



# Important E+M Equations

Maxwell's Laws:

$$\nabla \cdot \mathbf{D} = 4\pi \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = 0$$

$$\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}$$

$$\mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}$$

\*In linear, isotropic media:

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \epsilon = 1 + 4\pi \chi_e$$

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B}, \quad \mu = 1 + 4\pi \chi_m$$

Other Important Eqns:

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \quad (\text{Poynting vector})$$

$$\mathbf{g} = \frac{1}{4\pi c} \mathbf{E} \times \mathbf{H} \quad (\text{Momentum density})$$

$$U = \frac{1}{4\pi} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$$

(Energy Density)

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (\text{Continuity Eqn})$$

$$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) \quad (\text{Lorentz Force Law})$$

$$\mathbf{F} = \frac{q_1 q_2}{|\mathbf{r} - \mathbf{r}'|^2} \quad (\text{Coulomb's Law})$$

$$W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} dV \quad (\text{Work})$$

Electrostatics:

$$-\nabla \Phi = \mathbf{E} \quad (\text{Vector Potential})$$

$$\Phi(\mathbf{r}) = \sum_i \frac{q_i}{r - r_i}$$

$$= \int \frac{\rho}{|\mathbf{r} - \mathbf{r}'|} dV'$$

(Potential from charge distribution)

$$\mathbf{E} = \sum_i \frac{q_i (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}$$

$$= \int \frac{\rho (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'$$

(Electric Field)

$$V_{\text{dip}} = \frac{\hat{\mathbf{r}} \cdot \mathbf{P}}{r^2} = \frac{p \cos \theta}{r^2}$$

(Dipole potential)

Electrostatics:  $\vec{\tau} = \vec{p} \times \vec{E}$  (Torque on dipole)

$\rho_b = -\nabla \cdot \vec{P}$      $\sigma_b = \vec{P} \cdot \hat{n}$  (Bound Charge)

Magnetostatics:  $\vec{B} = \frac{1}{c} \int \frac{\vec{J} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'$  (Biot-Savart Law)

$\vec{B} = \nabla \times \vec{A}$  (Vector Potential)

$A(\vec{r}) = \frac{1}{c} \int \frac{J(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$      $\vec{\tau} = \vec{m} \times \vec{B}$

$\vec{A}_{\text{dip}} = \frac{\vec{m} \times \vec{r}}{cr^2}$  (Dipole Potential)

$\vec{J}_b = \nabla \times \vec{M}$      $\vec{K}_b = \vec{M} \times \hat{n}$  (Bound current)

Boundary Conditions:  $D_1^+ - D_2^+ = 4\pi\sigma_f$      $B_1^+ - B_2^+ = 0$

$E_1^+ - E_2^+ = 0$      $H_1^+ - H_2^+ = \frac{1}{c} \vec{K}_f$

$V_1 - V_2 = 0$      $A_1 - A_2 = 0$

Tensors/Relativity:  $\beta = \frac{v}{c}$      $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad F^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$      $\partial_\alpha F^{\alpha\beta} = 0$  (Maxwell Eqns)

$\frac{dp^\alpha}{d\tau} = m \frac{dU^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta$  (Lorentz Force)

Waves:  $n = \sqrt{\mu\epsilon}$

$k \propto \omega$  (from wave eqn)

# Electrodynamics Qualifier Preparation

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## 1 Mathematical Reminders

- Spherical
  - $h_1 = 1$
  - $h_2 = r$
  - $h_3 = r \sin \theta$
- Cylindrical
  - $h_1 = 1$
  - $h_2 = r$
  - $h_3 = 1$
- Unit Vector Cross Products:  $\hat{e}_u = \frac{\frac{\partial \vec{r}}{\partial u}}{\left| \frac{\partial \vec{r}}{\partial u} \right|}$  (see Schaum's)
- $\nabla^2 \frac{1}{z} = -4\pi \delta^3(\vec{z})$
- $\vec{\nabla} \cdot \left( \frac{\hat{z}}{z^2} \right) = 4\pi \delta^3(\vec{z})$
- To solve an inhomogeneous differential equation:
  - $\dot{x} + ax = b$  or  $\ddot{x} + a\dot{x} + bx = c$
  - $x(t) = \text{Homogeneous Solution} + \text{Non-homogeneous Solution}$
  - Homogeneous Solution is simple
  - Non-homogeneous solution: Whatever  $x$  should be so LHS=RHS – usually a constant
- Divergence Theorem:  $\oint_S \vec{E} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{E} d\tau$
- Stoke's Theorem:  $\int_S (\vec{\nabla} \wedge \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$

## 2 Electrostatics

The primary task of electrostatics is to find the electric field of a given stationary charge distribution. In principle, this is accomplished by:

- First calculating the potential,  $V$ , then solving for  $\vec{E}$
- Exploiting symmetry and using Gauss's law
- Calculating  $\vec{E}$  directly using Coulomb's law

One can solve Poisson's equation in a region with no charge distribution for the potential by:

- Method of Images
- Separation of Variables

## 2.3 Electric Potential

- NOT the same thing as potential energy

–  $W = QV$

- $V(\vec{r}) \equiv - \int_{\mathcal{O}}^{\vec{r}} \vec{E} \cdot d\vec{l}$

– where  $\mathcal{O} \equiv \infty$  for finite charge distributions

- $\vec{E} = -\vec{\nabla}V$

- Poisson's Equation:  $\nabla^2 V = -4\pi\rho$

– Derived from Gauss's Law

- Laplace's Equation:  $\nabla^2 V = 0$

– For regions with no charge

- Solution to Poisson's Equation:  $V(\vec{r}) = \int \frac{1}{z} dq$

- Obeys superposition principle:  $V = V_1 + V_2 + \dots$

- Boundary Conditions:

– Normal Component:  $E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp} = 4\pi\sigma$

– Parallel Component:  $\vec{E}_{\text{above}}^{\parallel} = \vec{E}_{\text{below}}^{\parallel}$

- Units:  $N \cdot m \cdot C^{-1} = J \cdot C^{-1} = V$

## 2.4 Work and Energy in Electrostatics

- Work Done to Move a Charge in an Electric Field:  $W = \int_a^b \vec{F} \cdot d\vec{l}$

– Force *you* must exert on the charge:  $\vec{F} = -Q\vec{E}$

– To bring a charge in from infinity:  $W = QV(\vec{r})$

- Energy of a Point Charge Distribution:  $W = \frac{1}{2} \sum_{i=1}^n q_i V(\vec{r}_i)$

- Energy of a Continuous Charge Distribution:

–  $W = \frac{1}{2} \int V dq$

–  $W = \frac{1}{8\pi} \int_{\text{all space}} E^2 d\tau$

- Work does not obey the superposition principle.

### 3.3 Separation of Variables

- Poisson's Equation:  $\nabla^2 V = -4\pi\rho$

- Properties of Separable Solutions:

- Completeness:  $f(y) = \sum_{n=1}^{\infty} C_n f_n(y)$

- Orthogonality:  $\int_0^a f_n(y) f_{n'}(y) dy = 0$  for  $n' \neq n$

- Method for Finding Separable Solution:

- Find differential equations for all components of separable solution and solve.
- Use boundary conditions and orthogonality to solve for all constants.

- General Solution to Laplace's Equation:

- Spherical Coordinates with azimuthal Symmetry:

$$V(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

where  $P_l(\cos \theta)$  is the Legendre Polynomial (see Schaum's for properties)

- Green's Function in Azimuthal Symmetry:

- $G(\vec{r} - \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos \gamma)$

where,  $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$

- If there is no azimuthal symmetry we use Associated Legendre Polynomial and Spherical Harmonics

$$Y_m^l(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi}$$

The properties of  $P_l^m$  is in Schaum's.

- Addition Theorem of Spherical Harmonics:

- $P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{l,m}^*(\theta', \phi') Y_{l,m}(\theta, \phi)$

where,  $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$

- Green's Function without Azimuthal Symmetry:

- $G(\vec{r} - \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{l,m}^*(\theta', \phi') Y_{l,m}(\theta, \phi)$

- Dipole Moment:

- $\vec{p} = \int \vec{r}' \rho(\vec{r}') dv'$

- Electrostatic Potential of a Point Dipole:

– Differential Equation:  $\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f$

• Interface Condition of Perpendicular Displacement Vector:  $D_{\text{above}}^\perp - D_{\text{below}}^\perp = 4\pi\sigma_f$

• Polarization for Linear Materials:  $\vec{P} = \chi_e \vec{E}$

– where  $\chi_e$  is the electric susceptibility

• Electric Permittivity:  $\epsilon = 1 + 4\pi\chi_e$

• Energy Stored in an Electric Field:  $W = \frac{1}{8\pi} \int \vec{E} \cdot \vec{D} d\tau$

## 5 Magnetostatics

• Lorentz Force Law:  $\vec{F} = Q \left[ \vec{E} + \frac{1}{c}(\vec{v} \wedge \vec{B}) \right]$

• **Magnetic forces do no work.**

• Current Units:  $C \cdot s^{-1} = A$

• Current Densities:

–  $\vec{I} = \lambda \vec{v} = \frac{q}{t}$

–  $\vec{K} = \sigma \vec{v} = \frac{d\vec{I}}{dl_\perp}$

–  $\vec{J} = \rho \vec{v} = \frac{d\vec{I}}{da_\perp}$

• Continuity Equation:  $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

–  $\vec{\nabla} \cdot \vec{J} = 0$  for magnetostatics

• Biot-Savart Law:  $\vec{B}(\vec{r}) = \frac{1}{c} \int \frac{\vec{v} \wedge \hat{z}}{r^2} dq$

• Divergence of Magnetic Field:  $\vec{\nabla} \cdot \vec{B} = 0$

• Ampere's Law:

– Differential Form:  $\vec{\nabla} \wedge \vec{B} = \frac{4\pi}{c} \vec{J}$

– Integral Form:  $\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_{\text{enc}}$

\* Derived from differential form using Stoke's Theorem

• The magnetic field outside of a solenoid is zero.

• Magnetic Vector Potential:

– Definition:  $\vec{B} = \vec{\nabla} \wedge \vec{A}$

- If your geometry has a piece missing try superposition with reverse current.
- $\vec{B} = \mu \vec{H}$ 
  - where  $0 \leq \mu \leq 1$  for diamagnetic materials and  $\mu \geq 1$  for paramagnetic materials
- Magnetization:  $\vec{M} = \chi_m \vec{H}$ 
  - where  $\chi_m \leq 0$  for diamagnetic materials
- Magnetic Permeability:  $\mu = 1 + 4\pi\chi_m$
- Electromagnetic Energy Density:  $u_{em} = \frac{1}{8\pi}(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H})$

## 7 Electrodynamics

- Ohm's Law:  $\vec{J} = \sigma \vec{f}$ 
  - where  $\sigma$  is the conductivity of the medium and  $\vec{f}$  is the force per unit charge
  - $V = IR$
- Power:  $P = VI$
- Electromotive Force:  $\mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{l}$
- Flux of  $\vec{B}$  through a closed surface  $S$ :  $\Phi = \int_S \vec{B} \cdot d\vec{a}$
- Faraday's Law:
  - Differential Form:  $\mathcal{E} = -\frac{1}{c} \frac{d\Phi}{dt}$
  - Integral Form:  $\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{d\Phi}{dt}$
- Lenz's Law: **Nature abhors a change in flux.**
  - An induced flux will flow in the direction that cancels a change in flux
  - Remember: Opposite currents repel
- Self Inductance:  $\Phi = LI$
- Energy Stored in a Magnetic Field:  $W = \frac{1}{8\pi} \int_{\text{all space}} \vec{B} \cdot \vec{H} d\tau$
- Maxwell's Equations

