Important E+M Equations

Maxwell's Laws: $\nabla \cdot D = 4\pi \rho_f$ $\qquad$ $\nabla \cdot B = 0$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = 0$$

$$\vec{D} = \vec{E} + 4\pi \vec{P} \qquad H = B - 4\pi \underline{M}$$

*In linear, isotropic media:

$$\vec{D} = \epsilon E, \quad \epsilon = 1 + 4\pi \chi_e$$

$$\vec{H} = \frac{1}{\mu} B, \quad \mu = 1 + 4\pi \chi_m$$

Other Important Eqns: $S = \frac{c}{4\pi} E \times H$ (Poynting vector)

$U = \frac{1}{4\pi}(E \cdot D + B \cdot H)$ (Energy Density)

$g = \frac{1}{4\pi c} E \times H$ (Momentum density)

$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$ (Continuity Eqn)

$F = q\left(E + \frac{v}{c} \times B\right)$ (Lorentz Force Law)

$F = \frac{q_1 q_2}{|r - r'|}$ (Coloumb's Law)

$W = \frac{1}{2}\int D \cdot E \, dV$ (Work)

Electrostates: $-\nabla \Phi = E$ (Vector Potential)

$$\Phi(r) = \sum_i' \frac{q_i}{r - r_i} = \int \frac{\rho}{|r - r'|} dV'$$

(Potential from charge distribution)

$$E = \sum_i' \frac{q_i (r - r_i')}{|r - r'|^3} = \int \frac{\rho(r - r')}{|r - r'|^3} dV'$$

(Electric Field)

$V_{dip} = \frac{\hat{r} \cdot \vec{p}}{r^2} = \frac{p\cos\theta}{r^2}$ (Dipole potential)

Electrostatics: $\vec{\tau} = \vec{p} \times \vec{E}$ (Torque on dipole)

$\rho_b = -\nabla \cdot P \qquad \sigma_b = P \cdot \hat{n}$ (Bound Charge)

Magnetostatics: $\vec{B} = \frac{1}{c} \int \frac{J \cdot (r-r')}{|r-r'|^3} dV'$ (Biot-Savart Law)

$\vec{B} = \vec{\nabla} \times \vec{A}$ (Vector Potential)

$A(\vec{r}) = \frac{1}{c} \int \frac{J(r')}{|r-r'|} dV' \qquad \tau = \vec{m} \times \vec{B}$

$\vec{A}_{dip} = \frac{\vec{m} \times \vec{r}}{c r^2}$ (Dipole Potential)

$\vec{J}_b = \nabla \times m \qquad K_b = m \times \hat{n}$ (Bound current)

Boundary Conditions: $D_1^\perp - D_2^\perp = 4\pi \sigma_f \qquad B_1^\perp - B_2^\perp = 0$

$E_1^\parallel - E_2^\parallel = 0 \qquad H_1^\parallel - H_2^\parallel = \frac{1}{c} \vec{K}_f$

$V_1 - V_2 = 0 \qquad A_1 - A_2 = 0$

Tensors / Relativity: $\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \qquad \mathcal{F}^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \qquad \partial_\alpha \mathcal{F}^{\alpha\beta} = 0$ (Maxwell Eqns)

$\frac{dp^\alpha}{d\tau} = m \frac{du^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta$ (Lorentz Force)

Waves: $n = \sqrt{\mu\epsilon}$

$k \propto \omega$ (from wave eqn)

Electrodynamics Qualifier Preparation

Sara D. Barber
Ahmed A. Hasib

1 Mathematical Reminders

- Spherical
  - $h_1 = 1$
  - $h_2 = r$
  - $h_3 = r \sin\theta$
- Cylindrical
  - $h_1 = 1$
  - $h_2 = r$
  - $h_3 = 1$
- Unit Vector Cross Products: $\hat{e}_u = \frac{\frac{\partial \vec{r}}{\partial u}}{\left|\frac{\partial \vec{r}}{\partial u}\right|}$ (see Schaum's)
- $\nabla^2 \frac{1}{\mathscr{r}} = -4\pi\delta^3(\vec{\mathscr{r}})$
- $\vec{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$
- To solve an inhomogeneous differential equation:
  - $\dot{x} + ax = b$ or $\ddot{x} + a\dot{x} + bx = c$
  - $x(t) =$ Homogeneous Solution + Non-homogeneous Solution
  - Homogeneous Solution is simple
  - Non-homogeneous solution: Whatever x should be so LHS=RHS – usually a constant
- Divergence Theorem: $\oint_S \vec{E} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{E} d\tau$
- Stoke's Theorem: $\int_S \left(\vec{\nabla} \wedge \vec{v}\right) \cdot d\vec{a} = \oint_{\mathcal{P}} \vec{v} \cdot d\vec{l}$

2 Electrostatics

The primary task of electrostatics is to find the electric field of a given stationary charge distribution. In principle, this is accomplished by:

- First calculating the potential, V, then solving for $\vec{E}$
- Exploiting symmetry and using Gauss's law
- Calculating $\vec{E}$ directly using Coulomb's law

One can solve Poisson's equation in a region with no charge distribution for the potential by:

- Method of Images
- Separation of Variables

2.3 Electric Potential

- NOT the same thing as potential energy
  - $W = QV$
- $\boxed{V(\vec{r}) \equiv -\int_{\mathcal{O}}^{\vec{r}} \vec{E} \cdot d\vec{l}}$
  - where $\mathcal{O} \equiv \infty$ for finite charge distributions
- $\boxed{\vec{E} = -\vec{\nabla} V}$
- Poisson's Equation: $\boxed{\nabla^2 V = -4\pi\rho}$
  - Derived from Gauss's Law
- Laplace's Equation: $\boxed{\nabla^2 V = 0}$
  - For regions with no charge
- Solution to Poisson's Equation: $\boxed{V(\vec{r}) = \int \frac{1}{\mathscr{r}} dq}$
- Obeys superposition principle: $V = V_1 + V_2 + \cdots$
- Boundary Conditions:
  - Normal Component: $E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = 4\pi\sigma$
  - Parallel Component:$\vec{E}^{\|}_{\text{above}} = \vec{E}^{\|}_{\text{below}}$
- Units: $N \cdot m \cdot C^{-1} = J \cdot C^{-1} = V$

2.4 Work and Energy in Electrostatics

- Work Done to Move a Charge in an Electric Field: $\boxed{W = \int_a^b \vec{F} \cdot d\vec{l}}$
  - Force *you* must exert on the charge: $\vec{F} = -Q\vec{E}$
  - To bring a charge in from infinity: $\boxed{W = QV(\vec{r})}$
- Energy of a Point Charge Distribution: $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)$
- Energy of a Continuous Charge Distribution:
  - $W = \frac{1}{2}\int V dq$
  - $\boxed{W = \frac{1}{8\pi} \int_{\text{all space}} E^2 d\tau}$
- Work does not obey the superposition principle.

3.3 Separation of Variables

- Poisson's Equation: $\nabla^2 V = -4\pi\rho$
- Properties of Separable Solutions:
  - Completeness: $f(y) = \sum_{n=1}^{\infty} C_n f_n(y)$
  - Orthogonality: $\int_0^a f_n(y) f'_n(y) dy = 0$ for $n' \neq n$
- Method for Finding Separable Solution:
  - Find differential equations for all components of separable solution and solve.
  - Use boundary conditions and orthogonality to solve for all constants.
- General Solution to Laplace's Equation:
  - Spherical Coordinates with azimuthal Symmetry:
    $$\boxed{V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)}$$
    where $P_l(\cos\theta)$ is the Legendre Polynomial(see Schaum's for properties)
- Green's Function in Azimuthal Symmetry:
  - $$\boxed{G(\vec{r} - \vec{r'}) = \frac{1}{|\vec{r} - \vec{r'}|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)}$$
    where, $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$
  - If there is no azimuthal symmetry we use Associated Legendre Polynomial and Spherical Harmonics
    $$\boxed{Y_m^l(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}}$$
    The properties of P_l^m is in Schaum's.
- Addition Theorem of Spherical Harmonics:
  - $$\boxed{P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{l,m}^*(\theta',\phi') Y_{l,m}(\theta,\phi)}$$
    where, $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$
- Green's Function without Azimuthal Symmetry:
  - $$\boxed{G(\vec{r} - \vec{r'}) = \frac{1}{|\vec{r} - \vec{r'}|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{l,m}^*(\theta',\phi') Y_{l,m}(\theta,\phi)}$$
- Dipole Moment:
  - $$\boxed{\vec{p} = \int \vec{r'} \rho(\vec{r'}) dv'}$$
- Electrostatic Potential of a Point Dipole:

- Differential Equation: $\boxed{\vec{\nabla} \cdot \vec{D} = 4\pi\rho_{\text{f}}}$

- Interface Condition of Perpendicular Displacement Vector: $D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = 4\pi\sigma_{\text{f}}$
- Polarization for Linear Materials: $\boxed{\vec{P} = \chi_e \vec{E}}$
  - where χ_e is the electric susceptibility
- Electric Permitivity: $\boxed{\epsilon = 1 + 4\pi\chi_e}$
- Energy Stored in an Electric Field: $W = \frac{1}{8\pi}\int \vec{E} \cdot \vec{D} d\tau$

5 Magnetostatics

- Lorentz Force Law: $\boxed{\vec{F} = Q\left[\vec{E} + \frac{1}{c}(\vec{v} \wedge \vec{B})\right]}$
- **Magnetic forces do no work.**
- Current Units: $C \cdot s^{-1} = A$
- Current Densities:
  - $\boxed{\vec{I} = \lambda \vec{v} = \frac{q}{t}}$
  - $\boxed{\vec{K} = \sigma \vec{v} = \frac{d\vec{I}}{dl_{\perp}}}$
  - $\boxed{\vec{J} = \rho \vec{v} = \frac{d\vec{I}}{da_{\perp}}}$
- Continuity Equation: $\boxed{\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}}$
  - $\vec{\nabla} \cdot \vec{J} = 0$ for magnetostatics
- Biot-Savart Law: $\boxed{\vec{B}(\vec{r}) = \frac{1}{c}\int \frac{\vec{v} \wedge \hat{\mathscr{r}}}{\mathscr{r}^2} dq}$
- Divergence of Magnetic Field: $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$
- Ampere's Law:
  - Differential Form: $\boxed{\vec{\nabla} \wedge \vec{B} = \frac{4\pi}{c}\vec{J}}$
  - Integral Form: $\boxed{\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_{\text{enc}}}$
    * Derived from differential form using Stoke's Theorem
- The magnetic field outside of a solenoid is zero.
- Magnetic Vector Potential:
  - Definition: $\boxed{\vec{B} = \vec{\nabla} \wedge \vec{A}}$

- If your geometry has a piece missing try superposition with reverse current.
- $\boxed{\vec{B} = \mu \vec{H}}$
  - where $0 \leq \mu \leq 1$ for diamagnetic materials and $\mu \geq 1$ for paramagnetic materials
- Magnetization: $\boxed{\vec{M} = \chi_m \vec{H}}$
  - where $\chi_m \leq 0$ for diamagnetic materials
- Magnetic Permeability: $\boxed{\mu = 1 + 4\pi\chi_m}$
- Electromagnetic Energy Density: $u_{\text{em}} = \frac{1}{8\pi}(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H})$

7 Electrodynamics

- Ohm's Law: $\boxed{\vec{J} = \sigma \vec{f}}$
  - where σ is the conductivity of the medium and $\vec{f}$ is the force per unit charge
  - $\boxed{V = IR}$
- Power: $\boxed{P = VI}$
- Electromotive Force: $\boxed{\mathcal{E} = \oint \vec{f}_{\text{mag}} \cdot d\vec{l}}$
- Flux of $\vec{B}$ through a closed surface $\mathcal{S}$: $\boxed{\Phi = \int_{\mathcal{S}} \vec{B} \cdot d\vec{a}}$
- Faraday's Law:
  - Differential Form: $\boxed{\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt}}$
  - Integral Form: $\boxed{\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c}\frac{d\Phi}{dt}}$
- Lenz's Law: **Nature abhors a change in flux.**
  - An induced flux will flow in the direction that cancels a change in flux
  - Remember: Opposite currents repel
- Self Inductance: $\Phi = LI$
- Energy Stored in a Magnetic Field: $\boxed{W = \frac{1}{8\pi} \int_{\text{all space}} \vec{B} \cdot \vec{H} d\tau}$
- Maxwell's Equations

Gauss's Law:	$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$	No Name:	$\vec{\nabla} \cdot \vec{B} = 0$
Faraday's Law:	$\vec{\nabla} \wedge \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$	Ampere's Law:	$\vec{\nabla} \wedge \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$

- **A changing electric field induces a magnetic field.**

- Maxwell Stress-Energy Tensor: $\boxed{T_{ij} = \frac{1}{4\pi}[E_i D_j + H_i B_j - \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})\delta_{ij}]}$
  - Momentum flux density
- The total force on the charges in volume V:
  - $\boxed{\vec{F} = \frac{d\vec{p}_{\text{mech}}}{dt} = \oint_S \sum_\beta T_{\alpha\beta} n_\beta da - \frac{1}{4\pi c}\frac{d}{dt}\int_V (\vec{E}\wedge\vec{B})dv}$
  - Momentum Stored in an Electromagnetic Field: $\vec{p}_{\text{em}} = \frac{1}{4\pi c}\int_V (\vec{E}\wedge\vec{B})dv = \frac{1}{c^2}\int \vec{S}dv$
  - the first integral is the momentum per unit time flowing in through the surface.
  - Differential Form: $\boxed{\frac{\partial}{\partial t}(\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \vec{\nabla}\cdot\vec{T}}$
  - Poynting vector has two quite different roles: $\vec{S}$ itself is the energy per unit area, per unit time, transported by the electromagnetic fields, while $\frac{1}{c^2}\vec{S}$ is the momentum per unit volume stored in those fields.
  - Similarly $T^{\alpha\beta}$ itself is the electromagnetic stress acting on a surface, and $-T^{\alpha\beta}$ describes the momentum transported by the fields.
- Angular Momentum in electromagnetic field:
  - $\boxed{\vec{l}_{\text{em}} = \vec{r}\wedge\vec{p}_{\text{em}}}$

9 Electromagnetic Waves

- Wave Number: $k = \frac{2\pi}{\lambda}$
- Period: $T = \frac{2\pi}{kv}$
- Frequency: $\nu = \frac{v}{\lambda}$
- Angular Frequency: $\omega = 2\pi\nu = kv$
  - Frequency is constant across interfaces
- Three Dimensional Wave Equation:
  - $\boxed{\nabla^2\vec{E} = \frac{\epsilon\mu}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}$
  - $\boxed{\nabla^2\vec{B} = \frac{\epsilon\mu}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}}$
- Monochromatic Plane Waves: Sinusoidal waves of frequency ω traveling in an arbitrary direction $\hat{k}$ polarized in the $\hat{n}$ direction
  - $\boxed{\tilde{\vec{E}}(\vec{r},t) = \tilde{E}_0 e^{(\vec{k}\cdot\vec{r} - \omega t + \delta)}\hat{n}}$
  - $\boxed{\tilde{\vec{B}}(\vec{r},t) = \sqrt{\epsilon\mu}\,\hat{k}\wedge\tilde{\vec{E}}}$
    - For transverse waves: $\hat{n}\cdot\hat{k} = 0$

- Length Contraction: $\boxed{\Delta x = \frac{\Delta x'}{\gamma}}$
  - Moving objects are shortened.
  - Only dimensions parallel to the velocity are contracted.
- General Lorentz Transformation: $\boxed{\left(\begin{array}{c|c} \gamma & -\gamma\vec{\beta} \\ \hline -\gamma\vec{\beta} & P_{\perp} + \gamma P_{\|} \end{array}\right) = \begin{pmatrix} ct' \\ \vec{x}' \end{pmatrix}}$

  where, $P_{\|} = \frac{\beta_i \beta_j}{\beta^2}$ and $P_{\perp} = \delta_{ij} - P_{\|}$
- Contravariant Vector: $x^{\mu} = (ct, \vec{x})$, $p^{\mu} = (E/c, \vec{p})$
- Covariant Vector: $x_{\mu} = (ct, -\vec{x})$, $p_{\mu} = (E/c, -\vec{p})$
- Minkowski Metric: $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
  - $x^{\mu} = g^{\mu\nu} x_{\nu}$
- Proper Length: $(\Delta s)^2 = (c\Delta\tau)^2 = (ct)^2 - x^2 - y^2 - z^2$
- Charge Conservation: $\boxed{\partial_{\mu} j^{\mu} = 0}$
  - where $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$ and $\boxed{j^{\mu} = (c\rho, \vec{j})}$
- Proper Velocity:
  - $\boxed{u'^{\alpha} = \frac{dx'^{\alpha}}{d\tau} = \frac{dx'^{\alpha}}{dt'} = (c, 0, 0, 0)}$
  - $\boxed{u^{\alpha} = \frac{dx^{\alpha}}{d\tau} = \frac{dx^{\alpha}}{dt/\gamma} = \gamma(c, \vec{v})}$
  - $\boxed{u^{\alpha} u_{\alpha} = c^2}$
- Proper Acceleration:
  - $\boxed{a^{\alpha} = \frac{du^{\alpha}}{d\tau} = \gamma^4 \hat{\beta}\dot{\vec{\beta}}(c, \vec{v}) + \gamma^2(0, \dot{\vec{v}})}$
  - $\boxed{a^{\alpha} u_{\alpha} = 0}$
- Relation of E-B fields in rest and lab frames:
  - $\boxed{\bar{E} = \bar{E}'_{\|} + \gamma\left(E'_{\perp} - \vec{\beta} \wedge \bar{B}'\right)}$
  - $\boxed{\bar{B} = \bar{B}'_{\|} + \gamma\left(B'_{\perp} + \vec{\beta} \wedge \bar{E}'\right)}$
  - $\boxed{\bar{B} = \vec{\beta} \wedge \bar{E}}$
- Field-Strength Tensor:
  - $\boxed{F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}}$
  - where $\boxed{F_{ij} = \epsilon_{ijk} B^k}$ and $\boxed{F_{0i} = -E_i}$

- Magnetic Dipole Moment:
  - $\boxed{\vec{m} = \frac{1}{2c}\int \vec{r} \wedge \vec{J}(\vec{r}) d^3r}$
- A point magnetic dipole at rest:
  - $\boxed{\rho'(\vec{r}') = 0 \quad \vec{J}'(\vec{r}') = -c\vec{m}'(t) \wedge \vec{\nabla}'\delta^3(\vec{r}' - \vec{r}'_m)}$
- A point magnetic dipole in motion:
  - $\boxed{\rho(\vec{r}) = -c(\vec{\beta} \wedge \vec{m}')\vec{\nabla}\delta^3(\vec{r} - \vec{r}_m(t))}$
  - $\boxed{\vec{J}(\vec{r}) = -c\vec{m}' \wedge \vec{\nabla}\delta^3(\vec{r} - \vec{r}_m(t))}$
  - A moving dipole has an electric dipole moment $\vec{P} = \vec{\beta} \wedge \vec{m}'$

13 Radiation

- Potential of a charge q located at r_0' in the rest frame:
  - $A'^{\alpha}(\vec{r}') = \frac{q}{|\vec{r}' - \vec{r_0}'|}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  - In the lab frame $\vec{r_0}'$ transforms to $\vec{r_0}$ as $\vec{r_0}_{\parallel} = \gamma^{-1}\vec{r_0}'_{\parallel}$ and $\vec{r_0}_{\perp} = \vec{r_0}'_{\perp}$ but in the lab frame the charge moves with a constant velocity v, so the position of the charge $\vec{r}_q(t) = \vec{r_0} + vt$
- Potential of the charge as a function of the current position of the charge:
  - $A^{\alpha}(\vec{r}) = \frac{\gamma q \begin{pmatrix} 1 \\ \beta \end{pmatrix}}{\sqrt{|r_{\perp} - r_{q\perp}(t)|^2 + \gamma^2|r_{\parallel} - r_{q\parallel}(t)|^2}}$

    where, $r_{q\perp} = r_{0\perp}$ and $r_{q\parallel} = r_{0\parallel} + vt$
- Potential of the charge as a function of its 'retarded' position:
  - $\boxed{A^{\alpha}(\vec{r}_{ret}) = \frac{q\begin{pmatrix} 1 \\ \beta \end{pmatrix}}{R(1 - \vec{\beta}\cdot\hat{n})} = \frac{qu^{\alpha}}{\gamma c R(1 - \vec{\beta}\cdot\hat{n})}}$

    where, $\vec{R} = \vec{r} - \vec{r}_q(t_{ret})$ and $\hat{n} = \frac{\vec{R}}{R}$
  - In covariant form: $\boxed{A^{\alpha}(x) = \frac{qu^{\alpha}}{Z^{\beta}u_{\beta}}}$

    where, $Z^{\beta} = x^{\beta} - x^{\beta}_{ret}(x)$ and $u^{\alpha} = u^{\alpha}(t_{ret}(x))$
- From Maxwell's Equation: $\Box A^{\alpha} = \frac{4\pi}{c}J^{\alpha}$
  - To solve this equation we need a Green's function such that $\Box_x G(x, x') = \delta^4(x - x')$
  - There are two sets of boundary conditions of interest and two Green's functions $D^{ret}(x, x')$ and $D^{adv}(x, x')$
  - $\boxed{D^{ret}(x, x') = \frac{\delta\left(x^0 - x'^0 - |\vec{r} - \vec{r}'|\right)}{4\pi|\vec{r} - \vec{r}'|}}$

E+M Qual Equation Sheet

① Vector Identities

$$A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B)$$

$$A\times(B\times C) = B(A\cdot C) - (A\cdot B)C$$

$$\nabla(fg) = f(\nabla g) + (\nabla f)g$$

$$\nabla(A\cdot B) = A\times(\nabla\times B) + B\times(\nabla\times A) + (A\cdot\nabla)B + (B\cdot\nabla)A$$

$$\nabla\cdot(fA) = f(\nabla\cdot A) + A\cdot(\nabla f)$$

$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$$

$$\nabla\times(fA) = f(\nabla\times A) - A\times(\nabla f)$$

$$\nabla\times(A\times B) = (B\cdot\nabla)A - (A\cdot\nabla)B + A(\nabla\cdot B) - B(\nabla\cdot A)$$

$$\nabla\cdot\nabla\times A = 0$$

$$\nabla\times(\nabla f) = 0$$

$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$$

② Maxwells Equations

*In free space:

$$\nabla\cdot E = \frac{\rho}{\epsilon_0} \qquad \nabla\cdot B = 0$$

$$\nabla\times E = \frac{-\partial B}{\partial t} \qquad \nabla\times B - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0\vec{J}$$

*In media:

$$\nabla\cdot B = 0 \qquad \nabla\times E + \frac{\partial B}{\partial t} = 0 \qquad D = \epsilon_0 E + P \overset{\text{(linear)}}{=} \epsilon_0(1+\chi_e)E$$

$$\nabla\cdot D = \rho \qquad \nabla\times H - \frac{\partial D}{\partial t} = \vec{J} \qquad H = \frac{1}{\mu_0}B - M$$

*Other equations:

$\nabla\cdot J + \frac{\partial\rho}{\partial t} = 0$ (Conservation of charge)

$S = E\times H$ (in media)

$\quad = E\times B$ (in free space)

$B = \nabla\times A$ (vector potential)

$E = -\nabla\Phi$ (scalar potential)

$F = q(E + v\times B)$ (Lorentz Force Law)

③ Problem Solving Techniques

⇒ Electrostatics

* By substituting $E = -\nabla\Phi$ into $\nabla \cdot E = \frac{\rho}{\epsilon_0}$, we generate the Poisson eqn

$$-\nabla^2 \Phi = \frac{\rho}{\epsilon_0},$$ of which the Laplace equation is a special case.

* In all cases, we can solve for Φ via the Green function method

↳ $$\Phi = \frac{1}{4\pi\epsilon_0}\int \rho(x')\, G_D\, d^3x' + \frac{-1}{4\pi}\oint_S \Phi(x')\, \frac{\partial G_D(x,x')}{\partial n'}\, da'$$

where G_D is typically found via method of images ($G_D = \frac{1}{|x-x'|}$ in most cases)

~~* For certain geometries, expansion via orthogonal functions is easier~~

↳ Cartesian coordinates

$$\Phi(x,y,z) = \sum_{m,n} A_{mn} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{mn} z)$$

where $$A_{mn} = \frac{4}{ab\sinh(\gamma_{nm} c)}\int_0^a dx \int_0^b dy\, V(x,y)\sin(\alpha_n x)\sin(\beta_m y)$$

↳ Spherical coordinates

$$\Phi = \frac{U(r)}{r} P(\theta) Q(\varphi)$$

$U(r) = Ar^{\ell+1} + Br^{-\ell}$ in all cases

* If the potential is azimuthally symmetric

$$\Phi(r,\theta,\varphi) = \sum_{\ell=0} (A_\ell r^\ell + B r^{-\ell-1}) P_\ell(\cos\theta)$$

* If the potential is not azimuthally symmetric

$$\Phi(r,\theta,\varphi) = \sum_{\ell=0}\sum_{m=-\ell}^{\ell} (A_{\ell m} r^\ell + B_{\ell m} r^{-\ell-1}) Y_{\ell m}(\theta,\varphi)$$

* Note that we can expand our known solution in terms of spherical harmonics

$$\frac{1}{|x-x'|} = 4\pi \sum_{\ell,m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell m}(\theta',\varphi') Y_{\ell m}(\theta,\varphi)$$

↳ Cylindrical coordinates

$\Phi(\rho,\varphi,z) = R(\rho)Q(\varphi)Z(z)$ where $Q(\varphi) = Ae^{-im\varphi} + Be^{im\varphi}$

$$Z(z) = Ce^{kz} + De^{-kz}$$

$$R(\rho) \sim AJ_\nu(\rho) + BN_\nu(\rho)$$

J_ν: Bessel fn. of 1st kind; N_ν: Bessel fn. of 2nd kind

⇒ Magnetostatics

* There are two relevant cases in solving magnetostatics problems

① $J(\vec{x}) = 0$! In this case we use the fictitious φ_m scalar potential and all our solutions from electrostatics will still work

② $J(\vec{x}) \neq 0$: $B = \nabla \times A$; $A = \int \frac{\mu_0}{4\pi} \frac{J(\vec{x}')}{|x-x'|} d^3x'$

E+M Midterm Study Guide

Basics

* Maxwells equations in SI are:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \quad \text{(Gauss Law)}$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \quad \text{(Faraday's Law)}$$

$$\nabla \cdot B = 0$$

$$\nabla \times B - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0 \vec{J} \quad \text{(Ampere's Law)}$$

* The Lorentz force law is: $F = q(E + \vec{v} \times \vec{B})$

Electrostatics

* A simple form of the force law is:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|x-x'|} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2 (x-x')}{|x-x'|^3}$$

$\Rightarrow$ Defining $E = \frac{F}{q}$

$$E(x) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x')(x-x')}{|x-x'|^3} d^3x' \quad \text{(Coulombs Law)}$$

* but since $E(x) = -\nabla \Phi$, and $\nabla \frac{1}{|x-x'|} = \frac{-(x-x')}{|x-x'|^3}$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x')}{|x-x'|} d^3x'$$

* From $\Phi(x)$, we know that the work done to position the charges in their current arrangement is

$$W = q\,\Delta\Phi$$

* From Stokes Law $\left(\oint_S (\nabla \times A) \cdot \hat{n}\, da = \oint_\ell A \cdot d\ell\right)$, we see that

$$(E_2 - E_1)\cdot \hat{n} = \frac{\sigma}{\epsilon_0} \Rightarrow E_\perp \text{ is discontinuous on a boundary}$$

$$E_2 - E_1 = 0 \Rightarrow E_\parallel \text{ is continuous across a boundary}$$

Electrostatics (cont.)

* Combining what we know about E and Φ

$$\Rightarrow \nabla \cdot (-\nabla \Phi) = \frac{\rho}{\epsilon_0}$$

$$-\nabla^2 \Phi = \frac{\rho}{\epsilon_0} \quad \text{(Poisson Equation)}$$

* Note: Laplace equation is special case where $-\nabla^2\Phi = 0$

$\Rightarrow$ This implies $\nabla^2 \frac{1}{|x-x'|} = 4\pi \delta^3(x-x')$ [To avoid discontinuity at origin]

Boundary Value Problems

* We have two different sets of boundary values we may consider
  1. Dirichlet Conditions $\rightarrow$ Φ known on boundary
  2. Nuemann Conditions $\rightarrow$ E known on boundary

* We account for surface terms in our equations via Green's Identities
  1. Green's First Identity
     * From the divergence theorem $\int_V (\nabla \cdot A)\, dV = \oint_S A \cdot \hat{n}\, ds$, if $A = \varphi \cdot \nabla \psi$

$$\Rightarrow \int_V (\varphi \nabla^2 \psi + \nabla\varphi \cdot \nabla\psi)\, d^3x = \oint \varphi \frac{d\psi}{dn}\, da$$

  2. Greens Theorem

$\Rightarrow$ If we interchange $\psi \leftrightarrow \varphi$ in the above and subtract the two we get

$$\int_V (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi)\, d^3x = \oint \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\right) da$$

* If we then specify $\psi = \frac{1}{|x-x'|}$, $\varphi = \Phi(x,y,z)$

$$\Rightarrow \Phi(x) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x')}{|x-x'|} d^3x' + \frac{1}{4\pi} \oint_S \Big(\underset{\substack{\uparrow \\ \text{Nuemann}}}{\frac{1}{R}\frac{\partial \Phi}{\partial n}} - \underset{\substack{\uparrow \\ \text{Dirichlet}}}{\Phi \frac{\partial}{\partial n}\left(\frac{1}{R}\right)} \Big) da$$

* Uniqueness theorem says we only need 1 set of B.C.'s to solve problem

Green's Functions

* Applying this technique to Poisson's Eqn, we already know:

$\frac{1}{|x-x'|}$ is a solution when $\rho = 4\pi\epsilon_0 \delta(x-x')$

$\Rightarrow \nabla^2\Phi = -4\pi\delta(x-x') \rightarrow \Phi$ is Green's function for this eqn

$$G(x,x') = \frac{1}{|x-x'|} + F(x,x') \quad \text{where} \quad \nabla^2 F(x,x') = 0$$

* For Dirichlet boundary conditions, we find

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')\, G_D(x,x')\, d^3x' + \frac{1}{4\pi}\oint_S \Phi(x') \frac{\partial G_D(x,x')}{\partial n'} da'$$

* For Neumann boundary conditions we find

$$\Phi(x) = \langle \Phi(x) \rangle_{\text{on } S} + \frac{1}{4\pi\epsilon_0}\int_V \rho(x')\, G_N(x,x')\, d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'} G_N(x,x')\, da'$$

* We typically solve for $G_D(x,x')$ or $G_N(x,x')$ via method of images

Orthogonal Functions

* For some geometries, it is easier to solve using orthogonal functions over the method of images

ex. $\nabla^2\Phi = 0$

* Solve by separation of variables in various coordinate systems

$\Rightarrow$ Cartesian:

$$\Phi(x,y,z) = \sum_{m,n} A_{mn} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{mn} z)$$

$$A_{mn} = \frac{4}{ab\sinh(\gamma_{mn} c)}\int_0^a dx\int_0^b dy\, V(x,y)\sin(\alpha_n x)\sin(\beta_m y)$$

$\Rightarrow$ Spherical:

$$\Phi(r,\theta,\varphi) = \frac{U(r)}{r} P(\theta) Q(\varphi)$$

$$\hookrightarrow U(r) = A r^{l+1} + B r^{-l}$$

* If azimuthally symmetric: $\Phi(r,\theta,\varphi) = \sum_{l=0} (A_l r^l + B r^{-l-1}) P_l(\cos\theta)$

* If no azimuthal symmetry: $\Phi(r,\theta,\varphi) = \sum_{l=0} \sum_{m=-l}^{l} (A_{lm} r^l + B_{lm} r^{-l-1}) Y_{lm}(\theta,\varphi)$

Orthogonal Functions (cont.)

* but we can also expand our known solutions in terms of these functions

$$\frac{1}{|x-x'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

⇒ Cylindrical:

$$\Phi(\rho, \theta, z) = R(\rho) Q(\varphi) Z(z)$$

$$\hookrightarrow Q(\varphi) = Ae^{-im\varphi} + Be^{im\varphi}$$

$$Z(z) = Ce^{kz} + De^{-kz}$$

$$R(\rho) \sim AJ_\nu(\rho) + BN_\nu(\rho)$$

where J_ν is Bessel function of first kind

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_j \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

$N_\nu(x)$ is Bessel function of second kind

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

E+M Final Study Guide

Electrostatics in Media

* In a vacuum, our electrostatic equations are:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$
$$\nabla \times E = 0$$

⇒ But presence of $\vec{E}$-field induces a dipole moment in the atom/molecules of the medium

↳ $\nabla \cdot D = \rho, \quad D = \epsilon_0 E + P$

↳ Alternatively, $\nabla \cdot E = \frac{1}{\epsilon_0} \rho_{eff}, \quad \rho_{eff} = \rho(x) - \nabla \cdot P$

* Note: The polarization P is defined as the avg. dipole moment per unit volume

⇒ We typically assume a linear media such that: $P = \epsilon_0 \chi_e E$

↳ $D = \epsilon_0 (1 + \chi_e) E$
$= \epsilon E$

> $\epsilon = \epsilon_0 (1 + \chi_e)$ is electric permitivity

↳ $\frac{\epsilon}{\epsilon_0}$ is the dielectric constant

* Given the above, our boundary conditions become: $\Delta D_\perp = \sigma$ (discontinuous)
$\Delta E_\parallel = 0$ (continuous)

Magnetostatics

Important quantities: $\vec{\mu}$ = magnetic dipole strength
$\vec{B}$ = magnetic induction / magnetic flux density
$\vec{H}$ = magnetic field
$\vec{N}$ = torque $= \vec{\mu} \times \vec{B}$

* Since we know there is no magnetic charge, we automatically know one of our magnetostatics equations is: $\nabla \cdot B = 0$

Magnetostatics (cont.)

