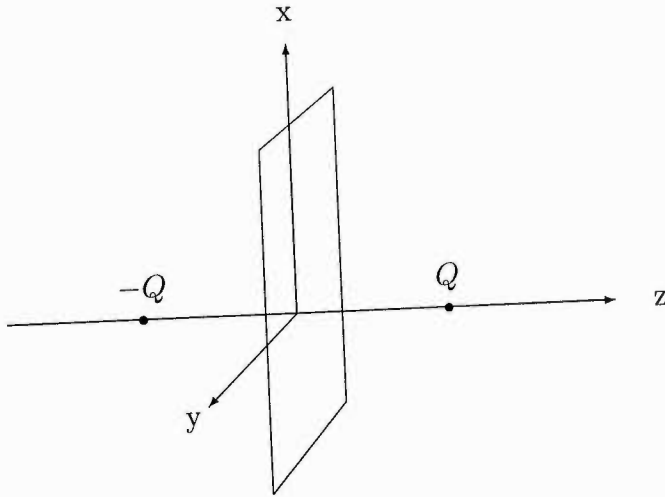
$$F' = \begin{pmatrix} 0 & -\gamma \omega d & -\gamma \omega d & 0 \\ \gamma \omega d & 0 & 0 & \beta \gamma \omega d \\ \gamma \omega d & 0 & 0 & \beta \gamma \omega d \\ 0 & -\beta \gamma \omega d & -\beta \gamma \omega d & 0 \end{pmatrix} \frac{2\lambda}{4\pi\epsilon_0 \rho}$$

$$\Rightarrow \vec{E}' = \frac{2\gamma\lambda}{4\pi\epsilon_0 \rho} \hat{y}$$

$$\vec{B}' = (-\beta \gamma \omega d, \beta \gamma \omega d, 0) \frac{2\lambda}{4\pi\epsilon_0 \rho}$$

$$\Rightarrow \vec{B}' = \frac{\hat{\phi} \beta \gamma 2\lambda}{c 4\pi\epsilon_0 \rho} = \frac{\vec{\beta}}{c} \times \vec{E}'$$

These can also be obtained from your answer on part (c)



6. This problem requires the use of Maxwell's stress tensor T_M^{ij} .

- (a) [3 pts] Compute Maxwell's stress tensor T_M^{ij} on the $z = 0$ plane for a system of two equal and opposite point charges ($\pm Q$) located on the z -axis at $\mathbf{r} = (0, 0, \pm b)$ as shown in the figure. For this application

$$T_M^{ij}|_{\text{Gaussian}} = \left(\frac{1}{4\pi} \right) \left[E^i E^j - \frac{\delta^{ij}}{2} E^2 \right],$$

or

$$T_M^{ij}|_{\text{SI}} = (\epsilon_0) \left[E^i E^j - \frac{\delta^{ij}}{2} E^2 \right].$$

- (b) [4 pts] Evaluate the surface integral

$$\int \int T_M^{ij} dA^j = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_M^{iz} dx dy,$$

over the $z = 0$ plane.

Hint: Use cylindrical polar coordinates to do the integral.

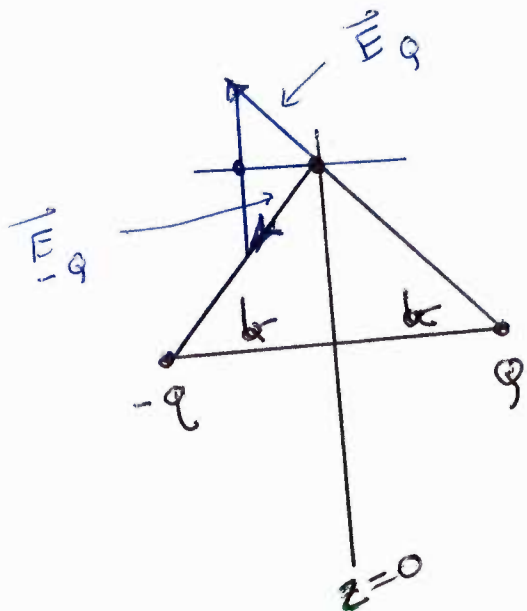
- (c) [3 pts] The following surface integral over the boundary of a closed volume V_3 is the total electromagnetic force on the E&M fields and their sources contained within that volume

$$F^i = \int_{\partial V_3} T_M^{ij} dA^j.$$

Use this fact to explain your answer to part (b).

(a) The integral is the net force on the $\epsilon + m$ fields + sources within the volume V_3 .

(b)



$$E_q = \frac{q}{4\pi\epsilon_0 (\rho^2 + b^2)}$$

$$\rho^2 = x^2 + y^2$$

$$\vec{E} = -2 \hat{z} \frac{q}{4\pi\epsilon_0 (\rho^2 + b^2)} \frac{b}{[\rho^2 + b^2]^{3/2}} = -\frac{2qb}{4\pi\epsilon_0 [\rho^2 + b^2]^{3/2}} \hat{z}$$

$$T_{zz} = \frac{\epsilon_0}{4\pi} \left[(E^z)^2 - \frac{1}{2} E^2 \right] = \boxed{\frac{\epsilon_0}{2(4\pi)} (E^z)^2}$$

$$= \frac{q^2 b^2}{(4\pi\epsilon_0) 2\pi [\rho^2 + b^2]^3}$$

$$T_{xx} = T_{yy} = \frac{\epsilon_0}{4\pi} \left[0 - \frac{1}{2} E^2 \right] = -T_{zz}$$

$$(c) \iint T_m^{i's} dA^s = \iint T_m^{iz} dx dy$$

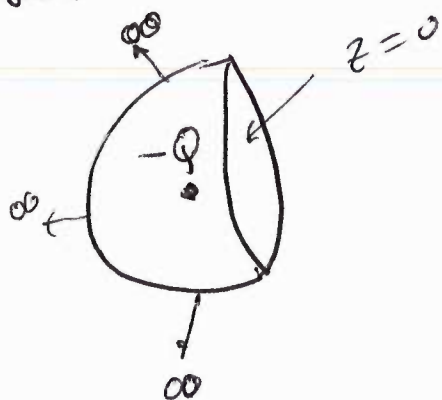
$$= \delta_z^i \frac{Q^2 b^3}{2\pi(4\pi\epsilon_0)} \iint \frac{dx dy}{[r^2 + b^2]^3}$$

$$= \delta_z^i \frac{Q^2 b^2}{2\pi(4\pi\epsilon_0)} \int_0^\infty \int_0^{2\pi} \frac{d\phi dr}{[r^2 + b^2]^3}$$

$$= \delta_z^i \frac{Q^2 b^2}{4\pi\epsilon_0} \left. \frac{-1}{4[r^2 + b^2]^2} \right|_0^\infty$$

$$\boxed{F^i = \delta_z^i \frac{Q^2}{4\pi\epsilon_0} \frac{1}{(2b)^2}}$$

(d) the integral is the same as integrating over the closed boundary containing the $-Q$ charge.



because $T^{i's} dA^s \rightarrow 0$
as $r \rightarrow \infty$!