Gauss's Law:	$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$	No Name:	$\vec{\nabla} \cdot \vec{B} = 0$
Faraday's Law:	$\vec{\nabla} \wedge \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$	Ampere's Law:	$\vec{\nabla} \wedge \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{J}$

- **A changing electric field induces a magnetic field.**



- Maxwell Stress-Energy Tensor:  $T_{ij} = \frac{1}{4\pi} [E_i D_j + H_i B_j - \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \delta_{ij}]$

– Momentum flux density

- The total force on the charges in volume V:

$$\vec{F} = \frac{d\vec{p}_{\text{mech}}}{dt} = \oint_S \sum_{\beta} T_{\alpha\beta} n_{\beta} da - \frac{1}{4\pi c} \frac{d}{dt} \int_V (\vec{E} \wedge \vec{B}) dv$$

– Momentum Stored in an Electromagnetic Field:  $\vec{p}_{\text{em}} = \frac{1}{4\pi c} \int_V (\vec{E} \wedge \vec{B}) dv = \frac{1}{c^2} \int \vec{S} dv$

– the first integral is the momentum per unit time flowing in through the surface.

– Differential Form:  $\frac{\partial}{\partial t} (\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \vec{\nabla} \cdot \vec{T}$

– Poynting vector has two quite different roles:  $\vec{S}$  itself is the energy per unit area, per unit time, transported by the electromagnetic fields, while  $\frac{1}{c^2} \vec{S}$  is the momentum per unit volume stored in those fields.

– Similarly  $T^{\alpha\beta}$  itself is the electromagnetic stress acting on a surface, and  $-T^{\alpha\beta}$  describes the momentum transported by the fields.

- Angular Momentum in electromagnetic field:

$$\vec{l}_{\text{em}} = \vec{r} \wedge \vec{p}_{\text{em}}$$

## 9 Electromagnetic Waves

- Wave Number:  $k = \frac{2\pi}{\lambda}$

- Period:  $T = \frac{2\pi}{kv}$

- Frequency:  $\nu = \frac{v}{\lambda}$

- Angular Frequency:  $\omega = 2\pi\nu = kv$

– Frequency is constant across interfaces

- Three Dimensional Wave Equation:

$$\nabla^2 \vec{E} = \frac{\epsilon\mu}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \frac{\epsilon\mu}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

- Monochromatic Plane Waves: Sinusoidal waves of frequency  $\omega$  traveling in an arbitrary direction  $\hat{k}$  polarized in the  $\hat{n}$  direction

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta)} \hat{n}$$

$$\vec{B}(\vec{r}, t) = \sqrt{\epsilon\mu} \hat{k} \wedge \vec{E}$$

\* For transverse waves:  $\hat{n} \cdot \hat{k} = 0$

- Length Contraction:  $\Delta x = \frac{\Delta x'}{\gamma}$

- Moving objects are shortened.
- Only dimensions parallel to the velocity are contracted.

- General Lorentz Transformation:  $\left( \begin{array}{c|c} \gamma & -\gamma\vec{\beta} \\ \hline -\gamma\vec{\beta} & P_{\perp} + \gamma P_{\parallel} \end{array} \right) = \left( \begin{array}{c} ct' \\ \vec{x}' \end{array} \right)$

where,  $P_{\parallel} = \frac{\beta_i \beta_j}{\beta^2}$  and  $P_{\perp} = \delta_{ij} - P_{\parallel}$

- Contravariant Vector:  $x^{\mu} = (ct, \vec{x})$ ,  $p^{\mu} = (E/c, \vec{p})$
- Covariant Vector:  $x_{\mu} = (ct, -\vec{x})$ ,  $p_{\mu} = (E/c, -\vec{p})$

- Minkowski Metric:  $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

- $x^{\mu} = g^{\mu\nu} x_{\nu}$

- Proper Length:  $(\Delta s)^2 = (c\Delta\tau)^2 = (ct)^2 - x^2 - y^2 - z^2$

- Charge Conservation:  $\partial_{\mu} j^{\mu} = 0$

- where  $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$  and  $j^{\mu} = (c\rho, \vec{j})$

- Proper Velocity:

- $u'^{\alpha} = \frac{dx'^{\alpha}}{d\tau} = \frac{dx'^{\alpha}}{dt'} = (c, 0, 0, 0)$

- $u^{\alpha} = \frac{dx^{\alpha}}{d\tau} = \frac{dx^{\alpha}}{dt/\gamma} = \gamma(c, \vec{v})$

- $u^{\alpha} u_{\alpha} = c^2$

- Proper Acceleration:

- $a^{\alpha} = \frac{du^{\alpha}}{d\tau} = \gamma^4 \hat{\beta} \hat{\beta}(c, \vec{v}) + \gamma^2(0, \hat{v})$

- $a^{\alpha} u_{\alpha} = 0$

- Relation of E-B fields in rest and lab frames:

- $\vec{E} = \vec{E}'_{\parallel} + \gamma(E'_{\perp} - \vec{\beta} \wedge \vec{B}')$

- $\vec{B} = \vec{B}'_{\parallel} + \gamma(B'_{\perp} + \vec{\beta} \wedge \vec{E}')$

- $\vec{B} = \vec{\beta} \wedge \vec{E}$

- Field-Strength Tensor:

- $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$

- where  $F_{ij} = \epsilon_{ijk} B^k$  and  $F_{0i} = -E_i$

- Magnetic Dipole Moment:

$$- \bar{m} = \frac{1}{2c} \int \bar{r} \wedge \bar{J}(\bar{r}) d^3r$$

- A point magnetic dipole at rest:

$$- \left[ \rho'(\bar{r}') = 0 \quad \bar{J}'(\bar{r}') = -c\bar{m}'(t) \wedge \bar{\nabla}'\delta^3(\bar{r}' - \bar{r}'_m) \right]$$

- A point magnetic dipole in motion:

$$- \left[ \rho(\bar{r}) = -c(\bar{\beta} \wedge \bar{m}') \bar{\nabla}\delta^3(\bar{r} - \bar{r}_m(t)) \right]$$

$$- \left[ \bar{J}(\bar{r}) = -c\bar{m}' \wedge \bar{\nabla}\delta^3(\bar{r} - \bar{r}_m(t)) \right]$$

- A moving dipole has an electric dipole moment  $\bar{P} = \bar{\beta} \wedge \bar{m}'$

### 13 Radiation

- Potential of a charge  $q$  located at  $r'_0$  in the rest frame:

$$- A'^{\alpha}(\bar{r}') = \frac{q}{|\bar{r}' - \bar{r}'_0|} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- In the lab frame  $\bar{r}'_0$  transforms to  $\bar{r}_0$  as  $\bar{r}_{0\parallel} = \gamma^{-1}\bar{r}'_{0\parallel}$  and  $\bar{r}_{0\perp} = \bar{r}'_{0\perp}$  but in the lab frame the charge moves with a constant velocity  $v$ , so the position of the charge  $\bar{r}_q(t) = \bar{r}_0 + vt$

- Potential of the charge as a function of the current position of the charge:

$$- A^{\alpha}(\bar{r}) = \frac{\gamma q \begin{pmatrix} 1 \\ \beta \end{pmatrix}}{\sqrt{|r_{\perp} - r_{q\perp}(t)|^2 + \gamma^2|r_{\parallel} - r_{q\parallel}(t)|^2}}$$

where,  $r_{q\perp} = r_{0\perp}$  and  $r_{q\parallel} = r_{0\parallel} + vt$

- Potential of the charge as a function of its 'retarded' position:

$$- \left[ A^{\alpha}(\bar{r}_{ret}) = \frac{q \begin{pmatrix} 1 \\ \beta \end{pmatrix}}{R(1 - \bar{\beta} \cdot \hat{n})} = \frac{qu^{\alpha}}{\gamma c R(1 - \bar{\beta} \cdot \hat{n})} \right]$$

where,  $\vec{R} = \vec{r} - \vec{r}_q(t_{ret})$  and  $\hat{n} = \frac{\vec{R}}{R}$

$$- \text{In covariant form: } \left[ A^{\alpha}(x) = \frac{qu^{\alpha}}{Z^{\beta}u_{\beta}} \right]$$

where,  $Z^{\beta} = x^{\beta} - x^{\beta}_{ret}(x)$  and  $u^{\alpha} = u^{\alpha}(t_{ret}(x))$

- From Maxwell's Equation:  $\square A^{\alpha} = \frac{4\pi}{c} J^{\alpha}$

- To solve this equation we need a Green's function such that  $\square_x G(x, x') = \delta^4(x - x')$

- There are two sets of boundary conditions of interest and two Green's functions  $D^{ret}(x, x')$  and  $D^{adv}(x, x')$

$$- \left[ D^{ret}(x, x') = \frac{\delta(x^0 - x'^0 - |\bar{r} - \bar{r}'|)}{4\pi|\bar{r} - \bar{r}'|} \right]$$

# E+m Qual Equation Sheet

## ① Vector Identities

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$$

$$A \times (B \times C) = B(A \cdot C) - (A \cdot B)C$$

$$\nabla(fg) = f(\nabla g) + (\nabla f)g$$

$$\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A$$

$$\nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot (\nabla f)$$

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

$$\nabla \times (fA) = f(\nabla \times A) - A \times (\nabla f)$$

$$\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A)$$

$$\nabla \cdot \nabla \times A = 0$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$$

## ② Maxwells Equations

\*In free space:  $\nabla \cdot E = \frac{\rho}{\epsilon_0}$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B - \frac{1}{c^2} \frac{\partial E}{\partial t} = \mu_0 \vec{J}$$

\*In media:  $\nabla \cdot B = 0$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$D = \epsilon_0 E + P = \epsilon_0 (1 + \chi_e) E \quad (\text{linear})$$

$$\nabla \cdot D = \rho$$

$$\nabla \times H - \frac{\partial D}{\partial t} = \vec{J}$$

$$H = \frac{1}{\mu_0} B - M$$

\*Other equations:  $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$  (Conservation of charge)

$$S = E \times H \quad (\text{in media})$$

$$= E \times B \quad (\text{in free space})$$

$$B = \nabla \times A \quad (\text{vector potential})$$

$$E = -\nabla \Phi \quad (\text{scalar potential})$$

$$F = q(E + v \times B) \quad (\text{Lorentz Force Law})$$



⇒ Magnetostatics

\* There are two relevant cases in solving magnetostatics problems

①  $\mathbf{J}(\vec{x}) = 0$  : In this case we use the fictitious  $\varphi_m$  scalar potential and all our solutions from electrostatics will still work

②  $\mathbf{J}(\vec{x}) \neq 0$  :  $\mathbf{B} = \nabla \times \mathbf{A}$ ;  $\mathbf{A} = \int \frac{\mu_0}{4\pi} \frac{\mathbf{J}(\vec{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'$