* using conservation of charge: $\frac{dq}{dt} = 0$

$$\Rightarrow \frac{dq}{dt} = 0 = \int \left(\frac{d\rho}{dt} + \frac{d\rho}{dx}\frac{dx}{dt} + \frac{d\rho}{dy}\frac{dy}{dt} + \frac{d\rho}{dz}\frac{dz}{dt} \right) dV$$

$$= \int \left(\frac{d\rho}{dt} + v \cdot \nabla \rho \right) dV$$

$$= \int \left(\frac{d\rho}{dt} + \nabla \cdot (\rho \vec{v}) - \cancel{\rho (\nabla \cdot \vec{v})}^{0} \right) dV \qquad (\text{use } J = \rho \vec{v})$$

$$\Rightarrow \frac{d\rho}{dt} + \nabla \cdot J = 0 \quad (\text{Continuity Equation})$$

* Immediately we define the following: $I = \int \vec{J} \cdot d\vec{a}$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \vec{x}}{|x|^3} \quad (\text{Biot-Savart Law})$$

$$\Downarrow$$

$$\vec{B}(x) = \frac{\mu_0}{4\pi} \int \frac{J(x') \times (x - x')}{|x - x'|^3} d^3x'$$

⇒ If we replace $\frac{(x-x')}{|x-x'|^3}$ by $\nabla \frac{1}{|x-x'|}$ we see that B is a pure curl

↳ $B = \frac{\mu_0}{4\pi} \nabla \times \int \frac{J(x')}{|x-x'|} d^3x'$

* Taking $\nabla \times B$, we see: $\nabla \times B = \mu_0 \vec{J}$ (differential form, Ampere's Law)

↳ Applying Stokes Thm leads to: $\oint B \cdot dl = \mu_0 I_{enc}$

* Summarizing, our magnetostatic equations are: $\nabla \cdot B = 0$

$$\nabla \times B = \mu_0 \vec{J}$$

⇒ To solve for B we consider 2 cases:

① $J(\vec{x}) = 0 \rightarrow$ Solve as we did for $\vec{E}$ via $B = -\nabla \varphi_m \Leftrightarrow \nabla^2 \varphi_m = 0$

↳ Laplace Equ w/ B.C.'s

② $J(\vec{x}) \neq 0 \rightarrow$ We use the fact that $\nabla \cdot \nabla \times A = 0$

↳ $B = \nabla \times A$, $A = \int \frac{\mu_0}{4\pi} \frac{J(x')}{|x-x'|} d^3x'$ (vector potential)

Magnetostatics (cont.)

* Note! Due to vector properties, $A \to A + \nabla \Psi$ w/o changing our results
  ↳ We typically choose the Coulomb Guage such that $\nabla \cdot A = 0$

$$\Rightarrow \nabla \times (\nabla \times A) = \mu_0 \vec{J}$$

$$\nabla(\cancel{\nabla \cdot A}^{0}) - \nabla^2 A = \mu_0 \vec{J}$$

$$-\nabla^2 A = \mu_0 \vec{J} \quad (\text{3 Poisson Eqns})$$

* Note! We choose Ψ to satisfy above via: $\nabla^2 \Psi = -\nabla \cdot \int \frac{\mu_0}{4\pi} \frac{J(x')}{|x-x'|} d^3x'$
  $= -f(x)$

⇒ We now proceed to solve problems via expansions in complete sets of orthogonal functions as before.

Magnetostatics in Media

* We proceed similarly to how we did in electrostatics

$$\Rightarrow J(\vec{x}) = \sum_i' q_i v_i \delta^3(x - x_i) \qquad L_i = m_i(x_i \times v_i)$$

$$M(x) = \sum_i' q_i (x_i \times v_i) \qquad M = \sum_i' \frac{q_i}{m_i} L_i$$

⇒ Remember that the magnetization $\underline{M}$ is defined as the magnetic moment density

↳ $\vec{\underline{M}} = \frac{\vec{m}}{V}$

↳ $A(x) = \frac{\mu_0}{4\pi} \int \frac{J(x') + \nabla' \times \underline{M}(x')}{|x-x'|} d^3x' \Rightarrow J_{eff} = J(x') + \nabla' \times \vec{M}(x')$

↳ $\nabla \times B = \mu_0 (J + \nabla \times \vec{M})$

* if $H = \frac{B}{\mu_0} + \underline{M}$

$$\nabla \times H = \vec{J}$$

* Again assuming a linear media $B = \mu H$, where μ = magnetic permeability
  ↳ $\mu = \mu_0$ in a vacuum
  $\mu > \mu_0$ in paramagnetic media
  $\mu < \mu_0$ in diamagnetic media

Media (cont.)

* At the intersection of two media, we now find:

$B_\perp$ is continuous $\qquad$ $H_\perp$ is discontinuous $\left(n \times [H_2 - H_1] = \vec{K} \right)$

$H_\parallel$ is continuous $\qquad$ ↳ $\vec{K}$ = surface current density

* Turning to our strategies to solve magnetostatics boundary value problems:

① Currents present in linear media ($B = \mu H$)

$$H = \frac{1}{\mu}(\nabla \times A)$$

$$\nabla^2 A = -\mu_0 \vec{J} \quad \text{(Solve Poisson Eqn)}$$

② $\vec{J} = 0$

$$\nabla \times H = 0 \Rightarrow H = -\nabla \Phi_m$$

$$-\mu \nabla^2 \Phi_m = 0 \quad \text{(Solve Laplace's Eqn)}$$

③ Hard Ferromagnet (Know $\vec{M}$, but $\vec{J} = 0$)

$$\nabla \cdot B = 0$$

$$\nabla \times H = 0 \rightarrow H = -\nabla \Phi_m$$

$$\Rightarrow \nabla \cdot \left[\mu_0 (\vec{H} + \vec{M}) \right] = 0$$

↳ $\nabla^2 \Phi_m = \nabla \cdot M$

$= -\rho_m \rightarrow$ Solve w/ electrostatic solutions

$$\Rightarrow \Phi = \frac{\vec{M} \cdot \vec{x}}{4\pi r^3} \quad (x \gg x')$$

* But we cannot ignore Faraday's Law

A changing magnetic flux generates an EMF

$$F = \int B \cdot dn \, da$$

$$\mathcal{E} = \int E' \cdot dl'$$

$$= \frac{dF}{dt}$$

$$\Rightarrow \oint_c E' \cdot dl' = -\frac{d}{dt} \int_s B \cdot \hat{n} \, da \rightarrow \oint_s \left(\nabla \times E + \frac{\partial B}{\partial t} \right) \cdot \hat{n} \, da = 0$$

Media (cont.)

* To determine the amount of energy stored in the fields:

$$W = \frac{1}{2}\int (E \cdot D)\, d^3x' \rightarrow \text{Electric Field}$$

$$W = \frac{1}{2}\int (B \cdot H)\, d^3x' \rightarrow \text{Magnetic Field}$$

Electrodynamics

* Our final, corrected Maxwell Equations are:

$$\nabla \cdot E = \rho/\varepsilon_0 \qquad \nabla \times E + \frac{\partial B}{\partial t} = 0 \qquad \nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla \cdot B = 0 \qquad \nabla \times B - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0 J$$

* In media, these equations become

$$\nabla \cdot B = 0 \qquad \nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla \cdot D = \rho \qquad \nabla \times H - \frac{\partial D}{\partial t} = \vec{J}$$

* To solve problems, we introduce potentials such that: $B = \nabla \times A$

$$E = -\nabla \Phi - \frac{\partial A}{\partial t}$$

* Note: Homogeneous equations are automatically satisfied by these substitutions

* But really, we replace $A' \rightarrow A + \nabla \Lambda$

$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

* Known as Lorenz guage when $\nabla \cdot A + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$

$$\Rightarrow \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 \vec{J}$$

> Wave Equation

* To solve for Green's Function of wave equation

$$DG = -4\pi\, \delta^3(x-x')\,\delta(t-t'), \quad D = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$\hookrightarrow G^{\pm}(x,t,x',t') = \frac{1}{|x-x'|}\,\delta\!\left(t' - \left[t \mp \frac{|x-x'|}{c}\right]\right)$$

$\Rightarrow$ use retarded Greens functions; all old formulas work but now must evaluate at $t = t' - \frac{|x-x'|}{c}$

Electrodynamics (cont.)

* The Poynting vector $\vec{S}$ represents energy flow w/in the system

$$S = E \times H$$

$$\hookrightarrow E_{field} = \frac{\epsilon_0}{2}\int_V (E^2 + c^2B^2)\, d^3x'$$

$$U = \frac{1}{2}(E\cdot D + B\cdot H) \quad \text{(Energy Density)}$$

$\Rightarrow$ Our conservation of energy law is:

$$\frac{\partial U}{\partial t} + \nabla\cdot S = -J\cdot E$$

* Similarly, for momentum:

$$\frac{d\vec{P}_{mech}}{dt} = \int_V (\rho E + J\times B)\, d^3x \quad \Rightarrow \quad \vec{g} = \frac{1}{c^2}\vec{S} = \frac{1}{c^2}(E\times H) \quad \text{(momentum density)}$$

* Defining the Maxwell stress tensor as: $T_{\alpha\beta} = \epsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(E\cdot E + c^2 B\cdot B)\,\delta_{\alpha\beta}\right]$

$$\hookrightarrow \frac{d}{dt}(\vec{P}_{mech} + \vec{P}_{field}) = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, d^3x$$

$$= \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta\, da$$

E + M Qualifier Topics List

January 2008

#1: Maxwell's Eqns, Electrodynamics, Media, Conservation Laws
#2: ~~Electro~~ Maxwell's Eqns, Relativity, Media, E, B, Plane-Cartesian
#3: Electrodynamics, Media, Work, Poynting Thm
#4: Waves, B, Stress Tensor
5: Electrodynamics, Media, Cylindrical or Spherical
6: 4-D Maxwell's Eqns

August 2008

1: 4-D Maxwell's Eqn's, Waves
2: Waves, Maxwell's Eqns, B
3: Relativity, Mechanics, Conservation Laws
4: Cylindrical, B, Mechanics, Forces
5: Magnetostatics, Cylindrical, H, Energy
6: Electrostatics, Plane-Cartesian,

Jan 2009

1: Electostatics, Media, E, D, Energy, Spherical
2: Magnetostatics, Media, Cylindrical, B, Potential, Plane-Cartesian
3: Electrostatics, Plane-Cartesian, Potential
4: Magnetostatics, Spherical
5: Electrostatics, Media, Stress Tensor, Spherical, E, B
6: Waves, 4-D Maxwells Eqns, Relativity

Aug 2009

1: Magnetostatics, Media, B, H, M, Spherical, Cylindrical
2: Electrostatics, Potential, E,
3: Electrostatics, Media, Cylindrical, D, E, P
4: Waves, 4-D Maxwell's Eqn's, Relativity
5: 4-D Maxwell's Eqns, Relativity, Plane-Cartesian
6: Stress Tensor, Electrodynamics, Cylindrical, Plane-Cartesian

Jan 2010

1: Electrostatics, Cylindrical, E, Force
2: Media, Electrostatics, Spherical, D, P, E
3: Electrostatics, Spherical, Potential
4: Electrodynamics, Forces, J
5: Waves, B, Poyting Thm
6: 4-D Maxwell Equations

Aug 2010

1: Electrodynamics Media, Cylindrical, D, E, P
2: Maxwell's Eqns, Waves
3: Electrostatics, Spherical, Potential, σ
4: Electrodynamics, Relativity, Radiation
5: Waves, Relativity, 4-D Maxwell's eqns
6: Electrodynamics, Retarded quantities, A, Radiation Field, Poynting Thm, Cylindrical or Spherical

Jan 2011

1: Electrostatics / Electrodynamics, Media, I, Plane - Cartesian
2: Radiation, Relativity, Waves
3: Electrostatics, Work, Plane, - Cartesian
4: Electrostatics, Spherical, Potential, E, σ
5: Waves, Polarization
6: Magnetostatics, Spherical, Potential, B

Aug 2011:

1: Electrostatics, Potential, Dipole, Spherical, σ
2: Maxwell's Eqns, 4-D Maxwell's Eqns, Poynting Thm
3: Magnetodynamics?, Forces, μ, B, Cylindrical or Spherical
4: Waves, Maxwell's Eqns, Antennae, B, $\mathcal{E}$
5: Relativity, Electrodynamics, Forces, Cylindrical 4-D E+M, Mechanics
6: 4-D Maxwell's Eqns, Plane - Cartesian, 4-D E+M

Jan 2012

1: Electrostatics, Spherical, E, Media
2: Maxwell's Equations
3: Electrostatics, E, B, Cylindrical, Poynting Thm
4: Electrostatics, Force, Media, Spherical, E, Potential, σ
5: Stress Tensor, Electrostatics, Plane-Cartesian, E
6: Waves, Relativity, 4-D E+M

Aug 2012

1: media, Electrodynamics, Cylindrical, B, J
2: Maxwell's Equations, Waves, E
3: Electrostatics, Spherical, Media, Potential
4: Waves, Polarization
5: Electrostatics, Relativity, E, B, cylindrical, 4-D E+M
6: Electrodynamics, Retarded Quantities, Plane-Cartesian, ρ, J, Potential, A

Jan 2013

1: Electrodynamics, Spherical, B
2: Media, Maxwell's Equations, Waves
3: 4-D Maxwell's Eqns, Relativity, E, B, Plane-Cartesian
4: Electrodynamics, Forces, Plane-Cartesian
5: ~~Electrodynam~~ Electrostatics, Spherical, Potential
6: Electrostatics, Work, Plane-Cartesian, Potential

Aug 2013

1: Waves
2: Magneto statics
3: Media, Electrostatics
4: Magnetostatics, Maxwell's Eqn's
5: Relativity, Electrodynamics
6: Stress Tensor

Jan 2014

1: Electrostatics, Media, Spherical, E, D, P
2: Electrostatics, Spherical, Potential, σ
3: Spherical, Electrodynamics, E, B, Poynting Thm
4: Waves, Media
5: Antenna, Waves, Radiation
6: Waves, Maxwell's Eqn's, Cylindrical

Aug 2014

1: Magnetostatics, Cylindrical, H, B
2: Electrostatics, Forces, Cylindrical, Spherical, E
3: Spherical, Media, Electrodynamics, E, Maxwell's Eqns
4: Cylindrical, Retarded Quantities, Magnetodynamics?, Potential, B
5: Waves
6: 4-D ~~Maxwell's Eqns~~ E+M, Relativity

Jan 2015:

1: Maxwell's Eqn's
2: Spherical, Media, Electrostatics, D, σ
3: 4-D E+M, 4-D Maxwell's Eqn's
4: Waves, Polarization
5: Spherical, Electrostatics, Potential
6: Waves, Media, Maxwell's Eqn's

Aug 2015

1: Charge densities, Spherical, Cylindrical
2: Waves, Media
3: Magneto statics, Cylindrical, B
4: Spherical, Green's Functions, Electrostatics, Potential
5: Cylindrical, Electrodynamics, Tensor, Relativity, E, B
6: 4-D E+M, Waves, Retarded Quantities

Jan 2016:

1: 4-D E+M, Relativity
2: Cylindrical, Magnetostatics, H
3: Media, Electrostatics, B
4: Cylindrical, Magnetostatics, E
5: Waves
6: Waves, Media, Polarization

Jan 2017:

1: Electrostatics, Media, E, Spherical
2: Magnetostatics, Spherical, B, H, Media
3: Maxwell's Eqns, Waves, Media
4: Radiation
5: Special Relativity, Tensors,
6: Relativistic E+M, Tensors, Maxwells Eqns

Aug 2017:

1: Electrostatics, Spherical, Media
2: Magneto statics, Cylindrical, Dipole
3: Waves, Media
4: Waves, Media, Polarization
5: Radiation
6: Relativity, Maxwells Eqns, Tensors

Jan 2018:

1: Electrostatics, Media, Cylindrical
2: Magnetostatics, B, Cylindrical
3: Waves, Media, Maxwells Eqns
4: 4-D E+M, Tensors, Maxwell's Eqns
5: Radiation, Retarded Potentials,
6: Stress Tensor, Maxwell's Eqns

Completed E&M Qual Problems

Exam	Q1	Q2	Q3	Q4	Q5	Q6
S2008		✓		✓	✓	
F2008					✓	
S2009						
F2009						
S2010					✓	✓
F2010	✓	✓	✓		✓	
S2011			✓			
F2011		✓				✓
S2012	✓	✓	✓			✓
F2012		✓			✓	
S2013						
F2013	✓	✓		✓	✓	
S2014						
F2014					✓	✓
S2015			✓		✓	
F2015	✓	✓	✓			
S2016	✓					
F2016						
S2017	✓		✓		✓	
F2017	✓	✓				✓
S2018	✓	✓				

1 Electrostatics

1. Spring 10 #1
2. Spring 10 #3
3. Fall 10 #3
4. Spring 10 #1
5. Spring 09 #3
6. Fall 09 #2
7. Fall 08 #6
8. Spring 07 #5
9. Fall 07 #1
10. Fall 07 #3
11. Fall 07 #5
12. Spring 06 #3
13. Fall 05 #2
14. Fall 05 #4
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2 Magnetostatics

1. Spring 09 #2
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3 Maxwell's Equations

1. Spring 08 #1
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4 Electrodynamics

1. Spring 08 #1
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5 Interaction Forces

1. Spring 10 #4
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6 Plane Waves

1. Spring 10 #5
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7 Dielectric Materials

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Electrodynamics Qualifier Examination

January 9, 2008

General Instructions: In all cases, be sure to state your system of units. Show all your work, write only on one side of the designated paper, and if you get stuck on one part, assume a result and proceed onward. The points given for each part of each problem are indicated. Each problem carries equal weight.

1. This problem refers to macroscopic electrodynamics in a general medium. No specific relation is assumed between the fields $\mathbf{E}$ and $\mathbf{D}$, nor between $\mathbf{B}$ and $\mathbf{H}$.

a) 1 pt. Write the microscopic Maxwell equations containing as source the total charge density ρ and the total current density $\mathbf{j}$.

b) 1 pt. Show that the Maxwell equations of part (a) imply the conservation of the total charge Q.

c) 3 pt. Separate ρ into its two parts, the free charge density ρ_f and the (bound) polarization charge density ρ_p, and rewrite the Maxwell equation containing ρ to obtain the macroscopic form containing the displacement field $\mathbf{D}$ and ρ_f.

d) 3 pt. Separate $\mathbf{j}$ into its parts, the free current density $\mathbf{J}_f$, the polarization current density $\mathbf{J}_p$, and the magnetization current density $\mathbf{J}_m$, and rewrite the Maxwell equation containing $\mathbf{j}$ to obtain the form containing the magnetic field intensity $\mathbf{H}$, the displacement field $\mathbf{D}$, and the free current density $\mathbf{J}_f$.

e) 2 pt. Show that the free charge Q_f is conserved through the continuity equation containing ρ_f and $\mathbf{J}_f$, and then show that therefore the polarization charge Q_p is conserved.

2. Consider an infinite slab of thickness d, carrying uniform charge density ρ, centered on the origin and extending in the x-y plane. Assume both the electric permittivity ε and the magnetic permeability μ have their vacuum values.

a) 2 pt. Find the electric field vector, $\mathbf{E}$, and magnetic flux density (magnetic induction), $\mathbf{B}$, everywhere. Do not just write the answer down, but in all cases clearly articulate your arguments and solution to receive credit.

For parts (b)–(e) consider an observer moving at velocity $\mathbf{v} = v_0\hat{\mathbf{x}}$. *Do not assume that* $v_0 \ll c$.

b) 1 pt. What is the current density, $\mathbf{J}'$, in the observer's frame of reference? [Hint: How does the charge density transform?]

c) 3 pt. Find the electric field vector, $\mathbf{E}'$, and magnetic flux density, $\mathbf{B}'$, everywhere in the observer's frame of reference. Do this by solving Ampére's and Gauss' law in the observer's frame.

d) 2 pt. Alternatively, obtain the same result by performing a Lorentz transformation on the fields found in part (a).

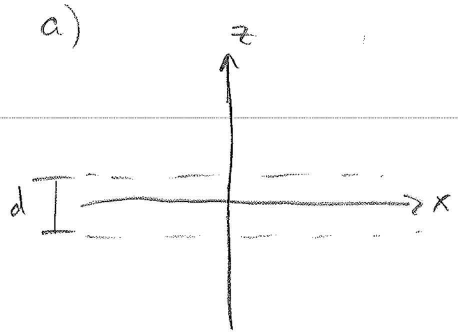
e) 2 pt. Show explicitly that $\mathbf{E}\cdot\mathbf{B}$ and $E^2 - B^2$ have the same value in both the rest frame of the slab and the observer's frame. Why is that? Is it possible to find a frame where $\mathbf{E} = \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$?

Jan 2008

E+M #2

Gaussian Units

a)



* Inside and outside slab must be considered seperately

* $\vec{B} = 0$ everywhere since there are no moving charges

Inside slab

Integral form of Gauss law states:

$$\int \vec{E} \cdot d\vec{a} = 4\pi Q_{enc}$$

$$E \cdot \int d\vec{a} = 4\pi \rho \cdot A \cdot (2|z|)$$ ⇐ Gaussian prism centered on origin of height $2z$ and bottom area A

$$E \cdot 2A = 4\pi \rho A \cdot 2z \, \hat{z}$$

$$E = 4\pi \rho z \, \hat{z}$$ ⇐ $+\hat{z}$ direction if $z > 0$, $-\hat{z}$ direction $z < 0$

Outside slab

$$\oint E \cdot d\vec{a} = 4\pi Q_{enc}$$

$$E \int d\vec{a} = 4\pi d A \rho \hat{z}$$

$$E \cdot 2A = 4\pi d A \rho \hat{z}$$

$$E = 2\pi d \rho \hat{z}$$ (follows same $\pm \hat{z}$ as above)

b) $v = V_0 \hat{x}$

$J' = L J$ where

$$L = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} c\rho \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

#2 (cont.)

b)

$$J' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cp \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma c p \\ -\beta\gamma c p \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad J' = -\beta\gamma c p \, \hat{x}$$

c) We proceed by transforming our coordinate system

$$x' = Lx$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \gamma ct - \beta\gamma x \\ \gamma x - \beta\gamma ct \\ y \\ z \end{bmatrix}$$

* From our transformations in parts b and c, we see

$z' = z$ and $p' = \gamma p$ $\Rightarrow$ previous results will be correct w/ coordinate substitutions

$$\Rightarrow \vec{E} = \begin{cases} 4\pi\gamma p'|z'|\,\hat{z}' & z < d \\ 2\pi d \gamma p' \hat{z} & z > d \end{cases}$$

#2 (cont.)

d) $F^{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -E_z & 0 & 0 & 0 \end{bmatrix}$

$F' = L F L^T$

$$= \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ \gamma E_z & -\beta\gamma E_z & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} E_z' &= \gamma \vec{E}_z \\ B_y' &= -\beta\gamma \vec{E}_z \end{aligned}$$

e) $E \cdot B$: * In rest frame

$$\vec{E} \cdot \vec{B} = 0 \quad (\vec{B} = \langle 0, 0, 0 \rangle)$$

* In moving frame

$$\vec{E} \cdot \vec{B} = 0 \quad (E_x, E_y, B_x, B_z = 0)$$

$E^2 - B^2$: * In rest frame

$$E^2 - B^2 = E^2 = E_z^2 \checkmark$$

* In moving frame

$$E^2 - B^2 = \gamma^2 E_z^2 - \beta^2\gamma^2 E_z^2$$

$$= E_z^2 \checkmark$$

3. Suppose there is a distribution of free charges (density ρ_f) with current density $\mathbf{J}_f$, in a medium with arbitrary $\mathbf{B}(\mathbf{H})$ and $\mathbf{D}(\mathbf{E})$, that is, there are two types of electric field, $\mathbf{D}, \mathbf{E}$, and two types of magnetic field, $\mathbf{B}, \mathbf{H}$.

a) 2 pt. Use the Lorentz force law to show that the rate at which the fields $\mathbf{E}$ and $\mathbf{B}$ do work on the charges in a volume V is given by

$$\frac{dW_f}{dt} = \int_V d^3x\, \mathbf{E} \cdot \mathbf{J}_f.$$

b) 5 pt. Use the result of part (a) and the macroscopic Maxwell equations to show Poynting's theorem,

$$\frac{dW_f}{dt} + \int_V d^3x\, \Upsilon + \oint_{\partial V} d\mathbf{a} \cdot \mathbf{S} = 0,$$

where ∂V is the surface enclosing the volume V, and $\mathbf{S}$ is the Poynting vector. Identify Υ, which in the absence of a medium is the rate of change of the electromagnetic energy density, $\partial u/\partial t$.

c) 3 pt. For a linear medium, with constant permittivity ε and constant permeability μ, state the relation between $\mathbf{D}$ and $\mathbf{E}$ and between $\mathbf{B}$ and $\mathbf{H}$, and show for such a medium the rate of change of electromagnetic energy density is given by

$$\Upsilon = \frac{\partial u}{\partial t}, \quad u = \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}\right),$$

in either SI or Heaviside-Lorentz units.

4. Consider a monochromatic plane electromagnetic wave of frequency ω propagating in vacuum in the z direction and polarized in the x direction, which impinges upon a perfect conductor at $z = 0$, as shown in the figure. The incident electric field is

$$\mathbf{E}_I(z,t) = \hat{\mathbf{x}} E_{0I} e^{i(kz-\omega t)}.$$

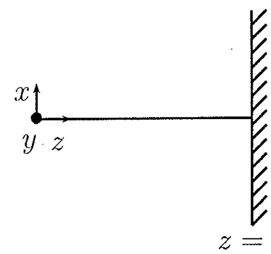


Figure 1: Plane wave normally incident on a perfectly conducting plane at $z = 0$.

a) 1 pt. Use Maxwell's equations to determine the relation between k and ω.

b) 1 pt. Use Maxwell's equations to determine the incident magnetic field, $\mathbf{B}_I(z,t)$.

c) 1 pt. What are the forms of the reflected wave $\mathbf{E}_R(z,t)$, $\mathbf{B}_R(z,t)$?

d) 2 pt. Apply the appropriate boundary conditions at the interface between the vacuum and the conductor to determine the reflected amplitudes E_{0R} and B_{0R} in terms of E_{0I}.

e) 1 pt. What is the phase of the incident and reflected electric fields? Are they in phase or out of phase at $z = 0$?

f) 2 pt. What is the force exerted on the conducting surface by the reflection of the plane wave? Answer this question by computing the momentum transferred from the field to the conductor.

g) 2 pt. Answer the same question by computing the discontinuity of the normal-normal component of the stress tensor across the interface, ΔT_{zz}.

Jan 2008 E+M #5 Gaussian

a) We determine the relationship by deriving the wave equation

$$\nabla \cdot D = 4\pi\rho_f \qquad \nabla \cdot B = 0$$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$

$$\nabla \times \left(\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t}\right) = 0$$

$$-\nabla^2 E + \nabla(\nabla \cdot E) + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times B) = 0$$

$$-\nabla^2 E + \frac{1}{\epsilon}\cancelto{0}{\nabla(\nabla \cdot D)} + \frac{\mu}{c}\frac{\partial}{\partial t}(\nabla \times H) = 0$$

$$-\nabla^2 E + \frac{\mu}{c}\frac{\partial}{\partial t}\left(\cancelto{0}{\frac{4\pi}{c}J_f} + \frac{1}{c}\frac{\partial D}{\partial t}\right) = 0$$

$$-\nabla^2 E + \frac{\mu\epsilon}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial E}{\partial t}\right) = 0$$

$$-\nabla^2 E + \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

$$-\frac{\partial^2}{\partial z^2}E + \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}E = 0$$

* Given $\hat{E} = E_0 \exp[i(kz - \omega t)]$

$$k^2 E_0 \exp[i(kz-\omega t)] - \frac{\mu\epsilon}{c^2}\omega^2 E_0 \exp[i(kz-\omega t)] = 0$$

$$\hookrightarrow \quad k^2 - \frac{\mu\epsilon}{c^2}\omega^2 = 0$$

$$k = \frac{\sqrt{\mu\epsilon}}{c}\omega$$

b) We use $\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$ to find B

$$\Rightarrow \quad \hat{B} = -c\int(\nabla \times E)\,dt$$

#5 (cont.)

b) $\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0 & 0 & 0 \end{vmatrix} = \langle 0, \partial_z E_0 \exp[i(kz-\omega t)], 0 \rangle$

$= ik E_0 \exp[i(kz-\omega t)] \hat{y}$

$\hookrightarrow B = \int -ikc E_0 \exp[i(kz-\omega t)] \hat{y} \, dt$

$= \frac{-ikc}{-i\omega} E_0 \exp[i(kz-\omega t)] \hat{y}$

$= \cancel{\frac{\sqrt{\mu\epsilon}\,\omega}{c}} \frac{c}{\omega} E_0 \exp[i(kz-\omega t)] \hat{y}$

$= \sqrt{\mu\epsilon}^{\,1} E_0 \exp[i(kz-\omega t)] \hat{y}$

$*$ in free space $\mu = \epsilon = 1$

$\Rightarrow \vec{B}_I = E_0 \exp[i(kz-\omega t)] \hat{y}$

c) The reflected waves will be of the form:

$\vec{E}_R = E_R \exp[i(-kz-\omega t)] \hat{x}$

$\vec{B}_R = n\hat{k} \times \vec{E}$

$= -\sqrt{\mu_0\epsilon_0}\, E_R \exp[i(-kz-\omega t)] \hat{y}$

$= -E_R \exp[i(-kz-\omega t)] \hat{y}$

#5 (cont.)

d) Our boundary conditions are:

$$① \; E_1^{\parallel} - E_2^{\parallel} = 0 \qquad ② \; D_1^{\perp} - D_2^{\perp} = 4\pi\sigma_f$$

$$② \; H_1^{\parallel} - H_2^{\parallel} = \frac{4\pi}{c}\vec{K}_f \, \hat{n} \qquad ④ \; B_1^{\perp} - B_2^{\perp} = 0$$

*Using boundary condition ①:

$$E_1^{\parallel} = E_I^{\parallel} + E_R^{\parallel} \qquad E_2^{\parallel} = E_T^{\parallel} = 0 \quad \text{b/c no field w/in conductor}$$

$$\hookrightarrow \; E_I = -E_R$$

$$B_R = -E_R \quad \text{(from previous work)}$$

$$\Rightarrow \vec{E}_R = -E_0 \exp[i(-kz - \omega t)]\,\hat{x}$$

$$\vec{B}_R = E_0 \exp[i(-kz - \omega t)]\,\hat{y}$$

e) We evaluate at $z = 0$ to determine phase difference

$$E_I = E_0 \exp[i(kz - \omega t)]$$

$$= E_0 \exp[-i\omega t]$$

$$E_R = -E_0 \exp[i(-kz - \omega t)]$$

$$= -E_0 \exp[-i\omega t]$$

$$= E_0 \exp[-i\omega t]\, e^{i\pi}$$

$$= E_I \, e^{i\pi} \quad \Rightarrow \text{out of phase by } \varphi = \pi$$

5. Consider a metallic conducting circular ring of inner and outer radii $r - \epsilon$ and r, respectively. Let the ring have thickness perpendicular to the radius h. Assume $h, \epsilon \ll r$. Let ρ be the resistivity and μ be the mass density of the material from which the ring is made. The ring rotates (on frictionless bearings) with angular frequency ω about an axis along a diameter of the ring and thus has mechanical energy $K(\omega) = \frac{1}{2}I\omega^2$, where $I = \frac{1}{2}(\mu 2\pi r h \epsilon)r^2$ is the moment of inertia of the ring. (Note that $\mu 2\pi r h \epsilon$ is the mass of the ring.) There is a uniform magnetic field $\mathbf{B}$ perpendicular to the axis of rotation. See Figure.

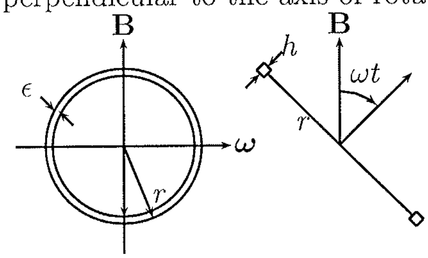


Figure 2: Top (left) and side (right) views of a conducting annulus, rotating with frequency ω about a diameter, with a magnetic field perpendicular to that axis of rotation.

a) 1 pt. Find the emf induced in the ring as a function of ω.

b) 3 pt. Find the average power dissipated by resistive heating in the ring as a function of ω, ρ, and B, and the relevant dimensional variables assuming that ω is constant.

c) 2 pt. If there is no external driving force keeping the ring rotating, the ring slows down due to this dissipation. Find the differential equation for $d\omega(t)/dt$ using $K(\omega)$ and the result of part (b). [The result of part (b) is correct if $d\omega/\omega \ll 1$.]

d) 2 pt. Solve this equation for ω as a function of time t in terms of the relevant variables such as μ, ρ, B, etc. Let ω_0 be the angular frequency at time $t = 0$.

e) 2 pt. If the annular ring were replaced by a solid disk of the same material with the same outer radius r and thickness h, would the disk take a longer time, a shorter time, or the same time as the annular ring to reach $\omega(t) = \omega_0/10$? You must support your answer with a physical argument to earn credit for this part.

6. This problem gives a covariant form of Maxwell's equations appropriate for describing both classical and quantum radiation.

a) 2 pt. Write down Maxwell's equations in vacuum (that is, no dielectric or magnetic media are present, only free charges and currents) in covariant form in terms of the field-strength tensor $F^{\mu\nu}$ and the electric four-current density j^{μ}. *State your units, and the metric you are using.* What is the relation between $F^{\mu\nu}$ and the electric and magnetic field $\mathbf{E}$ and $\mathbf{B}$, and between the current j^{μ} and the electric charge density ρ and the electric current density $\mathbf{j}$.

b) 2 pt. Show that one set of Maxwell's equations is satisfied if $F^{\mu\nu}$ is derivable from a four-vector potential,

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}.$$

What is the relation between A^{μ} and the ordinary scalar and vector potentials ϕ and $\mathbf{A}$?

c) 1 pt. Suppose A^{μ} is given in the Lorenz gauge, where

$$\partial_{\mu} A^{\mu} = 0.$$

What is the equation relating A^{μ} to the current density j^{μ}?

d) 1 pt. Show that the equation for A^{μ} derived in part (c) implies the conservation of electric charge.

e) 2 pt. Now consider the Coulomb gauge condition given by $\nabla \cdot \mathbf{A} = 0$. Show that this leads to

$$\nabla^2 A^0 = -\frac{4\pi}{c} J^0 \quad (\nabla^2 \Phi = -4\pi\rho),$$

in Gaussian units.

f) 2 pt. What is the corresponding equation satisfied by the vector potential $\mathbf{A}$ in the Coulomb gauge?

Electrodynamics Qualifier Examination

August 21, 2008

General Instructions: **In all cases, be sure to state your system of units.** Show all your work, write only on one side of the designated paper, and if you get stuck on one part, assume a result and proceed onward. The points given for each part of each problem are indicated. Each problem carries equal weight.

1. In relativistic notation, the field strength tensor $F^{\mu\nu}$ is given by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ in terms of the 4-vector potential $A^\mu = (\phi, \mathbf{A})$. Maxwell's equations become

$$\partial_\nu F^{\mu\nu} = kJ^\nu, \quad J^\mu = (c\rho, \mathbf{J}),$$

and k is a constant depending on the system of units adopted.

a) 1 pt. Write Maxwell's equations in terms of A^μ.

b) 2pts. Show that the field strength tensor is invariant under a gauge transformation,

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \lambda,$$

where λ is any function of space and time.

c) 1 pt. How does the form of Maxwell's equations found in part a) change if we exploit the gauge freedom to impose the Lorenz condition

$$\partial_\mu A^\mu = 0?$$

d) 2pts. Show that further gauge transformations are possible provided λ' satisfies

$$\partial^2 \lambda' \equiv \partial^\mu \partial_\mu \lambda' = 0.$$

e) 2pts. In empty space, $J^\mu = 0$, impose the further condition $A^0 = 0$ and rewrite the Lorenz gauge condition to obtain the radiation or Coulomb gauge condition. Is this gauge condition Lorentz invariant?

f) 2pts. Show that the plane-wave function

$$A^\mu(x) = a e^{ip\cdot x} \epsilon^\mu(p),$$

where $p^\mu = (\omega/c, \mathbf{k})$ is the propagation or wave vector, $x^\mu = (ct, \mathbf{x})$, $x \cdot p = x_\mu p^\mu$, a is a constant, and ϵ^μ is the polarization 4-vector, satisfies the Lorenz gauge condition provided ϵ^μ satisfies a particular condition. What is this condition? If this condition is satisfied, show that the empty-space Maxwell equation is satisfied provided there is a constraint on $p^2 = p^\mu p_\mu$. What does this constraint imply about the rest mass of the photon?