# E+M Midterm Study Guide

## Basics

\* Maxwell's equations in SI are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (\text{Gauss Law})$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \quad (\text{Faraday's Law})$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \quad (\text{Ampere's Law})$$

\* The Lorentz force law is:  $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

## Electrostatics

\* A simple form of the force law is:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2 (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$

$$\Rightarrow \text{Defining } \mathbf{E} = \frac{\mathbf{F}}{q}$$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}') (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x' \quad (\text{Coulomb's Law})$$

\* but since  $\mathbf{E}(\mathbf{x}) = -\nabla\Phi$ , and  $\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{-(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

\* From  $\Phi(\mathbf{x})$ , we know that the work done to position the charges in their current arrangement is

$$W = q \Delta\Phi$$

\* From Stokes Law ( $\oint_S (\nabla \times \mathbf{A}) \cdot \hat{n} da = \oint_L \mathbf{A} \cdot d\mathbf{l}$ ), we see that

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{n} = \frac{\sigma}{\epsilon_0} \Rightarrow E_{\perp} \text{ is discontinuous on a boundary}$$

$$\mathbf{E}_2 - \mathbf{E}_1 = \mathbf{0} \Rightarrow E_{\parallel} \text{ is continuous across a boundary}$$

## Electrostatics (cont.)

\* Combining what we know about  $E$  and  $\Phi$

$$\Rightarrow \nabla \cdot (-\nabla \Phi) = \frac{\rho}{\epsilon_0}$$
$$-\nabla^2 \Phi = \frac{\rho}{\epsilon_0} \quad (\text{Poisson Equation})$$

\* Note: Laplace equation is special case where  $-\nabla^2 \Phi = 0$

$$\Rightarrow \text{This implies } \nabla^2 \frac{1}{|x-x'|} = 4\pi \delta^3(x-x') \quad [\text{To avoid discontinuity at origin}]$$

## Boundary Value Problems

\* We have two different sets of boundary values we may consider

- ① Dirichlet Conditions  $\rightarrow \Phi$  known on boundary
- ② Neumann Conditions  $\rightarrow E$  known on boundary

\* We account for surface terms in our equations via Green's Identities

### ① Green's First Identity

\* From the divergence theorem  $\int_V (\nabla \cdot A) dV = \oint_S A \cdot \hat{n} ds$ , if  $A = \psi \cdot \nabla \chi$

$$\Rightarrow \int_V (\psi \nabla^2 \chi + \nabla \psi \cdot \nabla \chi) d^3x = \oint \psi \frac{d\chi}{dn} da$$

### ② Green's Theorem

$\Rightarrow$  If we interchange  $\chi \leftrightarrow \psi$  in the above and subtract the two we get

$$\int_V (\psi \nabla^2 \chi - \chi \nabla^2 \psi) d^3x = \oint (\psi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \psi}{\partial n}) da$$

\* If we then specify  $\chi = \frac{1}{|x-x'|}$ ,  $\psi = \Phi(x, y, z)$

$$\Rightarrow \Phi(x) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x')}{|x-x'|} d^3x' + \frac{1}{4\pi} \oint_S \left( \frac{1}{R} \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n} \left( \frac{1}{R} \right) \right) da$$

$\uparrow$  Neumann       $\uparrow$  Dirichlet

\* Uniqueness theorem says we only need 1 set of B.C.'s to solve problem



## Green's Functions

\* Applying this technique to Poisson's Eqn, we already know:

$\frac{1}{|x-x'|}$  is a solution when  $\rho = 4\pi\epsilon_0 \delta(x-x')$

$\Rightarrow \nabla^2 \Phi = -4\pi \delta(x-x') \rightarrow \Phi$  is Green's function for this eqn

$$G(x, x') = \frac{1}{|x-x'|} + F(x, x') \text{ where } \nabla^2 F(x, x') = 0$$

\* For Dirichlet boundary conditions, we find

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int_V \rho(x') G_D(x, x') d^3x' + \frac{1}{4\pi} \oint_S \Phi(x') \frac{\partial G_D(x, x')}{\partial n'} da'$$

\* For Neumann boundary conditions, we find

$$\Phi(x) = \langle \Phi(x) \rangle_{\text{on } S} + \frac{1}{4\pi\epsilon_0} \int_V \rho(x') G_N(x, x') d^3x' + \frac{1}{4\pi} \oint_S \frac{\partial \Phi}{\partial n'} G_N(x, x') da'$$

\* We typically solve for  $G_D(x, x')$  or  $G_N(x, x')$  via method of images

## Orthogonal Functions

\* For some geometries, it is easier to solve using orthogonal functions over the method of images

ex.  $\nabla^2 \Phi = 0$

\* Solve by separation of variables in various coordinate systems

$\Rightarrow$  Cartesian:

$$\Phi(x, y, z) = \sum_{m, n} A_{mn} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{mn} z)$$
$$A_{mn} = \frac{4}{ab \sinh(\gamma_{mn} c)} \int_0^a dx \int_0^b dy V(x, y) \sin(\alpha_n x) \sin(\beta_m y)$$

$\Rightarrow$  Spherical:

$$\Phi(r, \theta, \varphi) = \frac{U(r)}{r} P(\theta) Q(\varphi)$$

$$\hookrightarrow U(r) = A r^{\ell+1} + B r^{-\ell}$$

\* If azimuthally symmetric:  $\Phi(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + B_{\ell} r^{-\ell-1}) P_{\ell}(\cos \theta)$

\* If no azimuthal symmetry:  $\Phi(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1}) Y_{\ell m}(\theta, \varphi)$