2. Consider a monochromatic plane electromagnetic wave of frequency ω propagating in a non-magnetic dielectric (with index of refraction n_1), traveling in the z direction and polarized in the x direction, which impinges normally upon a second non-magnetic semi-infinite dielectric material (with index of refraction n_2), where the boundary between the two media occurs at $z = 0$, as shown in Fig. 1. The incident electric field is

$$\mathbf{E}_I(z,t) = \hat{\mathbf{x}} E_{0I} e^{i(kz-\omega t)}.$$

There are no free charges or currents in either medium.

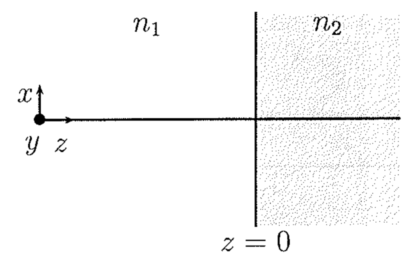


Figure 1: Plane wave normally incident on a surface separating two dielectric materials at $z = 0$. The medium in the the region $z < 0$ has index of refraction n_1 while the material in the region $z > 0$ has index of refraction n_2.

a) 1 pt. Use Maxwell's equations to determine the relation between k and ω in each region.

b) 1 pt. Use Maxwell's equations to determine the incident magnetic field, $\mathbf{B}_I(z,t)$, using the result of part b).

c) 1 pt. What are the forms of the reflected wave $\mathbf{E}_R(z,t)$, $\mathbf{B}_R(z,t)$ ($z < 0$), and of the transmitted wave $\mathbf{E}_T(z,t)$, $\mathbf{B}_T(z,t)$ ($z > 0$)?

d) 2pts. Apply the appropriate boundary conditions at the interface between the two media to obtain the equations determining the reflected amplitudes E_{0R} and B_{0R} and the transmitted amplitude E_{0T} and B_{0T} in terms of E_{0I}.

e) 2pts. Solve these equations for the reflection and transmission coefficients, $r = E_{0R}/E_{0I}$, $t = E_{0T}/E_{0I}$ in terms of the indices of refraction of the two media.

f) 2pts. Show that the averaged energy flux in a plane wave of amplitude E_0 moving in a medium with index of refraction n is given by (Gaussian units)

$$S = \frac{c}{8\pi} n |E_0|^2.$$

Show that the relative reflected and transmitted energy fluxes are

$$R = \frac{S_R}{S_I} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2, \quad T = \frac{S_T}{S_I} = \frac{4 n_1 n_2}{(n_1 + n_2)^2}.$$

g) 1 pt. Show that $R + T = 1$. Why is this as expected?

3. A *relativistic* particle of rest mass m and charge e is moving in a uniform (constant and static) magnetic field $\mathbf{B}$. The equations of motion for the particle momentum $\mathbf{p}$ and its energy E are (Gaussian units)

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c}\mathbf{v} \times \mathbf{B}, \quad \frac{dE}{dt} = 0.$$

a) 1 pt. Why is the particle energy conserved?

b) 1 pt. Express $\mathbf{p}$ in terms of m and the particle velocity $\mathbf{v}$, and E in terms of m and $\mathbf{v}$.

c) 3pts. Show that these equations of motion can be written as

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\omega} \times \mathbf{v},$$

and express $\boldsymbol{\omega}$ in terms of e, E, and $\mathbf{B}$. This says that the velocity vector precesses with angular velocity $\boldsymbol{\omega}$.

d) 3pts. Now suppose the motion is confined to the plane perpendicular to $\mathbf{B}$, that is, $\mathbf{B} \perp \mathbf{v}$. Then show that the particle moves with angular speed ω in a circle of radius R. Give an equation for R in terms of v, E, e, and B.

e) 2pts. Now give an equation relating the magnitude of the particle momentum p to the radius R found in part d). Thus show that a measurement of the radius of the orbit determines the particle momentum. If the velocity of the particle is independently known, we can then determine the mass m of the particle, according to the relation given in part b).

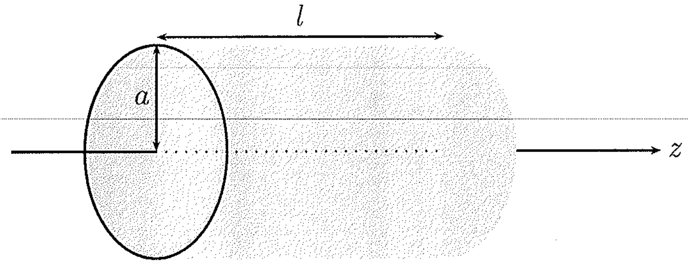


Figure 2: Hollow cylinder (radius a and length l) containing uniform gas flowing along the axis, the z direction, with velocity v. Protons are injected into the cylinder with velocity V parallel to the axis. As a result of magnetic forces, they are brought to a focus at a point on the z axis a distance p far from the cylinder, $p \gg l$.

4. Consider a hollow cylinder of radius a and length l filled with a completely ionized gas of uniform charge density ρ which is moving parallel to the axis of the cylinder with velocity v.

a) 3pts. Find the magnetic field (magnitude and direction) at a distance r from the axis of the cylinder, for $r < a$; assume that we are well inside the cylinder and that $l \gg r$ so that we can neglect edge effects. Assume that the gas is nonmagnetic.

b) 3pts. Suppose a beam of nonrelativistic protons of mass m and velocity V are sent into this cylinder with their initial velocities parallel to the z axis. Neglect electrostatic, edge effects, and collisions between protons and the gas. Show that while in the gas-filled cylinder, the protons experience a force pushing them toward the axis of the cylinder. Calculate the radial velocity V_r acquired by the protons when they exit the cylinder. Assume that the distance moved toward the axis while in the cylinder is negligible.

c) 2pts. After the protons leave the cylinder, they continue to move toward the z axis with constant radial velocity V_r. Calculate the time T required for the protons to reach the axis.

d) 2pts. As a result, the protons will travel through the cylinder and be focused at a point p on the z axis beyond the cylinder where $p \gg l$. Find p and show that it is independent of the initial distance of the protons from the axis when they enter the cylinder.

5. Consider an infinitely long, solid, nonmagnetic conducting rod of radius a centered on the z axis. An infinitely long, hollow, conducting cylinder with inner radius $b > a$ and outer radius d is coaxial with the rod. Let r be the radial distance perpendicular to the axis of the rod and the cylinder. The region between the conducting rod and the conducting cylinder (that is, $a < r < b$) is filled with a nonconducting, linear, isotropic magnetic material with a constant relative permeability $K = \mu/\mu_0$, where μ is the permeability of the material, and μ_0 is the permeability of free space ($\mu_0 = 1$ in Gaussian units).

   The rod carries a current I in the $+z$ direction while the concentric cylinder carries a current I in the $-z$ direction. We assume that the current density $\mathbf{j}$ is uniform and of the same magnitude in both the rod and the cylinder,

$$j = \frac{I}{\pi a^2} = \frac{I}{\pi(d^2 - b^2)}.$$

a) 3 pts. Calculate the magnetic field $H(r)$ for the four regions

$$\text{I: } r \leq a, \quad \text{II: } a \leq r \leq b, \quad \text{III: } b \leq r \leq d, \quad \text{IV: } d \leq r.$$

b) 3 pts. Calculate the magnetic flux (per unit length in the z direction) crossing a half-plane extending from the axis of the coaxial system and extending to infinity, that is, the surface defined by $x > 0$, $y = 0$, $-\infty < z < \infty$. Use this result to find the self-inductance L per unit length of the coaxial conductor.

c) 2 pts. Compute the magnetic energy U per unit length along the z axis stored in the region filled with the linear magnetic material, that is for region II, $a < r < b$.

d) 2 pts. Using the result from part c), show that the contribution to L coming from the region $a \leq r \leq b$, L_{II}, is consistent with the contribution from the same region that you calculated in part b) above. That is, compute $\frac{1}{2}L_{\text{II}}I^2$ and compare with the result of part c).

E+M #5

Aug 2008 Gaussian

a) We can compute the magnetic field via Ampere's Law (from $\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$)

$$\int \nabla \times B \, da = \int \frac{4\pi}{c}\vec{J} \cdot d\vec{a}$$

$$\int B \cdot d\vec{l} = \frac{4\pi}{c}\int \vec{J} \cdot da$$

$H = \vec{B} - 4\pi\vec{M}$

$B = \mu H$, $\mu = 1 + 4\pi\chi_m$

$M = \chi_m H$ *for linear, isotropic material

- We have 4 regions to consider:

Region I: $r \leq a$

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c}\int \left(\frac{I}{\pi a^2}\right) \cdot d\vec{a}$$

$$B \cdot \cancel{2\pi} r = \frac{{}^{2}\cancel{4\pi}}{c}\frac{I}{\cancel{\pi} a^2} \cdot \cancel{\pi} r^2$$

$$\vec{B} = \frac{2Ir}{ca^2}\hat{\varphi} \quad \text{(direction determined via symmetry)}$$

$$\vec{H} = \vec{B} - 4\pi\vec{M}$$

$$\vec{H} = \frac{2Ir}{ca^2}\hat{\varphi} - 0 \quad \text{(non-magnetic material)}$$

$$\vec{H} = \frac{2Ir}{ca^2}\hat{\varphi}$$

Region II: $a \leq r < b$

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c}\int \frac{I}{\pi a^2} \cdot d\vec{a}$$

$$B \cdot 2\pi r = \frac{4\pi}{c}\frac{I}{\pi a^2} \cdot \pi a^2$$

$$B = \frac{2I}{cr}\hat{\varphi}$$

* Because our magnetization is non-zero, $M = \chi_m H$, $H = \frac{1}{\mu}B$, $\mu = 1 + 4\pi\chi_m$

$$H = B - 4\pi M$$

$$H = B - 4\pi\chi_m H$$

$$H(1 + 4\pi\chi_m) = B$$

$$H\mu = B$$

$$H = \frac{1}{\mu}B$$

$$\Rightarrow H = \frac{2I}{\mu c r}\hat{\varphi}$$

#5 (cont.)

Region III: $b \leq r \leq d$

*Must separate $\vec{J}$ - integral to account for multiple currents

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c} \int \vec{J} \cdot da$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \left[\int \vec{J}_{rod} \cdot da_{rod} + \int \vec{J}_{shell} \cdot da_{shell} \right]$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \left[\frac{I}{\pi a^2} \cdot \pi a^2 + \frac{-I}{\pi(d^2-b^2)} \cdot \pi(r^2-b^2) \right]$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \left[I - \frac{I(r^2-b^2)}{d^2-b^2} \right]$$

$$\vec{B} = \frac{2I}{cr} \left[1 - \frac{r^2-b^2}{d^2-b^2} \right] \hat{\varphi}$$

* We again must consider magnetization in region II, following above work,

$$H = \frac{1}{\mu} B$$

$$= \frac{2I}{c\mu r} \left[1 - \frac{r^2-b^2}{d^2-b^2} \right] \hat{\varphi}$$

Region IV: $r \geq d$

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c} \int \vec{J} \cdot d\vec{a}$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \left[\int \vec{J}_{rod} \cdot da_{rod} + \int \vec{J}_{shell} \cdot da_{shell} \right]$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \left[\frac{I}{\pi a^2} (\pi a^2) + \frac{-I}{\pi(d^2-b^2)} (d^2-b^2)\pi \right]$$

$$\vec{B} = \frac{4\pi}{c} \left[I - I \right] \cdot \frac{1}{2\pi r}$$

$$\vec{B} = 0$$

$$\vec{H} = \frac{1}{\mu} \vec{B}$$

$$\vec{H} = 0$$

#5 (cont.)

b) *We know that the self-inductance L is related to the flux via: $\Phi = LI$

To get total flux of system, we sum the fluxes from the individual regions

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^\ell dz \int dx\, B$$

Region I: $r \le a$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^\ell dz \int_0^a \frac{2I\sqrt{x^2+y^2}}{ca^2}\,dx$$

$$= \int_0^r \frac{2I}{ca^2}\,x\,dx$$

$$= \frac{I}{ca^2}x^2\Big|_0^a$$

$$= \frac{I}{c}$$

Region II: $a \le r \le b$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^\ell dz \int_a^b \frac{2I}{c\sqrt{x^2+y^2}}\,dx$$

$$= \int_a^b \frac{2I}{cx}\,dx$$

$$= \frac{2I}{c}\ln(x)\Big|_a^b$$

$$= \frac{2I}{c}\left(\ln(b/a)\right)$$

Region III: $b \le r \le d$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^\ell dz \int_b^d \frac{2I}{cr}\left[1 - \frac{r^2-b^2}{a^2-b^2}\right]dx$$

$$= \int_b^d \frac{2I}{cx}\left[1 - \frac{x^2-b^2}{d^2-b^2}\right]dx$$

$$= \frac{2I}{c}\int\left[\frac{1}{x} - \frac{x}{d^2-b^2} + \frac{b^2}{x(d^2-b^2)}\right]dx$$

$$= \frac{2I}{c}\left[\ln(x) - \frac{1}{2}\frac{x^2}{d^2-b^2} + \frac{b^2}{d^2-b^2}\ln(x)\right]\Big|_b^d$$

#5 (cont.)

b) $\frac{\Phi}{l} = \frac{2I}{c}\left[\ln\left(\frac{d}{b}\right) - \frac{1}{2} + \frac{b^2}{d^2-b^2}\ln\left(\frac{d}{b}\right)\right]$

$= \frac{2I}{c}\left[\left(1+\frac{b^2}{d^2-b^2}\right)\ln\left(\frac{d}{b}\right) - \frac{1}{2}\right]$

Region IV : $r \geq d$

$B = 0 \Rightarrow \frac{\Phi}{l} = 0$

* Thus our overall flux per unit length is:

$\frac{\Phi}{l} = \frac{I}{c} + \frac{2I}{c}\left(\ln\left({}^{b}/_{a}\right)\right) + \frac{2I}{c}\left[\left(1+\frac{b^2}{d^2-b^2}\left(\ln\left(\frac{d}{b}\right)\right) - \frac{1}{2}\right]$

$= \cancel{\frac{I}{c}} + \frac{2I}{c}\ln\left({}^{b}/_{a}\right) + \frac{2I}{c}\ln\left(\frac{d}{b}\right) + \frac{2I}{c}\frac{b^2}{d^2-b^2}\ln\left(\frac{d}{b}\right) - \cancel{\frac{I}{c}}$

$= \frac{2I}{c}\left[\ln\left({}^{b}/_{a}\right) + \left(1+\frac{b^2}{d^2+b^2}\right)\ln\left(\frac{d}{b}\right)\right]$

$\frac{\Phi}{l} = \frac{LI}{l} \Rightarrow \frac{L}{l} = \frac{\Phi}{lI}$

$\Rightarrow \frac{L}{l} = \frac{2}{c}\left[\ln\left({}^{b}/_{a}\right) + \left(1+\frac{b^2}{d^2+b^2}\right)\ln\left(\frac{d}{b}\right)\right]$

c) $U = \frac{1}{8\pi}B^2$

$U = \frac{1}{8\pi}\left(\frac{2I}{cr}\hat{\varphi}\right)^2$

$= \frac{1}{8\pi}\frac{4I^2}{c^2r^2}$

$U = \int \frac{I^2}{4\pi c^2 r^2}\, r\, dr\, d\varphi\, dz$

$= \frac{lI^2}{2c^2}\int_a^b \frac{1}{r}\,dr$

$= \frac{lI^2}{2c^2}\ln\left({}^{b}/_{a}\right)$

$\frac{U}{l} = \frac{I^2}{2c^2}\ln\left({}^{b}/_{a}\right)$

#5 (cont.)

d) $\frac{1}{2} L_{II} I^2 = \frac{1}{2}\left(\frac{2}{c}\ln(b/a)\right) I^2$

$= \frac{I^2}{c}\ln(b/a)$

* off by factor $\frac{1}{2c}$, likely due to units being Gaussian v SI

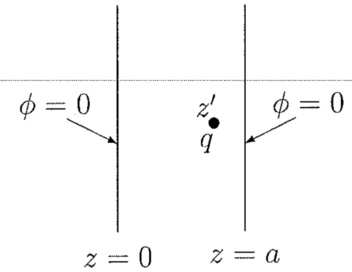


Figure 3: Geometry of point charge placed between grounded, parallel, conducting plates. The plates extend infinitely in the x and y directions.

6. Consider a point charge q placed between two parallel conducting plates, as shown in Fig. 3. The electrostatic potential ϕ vanishes at the two plates, located on the planes $z = 0$ and $z = a$.

a) 2pts. Because the physics has translational symmetry in the x-y plane, show that the potential at a point $\mathbf{r}$ between the plates due to a point charge at $\mathbf{r}'$ can be written in the form (Gaussian units)

$$\phi(\mathbf{r}) = 4\pi q \int \frac{(d^2\mathbf{r}_\perp)}{(2\pi)^2} e^{i\mathbf{r}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp} g(z, z'; k_\perp),$$

with $\mathbf{r}_\perp = (x, y)$, $\mathbf{r}'_\perp = (x', y')$, where the function g satisfies

$$\left(-\frac{\partial^2}{\partial z^2} + k_\perp^2\right) g(z, z'; k_\perp) = \delta(z - z').$$

What are the boundary conditions on $g(z, z'; k_\perp)$ at $z = 0$ and $z = a$?

b) 3pts. Solve this differential equation explicitly in closed form by solving it in two regions, I: $0 < z < z'$ and II: $z' < z < a$, and matching the solutions appropriately to reproduce the δ-function in the differential equation.

c) 2pts. What is the relationship between the electric field at the surface of a conductor and the surface charge density on the surface?

d) 3pts. Determine the normal component of the electric field just to the right of the plate at $z = 0$ (that is, at $z = 0 + \epsilon$) and just to the left of the plate at $z = a$ (that is, at $z = a - \epsilon$). By integrating this field over the surfaces, and using the result of part c), determine the total charge on each of the conducting surfaces. Is the sum of the charges on the two plates as expected?

E&M

January 2009

1 Capacitors

Consider a spherical capacitor which has the space between its plates filled with a dielectric of permittivity ϵ. The inner sphere has radius r_1 and the outer sphere has radius r_2. The total free charge on the capacitor is Q.

a) Find the electric displacement $\vec{\mathbf{D}}$, and the electric field $\vec{\mathbf{E}}$ at a radius r inside the dielectric. **(2.5-points)**

b) Find the electric energy density u_e inside the dielectric. **(2.5-points)**

c) Find the total energy U_e of the capacitor. **(2.5-points)**

d) Find the capacitance C, of the capacitor. **(2.5-points)**

2 Rods

In this problem you will determine the magnetic field produced by three different infinitely long, cylindrical conducting rods. The figures on the next page are useful for vizualizing the differences in the rods.

a) State whether you are using MKS or cgs units (**0-points**).

b) **Rod 1**: Consider an infinitely long, solid, cylindrical conducting rod (known as Rod 1) with radius $2a$ that is concentric with the z-axis and carries a uniform current density $+J_0$ in the $+z$ direction (Fig. 1). Let $r = (x^2+y^2)^{1/2}$ be the perpendicular distance to the z-axis and θ be the angle r makes with the positive x-axis. (See Fig. 1.) Find the magnitude and direction of the magnetic field $\vec{\mathbf{B}}_1$ produced by Rod 1 for all r. Give the direction in Cartesian coordinates using the unit vectors: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. **(3-points)**

c) **Rod 2**: Consider a second, infinitely long, solid, cylindrical conducting rod (i.e. Rod 2) that is parallel to the z-axis and has radius a. The axis of Rod 2 is centered at $(x, y) = (+a, 0)$. Rod 2 carries a uniform current density $-J_0$ in the $-z$ direction. Let ρ be the radial distance from the axis of the Rod 2 and let ϕ be the angle that ρ makes with the positive x-axis. (See Fig. 2.) Find the magnitude and direction of the magnetic field $\vec{\mathbf{B}}_2$ produced by Rod 2 for all values of ρ and ϕ. Give the direction in Cartesian coordinates using the unit vectors: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. **(1-points)**

d) **Rod 3**: Consider an infinitely long, cylindrical conducting rod (i.e. Rod 3) with radius $2a$ that is concentric with the z-axis and carries a uniform current density J_0 in the $+z$ direction. However in this conductor, an (infinitely long) hole of radius a is drilled parallel to the z-axis at the position $(x, y) = (+a, 0)$. (See Fig 3). Find the magnitude and direction of the magnetic field $\vec{\mathbf{B}}_3$ produced by Rod 3 on the x-axis (at $y = 0$) for all values of $x > 0$. Give the direction in the Cartesian cordinates using the unit vectors: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. **(6-points)**

3 Cubical Box

A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded. The top is made of a separate sheet of metal, insulated from the others, and held at constant potential V_0. In this problem you will find the potential **inside** the box.

a) Assume that Laplace's Equation is separable and, beginning with Laplace's Equation, write three ordinary, second-order, differential equations, one each for x, y, and z. How are these equations linked? **(2-points)**

b) Write the appropriate forms of the solutions to the three differential equations of part (a). **(2-points)**

c) Apply the boundary conditions to determine all constants for the solutions of part (b). **(4-points)**

d) From your results in part (c), construct the general solution for the potential $V(x, y, z)$ inside the box. **(2-points)**

4 Conducting Sphere

In this problem you are to prove that a perfectly conducting sphere acquires a magnetic dipole moment when placed in a uniform magnetic field $\vec{\mathbf{B}}_0$. Let the sphere have radius a. By perfectly conducting, we mean that there is no magnetic field in the interior of the sphere. As a result of the induced dipole moment, the magnetic field $\vec{\mathbf{B}}$ exterior to the sphere is no longer $\vec{\mathbf{B}}_0$. Determine the dipole moment as follows.

a) What are the boundary conditions on $\hat{\mathbf{n}} \cdot \vec{\mathbf{B}}$ and $\hat{\mathbf{n}} \times \vec{\mathbf{B}}$ at the surface of the sphere, where $\hat{\mathbf{n}}$ is the outward unit normal to the spherical surface? These boundary conditions involve the surface current density $\vec{\mathbf{K}}$, which will be determined below. **(1-point)**

b) Assume that the induced magnetic field is a pure magnetic dipole field, that is, in a spherical cordinate system with origin at the center of the sphere,

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 + \frac{3\vec{\mu} \cdot \vec{\mathbf{r}}\vec{\mathbf{r}} - \vec{\mu} r^2}{r^5}, \qquad r > a.$$

Use the boundary condition on $\hat{\mathbf{r}} \cdot \vec{\mathbf{B}}$ at $r = a_+$ to determine $\vec{\mu}$ in terms of a and $\vec{\mathbf{B}}_0$. **(3-points)**

c) Use the boundary condition on $\hat{\mathbf{r}} \times \vec{\mathbf{B}}$ at $r = a_+$ to determine the surface current density $\vec{\mathbf{K}}$ in terms of $\vec{\mathbf{r}}$ and $\vec{\mathbf{B}}_0$. **(3-points)**

d) Compute the magnetic dipole moment from the surface current according to

$$\vec{\mu} = \frac{1}{2c} \oint dS\, \hat{\mathbf{r}} \times \vec{\mathbf{K}}$$

where the integral extends over the surface of the sphere, and show that $\vec{\mu}$ coincides with the result found in part b. **(3-points)**

5 Stress Tensor

Consider a stationary solid sphere of radius a and uniform surface charge density σ. Assume a coordinate system for which the sphere is centered at the origin.

a) Specify the system of units you will be using. **(0-points)**

b) Determine the electromagnetic field everywhere on the x-y plane **(2-points)**

c) Write down the Maxwell Stress Tensor everywhere. **(4-points)**

d) Use the Maxwell Stress Tensor to determine the net force that the southern hemisphere ($z < 0$) exerts on the northern hemisphere ($z > 0$). **(4-points)**

6 Electromagnetic Waves

A monochromatic, plane polarized, plane electromagnetic wave traveling in the z-direction in the lab frame (in a vacuum, $\epsilon = \mu = 1$) can be written in the following 3+1 dimensional form:

$$\vec{\mathbf{E}} = E_0 \hat{\mathbf{i}} \exp^{i(kz-\omega t)},$$

$$\vec{\mathbf{B}} = B_0 \hat{\mathbf{j}} \exp^{i(kz-\omega t)},$$

a) Combine this $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ into a single electromagnetic field tensor $F^{\alpha\beta}$ and use Maxwell's equations in the 4-dimensional form

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta = 0$$

to find all constraints on the 4 constants E_0, B_0, k, and ω (i.e., the above wave won't satisfy Maxwell's equations for arbitrary values of all four of these parameters). **(2-points)**

b) What are the values of the invariants $F^{\alpha\beta}F_{\alpha\beta}$ and $\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$ for this wave? **(2-points)**

c) Use a Lorentz boost to find $F'^{\alpha\beta}$ in a frame moving in the +z direction with a speed ν. Don't forget to express your answer in terms of the moving coordinates t' and x'^i. **(2-points)**

d) How has the frequency and the wavelength of this wave changed in the moving frame? **(2-points)**

e) How has $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ changed in direction and/or magnitude? **(2-points)**

E&M

Fall 2009

1 Magnetic Materials

Assume the field inside a large piece of magnetic material is $\vec{\mathbf{B}}_0$ so that

$$\vec{\mathbf{H}}_0 = \frac{1}{\mu_0}\vec{\mathbf{B}}_0 - \vec{\mathbf{M}}$$

.

a) Consider a small spherical cavity that is hollowed out of the material. Find the field $\vec{\mathbf{B}}$, at the center of the cavity, in terms $\vec{\mathbf{B}}_0$ and $\vec{\mathbf{M}}$. Also find $\vec{\mathbf{H}}$ at the center of the cavity in terms of $\vec{\mathbf{H}}_0$ and $\vec{\mathbf{M}}$. (3 Points)

b) Do the same calculations for a long needle-shaped cavity running parallel to $\vec{\mathbf{M}}$. (3 Points)

c) Do the same calculations for a thin wafer-shaped cavity perpendicular to $\vec{\mathbf{M}}$. (4 Points)

Hint: Assume the cavities are small enough so that $\vec{\mathbf{M}}$, $\vec{\mathbf{B}}_0$ and $\vec{\mathbf{H}}_0$ are essentially constant. The field of a magnetized sphere is $\vec{\mathbf{B}} = \frac{2}{3}\mu_0\vec{\mathbf{M}}$ and the field inside a long solenoid is $\mu_0 K$ where K is the surface current density.

2 Space-charge-limited Thermionic Planar Diode

Consider a planar diode with a grounded, hot metallic cathode at $x = 0$ and a metallic anode plate at $x = H$, which is held at an electrical potential of V_p relative to ground. [Cathode and anode plates are infinite in the y and z directions.] The cathode is very hot and emits copious electrons such that the diode is "space charge limited", that is: the electric field **at the cathode is zero.** The current density J is constant and in the -x direction. [Ignore any transient effects.]

In this problem let : $V(x)$ be the electric potential, $E(x)$ be the electric field, s(x) be the velocity of an electron, $\rho(x)$ be the charge density, and m and -e be the mass and charge of an electron respectively.

a) State whether you are using MKS or cgs units. (1 Point)

b) Find $\rho(x)$ as a function of $V(x)$ and any other relevant variables. (2 Points)

c) Use Poisson's equation to find the differential equation for $V(x)$. (2 Points)

d) State the boundary conditions for $E(x)$ at $x = 0$ and $V(x)$, at $x = 0$ and $x = H$. (2 Points)

Work e) or f) on a separate sheet of paper and submit only the one you wish to be graded.

e) Solve for V(x) in terms of V_p and H using results of c) and d). ((3 Points for part e or f)

Hint: multiply both sides of your differential equation by $dV(x)/dx$ and recall that: $(dV/dx)(d^2V/dx^2) = \frac{1}{2}d(dV/dx)^2/dx$

If you have trouble using the above hint to complete part e), then try f).

f) Assume $V(x)$ is of the form: $Ax^n + Bx + C$ and solve for $V(x)$ in terms of V_p and H using parts c) and d) above. Find the current density J in terms of V_p and H. (3 Points for part e or f)

3 Wire

An infinitely-long, thin wire (radius b) is coated with a dielectric (relative dielectric constant $k = \epsilon/\epsilon_0$ with radius $a > b$). The metal wire has charge per unit length λ

a) Find the electric displacement $\vec{\mathbf{D}}$ everywhere. (2 points)

b) Find the electric field $\vec{\mathbf{E}}$ everywhere. (2 points)

c) Find the polarization $\vec{\mathbf{P}}$ everywhere. (3 points)

d) Find **all** the bound charge everywhere. (3 points)

4 Electromagnetic Waves

Consider a plane electromagnetic wave with propagation vector $\vec{\mathbf{k}}$ and angular frequency ω. Construct the four-vector $k^\mu = (\omega/c, \vec{\mathbf{k}})$. Use the metric $g_{\mu\nu} = diag(-1, 1, 1, 1)$

a)Verify that $k_\mu k^\mu = 0$. (2 points)

b) In terms of the position four-vector $x^\mu = (ct, \vec{\mathbf{r}})$, show that the plane wave propagation factor is

$$e^{ik_\mu x^\mu} = e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}.$$

(2 points)

c) Now use Lorentz transformations to show that radiation of frequency ω propagating at an angle θ with respect to the z-axis, will, to an observer moving with relative velocity $\nu = \beta c$ along the z axis, have the frequency

$$\omega' = \frac{1}{\sqrt{1-\beta^2}}\omega(1 - \beta\cos\theta).$$

(2 points)

d) Further show that the moving observer sees the radiation propagating at an angle θ' with repect to the z-axis, where

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta},$$

which is aberration. (2 points)

e) Find θ' explicitly if $|\beta| << 1$. (2 points)

5 Thin Infinite Sheet

a) Compute the 4-current $J^\alpha(x^\beta)$ and the E&M fields for a stationary, thin, and infinite sheet of charge located at $z = 0$ in the lab. Assume the surface charge density is a constant σ_0. (4 points)

b) Now assume you move with speed $v < c$ in the x-direction relative to the lab. What is the 4-current $J'^\alpha(x^\beta)$ and E&M field in your frame? (6 points)

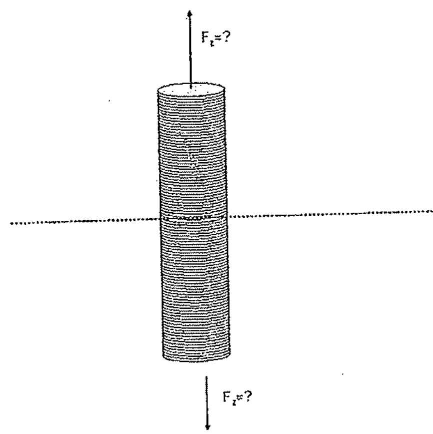


Figure 1: Stack of Disks for Stress Tensor Problem. Problem 6

6 Stress Tensor

Consider a long cylinder of radius a and length L made up of a stack of infinitesimally thin discs (See Figure). Assume the disks alternate between disks with charge density ρ and angular velocity $\omega\hat{z}$ and disks with charge density $-\rho$ and angular velocity $-\omega\hat{z}$.

a) Specify the system of units you will be using. (1 points)

b) write down an expression fo the charge and current density in any small volume (of dimension larger than the infinitesimal thickness of the disks). (1 points)

b) Find the electromagnetic field everywhere. (2 points)

c) Find the Maxwell Stress Tensor everywhere. (2 points)

d) Use your answer to part c to find the force of the top half of the cylinder on the bottom half. (2 points)

e) Is the force attractive or repulsive? (2 points)

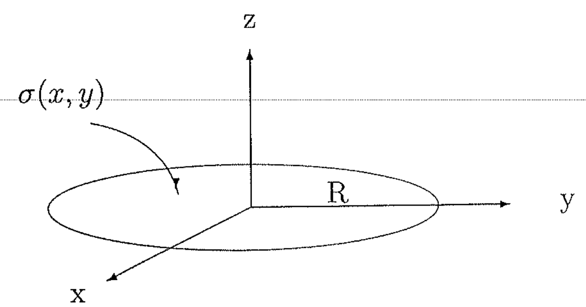
E & M Qualifier

January 14, 2010

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



Poorly written, field not symmetric about axis

1. Consider a thin nonconducting disk of radius R centered on the origin of a coordinate system, lying in the x-y plane, and carrying a surface charge density given by

$$\sigma = \sigma_o \frac{yR}{x^2 + y^2}.$$

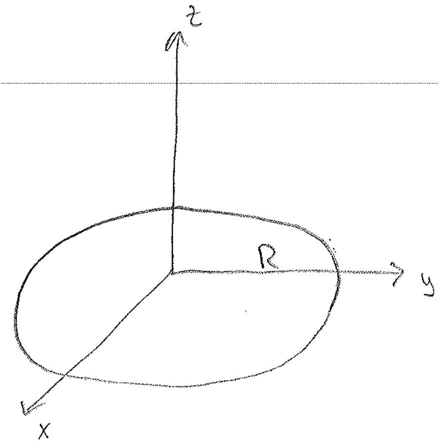
(a) {6 pts} Determine the electric field at a location $\vec{r} = z\hat{k}$.

(b) {3 pts} Give an approximation to your answer to part (a) that is valid for the $z >> R$.

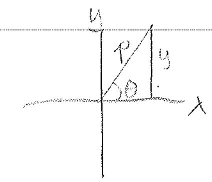
(c) {1 pts} Find the force on a charge q located at a position $\vec{r} = z\hat{k}$.

Jan 2010 # E+M #1 Gaussian

a)



$$\sigma = \sigma_0 \frac{yR}{x^2+y^2}$$



$$\sigma = \sigma_0 \frac{yR}{\rho^2}$$

$$= \sigma_0 \frac{\rho \sin\theta \, R}{\rho^2}$$

$$= \sigma_0 \frac{\sin\theta \, R}{\rho}$$

$$\int \nabla \cdot E = \int 4\pi \rho$$

$$\int \vec{E} \cdot d\vec{a} = 4\pi \int \sigma(r) \, \delta(z) \, \rho \, d\rho \, d\theta \, dz$$

$$E \cdot 2\pi \rho^2 = 4\pi \int \sigma_0 \sin\theta \, R \, \delta(z) \, \rho \, d\rho \, d\theta \, dz$$

$$= 4\pi \sigma_0 R \int \sin\theta \, \rho \, d\rho \, d\theta$$

$$= 2\pi \sigma_0 \rho \int \sin\theta \, d\theta$$

$$= -\pi \sigma_0 \rho \cos\theta \Big|_0^{2\pi}$$

$$= 0$$

2. Consider a linear, homogeneous, isotropic, and non-dissipative dielectric (i.e., a dielectric where $\mathbf{D} = \epsilon\mathbf{E}$ and ϵ is a constant) in the shape of a sphere of radius R with a point charge Q embedded at its center.

   (a) {2 pts} Find the electric displacement vector $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization density $\mathbf{P}$ inside the dielectric.

   (b) {2 pts} Find the bound charge volume density ρ_D inside the dielectric.

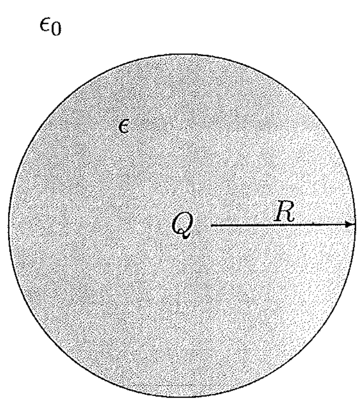
   (c) {1 pts} Find the total bound charge Q_D on the $r = R$ boundary of the dielectric.

   (d) {2 pts} Find the net charge (free plus bound) at the center of the dielectric.

   (e) {1 pts} Find the electric displacement vector $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization density $\mathbf{P}$, outside the dielectric sphere.

   (f) {2 pts} Are $\mathbf{D}$ and $\mathbf{E}$ continuous at $r = R$? If not explain why.

   (If you use Gaussian units you can put $\epsilon_0 = 1$.)



Jan 2010

E+M #2

Gaussian

a) *As a reminder, our electrostatic Maxwell's law in media are:

$$\nabla \cdot D = 4\pi \rho \qquad D = \vec{E} + 4\pi \vec{P}, \quad D = \epsilon \vec{E}$$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad P = \chi_e E \qquad \hookrightarrow \epsilon = 1 + 4\pi\chi_e$$

*We solve for D using Gauss Law

$$\int \nabla \cdot D = \int 4\pi \rho$$

$$D \cdot 4\pi r^2 = 4\pi Q$$

$$D = \frac{Q}{r^2}\hat{r}$$

$$\Rightarrow \vec{E} = \frac{D}{\epsilon} = \frac{Q}{\epsilon r^2}\hat{r}$$

$$\Rightarrow \vec{P} = \frac{\vec{D} - \vec{E}}{4\pi} = \frac{1}{4\pi}\left(\frac{Q}{r^2} - \frac{Q}{\epsilon r^2}\right)\hat{r} = \frac{Q}{4\pi r^2}\left(1 - \frac{1}{\epsilon}\right)\hat{r}$$

b) The bound charge density is defined as:

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{Q}{4\pi r^2}\left(1 - \frac{1}{\epsilon}\right)\right) = Q\,\delta(\vec{r})\left(1 - \frac{1}{\epsilon}\right)$$

c) To find the total surface bound charge,

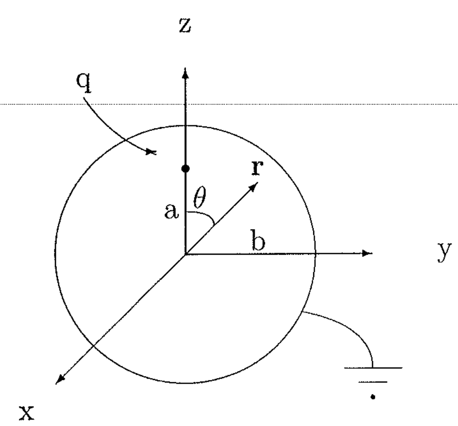
$$\sigma_b = P \cdot \hat{n} = P \cdot \hat{r} = \frac{Q}{4\pi r^2}\left(1 - \frac{1}{\epsilon}\right)\Big|_{r=R} = \frac{Q}{4\pi R^2}\left(1 - \frac{1}{\epsilon}\right)$$

#2 (cont.)

c) We must integrate over the surface to find:

$$Q_D = Q\left(1 - \frac{1}{6}\right)$$

d)



3. A thin grounded hollow conducting sphere of radius 'b' is centered at the origin. A point charge q is located on the z-axis at $z = a < b$ INSIDE the sphere.

   (a) {5 pts} Write the total potential for this system as a sum,

   $$\Phi = \Phi_{sphere} + \Phi_q,$$

   where Φ_q is the potential due to the point charge and Φ_{sphere} (in spherical polar coordinates) is the appropriate linear combination of Legendre polynomials $P_\ell(\cos(\theta))$. Evaluate the coefficients of the $P_\ell(\cos(\theta))$ in the Φ_{sphere} expansion. Recall that the Legendre polynomials are independent orthogonal functions satisfying

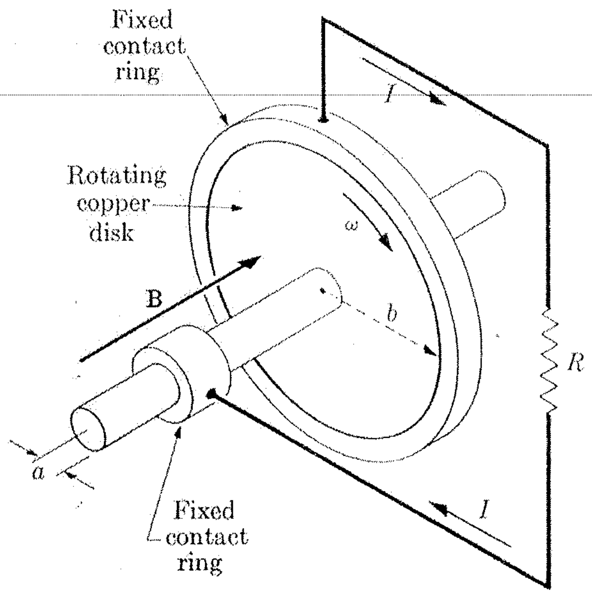
   $$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

   and

   $$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{\ell=0}^{\ell=\infty} \frac{(r_<)^\ell}{(r_>)^{\ell+1}} P_\ell(\cos(\gamma))$$

   where γ is the angle between the two directions $\mathbf{r}$ and $\mathbf{r}'$.

   (b) {5 pts} Show that your expression for Φ_{sphere} is equivalent to the potential of a point charge. Where is the point charge located and what is it's charge?



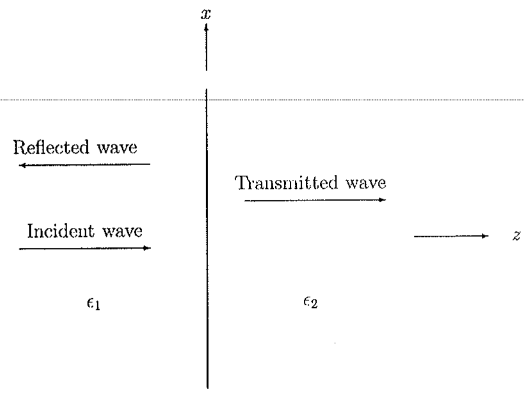
4. The Homopolar Generator consists of a flat copper disk of radius b and thickness t, mounted on an axle of radius a, which mechanically rotates the disk with angular speed ω in the presence of an orthogonal magnetic induction $\mathbf{B}$. A stationary contact ring with inner radius b and negligible resistance surrounds the rotating disk making good electrical and frictionless contact with it. As shown in the figure, the closed electrical circuit consists of the disk and a load resistor R connected by wires between the axle and the stationary contact ring. (Assume the load resistor R is much greater than the resistance of the disk, the contact ring, and the wires.) A constant magnetic induction $\mathbf{B}$ perpendicular to the disk (parallel to the rotation axis) exists between the radii a and b and is zero elsewhere in the circuit.

   (a) {4 pts} Find the current I that flows in the circuit as a function of B, a, b, ω, and R.

   (b) {2 pts}What is the magnitude of the current density $J(r)$ in the rotating disc.

   (c) {2 pts} What torque would you have to apply to the rotating wheel to keep ω from slowing down.

   (d) {2 pts} If σ is the conductivity of copper and t is the thickness of the disk, find the electrical resistance R_d of the disk between the radii a and b. Recall that the resistance of a small length $\Delta\ell$ of conducting material with cross sectional area A is $\Delta R = \Delta\ell/(\sigma A)$.



5. A plane-polarized harmonic ($e^{-i\omega t}$) plane electromagnetic wave traveling to the right in a homogeneous dielectric medium described by an dielectric constant ϵ_1, strikes a second homogeneous dielectric material described by dielectric constant $\epsilon_2 > \epsilon_1$ (see the figure). Assume that both materials have the same magnetic permeability μ_0 and that the incidence angle is 0^o (i.e., the wave is traveling perpendicular to the junction). Assume the incoming wave is polarized in the $\hat{x}$ direction and that its electric field amplitude is E_0, i.e., assume the incoming electric field is the real part of

$$\mathbf{E} = E_0\, e^{i(kz-\omega t)}\, \hat{x}.$$

(a) {3 pts} Give the magnetic induction $\mathbf{B}$ associated with the above incoming wave. Make sure your wave satisfies Maxwell's equations, e.g., give k as a function of ω, the direction of $\mathbf{B}$, and the amplitude of $\mathbf{B}$ as a function of E_0.

(b) {1 pts} Give similar expressions for the $\mathbf{E}$ and $\mathbf{B}$ components of the reflected and transmitted waves. Use E_0'' and E_0' for the respective amplitudes of reflected and transmitted waves.

(c) {2 pts} In general, what conditions must be satisfied at the junction between two materials by the electromagnetic fields $\mathbf{E}, \mathbf{B}, \mathbf{D}$, and $\mathbf{H}$, if Maxwell's equations are to be satisfied?

(d) {2 pts}Apply these junction conditions to the combined incoming, reflected, and transmitted wave to compute E_0'' and E_0' as functions of E_0 and the two dielectric constants ϵ_1 and ϵ_2.

(e) {2 pts} Evaluate the time averages of the Poynting vectors of the incident, reflected, and transmitted waves. Recall that

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} \equiv \frac{1}{4\pi} \mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

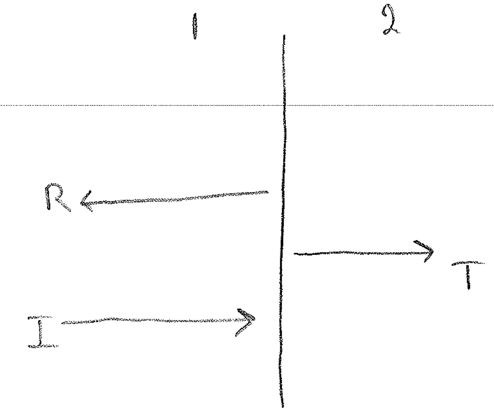
The sum of the magnitudes of the reflected and transmitted time averaged Poynting vectors should equal the magnitude of the incident wave's time averaged Poynting vector.

Jan 2010

E+M #5

Gaussian

a)



$$\vec{E} = E_0 \exp[i(k_1 z - \omega t)]\,\hat{x}$$

$$\vec{B} = n\hat{k}_1 \times \vec{E}$$

$$= \sqrt{\epsilon_1}\, E_0 \exp[i(k_1 z - \omega t)]\,\hat{y}$$

* The relevant Maxwell equations in Gaussian units are:

① $\nabla \cdot D = 4\pi \rho_f$ ③ $\nabla \cdot B = 0$

② $\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$ ④ $\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$

* Starting w/ equation ③:

$$\nabla \cdot B = \frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z$$

$$= \frac{\partial}{\partial y}\left(-\sqrt{\epsilon_1}\, E_0 \exp[i(k_1 z - \omega t)]\right)$$

$$= 0 \checkmark$$

* Now equation ②:

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$

$$\hookrightarrow \nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = \langle 0, +\partial/\partial z\, E_x, -\tfrac{\partial}{\partial y}E_x \rangle$$

$$= \langle 0, ik_1 \; E_0 \exp[i(k_1 z - \omega t)], 0 \rangle$$

#5 (cont.)

a) $\frac{1}{c}\frac{\partial}{\partial t}B = \frac{1}{c}\frac{\partial}{\partial t}\left(-\sqrt{\epsilon_1}\,E_0 \exp[i(k_1 z - \omega t)]\right)\hat{y}$

$= \frac{-1}{c}\sqrt{\epsilon_1}\, i\omega\, E_0 \exp[i(k_1 z - \omega t)]\,\hat{y}$

$\Rightarrow \nabla \times \vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} \overset{?}{=} 0$

$= -ik \quad E_0 \exp[i(kz - \omega t)] + -i\frac{\omega}{c}\sqrt{\epsilon_1}\,E_0 \exp[i(kz-\omega t)]$

$= -iE_0 \exp[i(k_1 z - \omega t)]\left(-k + \frac{\omega}{c}\sqrt{\epsilon_1}\right)$

↳ Only true if $k = \frac{\omega}{c}\sqrt{\epsilon_1}$

* Finally equation ④

$\nabla \times H - \frac{1}{c}\cdot\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$

$\vec{J}_f = 0$

$\nabla \times H = \nabla \times \frac{1}{\mu_0}\vec{B}$

$= \frac{1}{\mu_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & B_y & 0 \end{vmatrix} = \langle -\partial_z B_y, 0, \partial_x B_y \rangle$

$= \frac{1}{\mu_0}\frac{\partial}{\partial z}\left(\sqrt{\epsilon_1}\,E_0 \exp[i(k_1 z - \omega t)]\right)\hat{x}$

$= \frac{-\sqrt{\epsilon_1}}{\mu_0}E_0(ik_1)\exp[i(k_1 z - \omega t)]\,\hat{x}$

$-\frac{1}{c}\frac{\partial D}{\partial t} = \frac{-1}{c}\frac{\partial}{\partial t}\epsilon_1\vec{E}$

$= \frac{-\epsilon_1}{c}(-i\omega)E_0 \exp[i(k_1 z - \omega t)]\,\hat{x}$

↳ $\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{i\omega\epsilon_1}{c}E_0 \exp[i(k_1 z - \omega t)]\hat{x} + -i\frac{\sqrt{\epsilon_1}k}{\mu_0}E_0 \exp[i(k_1 z - \omega t)]$

$= 0$ if $\frac{\omega\epsilon_1}{c} - \frac{\sqrt{\epsilon_1}k}{\mu_0} = 0 \Rightarrow k = \frac{\omega\sqrt{\epsilon_1}}{c}$ as before ✓

#5 (cont.)

b) * For the transmitted wave:

$$\vec{E}' = E_0' \exp[i(k_2 z - \omega t)]\,\hat{x}$$

$$\vec{B}' = \sqrt{\epsilon_2}\, E_0' \exp[i(k_2 z - \omega t)]\,\hat{y}$$

$$k_2 = \frac{\omega\sqrt{\epsilon_2}}{c}$$

* For the reflected waves

$$\vec{E}'' = E_0'' \exp[i(k_1(-z) \omega t)]\,\hat{x}$$

$$\vec{B}' = +\sqrt{\epsilon_1}\, E_0'' \exp[i(-k_1 z - \omega t)]\,\hat{y}$$

$$k_1 = \frac{\omega\sqrt{\epsilon_1}}{c}$$

c) The general boundary conditions are:

$$D_1^{\perp} - D_2^{\perp} = 4\pi\sigma_f \qquad B_1^{\perp} - B_2^{\perp} = 0$$

$$E_1^{\parallel} = E_2^{\parallel} \qquad H_1^{\parallel} - H_2^{\parallel} = 4\pi \vec{K}_f$$

d) Applying these boundary conditions to our situation

$$E_1^{\parallel} - E_2^{\parallel} = 0$$

$$(\vec{E}_I^{\parallel} + \vec{E}_R^{\parallel}) - \vec{E}_T^{\parallel} = 0$$

$$[(E_I + E_R) - E_T] \times \hat{z} = 0 \qquad \Rightarrow \text{picks out } \hat{y} \text{ components}$$

$$-E_0 \exp[i(k_1 z - \omega t)] + E'' \exp[i(-k_1 z - \omega t)] - E' \exp[i(k_2 z - \omega t)] = 0$$

* if we set our boundry to be $z = 0$

$$-E_0 + E_0'' - E_0' = 0$$

#5 (cont.)

d) $H_1^{\parallel} - H_2^{\parallel} = \cancel{4\pi K_f} \rightarrow 0$

$\frac{1}{\mu_0}(B_1^{\parallel} - B_2^{\parallel}) = 0 \quad \Rightarrow \quad B_1^{\parallel} - B_2^{\parallel} = 0$

$\hookrightarrow \quad B_1^{\parallel} - B_2^{\parallel} = 0$

$B_I^{\parallel} + B_R^{\parallel} - B_T^{\parallel} = 0$

$[(B_I + B_R) - B_T] \times \hat{z} = 0 \quad \Rightarrow$ picks off x-component

$-E_0\sqrt{\epsilon_1}\exp[i(k_1 z - \omega t)] + E_0''\sqrt{\epsilon_1}\exp[i(-k_1 z - \omega t)] - E_0'\sqrt{\epsilon_2}\exp[i(k_2 z - \omega t)]$

$(-E_0 + E_0'')\sqrt{\epsilon_1} - \sqrt{\epsilon_2}\,E_0' = 0$

$\hookrightarrow \quad E_0' = \sqrt{\frac{\epsilon_1}{\epsilon_2}}(-E_0 + E_0'')$

*Using our equations to solve for E_0', E_0'' in terms of E_0:

$$E_0' = \sqrt{\frac{\epsilon_1}{\epsilon_2}}(-E_0 + E_0'')$$

$$= \sqrt{\frac{\epsilon_1}{\epsilon_2}}(-E_0 + (E_0' + E_0))$$

6. Maxwell's equations in 4 dimensions

(a) {2 pts} Write the Maxwell equations in the absence of polarizable materials using 4-vector notation, making use of the field strength tensor $F_{\mu\nu}$.

(b) {4 pts} Show that the equations of part (a) reduce to the usual form of Maxwell's equations in 3-vector notation.

(c) {2 pts} The Lagrangian density of the EM field is given by

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}, \qquad (SI)$$

or

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}. \qquad (Gaussian)$$

Recall that all repeated Greek indices are summed over 4-dimensions (1 time and 3 space). Show that the Lagrangian density is invariant under a gauge transformation $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \alpha(x)$, where α is an arbitrary function of spacetime $x \equiv (ct, \vec{x})$.

(d) {2 pts} If we add an interaction term $\mathcal{L} \rightarrow \mathcal{L} + \Delta\mathcal{L}$ where

$$\Delta\mathcal{L} = j^\mu A_\mu, \qquad (SI)$$

or

$$\Delta\mathcal{L} = \frac{1}{c} j^\mu A_\mu, \qquad (Gaussian)$$

to the Lagrangian– where j^μ is some spatially bounded and conserved 4-current density– how does the action $I \equiv \int \mathcal{L} d^4r$ change under a gauge transformation and do the resulting equations of motion change?

E+M #6

Jan 2010 Gaussian

a) In free space, 4-D Maxwell equations in tensor form are:

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \qquad \text{(Dual Tensor } (B \to E,\ E \to B))$$

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

b) To show these reduce to the 3 vector form:

$$\nabla \cdot E = 4\pi\rho \qquad \nabla \cdot B = 0$$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$$

It is easiest to show with the in homogeneous equations first

$$J^\beta = \begin{bmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{bmatrix} \qquad F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

* If $\beta = 0$:

$$\partial_\alpha F^{\alpha 0} = \frac{4\pi}{c} J^0$$

$$\partial_0 F^{00} + \partial_i F^{i0} = \frac{4\pi}{c}(c\rho)$$

$$0 + \nabla \cdot E = 4\pi\rho \checkmark$$

If $\beta = i$:
($i \in \{1,2,3\}$)

$$\partial_\alpha F^{\alpha i} = \frac{4\pi}{c} J^i$$

$$\partial_0 F^{0i} + \partial_j F^{ji} = \frac{4\pi}{c} J^i$$

$$\frac{1}{c}\frac{\partial E}{\partial t} + \nabla \times B = \frac{4\pi}{c}\vec{J} \checkmark$$

* Proceeding to the homogeneous eqns:

$$\mathcal{F}^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_x & E_y \\ B_y & E_x & 0 & E_z \\ B_z & E_y & E_z & 0 \end{bmatrix}$$

#6 (cont.)

b) $*$ If $\beta = 0$: $\quad \partial_\alpha F^{\alpha 0} = 0$

$$\partial_0 F^{00} + \partial_i F^{i0} = 0$$

$$0 + \nabla \cdot B = 0 \checkmark$$

$*$ if $\beta = i,\ i \in \{1,2,3\}$: $\quad \partial_\alpha F^{\alpha i} = 0$

$$\partial_0 F^{0i} + \partial_j F^{ji} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{B} - \nabla \times E = 0$$

$$-\frac{1}{c}\frac{\partial}{\partial t} B + \nabla \times E = 0 \checkmark$$

c) $*$ Remember that $\quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \quad F^{\alpha\beta} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$

$$\Rightarrow \mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$*$ if $A_\mu \to A_\mu + \partial_\mu \alpha(x)$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} \left(\partial_\mu [A_\nu - \partial_\nu \alpha(x)] - \partial_\nu [A_\mu - \partial_\mu \alpha(x)]\right)\left(\partial_\mu [A_\nu - \partial_\nu \alpha(x)] - \partial_\nu [A_\mu - \partial_\mu \alpha(x)]\right)$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} \left(\partial_\mu A_\nu - \cancel{\partial_\mu \partial_\nu \alpha(x)} - \partial_\nu A_\mu + \cancel{\partial_\nu \partial_\mu \alpha(x)}\right)\left(\partial_\mu A_\nu + \cancel{\partial_\mu \partial_\nu \alpha(x}\ - \partial_\nu A_\mu + \cancel{\partial_\nu \partial_\mu \alpha(x)}\right)$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\mu\nu} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \checkmark$$

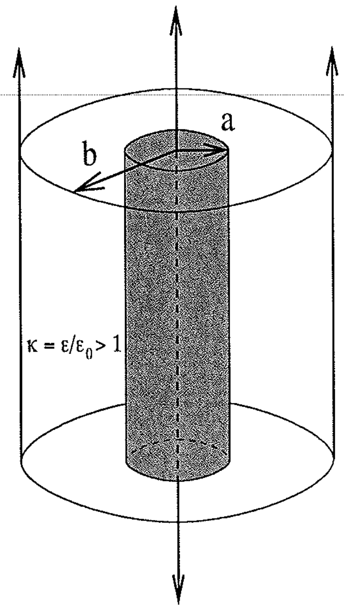
E & M Qualifier

August XX, 2010

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



1. A very long conducting wire of radius a, carrying free positive charge per unit length λ, is surrounded by a dielectric coating of outside radius b and relative dielectric constant $\kappa = \epsilon/\epsilon_0$.

   (a) {2 pts} Find the displacement vector **D** everywhere.

   (b) {2 pts} Find the electric field **E** everywhere.

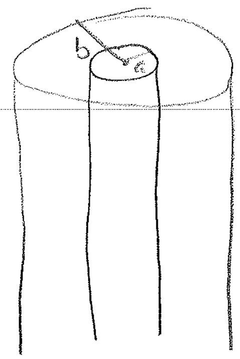
   (c) {2 pts} Find the polarization density **P** everywhere.

   (d) {2 pts} Find the bound volume charge density ρ_b and the bound surface charge density σ_b everywhere.

   (e) {2 pts} Show that the total charge densities, bound and free, produce the same **E** found in (a).

Aug 2010

E+M #1



$\Rightarrow$ Long conducting wire of radius a w/ free charge per unit length λ

$\Rightarrow$ dielectric coating of radius b, w/ $\kappa = \epsilon/\epsilon_0$

a) Gauss Law states:

$$\oint_S \vec{D}\cdot d\vec{a} = Q_{enc}\ , \quad Q_{enc} = \lambda L$$

* Long wire means D has azimuthal symmetry $\rightarrow$ points along $\hat{r}$

$$\Rightarrow \oint_S \vec{D}\cdot d\vec{a} = \lambda L$$

$$D\int_S d\vec{a} = \lambda L$$

$$D 2\pi r L = \lambda L$$

$$D = \frac{\lambda}{2\pi r}\hat{r} \quad \text{(outside wire)}$$

b/c our wire is conducting, all the charge flows on the surface

assumption of evenly distributed charge means this is an electrostatics problem and any field inside the conductor causes moving charges

$$\Rightarrow D = \begin{cases} 0 & r<a \\ \frac{\lambda}{2\pi r}\hat{r} & r\geq a \end{cases}$$

b) We know that $D = \epsilon E$

$$\Rightarrow E = \begin{cases} \frac{\lambda}{2\pi\epsilon_0 r} & r>b \\ \frac{\lambda}{2\pi\kappa\epsilon_0 r} & a<r<b \\ 0 & r<a \end{cases}$$

#1 (cont.)

c) We know that $\vec{P} = \vec{D} - \epsilon_0\vec{E}$ (from $D = \epsilon_0\vec{E} + \vec{P}$)

$$\Rightarrow P = \begin{cases} 0 & r > b \\ \frac{\lambda(1-\frac{1}{k})}{2\pi r}\hat{r} & a < r < b \\ 0 & r < a \end{cases}$$

d) The bound charge is given by: $\rho_b = -\nabla \cdot \vec{P}$

$$\begin{aligned} \rho_b &= -\nabla \cdot \vec{P} \\ &= -\frac{1}{s}\frac{\partial}{\partial s}\left(s\,\frac{\lambda(1-\frac{1}{k})}{2\pi s}\right) \\ &= -\frac{1}{s}\frac{\partial}{\partial s}\left(\frac{\lambda(1-1/k)}{2\pi}\right) \\ &= 0 \end{aligned}$$

$\rho_b = 0$ everywhere

The surface bound charge $\sigma_b = \vec{P} \cdot \hat{n}$

↳ at $s=b$, $P = \frac{\lambda(1-\frac{1}{k})}{2\pi b}\hat{s} \cdot \hat{n}$, $\hat{n} = \hat{s}$

$$\Rightarrow \sigma_b\big|_{s=b} = \frac{\lambda(1-\frac{1}{k})}{2\pi b}$$

$$\sigma_b\big|_{s=a} = \frac{-\lambda(1-\frac{1}{k})}{2\pi a}$$ (by similar logic as above, $\hat{n} = -\hat{s}$)

e) According to Gauss Law: $\int \vec{E} \cdot d\vec{a} = Q_{enc}/\epsilon_0$

*if $s > b$: $Q_{enc} = \lambda L - \frac{\lambda(1-\frac{1}{k})}{2\pi a}(2\pi a L) + \frac{\lambda(1-\frac{1}{k})}{2\pi b}(2\pi b L)$

$$= \lambda L - \cancel{\lambda L(1-\tfrac{1}{k})} + \cancel{\lambda L(1-\tfrac{1}{k})}$$

$$= \lambda L$$

↳ $\vec{E} = \frac{\lambda}{2\pi r \epsilon_0}\hat{r}$

*if $a < s < b$: $Q_{enc} = \lambda L - \frac{\lambda(1-\frac{1}{k})}{2\pi a}(2\pi a L)$

$$= \lambda L - (1-\tfrac{1}{k})\lambda L$$

$$= \frac{\lambda L}{k}$$

↳ $= \frac{\lambda}{2\pi r k \epsilon_0}\hat{r}$

*if $s < a$: $Q_{enc} = 0$

↳ $\vec{E} = 0$

All fields match!

2. (a) {2 pts} In a homogeneous, linear and isotropic conducting material whose electromagnetic properties (at low frequencies) are described by constant values of the permittivity, pearmeability, and conductivity respectively $\epsilon, \mu,$ and σ, show that Maxwell's equations require that the electric field satisfy

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0, \qquad (SI)$$

$$\nabla^2 \mathbf{E} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \mathbf{E}}{\partial t} = 0. \qquad (Gaussian)$$

(b) {2 pts} Given a plane polarized plane wave of angular frequency ω whose electric field is of the form

$$\mathbf{E}(z,t) = \text{Real}\left\{\hat{\imath} E_0 e^{i(kz-\omega t)}\right\},$$

evaluate k^2 as a function of ϵ, μ, σ, and ω.

(c) {2 pts} Find the real and imaginary parts of k assuming $\sigma >> \omega\epsilon$.

(d) {2 pts} Using your results from (c) find the skin depth δ of the conductor. The skin depth is defined by the depth at which the wave's amplitude decreases by e^{-1}, i.e.,

$$\frac{|\mathbf{E}(z+\delta,t)|}{|\mathbf{E}(z,t)|} = \frac{1}{e}$$

(e) {2 pts} Using Maxwell's equations, find the magnetic field $\mathbf{H}(x,t)$ associated with $\mathbf{E}(z,t)$ given in (b) and discuss their phase difference when $\sigma >> \omega\epsilon$.

Aug 2010

E+M #2

* Using SI units

a) We know that: $\nabla \times E + \frac{\partial B}{\partial t} = 0$

$$\nabla \times H - \frac{\partial D}{\partial t} = \vec{j} = \sigma \vec{E}$$

$$\frac{\partial}{\partial t}(\nabla \times H) = \nabla \times \frac{\partial H}{\partial t} - \frac{\partial^2 D}{\partial t^2} = \sigma \frac{\partial E}{\partial t}$$

$$\nabla \times \frac{\partial}{\partial t}\left(\frac{B}{\mu}\right) - \epsilon \frac{\partial^2 E}{\partial t^2} = \sigma \frac{\partial E}{\partial t}$$

$$\nabla \times (-\nabla \times E) - \mu\epsilon \frac{\partial E}{\partial t^2} = \mu\sigma \frac{\partial E}{\partial t}$$

$\times$ but $\nabla \times (-\nabla \times E) = -\nabla(\nabla \cdot E) \cdot \nabla^2 E$

$$-\nabla(\cancel{\nabla \cdot E}^{0}) \cdot \nabla^2 E - \mu\epsilon \frac{\partial E}{\partial t^2} = \mu\sigma \frac{\partial E}{\partial t}$$

b/c we are in a conducting material

$$\nabla^2 E - \epsilon\mu \frac{\partial^2 E}{\partial t^2} - \sigma\mu \frac{\partial E}{\partial t} = 0$$

b) Given $\vec{E} = \left(\hat{\imath} E_0 e^{i(kz - \omega t)}\right) Re$

$$\frac{\partial E}{\partial t} = Re\left[\hat{\imath} E_0 (-i\omega) e^{i(kz-\omega t)}\right]$$

$$\frac{\partial^2 E}{\partial t^2} = Re\left[-\hat{\imath} E_0 \omega^2 e^{i(kz-\omega t)}\right]$$

$$\nabla^2 E = Re\left[\hat{\imath} k^2 E_0 e^{i(kz-\omega t)}\right]$$

$$\Rightarrow -k^2 \cancel{E_0 e^{i(kz-\omega t)}} + \epsilon\mu\omega^2 \cancel{E_0 e^{i(kz-\omega t)}} + i\sigma\mu\omega \cancel{E_0 e^{i(kz-\omega t)}} = 0$$

$$-k^2 = -i\sigma\mu\omega - \epsilon\mu\omega^2$$

$$= \omega^2 \mu\epsilon\left[-\frac{i\sigma}{\epsilon\omega} - 1\right]$$

$$k^2 = \omega^2 \mu\epsilon\left[1 + \frac{i\sigma}{\epsilon\omega}\right]$$

#2 (cont.)

c) Assuming $\sigma >> w\epsilon \Rightarrow \frac{\sigma}{w\epsilon} >> 1$

$$\hookrightarrow k^2 \approx w^2 \mu\epsilon \left(\frac{i\sigma}{\epsilon w} \right)$$

$$\approx i\sigma\mu w$$

$$k \approx \sqrt{i\sigma\mu w}$$

$$\hookrightarrow \text{but } \sqrt{i} = \sqrt{e^{i\pi/2}} = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$\Rightarrow k \approx \sqrt{\sigma\mu w} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right)$$

d)
$$\frac{|E(z+\delta, t)|}{|E(z,t)|} = \frac{1}{e} = \frac{E_0 e^{i(k(z+\delta) - wt)}}{E_0 e^{i(kz - wt)}}$$

$$= \frac{\cancel{E_0} \cancel{e^{ikz}} e^{ik\delta} \cancel{e^{-iwt}}}{\cancel{E_0} \cancel{e^{ikz}} \cancel{e^{-iwt}}}$$

$$\hookrightarrow e^{-1} = e^{ik\delta}$$

$$\hookrightarrow -1 = ik\delta$$

$$-1 = i\left(\frac{i}{\sqrt{2}} \sqrt{\sigma\mu w} \right) \delta$$

$$\sqrt{\frac{2}{\sigma\mu w}} = \delta$$

e) From before, we know $B = \mu H$

$$\hookrightarrow \frac{\partial B}{\partial t} = -\nabla \times E$$

$$= \hat{j} \frac{\partial}{\partial t} \left(-ikE_0 e^{i(kz - wt)} \right)$$

??

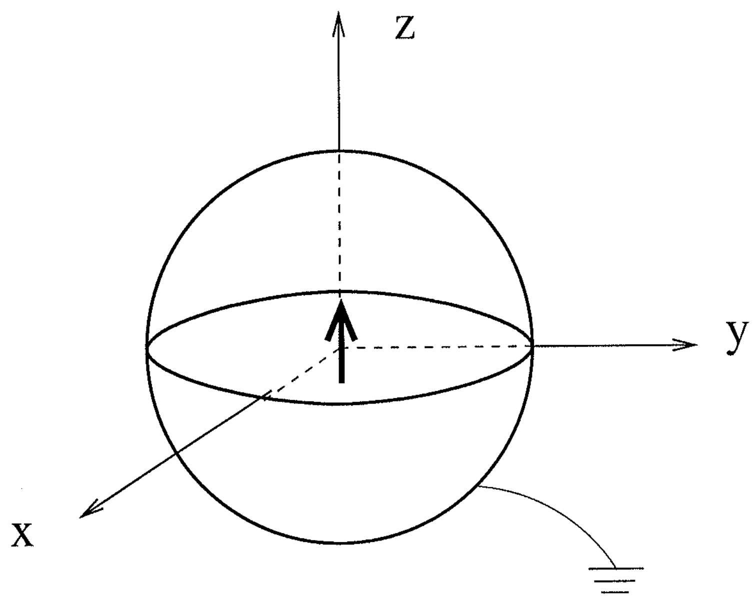
$$= \hat{j} - ik(-iw) E_0 e^{i(kz - wt)}$$

$$\vec{B} = \hat{j} kw E_0 e^{i(kz - wt)}$$

$$\approx \hat{j} \left(w\sqrt{\mu\epsilon} \sqrt{1 + \frac{i\sigma}{\epsilon w}} \right) w$$

$$\approx \hat{j} \left(w^2 \sqrt{\frac{i\sigma\mu}{w}} \right)$$

3. A point electric dipole with dipole moment $\mathbf{p} = p_0\hat{k}$ is located at the center of a hollow, grounded, conducting sphere.



(a) {2 pts} What are the boundary conditions satisfied by the electric field and electric potential in this problem?

(b) {5 pts} Compute the electrostatic potential inside the sphere.