## Orthogonal Functions (cont.)

\* but we can also expand our known solutions in terms of these functions

$$\frac{1}{|x-x'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

⇒ Cylindrical:

$$\Phi(\rho, \varphi, z) = R(\rho) Q(\varphi) Z(z)$$

$$\hookrightarrow Q(\varphi) = A e^{-im\varphi} + B e^{im\varphi}$$

$$Z(z) = C e^{kz} + D e^{-kz}$$

$$R(\rho) \sim A J_\nu(\rho) + B N_\nu(\rho)$$

where  $J_\nu$  is Bessel function of first kind

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_j \frac{(-1)^j}{j! \Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

$N_\nu(x)$  is Bessel function of second kind

$$N_\nu(x) = \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

# E+M Final Study Guide

## Electrostatics in Media

\* In a vacuum, our electrostatic equations are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = 0$$

⇒ But presence of  $\vec{E}$ -field induces a dipole moment in the atom/molecules of the medium

$$\hookrightarrow \nabla \cdot \mathbf{D} = \rho, \quad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\hookrightarrow \text{Alternatively, } \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho_{\text{eff}}, \quad \rho_{\text{eff}} = \rho(x) - \nabla \cdot \mathbf{P}$$

\* Note: The polarization  $\mathbf{P}$  is defined as the avg. dipole moment per unit volume

⇒ We typically assume a linear media such that:  $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$

$$\hookrightarrow \mathbf{D} = \epsilon_0 (1 + \chi_e) \mathbf{E} \quad \text{> } \epsilon = \epsilon_0 (1 + \chi_e) \text{ is electric permittivity} \\ = \epsilon \mathbf{E}$$

↳  $\frac{\epsilon}{\epsilon_0}$  is the dielectric constant

\* Given the above, our boundary conditions become:  $\Delta D_{\perp} = \sigma$  (discontinuous)

$$\Delta E_{\parallel} = 0 \text{ (continuous)}$$

## Magnetostatics

Important quantities:  $\vec{\mu}$  = magnetic dipole strength

$\vec{B}$  = magnetic induction / magnetic flux density

$\vec{H}$  = magnetic field

$$\vec{N} = \text{torque} = \vec{\mu} \times \vec{B}$$

\* Since we know there is no magnetic charge, we automatically know one of our magnetostatics equations is:  $\nabla \cdot \mathbf{B} = 0$

## Magnetostatics (cont.)

\* Using conservation of charge:  $\frac{dq}{dt} = 0$

$$\begin{aligned}\Rightarrow \frac{dq}{dt} = 0 &= \int \left( \frac{dp}{dt} + \frac{dp}{dx} \frac{dx}{dt} + \frac{dp}{dy} \frac{dy}{dt} + \frac{dp}{dz} \frac{dz}{dt} \right) dV \\ &= \int \left( \frac{dp}{dt} + \mathbf{v} \cdot \nabla p \right) dV \\ &= \int \left( \frac{dp}{dt} + \nabla \cdot (p\vec{v}) - p(\nabla \cdot \vec{v}) \right) dV \quad (\text{use } \mathbf{J} = p\vec{v}) \\ \Rightarrow \frac{dp}{dt} + \nabla \cdot \mathbf{J} &= 0 \quad (\text{Continuity Equation})\end{aligned}$$

\* Immediately we define the following:  $\mathbf{I} = \int \vec{J} \cdot d\vec{a}$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\mathbf{I} d\vec{l} \times \vec{r}}{|\mathbf{x}|^3} \quad (\text{Biot-Savart Law})$$

$$\vec{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x'$$

$\Rightarrow$  If we replace  $\frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$  by  $\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|}$  we see that  $\mathbf{B}$  is a pure curl

$$\hookrightarrow \mathbf{B} = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

\* Taking  $\nabla \times \mathbf{B}$ , we see:  $\nabla \times \mathbf{B} = \mu_0 \vec{J}$  (differential form, Ampere's Law)

$\hookrightarrow$  Applying Stokes Thm leads to:  $\oint \mathbf{B} \cdot d\vec{l} = \mu_0 I_{enc}$

\* Summarizing, our magnetostatic equations are:  $\nabla \cdot \mathbf{B} = 0$

$$\nabla \times \mathbf{B} = \mu_0 \vec{J}$$

$\Rightarrow$  To solve for  $\mathbf{B}$  we consider 2 cases:

①  $\mathbf{J}(\vec{x}) = 0 \rightarrow$  Solve as we did for  $\vec{E}$  via  $\mathbf{B} = -\nabla \phi_m \iff \nabla^2 \phi_m = 0$

$\hookrightarrow$  Laplace Equ w/ B.C.'s

②  $\mathbf{J}(\vec{x}) \neq 0 \rightarrow$  We use the fact that  $\nabla \cdot \nabla \times \mathbf{A} = 0$

$$\hookrightarrow \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{A} = \int \frac{\mu_0}{4\pi} \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \quad (\text{vector potential})$$

## Magnetostatics (cont.)

\* Note! Due to vector properties,  $A \rightarrow A + \nabla\gamma$  w/o changing our results

↳ We typically choose the Coulomb Gauge such that  $\nabla \cdot A = 0$

$$\Rightarrow \nabla \times (\nabla \times A) = \mu_0 \vec{J}$$

$$\nabla(\nabla \cdot A) - \nabla^2 A = \mu_0 \vec{J}$$

$$-\nabla^2 A = \mu_0 \vec{J} \quad (3 \text{ Poisson Eqns})$$

\* Note! We choose  $\gamma$  to satisfy above via:  $\nabla^2 \gamma = -\nabla \cdot \int \frac{\mu_0}{4\pi} \frac{\vec{J}(x')}{|x-x'|} d^3x'$   
 $= -f(x)$

⇒ We now proceed to solve problems via expansions in complete sets of orthogonal functions as before.

## Magnetostatics in Media

\* We proceed similarly to how we did in electrostatics

$$\Rightarrow \vec{J}(x) = \sum_i q_i v_i \delta^3(x-x_i)$$

$$\vec{L}_i = m_i (x_i \times v_i)$$

$$\vec{M}(x) = \sum_i q_i (x_i \times v_i)$$

$$\vec{M} = \sum_i \frac{q_i}{m_i} L_i$$

⇒ Remember that the magnetization  $\vec{M}$  is defined as the magnetic moment density

$$\hookrightarrow \vec{M} = \frac{\vec{m}}{V}$$

$$\hookrightarrow A(x) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(x') + \nabla' \times \vec{M}(x')}{|x-x'|} d^3x' \Rightarrow \vec{J}_{\text{eff}} = \vec{J}(x') + \nabla' \times \vec{M}(x')$$

$$\hookrightarrow \nabla \times B = \mu_0 (\vec{J} + \nabla \times \vec{M})$$

$$* \text{if } H = \frac{B}{\mu_0} + M$$

$$\nabla \times H = \vec{J}$$

\* Again assuming a linear media  $B = \mu H$ , where  $\mu =$  magnetic permeability

↳  $\mu = \mu_0$  in a vacuum

$\mu > \mu_0$  in paramagnetic media

$\mu < \mu_0$  in diamagnetic media

## Media (cont.)

\* At the intersection of two media, we now find:

$$\begin{aligned} B_{\perp} \text{ is continuous} & & H_{\perp} \text{ is discontinuous} & \left( n \times [H_2 - H_1] = \vec{K} \right) \\ H_{\parallel} \text{ is continuous} & & \rightarrow \vec{K} = \text{surface current density} \end{aligned}$$

\* Turning to our strategies to solve magnetostatics boundary value problems:

① Currents present in linear media ( $B = \mu H$ )

$$H = \frac{1}{\mu} (\nabla \times A)$$

$$\nabla^2 A = -\mu_0 \vec{J} \quad (\text{Solve Poisson Eqn})$$

②  $\vec{J} = 0$

$$\nabla \times H = 0 \Rightarrow H = -\nabla \Phi_m$$

$$-\mu \nabla^2 \Phi_m = 0 \quad (\text{Solve Laplace's Eqn})$$

③ Hard Ferromagnet (Know  $\vec{M}$ , but  $\vec{J} = 0$ )

$$\nabla \cdot B = 0$$

$$\nabla \times H = 0 \rightarrow H = -\nabla \Phi_m$$

$$\Rightarrow \nabla \cdot [\mu_0 (H + M)] = 0$$

$$\rightarrow \nabla^2 \Phi_m = \nabla \cdot M$$

=  $-\rho_m \rightarrow$  Solve w/ electrostatic solutions

$$\Rightarrow \underline{B} = \frac{\vec{M} \cdot \vec{x}}{4\pi r^3} \quad (x \gg x')$$

\* But we cannot ignore Faraday's Law

A changing magnetic flux generates an EMF

$$F = \int B \cdot dn \, da$$

$$\mathcal{E} = \int E' \cdot dl'$$

$$= \frac{dF}{dt}$$

$$\Rightarrow \oint_C E' \cdot dl' = -\frac{d}{dt} \int_S B \cdot \hat{n} \, da \rightarrow \oint_S (\nabla \times E + \frac{\partial B}{\partial t}) \cdot \hat{n} \, da = 0$$

## Media (cont.)

\* To determine the amount of energy stored in the fields:

$$W = \frac{1}{2} \int (\mathbf{E} \cdot \mathbf{D}) d^3x' \rightarrow \text{Electric Field}$$

$$W = \frac{1}{2} \int (\mathbf{B} \cdot \mathbf{H}) d^3x' \rightarrow \text{Magnetic Field}$$

## Electrodynamics

\* Our final, corrected Maxwell Equations are:

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0 \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

\* In media, these equations become

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{D} = \rho \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$$

\* To solve problems, we introduce potentials such that:  $\mathbf{B} = \nabla \times \mathbf{A}$

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

\* Note: Homogeneous equations are automatically satisfied by these substitutions

\* But really, we replace  $\mathbf{A}' \rightarrow \mathbf{A} + \nabla \Lambda$

$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

\* known as Lorenz gauge when  
 $\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0$

$$\Rightarrow \nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$$

> Wave Equation

\* To solve for Green's Function of wave equation

$$D G = -4\pi \delta^3(\mathbf{x}-\mathbf{x}') \delta(t-t'), \quad D = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

$$\rightarrow G^\pm(\mathbf{x}, t, \mathbf{x}', t') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} \delta\left(t-t' - \left[t \mp \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right]\right)$$

\(\Rightarrow\) use retarded Green's functions, all old formulas work but now  
must evaluate at  $t = t' - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$

## Electrodynamics (cont.)

\* The Poynting vector  $\vec{S}$  represents energy flow w/in the system

$$\vec{S} = \vec{E} \times \vec{H}$$

$$\hookrightarrow E_{\text{field}} = \frac{\epsilon_0}{2} \int_V (E^2 + c^2 B^2) d^3x'$$

$$U = \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \quad (\text{Energy Density})$$

$\Rightarrow$  Our conservation of energy law is:

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{S} = -\vec{J} \cdot \vec{E}$$

\* Similarly, for momentum:

$$\frac{d\vec{P}_{\text{mech}}}{dt} = \int_V (\rho \vec{E} + \vec{J} \times \vec{B}) d^3x \quad \Rightarrow \quad \vec{g} = \frac{1}{c^2} \vec{S} = \frac{1}{c^2} (\vec{E} \times \vec{H}) \quad (\text{momentum density})$$