(c) {3 pts} Compute the charge density σ on the inside surface on the grounded sphere.

Aug 2010

E+M #3

a) $\Phi = 0$ at the radius of the sphere (due to grounding) and at infinity

$\vec{E} \to 0$ at infinity

b) $\Phi = \dfrac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} + \dfrac{1}{4\pi\epsilon_0}\sum_{l=0} A_l r^l P_l(\cos\theta)$

$\Phi(\vec{r}=r) = \dfrac{pr\cos\theta}{4\pi\epsilon_0 r^3} + \dfrac{1}{4\pi\epsilon_0}\sum_{l} A_l r^l P_l(\cos\theta)$

*Since only $P_{l=1}$ is guaranteed to go to 0 at $\vec{r}=r$, A_l, $l\neq 1 = 0$

$$0 = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} + \frac{1}{4\pi\epsilon_0}A_1 r\cos\theta$$

$$0 = \frac{p}{r^2} + A_1 r$$

$$A_1 = \frac{-p}{r^3}$$

$$\Rightarrow \Phi = \frac{1}{4\pi\epsilon_0}\left(\frac{p}{r^2} - \frac{r}{a^3}\right)\cos\theta$$

c) $\int \vec{D}\cdot d\vec{a} = \sigma\Delta A$

$\vec{D}\Delta A = \sigma\Delta A$

$\hookrightarrow \sigma = -\epsilon_0 E(r=a)$

$E = -\nabla\Phi$

$= \dfrac{p}{4\pi\epsilon_0}\left(\dfrac{2}{a^3} + \dfrac{1}{a^3}\right)\cos\theta$

$= \dfrac{3p\cos\theta}{4\pi\epsilon_0 a^3}$

$\hookrightarrow \sigma = -\dfrac{3p\cos\theta}{4\pi a^3}$

4. (25 points) A 50 MeV electron ($m\gamma c^2 = 50$ MeV, $mc^2 = 0.5$ MeV) moving along the z-axis is decelerated and brought to a stop after traveling 10 cm in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{k}}$. (Recall $\gamma \equiv 1/\sqrt{1-\frac{v^2}{c^2}}$.)

   (a) {3 pts} Compute $\gamma(t)$ assuming the electron starts its deceleration at t=0.

   (b) {3 pts} How long does it take the electron to stop?

   (c) {3 pts} Compute the total energy radiated by the electron during the 10 cm stopping process.

   (d) {1 pts} What fraction of the electrons initial energy was lost to radiation?

Hint: The general Larmor formula for power radiated by an accelerating point charge is

$$P(t) = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0 c}\gamma^6[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2], \qquad (SI)$$

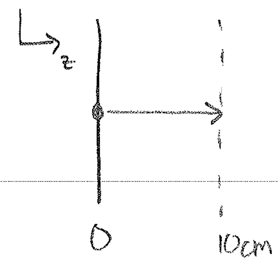
$$P(t) = \frac{2}{3}\frac{q^2}{c}\gamma^6[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2]. \qquad (Gaussian)$$

It might be useful to use $(\dot{\gamma\beta}) = \gamma^3\dot{\beta}$.

$$\begin{aligned} 1e &= 4.8\times 10^{-10}\text{statcoul} = 1.6\times 10^{-19}\text{coul}, \\ 1eV &= 1.6\times 10^{-12}\text{ergs} = 1.6\times 10^{-19}\text{J}. \end{aligned}$$

Aug 2010

E+M # 4



$\vec{E} = E_0 \hat{k}$

5. A plane-polarized electromagnetic wave traveling in vacuum is observed in the lab frame as

$$\mathbf{E} = \text{Real}\Big\{ E_0\, \hat{\imath}\, e^{i(kz-\omega t)} \Big\},$$

$$\mathbf{B} = \text{Real}\Big\{ B_0\, \hat{\jmath}\, e^{i(kz-\omega t)} \Big\},$$

where $k = \omega/c$ and

$$\begin{aligned} B_0 &= E_0/c, && (SI) \\ B_0 &= E_0. && (Gaussian) \end{aligned}$$

A relativistic particle moving in the z-direction ($\mathbf{v} = v_0\hat{k}$, where $\gamma >> 1$) encounters this wave. For this problem you are to find the form of the wave in the rest frame of the particle at the instant the wave is first encountered (before the particle's velocity is changed because of an interaction with the wave).

(a) {2 pts} Start by combining $\mathbf{E}$ and $\mathbf{B}$ into a single 4-tensor $F^{\alpha\beta}(x)$. This includes writing $(kz - \omega t) = \pm k_\alpha x^\alpha)$. The sign $\pm$ depends on your choice of Lorentz metrics $(-1,+,+,+)$ or $(1,-1,-1,-1)$. State which you are using.

(b) {2 pts} Give the Lorentz transformation $L^\alpha{}_\beta$ that transforms the lab frame into the particle's rest frame $x'^\alpha = L^\alpha{}_\beta\, x^\beta$.

(c) {2 pts} Apply your Lorentz transformation to $F^{\alpha\beta}(x)$ to find $F'^{\alpha\beta}(x')$, the electromagnetic 4-tensor in the particles rest frame.

(d) {2 pts} From your results in (c) give the 3-dimensional propagation direction of the wave in the particle's frame and the $\mathbf{E}'(x')$ and $\mathbf{B}'(x')$ fields.

(e) {2 pts} Compare the amplitudes and frequency of the wave as seen by the particle in its rest frame with those seen by a lab observer?

Why does Kantowski boost in x-direction?

Aug 2010

E+M #5

a) $E = \hat{i} E_0 e^{i(kz-\omega t)}$ $\quad k = \omega/c$

$B = \hat{j} B_0 e^{i(kz-\omega t)}$

* Use gaussian units so $B_0 = E_0$

* Use $(1,-1,-1,-1)$ metric

$$\Rightarrow \quad kz - \omega t = -k_\alpha x^\alpha$$
$$= -(k_0 ct + k_1 x + k_2 y + k_3 z)$$
$$= -(k^0 ct - k^1 x - k^2 y - k^3 z)$$
$$= -\frac{\omega}{c} ct + k^1 x + k^2 y + k^3 z$$
$$= -\omega t + kz \quad (k^1 = k^2 = 0,\ k^3 = k)$$

$$\Rightarrow F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} E_0 e^{\pm i k_\alpha x^\alpha}$$

b) For a particle moving in the z-direction,

$$L^\alpha{}_\beta = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

#5 (cont.)

c) $F' = L F L^T$

must transform as well

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} E_0 e^{\pm k_\alpha x^\alpha}$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ -\beta\gamma+\gamma & 0 & 0 & \gamma-\beta\gamma \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma+\beta\gamma & 0 & 0 \\ -\beta\gamma+\gamma & 0 & 0 & \gamma-\beta\gamma \\ 0 & 0 & 0 & 0 \\ 0 & \beta\gamma-\gamma & 0 & 0 \end{bmatrix} E_0 e^{\pm k'_\alpha x'^\alpha}$$

$$k' = L^\alpha{}_\beta k_\alpha \qquad\qquad x' = L^\alpha{}_\beta x_\alpha$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} w/c \\ 0 \\ 0 \\ k \end{bmatrix} \qquad = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\gamma w}{c} - \beta\gamma k \\ 0 \\ 0 \\ -\frac{w\beta\gamma}{c} + \gamma k \end{bmatrix} \qquad = \begin{bmatrix} \gamma ct - \beta\gamma z \\ x \\ y \\ -\beta\gamma ct + \gamma z \end{bmatrix}$$

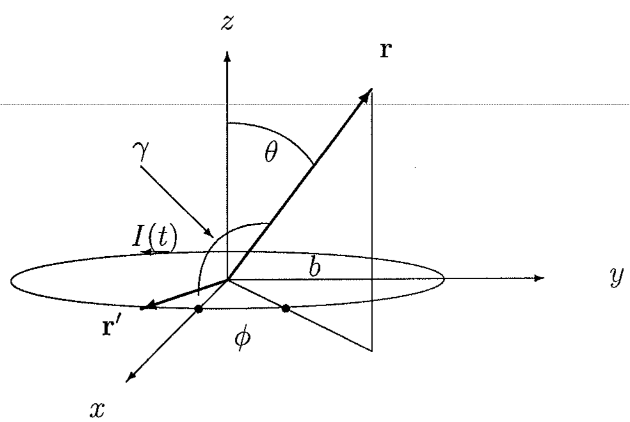
d) $\Rightarrow E'(x') = \hat{x}' E_0 (\gamma - \beta\gamma) e^{i\left(\left(\frac{-\gamma w\beta}{c} + \gamma k\right) z - w\left(\frac{\gamma w}{c} - \beta\gamma k\right)\right)}$

$\vec{B}' = \hat{y}' B_0 (\gamma - \beta\gamma) \exp\left[i\left(\left(-\frac{w\beta\gamma}{c} + \gamma k\right) z - \left(\frac{\gamma w}{c} - \beta\gamma k\right) t\right)\right]$

e) Amplitudes changed by factor of $(\gamma - \beta\gamma)$

$\hookrightarrow k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c} \Rightarrow \nu = \frac{ck}{2\pi}$

$\nu =$



6. A circular current loop of radius b lies in the x-y plane and is centered on the origin. If the current varies harmonically with time as $I(t) = I_0 \cos(\omega t)$, use the following to carry out steps (a) through (e):

$$\begin{aligned}
\mathbf{r}' &= b\left\{\sin\theta'(\cos\phi'\hat{\imath} + \sin\phi'\hat{\jmath}) + \cos\theta'\hat{k}\right\} \\
&= b\left(\cos\phi'\hat{\imath} + \sin\phi'\hat{\jmath}\right), \\
\mathbf{r}\cdot\mathbf{r}' &= b\,r\,\cos\gamma \\
&= b\,r\left\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right\} \\
&= b\,r\,\sin\theta\cos(\phi-\phi').
\end{aligned}$$

In the above $\mathbf{r}'$ is a point on the current loop with spherical-polar coordinates $r' = b,\ \theta' = \pi/2,\ 0 \le \phi' \le 2\pi$, and γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

(a) {2 pts} Compute the time dependent magnetic dipole moment $\mathbf{m}(t)$ of the current loop. Recall that $\mathbf{m}_{Gaussian} = \mathbf{m}_{SI}/c$.

(b) {2 pts} Give an integral expression for the retarded vector potential $\mathbf{A}(t, \mathbf{r})$.

(c) {2 pts} Approximate the integral found in (b) for $\mathbf{A}$ assuming $b << r$ and $b << c/\omega$. If you have made no mistakes your answer should agree with the potential for a point magnetic dipole, i.e., with:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\nabla\times\left\{\frac{\mathbf{m}(t-r/c)}{r}\right\}, \qquad SI$$

$$\mathbf{A} = \nabla\times\left\{\frac{\mathbf{m}(t-r/c)}{r}\right\}. \qquad Gaussian$$

(d) {2 pts} From your results for (c) or from the point magnetic dipole result, compute the radiation (far field) part of $\mathbf{E}$ by assuming $b << c/\omega << r$.

(e) {2 pts} Using only the radiation part, i.e., the part $\propto 1/r$, of $\mathbf{E}$ and $\mathbf{B}$, and the Poynting vector, compute the time averaged electromagnetic energy flux radiated away by the dipole as a function of the spherical polar coordinates (r, θ, ϕ). Recall that

$$\mathbf{B} = \frac{1}{c}\hat{r}\times\mathbf{E}, \qquad (SI)$$

$$\mathbf{B} = \hat{r}\times\mathbf{E}, \qquad (Gaussian)$$

for radiation coming from a source at the origin.

Aug 2010

E+M #6

a) $m(t) = \frac{1}{2c}\int \vec{r} \times \vec{J}$

$$= \frac{I(t)}{2c} \oint \vec{r} \cdot d\vec{r}$$

$*$ if $\vec{r} = b(\cos\hat{\imath} + \sin$

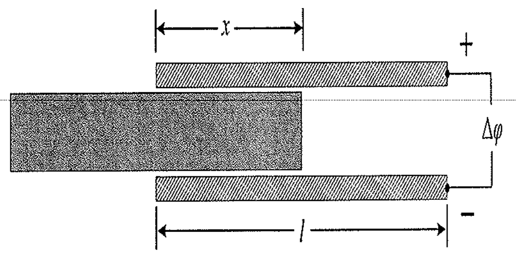
E & M Qualifier

January 13, 2011

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



1. A parallel plate capacitor has the region between its plates filled with a dielectric slab of dielectric constant $K = \epsilon/\epsilon_0$ and mass m. The plate dimensions are: width w, length ℓ, and plate separation d. The capacitor plates are connected to a battery of constant voltage V ($\Delta\phi = V$ in the figure). Neglect the fringe field and friction, and assume the slab is constrained to move in the plane parallel to the capacitor plates.

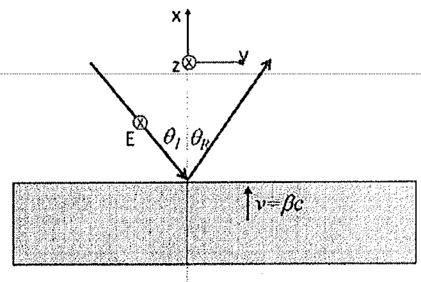
   (a) {2 pts} Compute the capacitance $C \equiv q/V$ of this capacitor as a function of x.

   (b) {2 pts} If the slab is withdrawn half way (to $x = \ell/2$) and held in place, what is the magnitude and direction of the force on the slab caused by the electric field?

   (c) {2 pts} At $x = \ell/2$ the slab is released and given a velocity v_0 to the right. Find the current supplied by the battery at the instant it is released.

   (d) {2 pts} At $x = \ell/2$ the slab is again released but with zero velocity. Describe the motion of the slab (in words). What is the maximum velocity achieved by the slab?

   (e) {2 pts} Sketch the displacement of the slab versus time.



2. This problem investigates the shifting frequency of electromagnetic radiation that is reflected off a moving target. Incident and reflected frequencies and angles are not the same if the target is moving.

   Assume that in the lab frame of reference, the target is a flat mirror traveling upward in the positive x-direction parallel to the mirror's normal with velocity $\mathbf{v} = \beta c\,\hat{\mathbf{x}}$ (see the figure). Also assume the wave is a linearly polarized plane wave traveling in vacuum towards the moving mirror at angle θ_I (relative to the mirror's normal). If the polarization is in the $\hat{z}$ direction, the incident electric field is given by

$$\mathbf{E}_I = E_0\,\hat{\mathbf{z}}\;e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega_I t)},$$

   with

$$\mathbf{k}_I = \frac{\omega_I}{c}(-\cos\theta_I\,\hat{\mathbf{x}} + \sin\theta_I\,\hat{\mathbf{y}}).$$

   (a) {2 pts} Write the Lorentz boost A as a function of β and $\gamma \equiv \sqrt{1-\beta^2}$ that transforms the Lab coordinates $\mathbf{r}$ and ct to coordinates $\mathbf{r}'$ and ct' co-moving with the mirror. Also give the inverse A^{-1} of the Lorentz boost A that transforms the moving coordinates $\mathbf{r}'$ and ct' into Lab coordinates $\mathbf{r}$ and ct.

   (b) {3 pts} By rewriting the above wave's phase in both reference frames, i.e.,

$$\mathbf{k}_I\cdot\mathbf{r} - \omega_I t = \mathbf{k}'_I\cdot\mathbf{r}' - \omega'_I t'$$

   as a function of the co-moving mirror coordinates $\mathbf{r}'$ and ct' (i.e., use A^{-1}) find $\mathbf{k}'_I$ and ω'_I as observed in the co-moving frame. These will be functions of β, γ, and θ_I as well as ω_I.

   (c) {2 pts} By writing the incident wave vector just obtained in the moving frame in the form

$$\mathbf{k}'_I = \frac{\omega'_I}{c}(-\cos\theta'_I\,\hat{\mathbf{x}} + \sin\theta'_I\,\hat{\mathbf{y}}),$$

   determine the incident angle θ'_I as seen by observers moving with the mirror (e.g., give $\cos\theta'_I$ as a function of θ_I, ω_I and the Lorentz parameters β, γ).

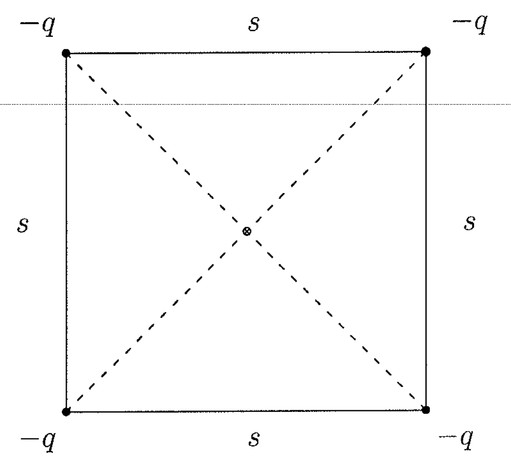
(d) {3 pts} If, as seen by observers moving with the mirror, the reflected wave has the same frequency as the incident wave $\omega'_R = \omega'_I$ and a reflection angle that is the same as the incidence angle $\theta'_R = \theta'_I$, i.e.,

$$\mathbf{k}'_R = \frac{\omega'_I}{c}(\cos\theta'_I\,\hat{\mathbf{x}} + \sin\theta'_I\,\hat{\mathbf{y}}),$$

what is the frequency ω_R of the reflected light as measured in the laboratory frame? Hint: again use

$$\mathbf{k}_R \cdot \mathbf{r} - \omega_R t = \mathbf{k}'_R \cdot \mathbf{r}' - \omega'_R t',$$

and the Lorentz boosts A.



3. Consider a square with sides of length s and charges -q at the corners as shown:

   (a) {2 pts} What is the potential at the center of the square if the potential is zero at ∞?

   (b) {2 pts} How much work does it take to bring in another charge -q from ∞ to the center of the square?

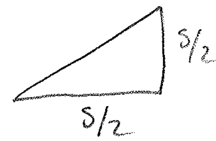
   (c) {3 pts} How much work does it take to assemble the original configuration of 4 negative charges (no charge at center)?

   (d) {3 pts} Now suppose that instead of the 4 charges being located at the corners of a square, a net charge of $-4q$ is distributed uniformly on the surface of a sphere of radius s. How much work does it take to bring in another charge q from ∞ to the center of the sphere?

Jan 2011 # E+M #3 Gaussian

a) We will assume the origin lies at the center of the square

$$\Rightarrow V = \sum_i \frac{q_i}{r_i}$$



$$= \sqrt{2\left(\frac{s}{2}\right)^2} = \sqrt{\frac{s^2}{2}}$$

$$= 4 \frac{-q\sqrt{2}}{s}$$

b) $W = \int \vec{F} \cdot d\vec{l}$

$$= q\,(V(b) - V(a))$$

$$= -q\left(-\frac{\sqrt{2}q}{s}\right)$$

$$= \frac{\sqrt{2}q^2}{s}$$

c) The work necessary to assemble the charge distribution is:

$W = \frac{1}{2}\sum_i q_i V_i$ where i is each individual charge

$$= \frac{1}{2}\left(0 + (-q)\left(\frac{-q}{s}\right) + (-q)\left(\frac{-q}{s} + \frac{-q}{\sqrt{2}s}\right) + -q\left(\frac{-q}{s} + \frac{-q}{s} + \frac{-q}{\sqrt{2}s}\right)\right)$$

$$= \frac{1}{2}\left(\frac{q^2}{s} + \frac{q^2}{s} + \frac{q^2}{\sqrt{2}s} + \frac{q^2}{s} + \frac{q^2}{s} + \frac{q^2}{\sqrt{2}s}\right.$$

$$= \frac{2q^2}{s} + \frac{q^2}{\sqrt{2}s}$$

d) From symmetry, and the fact that the charge is distributed evenly over the surface of the sphere,

$$\vec{E}^s = \begin{cases} -\frac{4q}{r^2} & r > s \\ 0 & r < s \end{cases}$$

* We also know that

$$V = -\int_\infty^0 E \cdot d\vec{l} = -\left[\int_\infty^s E_{out} \cdot dl - \int_s^0 E_{in} \cdot dl\right]$$

#3 (cont.)

d) $$V = -\int_{\infty}^{s} \frac{-4q}{r^2}\,dr$$

$$= -\left[-4q\left(\frac{-1}{r}\Big|_{\infty}^{s}\right)\right]$$

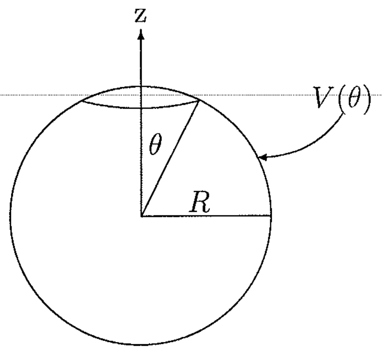
$$= 4q\left(\frac{-1}{s} - \cancel{\frac{-1}{\infty}}^{\,0}\right)$$

$$= \frac{-4q}{s}$$

$$W = -q\,V(0)$$

$$= -q\left(\frac{-4q}{s}\right)$$

$$= \frac{4q^2}{s}$$



4. Consider an isolated spherical surface of radius R centered on the origin, that is kept at a known potential $V(\theta)$, i.e.,

$$\Phi(r = R, \theta) = V(\theta)$$

where (r, θ, ϕ) are the usual spherical polar coordinates, i.e., θ is measured with respect to a z-axis passing through the center of the sphere and ϕ is the azimuthal angle about the z-axis measured from the x axis.

(a) {2 pts} Write down expressions for the general solution to $\nabla^2\Phi(r, \theta) = 0$ for the electrostatic potential as a linear combination of Legendre polynomials in the respective regions $0 \leq r < R$ and $r > R$. Assume that the potential vanishes at $r \to \infty$ and has azimuthal symmetry i.e., no dependence on the angle ϕ. Do not include terms that must vanish. Do not attempt to evaluate the constants that appear in the linear combination but do give the correct r dependence of each term.

(b) {2 pts} What boundary conditions must your two expressions satisfy at the junction $r = R$ to have a unique solution to Maxwell's equations?

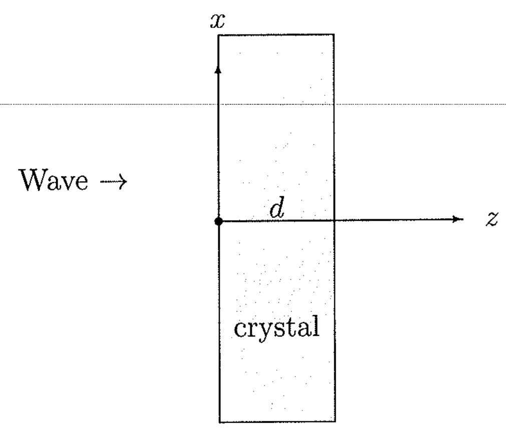
(c) {2 pts} If the particular surface potential imposed is

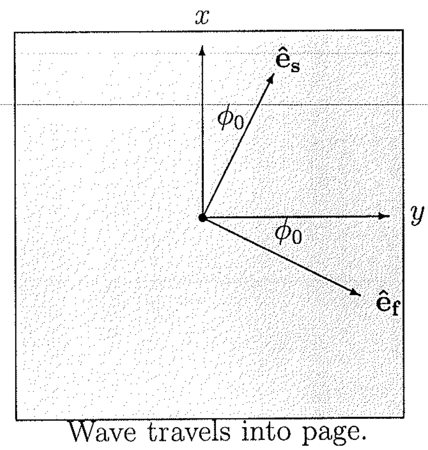
$$\Phi(r = R, \theta) = V_0 \cos\theta$$

where V_0 is a constant, what is the explicit form of your potential for both regions $r \leq R$ and $r > R$?

(d) {2 pts} Determine the resulting electric field on both sides of the r=R surface.

(e) {2 pts} What is the surface charge density $\sigma(\theta)$ on the spherical shell at r=R.





5. A plane polarized monochromatic light wave traveling in the $+z$ direction enters a large flat slab of transparent crystal of thickness d, located between $z = 0$ and $z = d$. This crystal has the property that the index of refraction depends on the direction of polarization as follows: Plane waves traveling in the z direction but polarized in the direction

$$\hat{\mathbf{e}}_s = \cos\phi_0\hat{\mathbf{x}} + \sin\phi_0\hat{\mathbf{y}},$$

travel with speed $v_s = c/n_s < c$ but those polarized in the orthogonal direction

$$\hat{\mathbf{e}}_f = -\sin\phi_0\hat{\mathbf{x}} + \cos\phi_0\hat{\mathbf{y}},$$

travel with the faster speed $v_f = c/n_f < c$ where $n_s = n_f + \delta n$.

Assume the wave, just after entering the crystal (i.e., for very small $z << \lambda < d$), is polarized in the y direction and hence has the form

$$\mathbf{E}(z \approx 0, t) = E_0\,\hat{\mathbf{y}}\,e^{-i\omega t}.$$

(a) {4 pts} Prove that in general the initial plane wave becomes elliptically polarized when it reaches $z = d$ by deriving the following expression

$$\mathbf{E}(z = d, t) = [E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}]\;e^{i(\bar{k}d-\omega t)},$$

where

$$\bar{k} \equiv \frac{\omega}{c}\left(\frac{n_s + n_f}{2}\right),$$

and

$$E_x = iE_0\sin 2\phi_0\sin\delta,$$
$$E_y = E_0(\cos\delta - i\cos 2\phi_0\sin\delta),$$

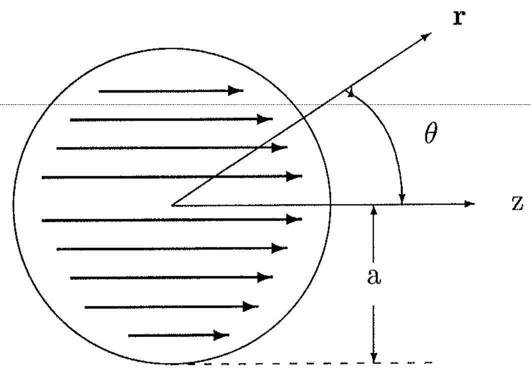
with

$$\delta \equiv \frac{\omega d}{2c}\delta n.$$

Hint: Write the wave at z=0 as a combination of slow and fast plane polarized parts using $\hat{\mathbf{y}} = \sin\phi_0\hat{\mathbf{e}}_s + \cos\phi_0\hat{\mathbf{e}}_f$.

(b) {3 pts} For what values of δ and θ_0 will the wave emerge from the crystal as a circularly polarized wave? ($E_x/E_y = \pm i$).

(c) {3 pts} For what minimum crystal thicknesses $d = d_{min}$ will the wave emerge as a plane polarized wave (E_x/E_y = real) and what will its polarization direction be?



6. A permanent magnet in the shape of a solid sphere of radius a is oriented on the z-axis as shown in the figure. The magnetization of the magnet is given by $\vec{M} = M_0\hat{z}$. [Recall that $\nabla \times \mathbf{H} = 0$ implies the existance of a magnetic scalar potential $\Phi_m(r, \theta)$ related to the magnetic field by $\mathbf{H} = -\vec{\nabla}\Phi_m(r, \theta)$.]

   (a) {4 pts} Compute the scalar magnetic potential $\Phi_m(r, \theta)$ at all points $r < a$ and $r > a$.

   (b) {3 pts} Compute the magnetic Field $\mathbf{H} = -\vec{\nabla}\Phi_m(r, \theta)$ at all points $r < a$ and $r > a$.

   (c) {3 pts} Compute the magnetic induction $\mathbf{B}$, where

$$\begin{aligned} \mathbf{B}/\mu_0 &= \mathbf{H} + \mathbf{M}, && (SI) \\ \mathbf{B} &= \mathbf{H} + 4\pi\mathbf{M}, && (Gaussian) \end{aligned}$$

   at all points $r < a$ and $r > a$.

   Hints: The magnetic potential is axial symmetric about the z-axis and satisfies the Laplace equation at all points except $r = a$. Legendre polynomials are useful.

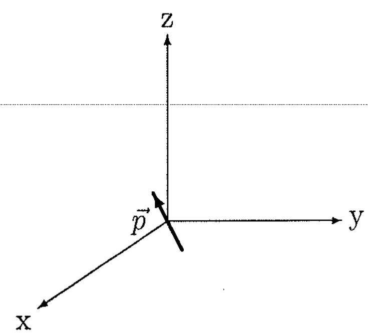
E & M Qualifier

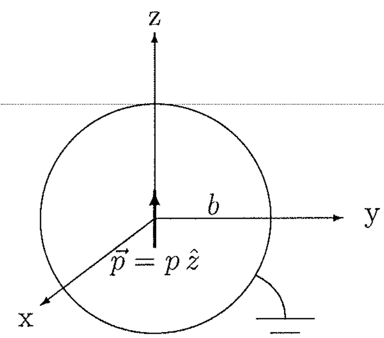
August 18, 2011

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).





1. (a) {3 pts} Give the potential for a static point electric dipole, with dipole moment $\mathbf{p}$, located at the origin and pointing in an arbitrary direction (see Figure).

   (b) {3 pts} If the dipole moment points in the z-direction ($\mathbf{p} = p\,\hat{\mathbf{z}}$) and is surrounded by a thin grounded conducting sphere of radius b (see Figure), what is the electrostatic potential inside the sphere?

   (c) {4 pts}Compute the static electric charge density that exists on the inner surface of the thin conducting sphere?

Aug 2011

E + M #1

Gaussian

a) The general potential for an electric dipole is:

$$\Phi = \frac{\vec{p} \cdot \vec{r}}{r^3}$$

b) Since our conducting spherical shell is grounded, we proceed by the method of images to determine

2. (a) {3 pts} Use Maxwell's equations to derive the continuity equation (in differential form) relating charge ρ and current density $\mathbf{J}$.

(b) {2 pts} Use the divergence theorem and the results of part (a) to derive the conservation of charge, $\dot{Q} = 0$, for a bounded charge distribution.

(c) {2 pts} Show that the continuity equation can be written in 4-vector form using the 4-current J^{μ}. Define all symbols you use.

(d) {3 pts} Use Maxwell's equations to derive Poynting's theorem, the equation analogous to the continuity of charge equation relating the Poynting vector $\mathbf{S}$ and the energy density u in the $\mathbf{E}$ and $\mathbf{B}$ fields, that represents conservation of electromagnetic energy. Assume the electric and magnetic fields are in vacuum, i.e., no charges, currents, or polarizable materials are present.

Aug 2011

E+M #2

Gaussian

a) We know: $\nabla \cdot D = 4\pi\rho \qquad \nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$

$\nabla \cdot B = 0 \qquad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}$

Continuity Eqn: $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$

$$\frac{\partial}{\partial t} \nabla \cdot D = 4\pi \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \left(\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} \right) = \frac{4\pi}{c} \nabla \cdot J$$

$$\nabla \cdot \nabla \times H - \frac{1}{c}\frac{\partial}{\partial t} \nabla \cdot D = \frac{4\pi}{c} \nabla \cdot J$$

$$\cancelto{0}{\nabla \cdot \nabla \times H} - \frac{4\pi}{c}\frac{\partial \rho}{\partial t} = \frac{4\pi}{c} \nabla \cdot J$$

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot J$$

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot J \ ✓$$

b) In general, the Divergence theorem says: $\int_V \vec{\nabla} \cdot \vec{A} \, dV = \oint \vec{A} \cdot d\vec{a}$

↳ If we want to show conservation of charge, we define a surface that encloses all of our charge with $J_{bound} = 0$

$$\Rightarrow Q_{enc} = \frac{1}{4\pi} \int \rho \cdot dV$$

$$\dot{Q} = 0 = \frac{1}{4\pi} \frac{\partial}{\partial t} \int \rho \cdot dV$$

$$= \frac{-1}{4\pi} \int \nabla \cdot J \, dV$$

$$= \frac{-1}{4\pi} \int \cancelto{0}{J \cdot d\vec{a}}$$

$$= 0 \ ✓$$

c) $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}$

* if $J^0 = c\rho$, $J^i = j_i$, $i \in \{x, y, z\}$

↳ $\frac{\partial}{\partial x^\mu} J^\mu = 0$ ✓

#2 (cont.)

d) In Gaussian units: Poynting vector: $\vec{S} = \frac{c}{4\pi} E \times H$

Energy density: $u = \frac{1}{4\pi}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$

$\vec{E} \cdot \left(\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t}\right) = \vec{E} \cdot \frac{4\pi}{c}\vec{J}$

$\vec{E} \cdot \nabla \times \vec{H} - \frac{1}{c}\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c} \cancel{\vec{E} \cdot \vec{J}}$ 0 b/c no current

$-\nabla \cdot (E \times H) + H \cdot (\nabla \times E) - \frac{1}{c} E \cdot \frac{\partial D}{\partial t} = 0$

$-\nabla \cdot (E \times H) + H \cdot \left(-\frac{1}{c}\frac{\partial B}{\partial t}\right) - \frac{1}{c} E \cdot \frac{\partial D}{\partial t} = 0$

$-\nabla \cdot (E \times H) - \frac{1}{c}\left[H \cdot \frac{\partial B}{\partial t} + E \cdot \frac{\partial D}{\partial t}\right] = 0$

$-\nabla \cdot (E \times H) - \frac{1}{c}\frac{\partial}{\partial t}(E \cdot D + H \cdot B) = 0$ ← b/c ϵ, μ have no time dependence

$E = \epsilon D, \quad H = \frac{1}{\mu} B$

* multiplying by $\frac{-c}{4\pi}$

$\nabla \cdot \vec{S} + \frac{\partial u}{\partial t} = 0$ ✓

Aug 2011 | $E+M$ #2 | Gaussian

a) The generalized Maxwell Eqns are:

$$\nabla\cdot D = 4\pi\rho_f \qquad \nabla\cdot B = 0$$

$$\nabla\times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla\times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$$

$$\nabla\cdot\left(\nabla\times H - \frac{1}{c}\frac{\partial D}{\partial t}\right) = \nabla\cdot\frac{4\pi}{c}\vec{J}_f$$

$$\cancelto{0}{\nabla\cdot(\nabla\times H)} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot D) = \frac{4\pi}{c}\nabla\cdot\vec{J}_f$$

$$-\cancel{\frac{4\pi}{c}}\frac{\partial}{\partial t}\rho_f = \cancel{\frac{4\pi}{c}}\nabla\cdot\vec{J}_f$$

$$-\frac{\partial}{\partial t}\rho_f = \nabla\cdot\vec{J}_f \checkmark$$

b) The divergence theorem states: $\int\nabla\cdot A\,dV = \oint A\cdot d\vec{a}$

$$\int\nabla\cdot\vec{J}_f\,dV = -\frac{\partial}{\partial t}\int\rho_f\,dV$$

$$\oint\vec{J}_f\cdot d\vec{a} = -\frac{\partial}{\partial t}Q$$

*if our arbitrary surface totally encloses region of $\vec{J}_f$, the flux is 0

$$0 = \dot{Q} \checkmark$$

c) We define J^μ as follows

$$J^\mu = \begin{bmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{bmatrix} \Rightarrow 0 = \frac{1}{c}\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}J_i$$

$$= \frac{1}{\partial x^\mu}J^\mu \checkmark$$

#2 (cont.)

d) * Remember that $\vec{S} = \frac{c}{4\pi} E \times H$

$$U = \frac{1}{4\pi}(E \circ D + B \cdot H)$$

$$\Rightarrow \quad E \circ \left(\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} \right) = E \circ 0 \quad \text{(b/c in vaccum)}$$

$$E \circ \nabla \times H - \frac{1}{c}\frac{\partial}{\partial t} E \circ D = 0$$

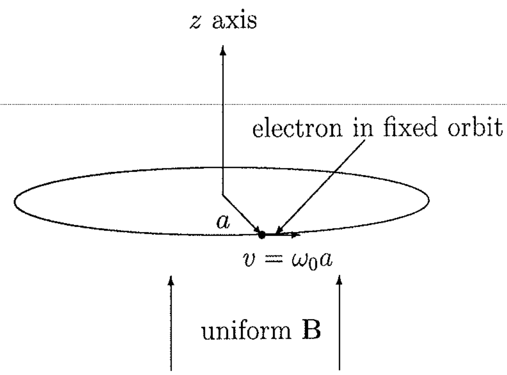
$$-\nabla \circ (E \times H) + H \circ (\nabla \times E) - \frac{1}{c}\frac{\partial}{\partial t} E \cdot D = 0$$

$$-\nabla \circ (E \times H) + H \circ \left(-\frac{1}{c}\frac{\partial B}{\partial t} \right) - \frac{1}{c}\frac{\partial}{\partial t} E \cdot D = 0$$

$$-\nabla \circ (E \times H) - \frac{1}{c}\frac{\partial}{\partial t}(E \cdot D + H \cdot B) = 0$$

$$\frac{c}{4\pi}\left(\nabla \circ (E \times H) + \frac{1}{4\pi}\frac{\partial}{\partial t}(E \cdot D + H \cdot B) \right) = 0$$

$$\nabla \circ S + \frac{\partial}{\partial t} U = 0 \checkmark$$

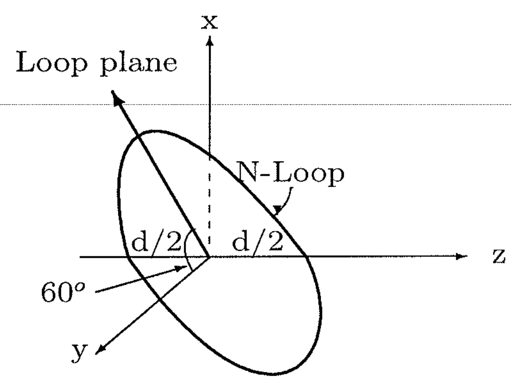


3. An electron is <u>constrained</u> to revolve (without friction) in a circular orbit of radius a and does so with an initial angular velocity ω_0 (assume $\omega_0 a << c$).