\* Defining the Maxwell stress tensor as:  $T_{\alpha\beta} = \epsilon_0 [E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2} (\vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B}) \delta_{\alpha\beta}]$

$$\begin{aligned} \hookrightarrow \frac{d}{dt} (\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}) &= \sum_{\beta} \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta} d^3x \\ &= \oint_S \sum_{\beta} T_{\alpha\beta} n_\beta da \end{aligned}$$



# E + M Qualifier Topics List

## January 2008

- #1: Maxwell's Eqs, Electrodynamics, Media, Conservation Laws
- #2: ~~Electro~~ Maxwell's Eqs, Relativity, Media, E, B, Plane-Cartesian
- #3: Electrodynamics, Media, Work, Poynting Thm
- #4: Waves, B, Stress Tensor
- 5: Electrodynamics, Media, Cylindrical or Spherical
- 6: 4-D Maxwell's Eqs

## August 2008

- 1: 4-D Maxwell's Eqn's, Waves
- 2: Waves, Maxwell's Eqs, B
- 3: Relativity, Mechanics, Conservation Laws
- 4: Cylindrical, B, Mechanics, Forces
- 5: Magnetostatics, Cylindrical, H, Energy
- 6: Electrostatics, Plane-Cartesian,

## Jan 2009

- 1: Electrostatics, Media, E, D, Energy, Spherical
- 2: Magnetostatics, Media, Cylindrical, B, Potential, Plane-Cartesian
- 3: Electrostatics, Plane-Cartesian, Potential
- 4: Magnetostatics, Spherical
- 5: Electrostatics, Media, Stress Tensor, Spherical, E, B
- 6: Waves, 4-D Maxwell's Eqs, Relativity

## Aug 2009

- 1: Magnetostatics, Media, B, H, m, Spherical, Cylindrical
- 2: Electrostatics, Potential, E,
- 3: Electrostatics, Media, Cylindrical, D, E, P
- 4: Waves, 4-D Maxwell's Eqn's, Relativity
- 5: 4-D Maxwell's Eqs, Relativity, Plane-Cartesian
- 6: Stress Tensor, Electrodynamics, Cylindrical, Plane-Cartesian

### Jan 2010

- 1: Electrostatics, Cylindrical,  $E$ , Force
- 2: Media, Electrostatics, Spherical,  $D$ ,  $P$ ,  $E$
- 3: Electrostatics, Spherical, Potential
- 4: Electrodynamics, Forces,  $J$
- 5: Waves,  $B$ , Poynting Thm
- 6: 4-D Maxwell Equations

### Aug 2010

- 1: Electrodynamics Media, Cylindrical,  $D$ ,  $E$ ,  $P$
- 2: Maxwell's Eqs, Waves
- 3: Electrostatics, Spherical, Potential,  $\sigma$
- 4: Electrodynamics, Relativity, Radiation
- 5: Waves, Relativity, 4-D Maxwell's eqns
- 6: Electrodynamics, Retarded quantities,  $A$ , Radiation Field, Poynting Thm, Cylindrical or Spherical

### Jan 2011

- 1: Electrostatics / Electrodynamics, Media,  $I$ , Plane-Cartesian
- 2: Radiation, Relativity, Waves
- 3: Electrostatics, Work, Plane-Cartesian
- 4: Electrostatics, Spherical, Potential,  $E$ ,  $\sigma$
- 5: Waves, Polarization
- 6: Magnetostatics, Spherical, Potential,  $B$

### Aug 2011

- 1: Electrostatics, Potential, Dipole, Spherical,  $\sigma$
- 2: Maxwell's Eqs, 4-D Maxwell's Eqs, Poynting Thm
- 3: Magnetodynamics?, Forces,  $\mu$ ,  $B$ , Cylindrical or Spherical
- 4: Waves, Maxwell's Eqs, Antennae,  $B$ ,  $E$
- 5: Relativity, Electrodynamics, Forces, Cylindrical 4-D EM, Mechanics
- 6: 4-D Maxwell's Eqs, Plane-Cartesian, 4-D EM

## Jan 2012

- 1: Electrostatics, Spherical,  $E$ , Media
- 2: Maxwell's Equations
- 3: Electrostatics,  $E$ ,  $B$ , Cylindrical, Poynting Thm
- 4: Electrostatics, Force, Media, Spherical,  $E$ , Potential,  $\sigma$
- 5: Stress Tensor, Electrostatics, Plane-Cartesian,  $E$
- 6: Waves, Relativity, 4-D EM

## Aug 2012

- 1: Media, Electrodynamics, Cylindrical,  $B$ ,  $J$
- 2: Maxwell's Equations, Waves,  $E$
- 3: Electrostatics, Spherical, Media, Potential
- 4: Waves, Polarization
- 5: Electrostatics, Relativity,  $E$ ,  $B$ , cylindrical, 4-D EM
- 6: Electrodynamics, Retarded Quantities, Plane-Cartesian,  $\rho$ ,  $J$ , Potential,  $A$

## Jan 2013

- 1: Electrodynamics, Spherical,  $B$
- 2: Media, Maxwell's Equations, Waves
- 3: 4-D Maxwell's Eqns, Relativity,  $E$ ,  $B$ , Plane-Cartesian
- 4: Electrodynamics, Forces, Plane-Cartesian
- 5: ~~Electrodynamics~~ Electrostatics, Spherical, Potential
- 6: Electrostatics, Work, Plane-Cartesian, Potential

## Aug 2013

- 1: Waves
- 2: Magnetostatics
- 3: Media, Electrostatics
- 4: Magnetostatics, Maxwell's Eqns
- 5: Relativity, Electrodynamics
- 6: Stress Tensor