   (a) {3 pts} What is the magnetic dipole moment due to the electron's motion?

   (b) {5 pts}
   A uniform magnetic field **B**, parallel to the angular momentum of the electron is slowly turned on. Derive an expression for $\delta\omega(B)$, the change in angular speed of the electron as a function of the magnetic induction B (Hint: Use Faraday's Law of induction and Newtonian mechanics).

   (c) {2 pts} Does $\delta\omega$ increase or decrease the magnetic dipole moment?



4. In vacuum, a plane electromagnetic wave of angular frequency $\omega = c\,k = 2\pi\, c/\lambda$ travels parallel to the z-axis. The wave has an electric field given by

$$\mathbf{E} = \hat{\mathbf{j}}\ E_0 e^{i(kz-\omega t)},$$

A small N-loop (i.e., an N-turn circular coil) of diameter d very much smaller than the wavelength λ ($d << \lambda$) acting as an antenna is located with its center at the origin. It is oriented so that a diameter of the coil lies along the z-axis and the plane of the coil makes an angle $\theta = 60^o$ with the y-axis.

(a) {3 pts} Use Maxwell's equations to obtain the **B** field associated with the above wave?

(b) {3 pts} Compute the Magnetic flux through the N turn coil as a function of time.

(c) {4 pts} What is the peak EMF induced in the antenna?

5. A charged particle with charge q and mass m starts from rest on the inner plate (radius a) of a cylindrical capacitor and is accelerated towards the outer plate (radius b). Orient the coordinates so that q starts at $(x, y) = (a, 0)$ and is accelerated along the +x direction until it reaches $(x, y) = (b, 0)$.

   (a) {2 pts} If the charge/length on the capacitor λ_0 is constant find the electric field causing the acceleration by using Gauss's law. Assume the capacitor is very long compared to the radii a and b and assume the electric field can is cylindrically symmetric.

   (b) {4 pts} Write down the 4-D [or (3+1) D] special relativistic version of Newton's equations for the motion of a point charge experiencing the Lorentz force (the force due to an arbitrary external **E** and **B** field).

$$\frac{dp^\mu}{d\tau} = ?$$

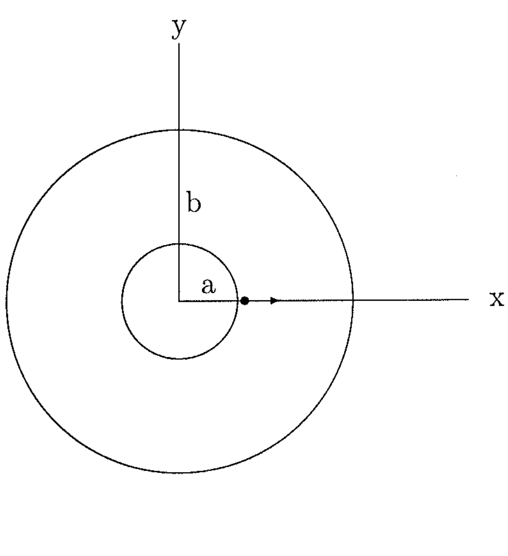
   or

$$\frac{dm\gamma c^2}{dt} = ? \qquad \text{and} \qquad \frac{d\vec{p}}{dt} = ?$$

   Be sure to define p^μ and $\vec{p}$ as well as the Lorentz force terms that appear on the right hand sides of the above equations.

   (c) {4 pts} Apply your answer to part (b) to the field you found in part (a) and integrate your dynamical equations to obtain the particle's energy when it reaches the outer plate. You can easily compute $\gamma(x)$ from Newton's equations even though the acceleration is not constant.

   You are **not** asked to compute $x(t)$ or $x(\tau)$ nor how long it took to reach $x = b$.



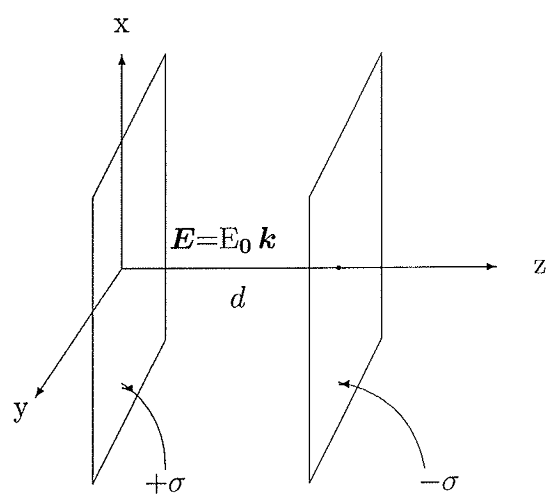
6. A uniform static electric field $\mathbf{E} = E_0\mathbf{k}$ exists between two large thin conducting metal plates. The positive plate is at $z = 0$ and the negative plate is at $z = a$. You can assume the plates are infinitely large in the x-y directions.

(a) {1 pts} Use Maxwell's equations to relate the value of E_0 to the surface charge density $\pm\ \sigma$ on the plates.

(b) {4 pts} Lorentz transform $F^{\alpha\beta}$ to obtain the $\mathbf{E}$ and $\mathbf{B}$ fields seen by an observer moving between the plates with velocity c/2 $\hat{\imath}$?

(c) {2 pts} What are the 4-current densities J^σ of the plates in the rest frame and in a frame moving with the observer?

(d) {3 pts} Show that Maxwell's inhomogeneous equations are satisfied by your fields and charge-currents in the moving frame. (Hint: Using $\mathbf{E}$ and $\mathbf{B}$ rather than $F^{\alpha\beta}$ is probably easier.)



Aug 2011

E+M #6

Gaussian

a) Using Gauss Law, we see

$$\int \vec{E}\cdot d\vec{a} = 4\pi q_{enc}$$

$$E\cdot 2A = 4\pi\sigma A$$

$$E = 2\pi\sigma \quad \text{(for one plate)}$$

$$\hookrightarrow \vec{E}_0 = 4\pi\sigma$$

b) $F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$ $\qquad \Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

* In matrix form $F' = \Lambda F \Lambda^T$

$$F' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & +\beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ \gamma E_z & -\beta\gamma E_z & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{E}' = \gamma\vec{E}$$

$$\vec{B}' = -\beta\gamma E_z\,\hat{y}$$

* If $\beta = \frac{v}{c} = \frac{1}{2}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{1}{4}}}$

$$= \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2}{\sqrt{3}}$$

#6 (cont.)

c) The 4-current vector J^α is defined as $J^\alpha = \langle c\rho, \vec{J} \rangle$ and transforms according to: $J'^\beta = \Lambda^\beta_{\ \alpha} J^\alpha$

$$J^\alpha = \langle c[\sigma\delta(z) - \sigma\delta(z-d)], 0 \rangle$$

$$J' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\rho \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J' = \begin{bmatrix} \gamma c\rho \\ -\beta\gamma c\rho \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c\rho' \\ J'_x \\ J'_y \\ J'_z \end{bmatrix} \Rightarrow \begin{array}{l} \rho' = \gamma\sigma(\delta(z) - \delta(z-d)) \\ \vec{J} = \langle -\beta\gamma c\sigma(\delta(z) - \delta(z-d)), 0, 0 \rangle \end{array}$$

d) The inhomogeneous Maxwell equations in Gaussian units are:

$$\nabla \cdot E' = 4\pi\rho'$$

$$\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$$

– To test Gauss Law

$$\nabla \cdot E' = \frac{\partial}{\partial z} \frac{2}{\sqrt{3}}(4\pi\sigma)$$

$$= \frac{8\pi}{\sqrt{3}} \frac{\partial}{\partial z}\left(\sigma[\Theta(z) - \Theta(z-d)]\right) \leftarrow$$ Assumption works, not sure why since σ only exists on plates

$$= \frac{8\pi}{\sqrt{3}}\sigma(\delta(z) - \delta(z-d))$$

$$= 4\pi\left(\frac{2}{\sqrt{3}}\sigma[\delta(z) - \delta(z-d)]\right)$$

$$= 4\pi\rho' \checkmark$$

#6 (cont)

d) - Evaluating Ampere's Law

$$\nabla \times B' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & -\beta\gamma E_z & 0 \end{vmatrix} = \langle -\partial_z(\beta\gamma E_z),\ 0,\ \partial_x(-\beta\gamma E_z) \rangle$$

$$= \langle -\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{3}} 4\pi\sigma[\Theta(z)-\Theta(z-d)]\right),\ 0,\ \frac{\partial}{\partial x}\left(\frac{-1}{\sqrt{3}} 4\pi\sigma[\Theta(z)-\Theta(z-d)]\right) \rangle$$

$$= \langle -\frac{4\pi\sigma}{\sqrt{3}}(\delta(z)-\delta(z-d)),\ 0,\ 0 \rangle$$

$$= \langle -\frac{4\pi}{c}\left(\frac{c\sigma}{\sqrt{3}}[\delta(z)-\delta(z-d)]\right),\ 0,\ 0 \rangle$$

$$= \langle -\frac{4\pi}{c}(\beta\gamma c E_z),\ 0,\ 0 \rangle$$

$$= -\frac{4\pi}{c} \langle J'_x,\ 0,\ 0 \rangle \checkmark$$

(*Note: $-\frac{1}{c}\frac{\partial E}{\partial t} = 0$ b/c no time dependence in formula)

E & M Qualifier

January 11, 2012

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).

1. Dielectric Sphere

A dielectric sphere of radius R is polarized so that $\mathbf{P} = (K/r)\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ is the unit radial vector. Assume the sphere is in an empty vacuum and that the sphere's dielectric material is linear and isotropic, calculate

(a) (3 pts) the volume and the surface densities of bound charge,

(b) (2 pts) the volume density of free charge,

(c) (2 pts) the electric field inside the sphere,

(d) (3 pts) the electric field outside the sphere.

Your answers should be given in terms of $K, \chi_E, \epsilon_0, \epsilon$, and/or ϵ_r. Recall that for linear isotropic materials:

In SI units,

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \epsilon_0\chi_E\mathbf{E}$$

$$\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = 1 + \chi_E$$

In Gaussian units,

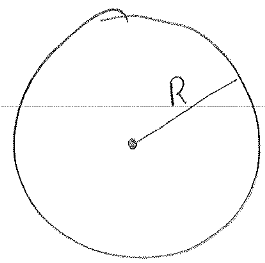
$$\mathbf{D} = \epsilon\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}$$

$$\mathbf{P} = \chi_E\mathbf{E}$$

$$\epsilon = 1 + 4\pi\chi_E = \epsilon_r$$

Jan 2012

E + M #1



$$\vec{P} = \frac{K}{r}\hat{r}$$

Given a polarized, dielectric sphere (dielectric is linear + isotropic) in a vacuum;

a) Our surface charge density is:

$$\sigma_b = \vec{P}\cdot\hat{n}\,\Big|_{surface}$$
$$= \frac{K}{r}\hat{r}\cdot\hat{r}\,\Big|_{r=R}$$
$$= \frac{K}{R}$$

The volume bound charge density is:

$$\rho_b = -\nabla\cdot\vec{P}$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{K}{r}\right)$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(kr)$$
$$= -\frac{k}{r^2}$$

b) We can determine the volume density of free charge by:

$$-\nabla\cdot E = \frac{\rho_f}{\epsilon_0}$$

$\Rightarrow$ To determine the electric field inside the material, we use

$$\vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P}$$
$$(\epsilon - \epsilon_0)E = \vec{P}$$
$$\vec{E} = \frac{1}{\epsilon - \epsilon_0}\vec{P}$$
$$= \frac{K}{r(\epsilon - \epsilon_0)}\hat{r}$$

Work for part C HERE

#1 (cont.)

b) $$-\nabla \cdot E = \frac{\rho_f}{\epsilon_0}$$

$$-\nabla \cdot \frac{K}{(\epsilon-\epsilon_0)r}\hat{r} = \frac{\rho_f}{\epsilon_0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Kr}{\epsilon-\epsilon_0}\right) = \frac{\rho_f}{\epsilon_0}$$

$$\frac{K}{r^2(\epsilon-\epsilon_0)} = \frac{\rho_f}{\epsilon_0}$$

$$\hookrightarrow \rho_f = \frac{K\epsilon_0}{r^2(\epsilon-\epsilon_0)}$$

$$= \frac{K}{r^2(-1+\epsilon/\epsilon_0)}$$

$$= \frac{K}{r^2(\epsilon_r-1)}$$

d) From Gauss' Law, we know

$$\oint_S E \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

To get q_{enc}:

$$q_{enc} = \int \sigma_b \, da + \int \rho_f \, dV + \int \rho_b \, dV$$

$$= \int_0^{2\pi}\int_0^{\pi} R^2 \frac{K}{R} \sin^2\theta \, dr \, d\theta \, d\varphi + \int_0^R \int_0^{2\pi}\int_0^{\pi} \frac{K}{r^2(\epsilon_r-1)} r^2 \sin\theta \, dr \, d\theta \, d\varphi + \int_0^R\int_0^{2\pi}\int_0^{\pi} \frac{-k}{r^2} r^2 \sin\theta \, d\theta \, d\varphi \, dr$$

$$= KR \cdot 4\pi + \frac{KR}{\epsilon_r - 1} \cdot 4\pi - 4\pi KR$$

$$= \frac{4\pi KR}{\epsilon_r - 1}$$

$$\oint E \cdot d\vec{a} = \frac{4\pi KR}{\epsilon - \epsilon_0}$$

$$4\pi r^2 E = \frac{4\pi KR}{\epsilon - \epsilon_0}$$

$$E = \frac{KR}{R(\epsilon-\epsilon_0)} \cdot \frac{1}{r^2} \hat{r}$$

2. Gauge Transformation

   (a) (2 pts)
   Define the vector potential $\mathbf{A}$ and the scalar potential Φ using Maxwell's equations. (i.e. give their relationships to the $\mathbf{E}$ and $\mathbf{B}$ fields.)

   (b) (3 pts) Show that when $\mathbf{A}$ and Φ undergo the gauge transformations,

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \qquad (SI) \text{ and } (Gaussian)$$

$$\Phi' = \Phi - \frac{\partial\Lambda}{\partial t}, \qquad (SI)$$

   or

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}, \qquad (Gaussian)$$

   where Λ is an arbitrary scalar, $\mathbf{B}$ and $\mathbf{E}$ are unaffected.

   (c) Two gauges used in solid-state physics for static, uniform magnetic fields $\mathbf{B}$ (i.e., constant in direction, magnitude, and time) are the Landau gauge and the circular gauge. Examples for $\mathbf{B} = B_0\hat{z}$ of each gauge respectively are:

$$\mathbf{A} = (A_x, A_y, A_z) = (0, B_0\, x, 0)$$

   and

$$\mathbf{A}' = (A'_x, A'_y, A'_z) = (-B_0 y/2, B_0 x/2, 0),$$

   with

$$\Phi = 0,$$

   for both gauges.

   i. (2 pts) Show that $\mathbf{A}$ and $\mathbf{A}'$ with $\Phi = \Phi' = 0$ describe the same $\mathbf{E}$ and $\mathbf{B}$ fields.

   ii. (3 pts) Find the scalar function Λ that produces the gauge transformation from $\mathbf{A}$ to $\mathbf{A}'$ in part (c).

Jan 2012

E+M #2

Gaussian

a) $\vec{\nabla} \times \vec{A} = \vec{B}$

$-\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t} = \vec{E}$

b) * If we now define $\vec{A}' = \vec{A} + \nabla \Lambda$, $\Phi' = \Phi - \frac{\partial \Lambda}{c \partial t}$

$\vec{\nabla} \times \vec{A}' = \nabla \times (\vec{A} + \nabla \Lambda)$

$= \nabla \times A + \nabla \times \cancel{(\nabla \Lambda)}^{0}$

$= \vec{\nabla} \times \vec{A}$ ✓

$-\nabla \Phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} = -\nabla\left(\Phi - \frac{\partial \Lambda}{c \partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{A} + \nabla \Lambda)$

$= -\nabla \Phi + \cancel{\frac{1}{c}\frac{\partial}{\partial t}\nabla \Lambda} - \frac{1}{c}\frac{\partial A}{\partial t} - \cancel{\frac{1}{c}\frac{\partial}{\partial t}\nabla \Lambda}$

$= -\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t}$ ✓

c) i) * Working with A, Φ

$\vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t}$

$= 0$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$= \langle -\frac{\partial}{\partial z} B_0 x, 0, \frac{\partial}{\partial x} B_0 x \rangle$

$= B_0 \hat{z}$

* Working with A', Φ'

$\vec{E} = -\nabla \Phi' - \frac{1}{c}\frac{\partial A'}{\partial t}$

$= 0$

$\vec{B} = \nabla \times A'$

$= \langle -\frac{\partial}{\partial z}\frac{B_0}{2}x, \frac{\partial}{\partial z}\frac{B_0}{2}y, \frac{\partial}{\partial x}\frac{B_0}{2}x + \frac{\partial}{\partial y}\frac{B_0}{2}y \rangle$

$= \langle 0, 0, B_0 \rangle$

* Both sets match

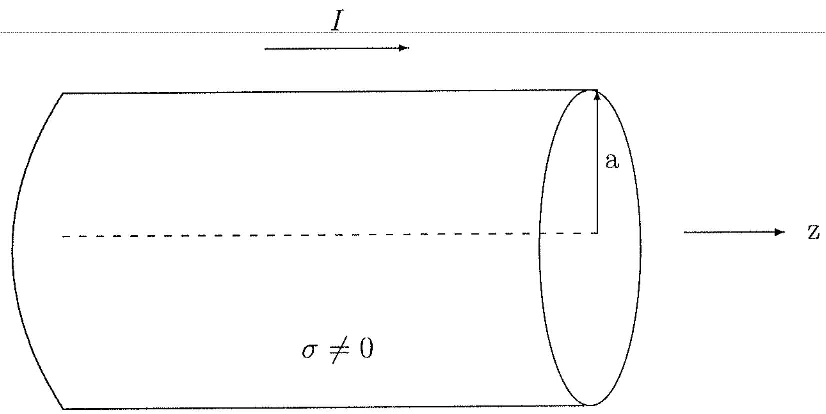
ii) $A' = A + \nabla \Lambda \implies \nabla \Lambda = A' - A$

* Note: $\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

$\hookrightarrow \nabla \Lambda = \langle -\frac{B_0}{2} y, -\frac{B_0}{2} x, 0 \rangle$

$\Lambda = -\frac{B_0}{2} xy + \text{constant}$

3. Poynting Vector



A straight metal wire of conductivity σ and cross-sectional area $A = \pi a^2$ carries a uniform, steady current I.

(a) (2 pts) Calculate $\mathbf{E}$ at the surface of the wire.

(b) (2 pts) Calculate $\mathbf{B}$ at the surface of the wire.

(c) (1 pts) Calculate the direction and magnitude of the Poynting vector at the surface of the wire.

(d) (3 pts) Integrate the normal component of the Poynting vector over the surface of the wire for a segment of length L.

(e) (2 pts) compare your result for (d) with the Joule heat produced in this segment.

The Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

Jan 2012

E+M #3

a) From Ohms Law, we know $\vec{J} = \sigma\vec{E}$, and that the total current I is related to $\vec{J}$ via $I = \oint \vec{J}\cdot d\vec{a}$

$$\Rightarrow \quad I = \oint \sigma E \cdot d\vec{a}$$

$$I = \sigma \oint E \cdot d\vec{a}$$

$$I = \sigma E \cdot \pi a^2$$

$$\frac{I}{\sigma \pi a^2}\hat{z} = \vec{E}$$

b) We can calculate $\vec{B}$ using Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I_{enc}$$

$$B \cdot 2\pi a = \frac{4\pi}{c} I$$

$$\vec{B} = \frac{2I}{ca}\hat{\varphi} \quad (\text{from RHR})$$

c) In Gaussian units, $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$, $B = H + 4\pi M$

$$\vec{S} = \frac{c}{4\pi}\begin{vmatrix} \hat{\rho} & \hat{\varphi} & \hat{z} \\ 0 & 0 & \frac{I}{\pi\sigma a^2} \\ 0 & \frac{2I}{ca} & 0 \end{vmatrix}$$

$$= \left\langle -\frac{2I^2}{\pi c\sigma a^2}, 0, 0 \right\rangle$$

$$= \frac{-2I^2}{\pi c\sigma a^2}\cdot\frac{c}{4\pi}$$

$$= \frac{-I^2}{2\pi^2\sigma a^3}\hat{\rho}$$

#3 (cont.)

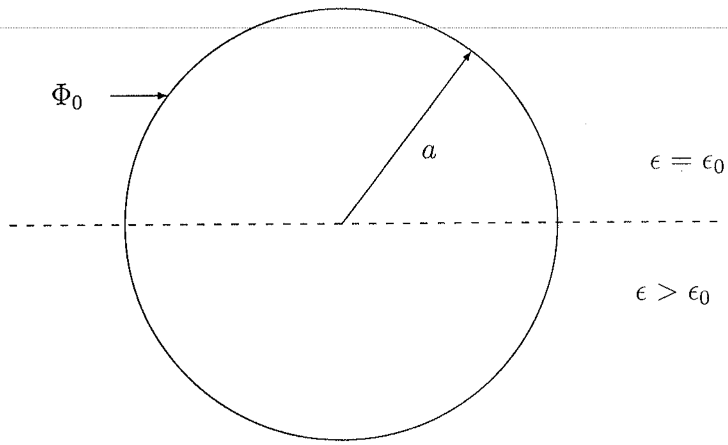
d) We want to calculate $\oint S \cdot d\vec{a}$

$$\oint \vec{S} \cdot d\vec{a} = S \oint d\vec{a}$$

$$= \frac{-I^2}{2\pi^2 \sigma a^3} (2\pi a L)$$

$$= \frac{-I^2 L}{\sigma a^2}$$

4. Half Submerged Conducting Sphere



An originally uncharged thin spherical conducting shell of radius a is brought to a potential Φ_0. The shell floats half submerged in a dielectric liquid of dielectric constant $k = \epsilon_r \equiv \epsilon/\epsilon_0$.

Determine the following:

(a) (2 pts) The electric potential Φ everywhere **outside** the shell,

(b) (2 pts) The electric field $\mathbf{E}$ everywhere **outside** the shell,

(c) (2 pts) The free surface charge density σ on the shell,

(d) (4 pts) The net electrostatic force $\mathbf{F}$ acting on the shell.

Jan 2012

E+M #4

5. Capacitor Plates

Consider a very large parallel plate capacitor with the positive plate at $z = d/2$, the negative plate at $z = -d/2$ and no dielectric material in between. If the respective surface charge densities are $\pm\sigma$ compute the *force/area on the positive plate* in the following two ways:

(a) (4 pts) Calculate it directly from σ and the electric field $\mathbf{E}$. Give a logical explanation of why your answer is correct.

(b) (6 pts) Calculate it using the Maxwell stress tensor

$$T_M^{ij} = \epsilon_0 \left[E^i E^j - \frac{1}{2}\delta^{ij}\vec{E}\cdot\vec{E} \right], \qquad (SI)$$

$$T_M^{ij} = \frac{1}{4\pi} \left[E^i E^j - \frac{1}{2}\delta^{ij}\vec{E}\cdot\vec{E} \right]. \qquad (Gaussian)$$

6. E&M Waves

A monochromatic, plane polarized, plane electromagnetic wave traveling in the z-direction in the lab (in a vacuum) can be written in the following 3+1 dimensional form:

$$\mathbf{E} = E_0 \, \hat{\mathbf{x}} \, e^{i(k\,z-\omega\,t)},$$

$$\mathbf{B} = B_0 \, \hat{\mathbf{y}} \, e^{i(k\,z-\omega\,t)}.$$

(a) (3 pts) Combine this $\mathbf{E}$ and $\mathbf{B}$ into a single electromagnetic field tensor $F^{\alpha\beta}$ and use Maxell's equations in the 4-dimensional form

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0,$$

$$\partial_\alpha F^{\alpha\beta} = 0$$

to find all constraints on the 4 constants E_0, B_0, k, and ω (i.e., the above wave won't satisfy Maxwell's equations for arbitrary values of all four of these parameters). Depending on your choice of conventions: $x^\alpha = (x^0, x^1, x^2, x^3)$ with $x^0 = ct$ or $x^\alpha = (x^1, x^2, x^3, x^4)$ with $x^4 = ct$ and $x^1 = x, x^2 = y, x^3 = z$.

(b) (1 pts) What are the values of the invariants $F^{\alpha\beta}F_{\alpha\beta}$ and $\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$ for this wave?

(c) (3 pts) Use a Lorentz boost to find $F'^{\alpha\beta}$ in a frame moving in the $+z$ direction with a speed v. Don't forget to express your answer in terms of the moving coordinates ct' and x', y', z'.

(d) (2 pts) What is the frequency and the wavelength of this wave in the moving frame?

(e) (1 pts) How have the electric and magnetic fields changed in direction and/or magnitude?

Jan 2012

E+M #6

Gaussian

a) $\vec{E} = \langle E_0, 0, 0 \rangle \exp[i(kz-\omega t)]$

$\vec{B} = \langle 0, B_0, 0 \rangle \exp[i(kz-\omega t)]$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} \exp[i(kz-\omega t)]$$

* for $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$, the only non-zero terms are: F_{01}, F_{13}

$$\partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(-B_0 \exp[i(kz-\omega t)]\right) + \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial z}\left(E_0 \exp[i(kz-\omega t)]\right) = 0$$

$$-\frac{i\omega}{c} B_0 \exp[i(kz-\omega t)] + ik E_0 \exp[i(kz-\omega t)] = 0$$

$$k E_0 = \frac{\omega}{c} B_0 \quad ①$$

* for $\partial_\alpha F^{\alpha\beta} = 0$, $\beta = 1$ to get both E_0, B_0

$$\partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(E_0 \exp[i(kz-\omega t)]\right) + 0 + 0 + \frac{\partial}{\partial z}\left(B_0 \exp[i(kz-\omega t)]\right) = 0$$

$$\frac{-i\omega}{c} E_0 \exp[i(kz-\omega t)] + ik B_0 \exp[i(kz-\omega t)] = 0$$

$$\hookrightarrow \frac{\omega}{c} E_0 = k B_0 \quad ②$$

* Combining ① and ② yields

$$\frac{\omega}{c} E_0 = k\left(\frac{ck}{\omega} E_0\right)$$

$$\frac{\omega^2}{c^2} = k^2 \Rightarrow \boxed{k = \pm \frac{\omega}{c}}$$

$$\frac{\omega}{c} E_0 = \pm \frac{\omega}{c} B_0 \Rightarrow \boxed{E_0 = \pm B_0}$$

#6 (cont.)

b) $$F^{\alpha\beta}F_{\alpha\beta} = \begin{bmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & E_0 & 0 & 0 \\ -E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} E_0^2 & 0 & 0 & -E_0B_0 \\ 0 & E_0^2-B_0^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_0B_0 & 0 & 0 & -B_0^2 \end{bmatrix}$$ * but we want the trace of this matrix

$$= 2(E_0^2 - B_0^2)$$ *Should be $2(B_0^2 - E_0^2)$

$$\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} = 2F^{\gamma\delta}F_{\gamma\delta}$$

$$= \begin{bmatrix} 0 & 0 & -B_0 & 0 \\ 0 & 0 & 0 & 0 \\ B_0 & 0 & 0 & E_0 \\ 0 & 0 & -E_0 & 0 \end{bmatrix} \begin{bmatrix} 0 & E_0 & 0 & 0 \\ -E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} 2$$

$$= 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2B_0E_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$= 0$, makes sense b/c $\hat{x}\cdot\hat{y} = 0$ *Should be $-8(B\cdot E)$

#6 (cont.)

c) $$L = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$F' = L^T F L$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma+\beta\gamma & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \beta\gamma-\gamma & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} E_0 \exp[i(kz-\omega t)]$$

$$= \begin{bmatrix} 0 & -\gamma+\beta\gamma & 0 & 0 \\ \gamma-\beta\gamma & 0 & 0 & \gamma-\beta\gamma \\ 0 & 0 & 0 & 0 \\ 0 & -\gamma+\beta\gamma & 0 & 0 \end{bmatrix} E_0 \exp[i(kz-\omega t)]$$

* but we must now transform the coordinates, which we can rewrite according to

$$kz - \omega t = -k_\sigma x^\sigma, \qquad k_\sigma = \frac{\omega}{c}\langle ct, 0, 0, -z\rangle$$
$$= k\langle 1, 0, 0, -1\rangle$$

$$k'_\lambda x'^\lambda = k_\sigma B^{-1\sigma}{}_\lambda$$

$$= [1\ 0\ 0\ -1] \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix}$$

$$= [\gamma(1-\beta)\quad 0 \quad 0 \quad -\gamma(1-\beta)]$$

$$\Rightarrow k' = k\gamma(1-\beta) \quad \Rightarrow \quad \frac{\omega'}{c} = \frac{\omega}{c}\gamma(1-\beta)$$

#6 (cont.)

c) Since wave will have same form in both frames

$$E' = E_0 \gamma(1-\beta) \exp[i(k'z' - w't')]$$

$$B' = B_0 \gamma(1-\beta) \exp[i(k'z' - w't')]$$

d) where $k' = k\gamma(1-\beta)$, $w' = w\gamma(1-\beta) = w\sqrt{\frac{1-\beta}{1+\beta}} \Rightarrow \lambda' = \lambda\sqrt{\frac{1+\beta}{1-\beta}}$

e) Direction is same but magnitude has changed

$$E_0' = E_0 \gamma(1-\beta)$$

$$B_0' = B_0 \gamma(1-\beta)$$

$$= E_0 \gamma(1-\beta)$$

E & M Qualifier

August 16, 2012

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
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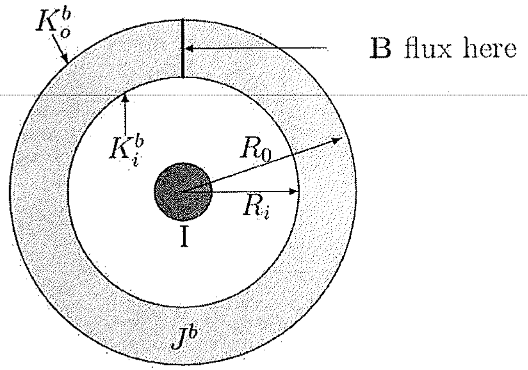


1. A long wire of radius R_{wire} carries a current I and is surrounded by a long hollow iron cylinder. The inner radius of the cylinder is R_i and the outer radius is R_o ($R_{wire} < R_i < R_o$, see the figure, assume the current flows out of the page).

   (a) (2 pts) Compute the flux of $\mathbf{B}$ through a rectangular section of the iron cylinder L meters long and $R_o - R_i$ wide.

   (b) (3 pts) Find the bound surface current densities flowing along the inner and outer iron surfaces, respectively K_i^b and K_o^b, and find the direction of these currents relative to the current in the wire.

   (c) (2 pts) Find the bound volume current density J^b inside the iron.

   (d) (3 pts) Find $\mathbf{B}$ at distances $r > R_0$ from the wire. Would this value of $\mathbf{B}$ be affected if the iron cylinder were removed?

   Recall that the magnetization $\mathbf{M}$ is related to the magnetic field strength $\mathbf{H}$ and the susceptibility χ_m by

$$\begin{aligned}
\mathbf{M} &= \chi_m^{SI}\mathbf{H} && \text{in SI units} \\
&= \chi_m^{G}\mathbf{H} && \text{in Gaussian units} \\
\mathbf{B} &= \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0(1+\chi_m^{SI})\mathbf{H} && \text{in SI units} \\
&= (\mathbf{H}+4\pi\mathbf{M}) = (1+4\pi\chi_m^{G})\mathbf{H} && \text{in Gaussian units}
\end{aligned}$$

   For all substances $4\pi\chi_m^G = \chi_m^{SI}$. For iron χ_m is in the range 10 to 1000.

2. (a) (3 pts) From Maxwell's Equations, derive the wave equation for $\mathbf{E}$ with no sources ($\rho = 0, \mathbf{J} = 0$) in a homogeneous, isotropic, linear medium of permittivity ϵ and permeability μ.

(b) (1 pts) Show that if $\mathbf{E} = E(t, z)\,\hat{\mathbf{y}}$, the wave equation reduces to

$$\begin{aligned} \frac{\partial^2 E}{\partial z^2} &= \epsilon\mu \frac{\partial^2 E}{\partial t^2}, && \text{in SI units} \\ \frac{\partial^2 E}{\partial z^2} &= \frac{\epsilon\mu}{c^2} \frac{\partial^2 E}{\partial t^2}. && \text{in Gaussian units} \end{aligned}$$

(c) (4 pts) By introducing the change of variables

$$\begin{aligned} \xi &= t + \sqrt{\epsilon\mu}\, z, && \text{in SI units} \\ \xi &= ct + \sqrt{\epsilon\mu}\, z, && \text{in Gaussian units} \\ \eta &= t - \sqrt{\epsilon\mu}\, z, && \text{in SI units} \\ \eta &= ct - \sqrt{\epsilon\mu}\, z, && \text{in Gaussian units} \end{aligned}$$

show that the wave equation assumes a form that is easily integrated.

(d) (2 pts) Integrate the equation to obtain

$$E(z, t) = E_1(\xi) + E_2(\eta),$$

where E_l and E_2 are arbitrary functions.

Aug 2012

E+M #2

Gaussian

a) Maxwells eqns: $\nabla \cdot D = \cancel{4\pi\rho}^{\,0}$ $\qquad \nabla \cdot B = 0$

$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \cancel{J}^{\,0}$

To derive wave eqn $\left(-\nabla^2 E = -\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2} E \right)$

$$\nabla \times (\nabla \times E) + \frac{1}{c}\nabla \times \frac{\partial B}{\partial t} = 0$$

$$\nabla(\cancel{\nabla \cdot E}^{\,0}) - \nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mu H) = 0$$

$$-\nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mu}{c}\frac{\partial D}{\partial t} \right) = 0$$

$$-\nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mu\epsilon}{c}\frac{\partial E}{\partial t} \right)$$

$$-\nabla^2 E + \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t} = 0 \checkmark$$

b) If $\vec{E} = E(t,z)\,\hat{y}$

$$-\nabla^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

$$\hookrightarrow -\nabla^2 E = \frac{\partial^2}{\partial z^2}\vec{E}$$

$$-\frac{\partial^2 E}{\partial z^2} + \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \Rightarrow \frac{\partial^2 E}{\partial z^2} = \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2}$$

c) Introducing the following variable substitutions: $\zeta = ct + \sqrt{\epsilon\mu}\,z$

$\eta = ct - \sqrt{\mu\epsilon}\,z$

$$\frac{\partial}{\partial z} = \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}$$

$$= \sqrt{\mu\epsilon}\frac{\partial}{\partial \zeta} - \sqrt{\mu\epsilon}\frac{\partial}{\partial \eta} = \sqrt{\mu\epsilon}\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \eta} \right)$$

$$\frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial \zeta}{\partial t}\frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}$$

$$= c\frac{\partial}{\partial \zeta} + c\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta}$$

#2 (cont.)

c) $\Rightarrow \; -\frac{\partial^2}{\partial z^2} = \mu\epsilon\left(-\frac{\partial^2}{\partial \zeta} + 2\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \eta} - \frac{\partial^2}{\partial \eta^2}\right)$

$\frac{\mu\epsilon}{c^2}\frac{\partial}{\partial t^2} = \mu\epsilon\left(\frac{\partial^2}{\partial \zeta^2} + 2\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \eta} - \frac{\partial^2}{\partial \eta^2}\right)$

$\Rightarrow \; -\frac{\partial^2 E}{\partial z^2} + \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2} = 4\mu\epsilon\frac{\partial^2}{\partial \zeta \partial \eta} E = 0$

d)

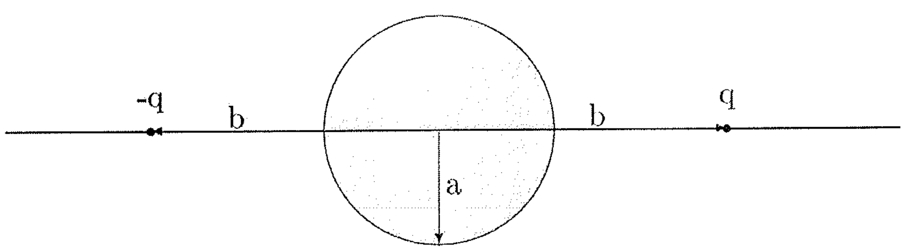
3. Two charges $\pm q$ are on opposite sides of a dielectric sphere (ϵ = constant) as shown in the figure. The three objects are on a common axis, the sphere is of radius a and the two charges are a distance $b > a$ from the sphere's center.

   (a) (2 pts) Give the form of potential $\Phi(r, \theta)$ inside the sphere ($r < a$) as a series of Legendre polynomials, $P_\ell(\cos\theta)$, with coefficients A_ℓ. Give the correct r dependence of each term and do not include ℓ values that vanish from symmetry.

   (b) (2 pts) Give the form of the potential $\Phi(r, \theta)$ outside the sphere ($r > a$) as the sum of two terms; one a series of Legendre polynomials with coefficients B_ℓ caused by the polarization charges on the dielectric, and the other term caused by the two point charges. In the series part keep only non-vanishing ℓ values and give the correct r dependence of each term.

   (c) (3 pts) In the outside region where $r > a$, expand the part of the part of the potential caused by the point charges as a single series in P_ℓ. Give two explicit forms of this series, one good for $a < r < b$ and one good for $r > b$.

   (d) (3 pts) You do not have to evaluate the constants A_ℓ and B_ℓ but write down the two sets of equations from which you can determine them (the boundary matching conditions).



4. The reflection of a circularly polarized plane wave at a metallic boundary.

   (a) (2 pts) Give expressions for the **E** and **B** fields of a monochromatic, right circularly polarized plane wave traveling in vacuum. Use rectangular Cartesian coordinates, assume the angular frequency is ω, assume the polarization plane is the x–y plane, and assume the propagation direction is in the positive z direction.

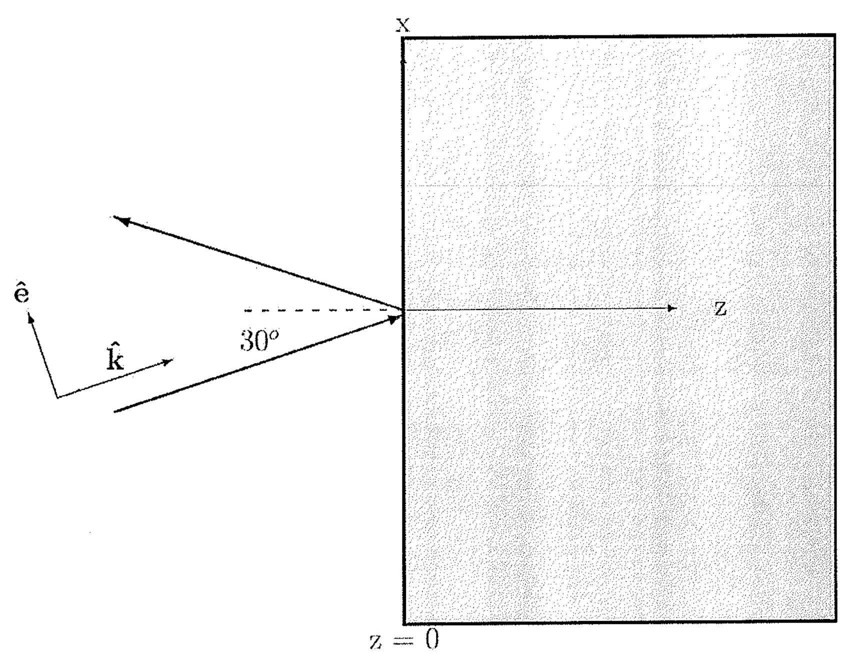
   (b) (1 pt) Explain in words what is meant by a monochromatic right circularly polarized wave.

   (c) (2 pts) Rewrite your **E** and **B** fields of part (a) assuming the propagation direction is 30° above the $\hat{\mathbf{z}}$ direction as shown in the figure. You can use unit vectors $\hat{\mathbf{e}}$ and $\hat{\mathbf{k}}$ in your expressions but be sure to define what they are in terms of the coordinate directions $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

   (d) (2 pts) If the wave of part (c) strikes a flat perfectly conducting surface at $z = 0$ it will be reflected. What boundary conditions are satisfied by the combined **E** and **B** fields of the incoming and reflected waves at the $z = 0$ junction?

   (e) (2 pts) Give expressions for the reflected **E** and **B** fields. Make sure they satisfy your junction conditions of part (d).

   (f) (1 pt) Is the reflected wave right or left circularly polarized?



5. An infinitely long, uniformly charged wire of radius a and total charge per unit length λ, is at rest on the z-axis of the lab frame.

(a) (2 pts) Compute the electric field $\mathbf{E}(x, y, z)$ interior and exterior to the wire in the lab frame by solving Gauss's law in that frame.

(b) Complete the next 4 steps to compute $\mathbf{E}'(x', y', z')$ and $\mathbf{B}'(x', y', z')$ in a frame moving in the positive z-direction with speed v.

i. (2 pts) Give the Lorentz boost $x'^{\sigma} = L^{\sigma}_{\mu} x^{\mu}$ ($\mathbf{x}' = \mathbf{Lx}$) from the Lab to the moving frame (take $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$).

ii. (2 pts) Construct the electromagnetic field tensor $F^{\alpha\beta}$ from the electric field you found in part (a).

iii. (2 pts) Use your lorentz boost to compute the electromagnetic field tensor $F'^{\alpha\beta} = L^{\alpha}_{\mu} L^{\beta}_{\nu} F^{\mu\nu}$ ($\mathbf{F}' = \mathbf{LFL^T}$) in the moving frame.

iv. (2 pts) From your $F'^{\alpha\beta}$ give the answer to (b).

Hint: Recall that in both SI and Gaussian units $F^{\sigma\mu} = -F^{\mu\sigma}$ and $F^{0i} = -E^i$. In Gaussian units $F^{12} = -B^z$, $F^{23} = -B^x$ and $F^{13} = B^y$, but in SI units $F^{12} = -c\, B^z$, $F^{23} = -c\, B^x$ and $F^{13} = c\, B^y$

a) We can use Gauss' Law to find $\vec{E}$

$$\int \nabla \cdot E = \int 4\pi \rho$$

$$\oint E \cdot d\vec{a} = 4\pi Q_{enc}$$

* If we are at $r > a$:

$$\oint \vec{E} \cdot d\vec{a} = 4\pi \lambda L$$

$$E \cdot 2\pi L r = 4\pi \lambda L$$

$$\vec{E} = \frac{2\lambda}{r} \hat{r}$$

$$= \frac{2\lambda}{\sqrt{x^2+y^2}} \langle \cos\varphi, \sin\varphi, 0 \rangle \quad \text{(in Cartesian)}$$

* If $r < a$:

$$Q_{enc} = \frac{\lambda r^2 L}{a^2} \quad \text{since wire is uniformly charged}$$

$$\hookrightarrow \oint \vec{E} \cdot d\vec{a} = 4\pi \frac{\lambda r^2 L}{a^2}$$

$$E \cdot 2\pi L r = \frac{4\pi L \lambda r^2}{a^2}$$

$$\vec{E} = \frac{2\lambda r}{a^2} \hat{r}$$

$$= \frac{2\lambda \sqrt{x^2+y^2}}{a^2} \langle \cos\varphi, \sin\varphi, 0 \rangle$$

$$\Rightarrow E = \begin{cases} \dfrac{2\lambda \sqrt{x^2+y^2}}{a^2} \langle \cos\varphi, \sin\varphi, 0 \rangle & \text{if } r < a \\ \dfrac{2\lambda}{\sqrt{x^2+y^2}} \langle \cos\varphi, \sin\varphi, 0 \rangle & r > a \end{cases}$$

#5 (cont.)

b) i) For a boost of velocity v in $\hat{z}$ direction:

$$L = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$\Rightarrow x'^{\sigma} = L^{\sigma}{}_{\mu} x^{\mu}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma ct - \beta\gamma z \\ x \\ y \\ -\beta\gamma ct + \gamma z \end{bmatrix}$$

ii) $$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\cos\varphi & -\sin\varphi & 0 \\ \cos\varphi & 0 & 0 & 0 \\ \sin\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{cases} \frac{2\lambda\sqrt{x^2+y^2}}{a^2} & r<a \\ \frac{2\lambda}{\sqrt{x^2+y^2}} & r>a \end{cases}$$

iii) $$F'^{\alpha\beta} = L^{\alpha}{}_{\mu} L^{\beta}{}_{\nu} F^{\mu\nu}$$

$$= L F L^{T}$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -\cos\varphi & -\sin\varphi & 0 \\ \cos\varphi & 0 & 0 & 0 \\ \sin\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma\cos\varphi & -\gamma\sin\varphi & 0 \\ \cos\varphi & 0 & 0 & 0 \\ \sin\varphi & 0 & 0 & 0 \\ 0 & \beta\gamma\cos\varphi & \beta\gamma\sin\varphi & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

#5 (cont.)

b) iii) $F'^{\alpha\beta} = \begin{bmatrix} 0 & -\gamma\cos\varphi & -\gamma\sin\varphi & 0 \\ \gamma\cos\varphi & 0 & 0 & -\beta\gamma\cos\varphi \\ \gamma\sin\varphi & 0 & 0 & -\beta\gamma\sin\varphi \\ 0 & \beta\gamma\cos\varphi & \beta\gamma\sin\varphi & 0 \end{bmatrix}$

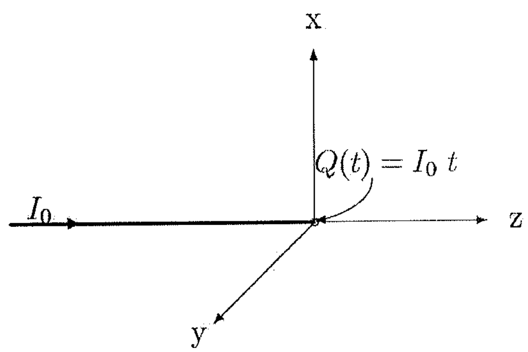
iv) $E' = \begin{cases} \dfrac{\gamma 2\lambda\sqrt{x^2+y^2}}{a^2} \langle \cos\varphi, \sin\varphi, 0 \rangle & r < a \\ \dfrac{\gamma 2\lambda}{\sqrt{x^2+y^2}} \langle \cos\varphi, \sin\varphi, 0 \rangle & r > a \end{cases}$

$B' = \begin{cases} \dfrac{\beta\gamma 2\lambda\sqrt{x^2+y^2}}{a^2} \langle \cos\varphi, -\sin\varphi, 0 \rangle & r < a \\ \dfrac{\beta\gamma 2\lambda}{\sqrt{x^2+y^2}} \langle \cos\varphi, -\sin\varphi, 0 \rangle & r > a \end{cases}$

6. In the absence of polarizable and/or magnetizable material (i.e., only free charges and currents present) Maxwell's equations, in Gaussian units and in the Lorentz gauge, reduce to the inhomogeneous wave equation:

$$\Box \begin{Bmatrix} \Phi \\ A^x \\ A^y \\ A^z \end{Bmatrix} = \frac{4\pi}{c} \begin{Bmatrix} c\rho \\ J^x \\ J^y \\ J^z \end{Bmatrix}, \quad \text{where } \Box \equiv \left(\frac{\partial}{c\partial t}\right)^2 - \nabla^2.$$

A time dependent charge $Q(t) = I_0\, t,\ t \geq 0$ is fixed at the origin



of a cylindrical polar coordinate system (ρ, ϕ, z) The charge increases linearly with time because a constant current I_0 flows in along a thin wire attached to the charge on its left, see the figure. Assume the wire carries no current for $t < 0$, however, at $t = 0$ a current I_0 abruptly starts flowing in the +z direction and remains constant for $t \geq 0$. Assume the wire remains neutral as the charge at the origin grows. Find the following quantities at time t for points (ρ, ϕ, z):

(a) (2 pts) The charge density $\rho(t, \rho, \phi, z)$,

(b) (2 pts) The current density $\mathbf{J}(t, \rho, \phi, z)$,

(c) (2 pts) The retarded scalar potential $\Phi(t, \rho, \phi, z)$,

(d) (4 pts) The retarded vector potential $\mathbf{A}(t, \rho, \phi, z)$.

Hints: Parts (a) and (b) require the use of $\delta(x)$-functions and Heaviside step functions $\Theta(x) \equiv 1, 0$ respectively for $x > 0$ or $x < 0$. The retarded Green's function for the $\Box$ operator is:

$$G^{ret}(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta\left(t - t' - |\mathbf{r} - \mathbf{r}'|/c\right)}{4\pi\, |\mathbf{r} - \mathbf{r}'|},$$

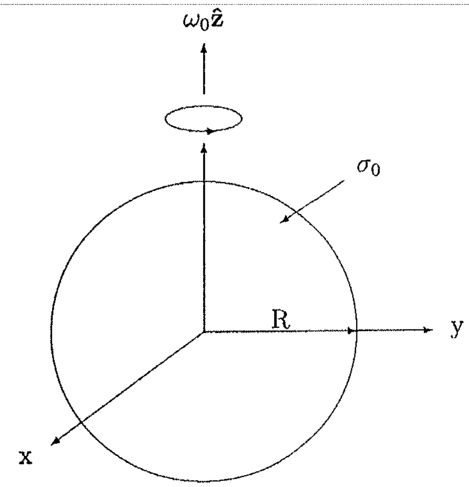
which gives retarded potentials

$$\Big(\Phi(t, \mathbf{r}), \mathbf{A}(t, \mathbf{r})\Big)^{ret} = \frac{1}{c} \int \frac{\Big(c\rho(t - |\mathbf{r} - \mathbf{r}'|/c, \mathbf{r}'), \mathbf{J}(t - |\mathbf{r} - \mathbf{r}'|/c, \mathbf{r}')\Big)}{|\mathbf{r} - \mathbf{r}'|} d^3r'.$$

For part (d) you might need the integral

$$\int \frac{dX}{\sqrt{X^2 + a^2}} = \ln(\sqrt{X^2 + a^2} + X).$$

Spring 2013 Qualifier



1. A rotating thin, non-conducting, sphere of radius R is covered with a uniform surface charge density σ_0. If the angular velocity is in the $+\hat{\mathbf{z}}$ direction and has a constant magnitude ω_0:

   (a) [3 pts] Compute the surface current density $\mathbf{K}(\theta, \phi)$ (magnitude and direction) as a function of R, σ_0, and ω_0.

   (b) [3 pts] Compute the magnetic dipole moment $\mathbf{m}_0$ of the rotating sphere.

   (c) [4 pts] The magnetic induction <u>exterior</u> to the sphere turns out to be a simple magnetic dipole field. Compute $\mathbf{B}(r, \theta, \phi)$ for $r > R$ assuming the vector potential is of the form:

$$\mathbf{A}(\mathbf{r})_G = \frac{\mathbf{m}_0 \times \hat{\mathbf{r}}}{r^2} \qquad \text{Gussian units,}$$

$$\mathbf{A}(\mathbf{r})_{SI} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}_0 \times \hat{\mathbf{r}}}{r^2} \qquad \text{SI units.}$$

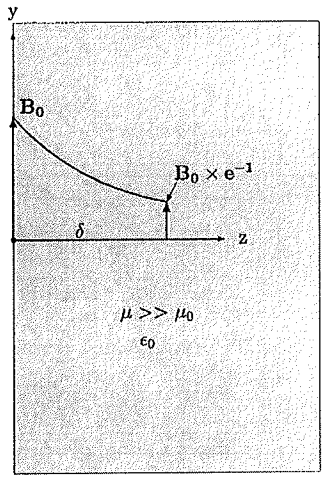


Figure 1: In Gaussian units $\epsilon_0 \to 1$ and $\mu_0 \to 1$

2. Within a transformer, oscillating magnetic fields and their associated electric fields penetrate into the transformer's iron core producing "eddy" currents which heat and frequently destroy the transformer. In this problem you are to analyze the depths to which these currents penetrate and the phase difference between the driving harmonic **B** field and the lagging eddy current.

   A large slab of permeable ($\mu >> \mu_0$) conductor with conductivity $\sigma > 0$ and with negligible permittivity ($\epsilon = \epsilon_0$) is located in the x-y plane at $z \geq 0$ as shown in the figure. A low frequency wave, $\omega << \sigma/\epsilon_0$, whose magnetic induction is the real part of

$$\mathbf{B} = B_0 e^{i(kz-\omega t)}\hat{\mathbf{y}},$$

   diminishes as z increases because k is complex.

   (a) [2 pts] Give Maxwell's 4 macroscopic equations appropriate for this material ($\rho = 0, \mathbf{D} = \epsilon_0\mathbf{E}, \mathbf{B} = \mu\mathbf{H}$, and $\mathbf{J} = \sigma\mathbf{E}$).

   (b) [2 pts] Use Maxwell's equations to find the complex wave number k as a function of ω, σ, μ, and ϵ_0.

   (c) [3 pts] The depth at which the amplitude reaches e^{-1} times its original value is called the skin depth, δ. The skin depth diminishes with the wave's frequency. For low frequency waves ($\omega << \sigma/\epsilon_0$) determine δ.

   (d) [3 pts] For low frequency waves compute the phase lag of the eddy current density **J** relative to the magnetic induction **B**.

3. (a) [1 pts] Give the 4-current $J^\alpha(x^\beta)$ for the static surface charge density σ_0 shown in the figure (a thin uniform and infinite sheet of charge located at $z = 0$ in the lab).

   (b) [2 pts] Give the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ caused by the static surface charge.

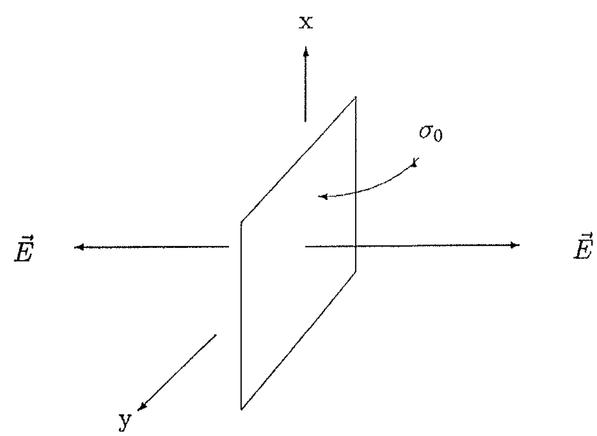
   (c) [2 pts] Compute the 4-current $J'^\alpha(x'^\beta)$ in a frame that moves with speed $v < c$ in the positive x-direction relative to the lab (v isn't necessarily small).

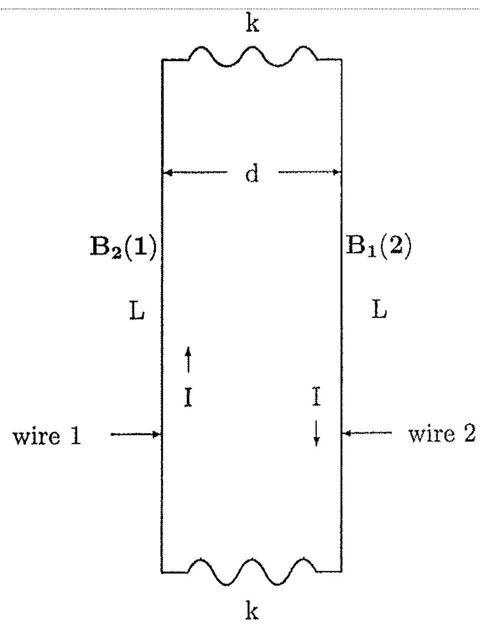
   (d) [1 pts] What is the surface charge density σ' in the moving frame?

   (e) [1 pts] What is the electric field $\mathbf{E}'$ in the moving frame?

   (f) [1 pts] What is the surface current density $\mathbf{K}'$ in the moving frame?

   (g) [2 pts] What is the magnetic induction $\mathbf{B}'$ in the moving frame?





4. A current balance consists of two very long rigid parallel wires of lengths L that are connected at each end by springs (see the figure). The spring constant of both springs is k and the equilibrium distance between the wires, d(I), depends on the current. Assume $d << L$.

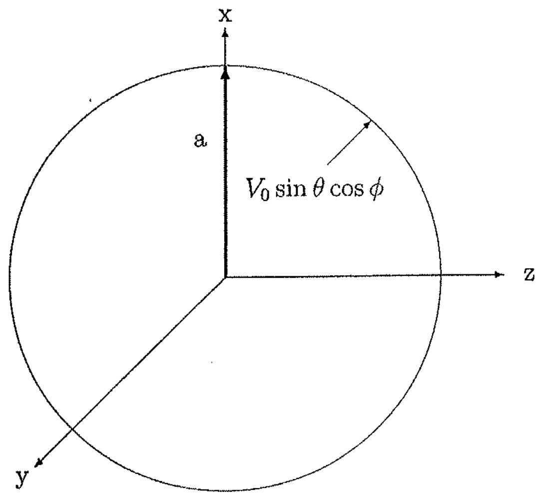
   (a) [2 pts] If a current I flows through the closed circuit of the 2 wires and 2 springs, find an expression for the magnetic induction $B_1(2)$ created by the first wire at the location of the second. What is the direction of this $B_1(2)$ field (give the direction as up, down, left, right, into, or out of the page)?

   (b) [2 pts] Find an expression for the magnetic force $\mathbf{F}_1(2)$ on the second wire due to the $B_1(2)$. What is the direction of this force?

   (c) [2 pts] Find an expression for the magnetic force $\mathbf{F}_2(1)$ on the first wire due to the magnetic induction created by the second. What is the direction of this force?

   (d) [4 pts] Are the springs stretched or compressed from equilibrium? Using the above results, find an expression for the current as a function of the amount the springs are stretch/compressed.

Similar to Jan 2015 #5



5. In spherical polar coordinates the solution to the Laplace equation $\nabla^2\Phi(r,\theta,\phi) = 0$, for a spherical region $r_1 < r < r_2$ can be expanded in terms of spherical harmonics in the following form:

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\ell=\infty} \left(A_{\ell,m} r^\ell + \frac{B_{\ell,m}}{r^{\ell+1}} \right) Y_\ell^m(\theta,\phi),$$

where $A_{\ell,m}$ and $B_{\ell,m}$ are constants.

(a) [3 pts] If the potential $\Phi(r,\theta,\phi)$ is given on a sphere $r = a$ but satisfies the laplace equation everywhere else, what is the form of the potential inside $(0 \le r < a)$ the sphere? Outside $(a < r < \infty)$ the sphere?

(b) [7 pts] For the particular potential given in the figure, $\Phi(r = a,\theta,\phi) = V_0 \sin\theta\cos\phi$, what is the potential inside the sphere? Outside $(a < r < \infty)$ the sphere?

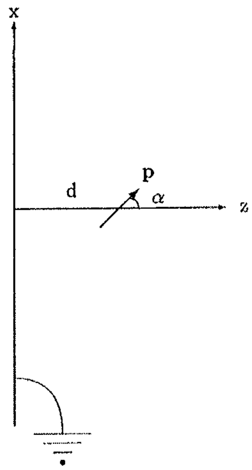
Recall that the spherical harmonics are ortho-normal on the sphere and for $\ell = 1$

$$\begin{aligned} Y_1^{-1}(\theta,\phi) &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}, \\ Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\ Y_1^1(\theta,\phi) &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}. \end{aligned}$$

Jan 2013

E+M #5

Gaussian



6. A point dipole with dipole moment $\mathbf{p} = p(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}})$ is located on the z-axis a distance d from a large flat grounded conducting plate situated in the z=0 plane (see the figure).

   (a) [2 pts] The part of the total potential in the region $z > 0$ caused by the induced surface charge on the grounded conductor at $z = 0$ is the same as the potential of an image dipole. What is the dipole moment $\mathbf{p}_i$ of the image dipole and where is it located?

   (b) [3 pts] What is the total electrostatic potential in the region $z \geq 0$?

   (c) [3 pts] How much work must be done to remove the dipole from $z = d$ to $z = +\infty$?

   (d) [2 pts] When at $z = d$ what force does the dipole experience?

   Hint: The electrostatic potential caused by an ideal point dipole located at the origin ($\mathbf{r} = 0$) with dipole moment $\mathbf{p} = p^x\hat{\mathbf{x}} + p^y\hat{\mathbf{y}} + p^z\hat{\mathbf{z}}$ is

$$\Phi_G(\mathbf{r}) = \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \qquad \text{Gaussian units}$$

$$\Phi_{SI}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \qquad \text{SI units}$$

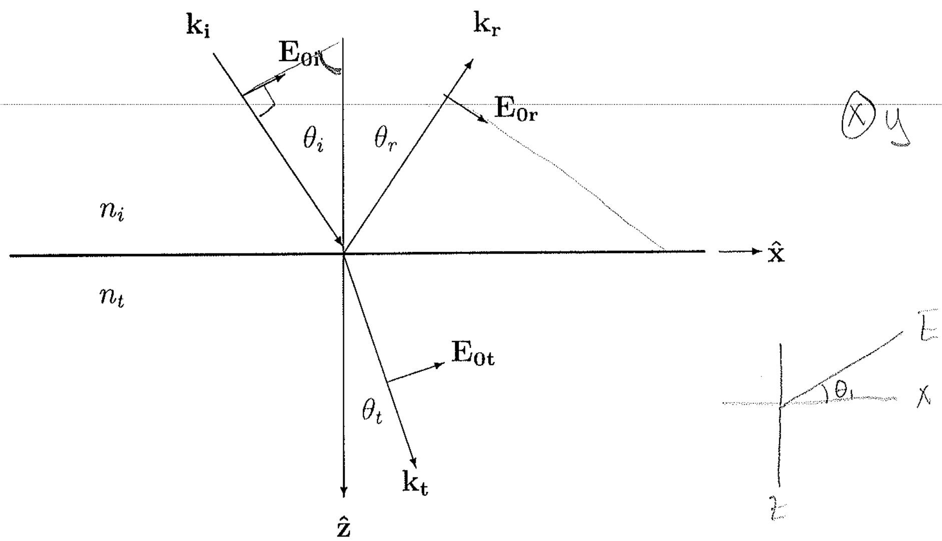
E & M Qualifier

August 15, 2013

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias **(NOT YOUR REAL NAME)** on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. **do not** staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



1. A plane monochromatic electromagnetic wave of wave vector $\mathbf{k}_i$, amplitude $\mathbf{E}_{0i}$, and angular frequency ω is incident at the planar interface of two dielectric, non-magnetic ($\mu = \mu_0$ in SI units), non-absorbing media (i.e., have real indices of refraction n_i and n_t). The angle of incidence is equal to θ_i. Part of the incident wave is reflected at an angle $\theta_r = \theta_i$ and part of it is transmitted into the second medium at a transmission angle θ_t. Assume the electric fields of the incident, reflected, and refracted waves lie in the plane of incidence as shown in the figure. Assume coordinates are chosen so that the dielectric interface is the $z = 0$ plane and the polarization is in the x-z plane.

   (a) [2 pts] Use Maxwell's equations to derive an expression for the magnetic induction $\mathbf{B}$ associated with a plane monochromatic electromagnetic wave whose electric field is

   $$\mathbf{E} = \mathbf{E}_0 \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

   traveling in a homogeneous material described by a real index of refraction n. Give the relationship of $|\mathbf{k}|$ to ω.

   (b) [2 pts] From the above figure give $\mathbf{k}_i, \mathbf{k}_r$, and $\mathbf{k}_t$ in terms of their $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components, and evaluate $\mathbf{k} \cdot \mathbf{r}$ in the $z = 0$ plane.

   (c) [1 pts] From the above figure give $\mathbf{E}_{0i}, \mathbf{E}_{0r}$, and $\mathbf{E}_{0t}$ in terms of their $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components.

   (d) [1 pts] State the 4 boundary conditions satisfied by the fields $\mathbf{E}, \mathbf{B}, \mathbf{H}$, and $\mathbf{D}$ at the above $z = 0$ junction.

   (e) [1 pts] Use one of these junction conditions to prove Snell's law, $n_t \sin\theta_t = n_i \sin\theta_i$ (only 2 of the 4 are independent).

   (f) [3 pts] Use two of the junction conditions to determine the ratio of the magnitude of the amplitudes of the reflected and transmitted to the incident electric fields, i.e., evaluate $|\mathbf{E}_{0r}|/|\mathbf{E}_{0i}|$ and $|\mathbf{E}_{0t}|/|\mathbf{E}_{0i}|$ as shown in the figure.

Aug 2013

E+M #1

SI

* Remember:

$$\nabla \cdot D = \rho_f \qquad \nabla \cdot B = 0$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \qquad \nabla \times H - \frac{\partial D}{\partial t} = \vec{J}_f$$

$$D = \epsilon E \qquad n = \sqrt{\mu_r \epsilon_r} = c/v$$

$$B = \mu H \qquad \mu = \mu_0 \Rightarrow \mu_r = 1$$

a) To determine the relationship b/w $|k|$ and ω, we derive the wave equation

$$\nabla \times \left(\nabla \times E + \frac{\partial B}{\partial t} \right) = 0$$

$$\nabla \times \nabla \times E + \frac{\partial}{\partial t} \nabla \times B = 0$$

$$-\nabla^2 E + \nabla(\nabla \cdot E) + \frac{\partial}{\partial t} \mu (\nabla \times H) = 0$$

$$-\nabla^2 E + \frac{\partial}{\partial t} \mu \left(\frac{\partial}{\partial t} D \right) = 0 \qquad \text{(no free charges / currents)}$$

$$-\nabla^2 E + \mu \epsilon \frac{\partial^2}{\partial t^2} E = 0$$

* Given $\vec{E} = E_0 \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$

$$\frac{\partial^2}{\partial t^2} E = -\omega^2 E_0 \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$$

$$-\nabla^2 E = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_0 \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$$

$$= |k|^2 \exp[i(k \cdot r - \omega t)]$$

$$|k|^2 - n^2 \omega^2 = 0 \Rightarrow |k| = n\omega$$

b) $\vec{k}_i = \langle \sin\theta_i, 0, \cos\theta_i \rangle |k_i|$

$\vec{k}_r = \langle \sin\theta_r, 0, -\cos\theta_r \rangle |k_r|$

$\vec{k}_t = \langle \sin\theta_t, 0, \cos\theta_t \rangle |k_t|$

#1 (cont.)

b) $k_i \cdot r = k_i x \sin\theta_i$

$k_r \cdot r = k_r x \sin\theta_r$ $\quad r = \langle x, y, 0 \rangle$ b/c in $z=0$ plane

$k_t \cdot r = k_t x \sin\theta_t$

c) $\vec{E}$ must be perpendicular to $\vec{k}$

$\vec{E}_i = E_{0,i} \langle \cos\theta_i,\ 0,\ -\sin\theta_i \rangle$

$\vec{E}_r = E_{0,r} \langle \cos\theta_r,\ 0,\ \sin\theta_r \rangle$

$\vec{E}_t = E_{0t} \langle \cos\theta_t,\ 0,\ -\sin\theta_t \rangle$

d) Our boundary conditions are:

① $D_1^{\perp} - D_2^{\perp} = \sigma_f$ ③ $B_1^{\perp} - B_2^{\perp} = 0$

② $E_1^{\parallel} - E_2^{\parallel} = 0$ ④ $H_1^{\parallel} - H_2^{\parallel} = k_f \cdot \frac{1}{\mu_0}$

* However, since there are no free charges

$D \cdot \hat{n}$, $B \cdot \hat{n}$, $E \times \hat{n}$, $H \times \hat{n}$ are all continuous

e) Using boundary condition ①:

$\epsilon_1 E_1 \cdot \hat{z} = \epsilon_2 E_2 \cdot \hat{z}$

$\epsilon_1 (E_i + E_r) \cdot \hat{z} = \epsilon_2 (E_t \cdot \hat{z})$

$\epsilon_1 (-E_{0,i} \sin\theta_1 + E_{0,r} \sin\theta_r) \exp[i(\underline{x k_i \sin\theta_i} - \omega t)] = \epsilon_2 E_t \sin\theta_t \exp[i(\underline{x k_t \sin\theta_t} - \omega t)]$

$x k_i \sin\theta_i = x k_t \sin\theta_t$

$n_i \sin\theta_i = n_t \sin\theta_t$

#1 (cont.)

f) * From part e we know

$$\epsilon_1 (E_r - E_i) \sin\theta_i = \epsilon_2 E_t \sin\theta_t$$

$$n_i^2 \cancel{\epsilon_0} (E_r - E_i) \cancel{\sin\theta_i} = n_t^2 \cancel{\epsilon_0} E_t \cancel{\sin\theta_t}$$

* using our $E \times \hat{n}$ boundary condition:

$$(E_i + E_r)\cos\theta_i = E_t \cos\theta_t$$

* Thus:

① $n_i (E_r - E_i) = n_t E_t$

② $(E_i + E_r)\cos\theta_i = E_t \cos\theta_t$

* Solving for E_r:

$$(E_i + E_r)\cos\theta_i = \frac{n_i}{n_t}(E_r - E_i)\cos\theta_t$$

$$E_i\left(\cos\theta_i + \frac{n_i}{n_t}\cos\theta_t\right) = E_r\left(\frac{n_i}{n_t}\cos\theta_t - \cos\theta_i\right)$$

$$\frac{E_r}{E_i} = \frac{n_i/n_t \cos\theta_t - \cos\theta_i}{\cos\theta_i - n_i/n_t \cos\theta_t} = \frac{n_i\cos\theta_t - n_t\cos\theta_i}{n_t\cos\theta_i - n_i\cos\theta_t}$$

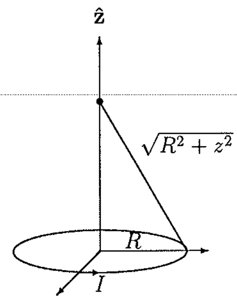
* Solving for E_t

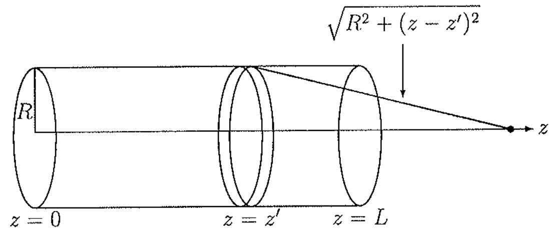
$$E_r = \frac{n_t}{n_i}E_t + E_i$$

$$\left(2E_i + \frac{n_t}{n_i}E_t\right)\cos\theta_i = E_t\cos\theta_t$$

$$2E_i\cos\theta_i = E_t\left(\cos\theta_t - \frac{n_t}{n_i}\cos\theta_i\right)$$

$$\frac{E_t}{E_i} = \frac{\cos\theta_t - \frac{n_t}{n_i}\cos\theta_i}{2\cos\theta_i}$$





2. (a) [3 pts] A circular loop of radius R, centered on the origin and $z=0$ plane, carries a current I. Find the magnetic field B <u>on the axis</u> of the loop as a function of the distance z from the center of the loop.

   (b) [4 pts] Use the result of part (a) to find B <u>along the axis</u> of a solenoid of radius R and length L, uniformly wound with $n = N/L$ turns per unit length.

   (c) [3 pts] Assume that instead of a solenoid you had a cylinder of radius R and length L made out of a piece of uniformly magnetized iron with magnetization $\mathbf{M}$ pointing along the axis of the solenoid. Use the solution of part (b) to calculate the magnetic field strength $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ <u>along the axis</u> of the cylinder, both inside and outside.

HINTS:

$$\int \frac{dw}{[R^2+w^2]^{3/2}} = \frac{w}{\sqrt{R^2+w^2}} + \text{constant.}$$

The bound volume and surface current densities associated with a smooth magnetization density are respectively

$$\mathbf{J}_b|_{SI} = \nabla \times \mathbf{M},$$

and

$$\mathbf{K}_b|_{SI} = \mathbf{M} \times \mathbf{n},$$

where $\mathbf{n}$ is the outward unit normal at the magnet's boundary. The Gaussian expressions for $\mathbf{J}_b$ and $\mathbf{K}_b$ contain an additional factor of c in the numerators.

Aug 2013

E+M #2

SI

a) We can use Biot-Savart Law to determine $\vec{B}$

$$B = \frac{\mu_0}{4\pi}\int \frac{\vec{J} \times (\vec{r}-\vec{r}')}{|r-r'|^3}\, dr'$$

$$B = \frac{\mu_0}{4\pi}\int \frac{I\,dl\,\hat{\varphi} \times (r-r')}{(R^2+z^2)^{3/2}}$$

* evaluate cross product:

$$\begin{vmatrix} \hat{r} & \hat{\varphi} & \hat{z} \\ 0 & dl & 0 \\ R & 0 & z \end{vmatrix} = \langle z\,dl,\ 0,\ R\,dl \rangle$$

$$B = \frac{\mu_0}{4\pi}\int \frac{I R\, dl}{(R^2+z^2)^{3/2}}\,\hat{z}$$ (ignore $\hat{r}$ component due to symmetry

$$= \frac{\mu_0}{4\pi} \frac{2\pi R^2 I}{(R^2+z^2)^{3/2}}\,\hat{z}$$

$$= \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\,\hat{z}$$

b) * Typically, when going from current loop to solenoid, $I \rightarrow I n\, dz$. Also, because our solenoid is of a finite length, $z \rightarrow z - z'$

$$\hookrightarrow dB = \frac{\mu_0 n I\, dz'\, R^2}{2(R^2-(z-z')^2)^{3/2}}\,\hat{z}$$

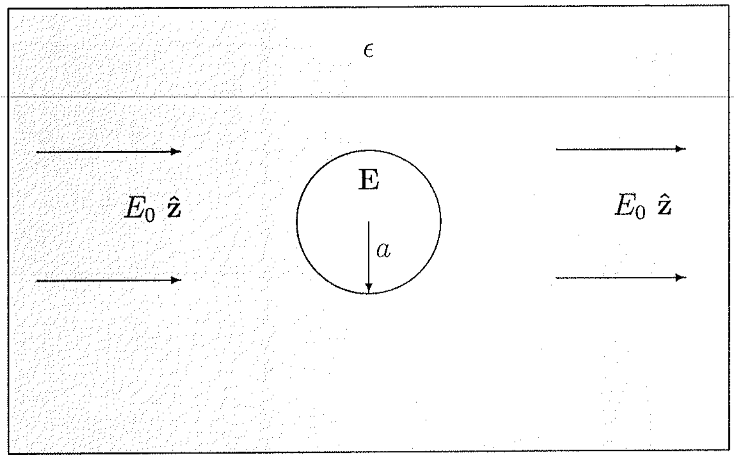
$$B = \frac{\mu_0 n I}{2}\int_0^L \frac{R^2\, dz'\,\hat{z}}{(R^2-(z-z')^2)^{3/2}}$$

$$= \frac{\mu_0 n I R^2}{2}\left[\frac{z'}{(R^2+(z-z')^2)}\Big|_0^L \right]$$

$$= \frac{\mu_0 n I R^2 L}{2(R^2+(z-L)^2)^{1/2}}$$

#2 (cont.)

c)



3. A small spherical vacuum bubble of radius a exists inside an otherwise homogeneous dielectric material whose electric polarization properties are described by a constant permitivity ϵ. Assume the bubble is centered on the origin and that the electric field far from vacuum bubble but still in the homogeneous material is of the form $\mathbf{E} = E_0\,\hat{\mathbf{z}}$. Because of the axial symmetry the electrostatic potential $\Phi(r, \theta)$ for this problem can be written as a linear combination of constants times Legendre polynomials.

   (a) [2 pts] What are the boundary conditions satisfied by the fields $\mathbf{E}$ and $\mathbf{D}$ fields , and the potential Φ at the junction $r = a$?

   (b) [2 pts] Give the electrostatic potential inside the bubble as a combination of constants and Legendre polynomials (keep only non-vanishing ℓ terms).

   (c) [2 pts] Give the electrostatic potential outside the bubble as a combination of constants and Legendre polynomials (keep only non-vanishing ℓ terms).

   (d) [2 pts] Use the boundary conditions at the $r = a$ junction from part (a) to evaluate the non-vanishing constants in parts (b) and (c).

   (e) [2 pts] Express the electric field outside the bubble as an electric dipole field plus the uniform field $E_0\,\hat{\mathbf{z}}$ and give the value of the dipole moment.

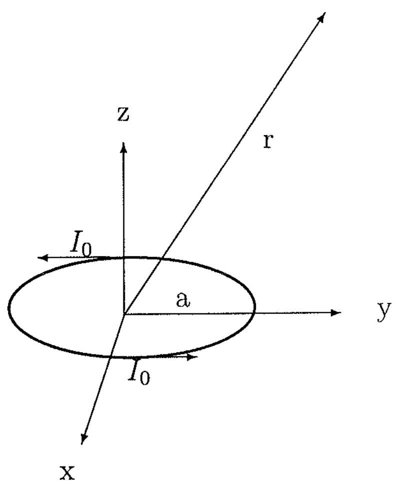
4. (a) [2 pts] Write down any vector potential that produces the uniform magnetic induction

$$\mathbf{B} = B_0\,\hat{\mathbf{z}}.$$

(b) [4 pts] What is the magnetic induction $\mathbf{B}$ and an associated vector potential $\mathbf{A}$ ($\mathbf{B} = \nabla \times \mathbf{A}$) produced by a very long wire located on the z-axis and carrying a current I_0 in the +z direction?

(c) [4 pts] A small circular loop of wire of radius a, centered at the origin and lying in the $z = 0$ plane, carries a current I_0 as shown in the figure. Derive an approximate expression for the vector potential at large distances ($r >> a$) from the loop. Recall that

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r}\left\{1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \mathcal{O}\left(\frac{r'}{r}\right)^2\right\}$$



Aug 2013

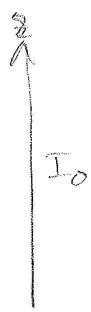
E+M #4

a) We know that $\nabla \times A = B$, therefore if $B_z = B_0 \hat{z}$

$$B_z = \partial_x A_y - \partial_y A_x \qquad \Rightarrow \text{ set } A_z = 0$$
$$= \frac{\partial}{\partial x} \frac{B_0}{2} x - \frac{\partial}{\partial y} \frac{-B_0}{2} y$$
$$= \frac{B_0}{2} + \frac{B_0}{2}$$
$$= B_0 \checkmark$$

$$\Rightarrow \quad A = \langle \frac{-B_0}{2} y, \ \frac{B_0}{2} x, \ 0 \rangle$$

b) * We first solve for B



$$\int B \cdot dl = \frac{4\pi}{c} I_0$$
$$B \cdot 2\pi r = \frac{4\pi}{c} I_0$$
$$B = \frac{2 I_0}{c r} \hat{\varphi}$$
$$= \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{A} = \langle \frac{1}{\rho} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}, \ \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}, \ \frac{1}{\rho} \left(\frac{\partial}{\partial \rho}(A_\varphi) - \frac{\partial A_\rho}{\partial \varphi} \right) \rangle$$

↳ set $A_\varphi = 0$; $A_\rho A_z$ has no φ dependence

$$= \frac{2 I_0}{c} \langle \frac{1}{\rho} \cancel{\partial_\varphi A_z}, \ \partial_z A_\rho - \partial_\rho A_z, \ \frac{1}{\rho} \cancelto{0}{\partial_\varphi A_\rho} \rangle$$
$$= \frac{2 I_0}{c} \langle 0, \ \partial_z A_\rho - \partial_\rho A_z, \ 0 \rangle$$

↳ $\frac{1}{r} = \partial_z A_\rho - \partial_\rho A_z$

*since an arbitrary A works, set $A_\rho = 0$,

$$\int \frac{1}{r} d\rho = \int \partial_\rho A_z \, d\rho$$
$$\ln(r) = A_z$$

$$\Rightarrow \quad A = \frac{2 I_0}{c} \langle 0, 0, \ln(r) \rangle$$

5. An infinitely long, uniformly charged wire of radius a and total charge per unit length λ, is at rest on the z-axis of the lab frame.

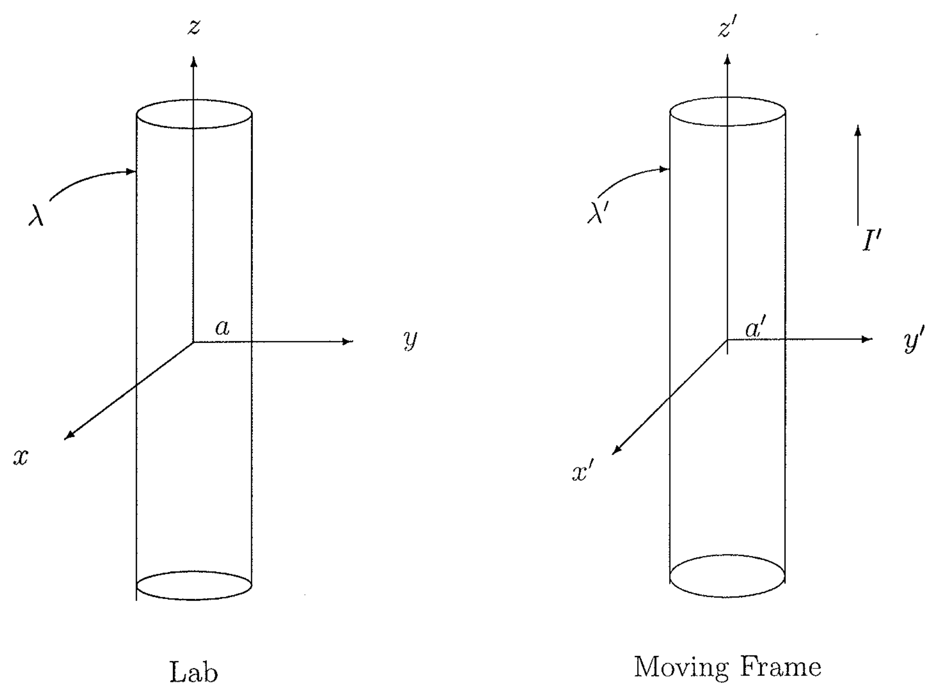
   (a) [2 pts] Compute the electric field $\mathbf{E}(x, y, z)$ exterior to the wire in the lab frame by solving Gauss's law in that frame. What is the magnetic induction $\mathbf{B}(x, y, z)$ in this frame?

   (b) [2 pts] If you are moving in the lab's negative z direction with speed v how are your spatial and time coordinates related to those of the lab's? To answer this question simply give the Lorentz boost $x'^{\sigma} = L^{\sigma}{}_{\mu} x^{\mu}$ that relates the two sets of coordinates.

   (c) [2 pts] In your frame what is the radius a' of the wire? What is the charge/length λ' of the wire and what is the current I' in the wire?

   (d) [1 pts] Combine the E and B fields in the lab into a single electromagnetic field tensor $F^{\alpha\beta}$ using $F^{\sigma\mu} = -F^{\mu\sigma}$ and $F^{0i} = -E^{i}$. In Gaussian units $F^{12} = -B^{z}$, $F^{23} = -B^{x}$ and $F^{13} = B^{y}$, and in SI units $F^{12} = -c\,B^{z}$, $F^{23} = -c\,B^{x}$ and $F^{13} = c\,B^{y}$, .

   (e) [3 pts] What electric field $\mathbf{E}\,'(x', y', z')$ and what magnetic induction $\mathbf{B}\,'(x', y', z')$ will you measure exterior to the wire in your frame? To answer this part you can use your answers for part (c) or you can compute $F' = LFL^{T}$.



Aug 2013

E+M #5

a) From Gauss Law, we know that

$$\int \vec{E}\cdot d\hat{a} = \frac{q}{\epsilon_0}$$

$$E\int da = \frac{\lambda L}{\epsilon_0}$$

$$E(2\pi r L) = \frac{\lambda L}{\epsilon_0}$$

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\hat{r}$$

* However, we most convert this from cylindrical coordinates to cartesian

$$\Rightarrow \quad r = \sqrt{x^2+y^2}$$

$$\hat{r} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$$

$$\hookrightarrow \quad \vec{E}(x,y,z) = \frac{\lambda}{2\pi\epsilon_0}\frac{\langle \cos\varphi, \sin\varphi, 0\rangle}{\sqrt{x^2+y^2}}$$

* Since there is no moving charge in the wire, $\vec{B}=0$

b) In a moving frame, we know coordinates are transformed according to:

$$x'^{\sigma} = L^{\sigma}{}_{\mu}x^{\mu} \quad \Longleftrightarrow \quad x' = L_z x$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma ct + \gamma\beta z \\ x \\ y \\ \gamma_\beta ct + \gamma z \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} t' &= \frac{\gamma}{c}(ct+\beta z) \\ x' &= x \\ y' &= y \\ z' &= \gamma(ct\beta + z) \end{aligned}$$

#5 (cont.)

c) * Since the radius of the wire only depends on x and y, it is unchanged moving from the unprimed frame to the primed frame.

* We can transform the current vector to find λ' and I'

$$\vec{J} = \begin{bmatrix} c\lambda \\ J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} c\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{J}' = L_z J$$

$$= \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} c\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c\lambda\gamma \\ 0 \\ 0 \\ \gamma\beta c\lambda \end{bmatrix} \Rightarrow \begin{array}{l} c\lambda' = c\lambda\gamma \rightarrow \lambda' = \gamma\lambda \\ J_z' = \gamma\beta c\lambda \end{array}$$

$$I' = \oint \vec{J}' \cdot d\vec{a}'$$

$$= J_z \cdot \pi a^2$$

$$= \gamma\beta c\lambda\pi a^2$$

d) In the lab (unprimed) frame, our field tensor is:

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & 0 \\ E_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} E_x = \dfrac{\lambda\cos\varphi}{2\pi\epsilon_0\sqrt{x^2+y^2}} \\[2ex] E_y = \dfrac{\lambda\sin\varphi}{2\pi\epsilon_0\sqrt{x^2+y^2}} \end{array}$$

#5 (cont.)

e) The field tensor is transformed according to:

$$F' = LFL^T$$

$$= \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & 0 \\ E_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & 0 \\ \gamma E_x & 0 & 0 & \gamma\beta E_x \\ \gamma E_y & 0 & 0 & \gamma\beta E_y \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma E_x & -\gamma E_y & 0 \\ \gamma E_x & 0 & 0 & \gamma\beta E_x \\ \gamma E_y & 0 & 0 & \gamma\beta E_y \\ 0 & -\gamma\beta E_x & -\gamma\beta E_y & 0 \end{bmatrix}$$

$\Rightarrow E_x' = \gamma E_x \qquad E_z' = 0$

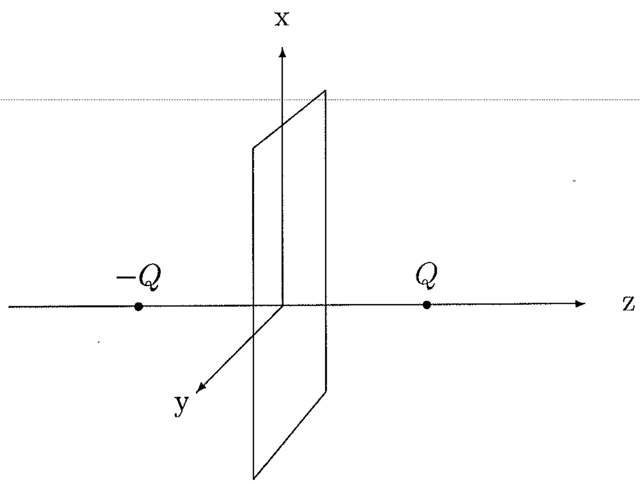
$E_y' = \gamma E_y$

$B_y' = \frac{\gamma\beta}{c} E_x \qquad B_z' = 0$

$B_x' = -\frac{\gamma\beta}{c} E_y$

$$\Rightarrow \vec{E} = \frac{\gamma\lambda}{2\pi\epsilon_0\sqrt{x^2+y^2}} \langle \cos\varphi\,\hat{x}, \sin\varphi\,\hat{y}, 0 \rangle$$

$$\vec{B} = \frac{\gamma\beta\lambda}{c\,2\pi\epsilon_0\sqrt{x^2+y^2}} \langle \sin\varphi\,\hat{x}, -\cos\varphi\,\hat{y}, 0 \rangle$$



6. This problem requires the use of Maxwell's stress tensor T_M^{ij}.

   (a) [3 pts] Compute Maxwell's stress tensor T_M^{ij} on the $z = 0$ plane for a system of two equal and opposite point charges ($\pm Q$) located on the z-axis at $\mathbf{r} = (0, 0, \pm b)$ as shown in the figure. For this application

   $$T_M^{ij}|_{\text{Gaussian}} = \left(\frac{1}{4\pi}\right)\left[E^i E^j - \frac{\delta^{ij}}{2}E^2\right],$$

   or

   $$T_M^{ij}|_{\text{SI}} = (\epsilon_0)\left[E^i E^j - \frac{\delta^{ij}}{2}E^2\right].$$

   (b) [4 pts] Evaluate the surface integral

   $$\int\!\!\int T_M^{ij}\, dA^j = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_M^{iz}\, dx\, dy,$$

   over the $z = 0$ plane.
   Hint: Use cylindrical polar coordinates to do the integral.

   (c) [3 pts] The following surface integral over the boundary of a closed volume V_3 is the total electromagnetic force on the E&M fields and their sources contained within that volume

   $$F^i = \int_{\partial V_3} T_M^{ij}\, dA^j.$$

   Use this fact to explain your answer to part (b).

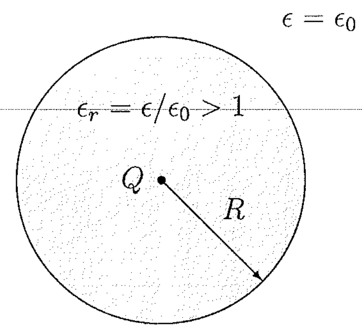
E & M Qualifier

January 9, 2014

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias **(NOT YOUR REAL NAME)** on every page,
4. supply 3 numbers for every page (Problem: #, Page: #, of: #). If you took 4 pages to work the 5^{th} problem then the 2^{nd} page would be (Problem: 5, Page: 2, of: 4)
5. start each problem by stating your units e.g., SI or Gaussian,
6. **do not** staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



1. A point charge Q, is embedded at the center of a sphere of linear dielectric material with electric susceptibility χ_e, and radius R. If the sphere is centered on the origin, calculate the following:

   (a) [2 pts] The magnitude and direction of the electric displacement $\mathbf{D}$ inside and outside the sphere.

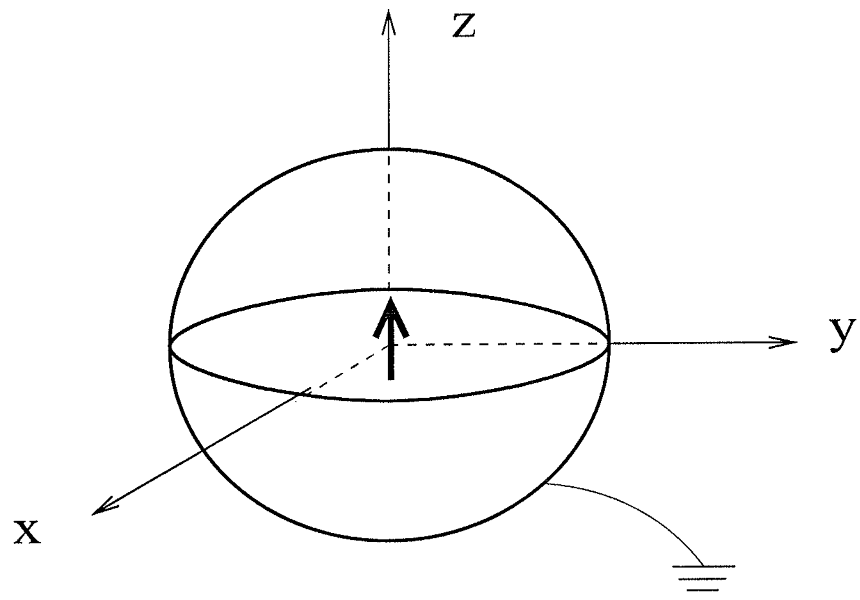
   (b) [2 pts] The magnitude and direction of the electric field $\mathbf{E}$ inside and outside of the sphere.

   (c) [2 pts] The magnitude and direction of the electric polarization $\mathbf{P}$ inside and outside the sphere.

   (d) [2 pts] The bound surface charge density σ_b on the sphere and the bound volume charge density ρ_b inside the sphere.

   (e) [2 pts] The total bound charge on the sphere's surface and the total bound charge inside the sphere.

2. A point electric dipole with dipole moment $\mathbf{p} = p_0\,\hat{\mathbf{z}}$ is located at the center of a thin hollow, grounded, conducting sphere of radius R.



(a) [2 pts] For general boundary conditions like those given in the figure, i.e., sources are located at the origin and the potential is given at $r = R$, a solution to the Laplace equation in spherical-polar coordinates with azimuthal symmetry ($\nabla^2\Phi(r,\theta) = 0$ with no ϕ dependence) can be written as a sum of Legendre polynomials

$$\Phi(r,\theta) = \sum_{\ell} f_\ell(r) P_\ell(\cos\theta).$$

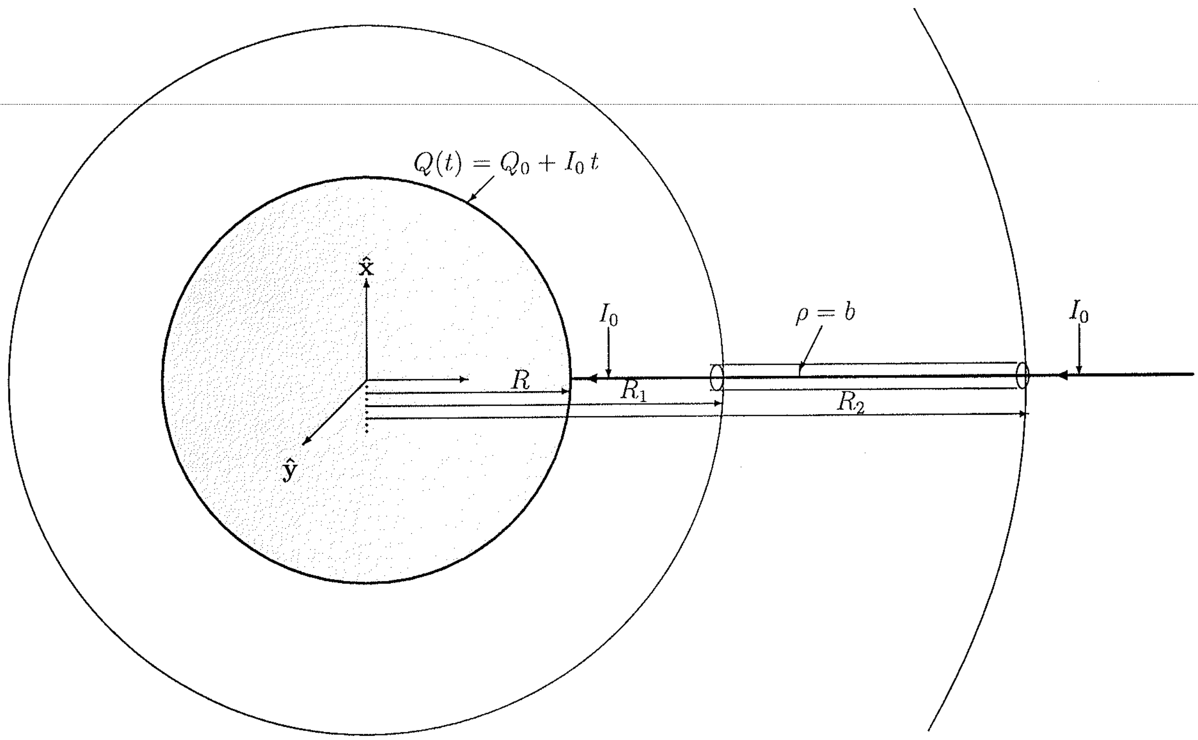
Give the r dependence of each function $f_\ell(r)$.

(b) [2 pts] For the explicit boundary conditions shown in the figure what are the limiting values of the electrostatic potential at $r = R$ and at $r \sim 0$?

(c) [2 pts] What constraints do the boundary conditions in (b) place on the functions $f_\ell(r)$?

(d) [2 pts] Give the electrostatic potential inside the sphere.

(e) [2 pts] Compute the charge density σ on the inside surface on the grounded sphere.



3. A very thin straight wire (on the z-axis) carries a constant current I_0 from infinity radially inward to a spherical conducting shell of radius R to which the wire is attached. Assume the time dependent charge on the surface of the shell is uniformly distributed and can be approximated by $Q(t) = Q_0 + I_0 t$. This time dependent charge causes the electric field in the space outside the sphere to increase and hence the energy density stored in electric field to increase. Neglect retardation effects when answering the following.

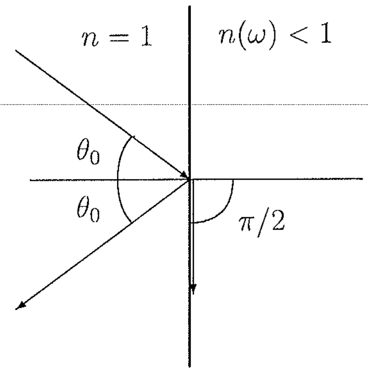
   (a) [2 pts] Calculate the electric field $\mathbf{E}(t, r)$ as a function of distance for $r > R$ from the center of the spherical shell due to $Q(t)$.

   (b) [2 pts] Calculate the energy $U(t)$ stored in the electric field in the region $R_1 \le r \le R_2$ where $R < R_1$.

   (c) [2 pts] Calculate the magnetic field $\mathbf{B}(r, \rho)$ caused by I_0 at the surface of a thin cylinder $(\rho = b \ll R)$ that surrounds the wire on the z-axis $(z > R)$.

   (d) [2 pts] Use your $\mathbf{E}(t, r)$ and $\mathbf{B}(r, \rho)$ fields to calculate the Poynting vector on the surface of the cylinder. Assume the cylinder is so small in diameter that the electric field is approximately tangent to the cylinder's surface.

   (e) [2 pts] Use the Poynting vector to show that the rate energy leaves the part of the wire between $R_1 \le r \le R_2$ equals the rate of change of the energy stored in the electric field calculated in (b).



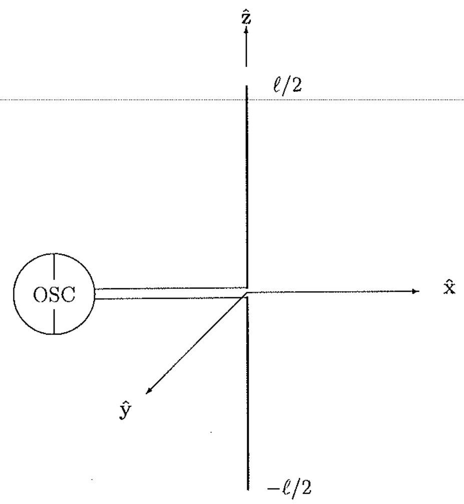
4. X-rays of a fixed angular frequency ω which are incident on a metallic surface at an angle θ (relative to the normal) that is greater than a critical angle $\theta_0(\omega)$ are totally reflected.

   (a) [2 pts] Assume you know the index of refraction of the metal is $n(\omega)$ and that it's value is less than 1. Use Snell's law to calculate the critical angle $\theta_0(\omega)$.

   (b) [3 pts] Assume the electrons in the metal behave as if they are as completely free as electrons in a plasma. As the wave penetrates into the metal each free electron is accelerated by the x-ray's time dependent electric field (a wave $\propto E_0 e^{-i\omega t}$). Calculate the electron's steady state motion as a function of time in response to x-ray's electric field.

   (c) [2 pts] Knowing the steady steady state motion of the electrons from (b) calculate each electron's contribution to the polarization density as a function of the electric field.

   (d) [3 pts] Assuming the metal contains n_e free electrons per unit volume, you can now calculate the index of refraction $n(\omega)$ caused by the free electrons. Assume that $\mu = \mu_0$ and that only the electrons are contributing to ϵ.



5. Consider a half-wave antenna (length $\ell = \lambda/2$, see the figure) centered on the origin and aligned with the z-axis . The antenna is driven by an alternating signal ($\omega/k = \nu\lambda = c$)applied to its center which produces a current in the antenna given by

$$\begin{aligned} I(z,t) &= I_0 \cos(kz)\sin(\omega t), |z| \leq \ell/2 \\ &= 0, |z| > \ell/2, \end{aligned}$$

(a) [2 pts] Give a 1-dimensional integral expression for the retarded vector potential $\mathbf{A}(t, \mathbf{r}) = A^z(t, r, \theta)\ \hat{\mathbf{z}}$ for this antenna using

$$\mathbf{A}(t,\mathbf{r}) = \left(\frac{\mu_0}{4\pi}\right) \int \frac{\mathbf{J}(t - |\mathbf{r} - \mathbf{r}'|/c, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r', \qquad \left(\frac{\mu_0}{4\pi}\right) \longrightarrow \frac{1}{c} \text{ in Gaussian units.}$$

(b) [3 pts] Evaluate your integral from (a) assuming $|\mathbf{r}| \gg \ell$,i.e., assume

$$|\mathbf{r} - \mathbf{r}'| = r - z'\cos\theta + \mathcal{O}\left(\frac{1}{r}\right), \text{ and } \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right).$$

You only need to keep these terms if you want to find the radiation part of $\mathbf{B}$. The form of the integral you will need to evaluate is

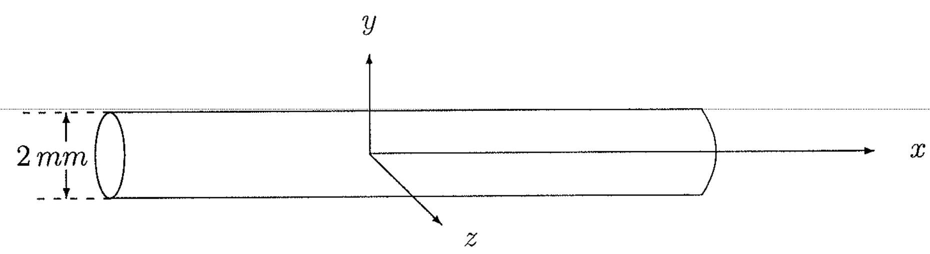
$$\int_{-\pi/2}^{\pi/2} \cos(x)\sin(a + bx) = \frac{2\cos(\frac{\pi}{2}b)\sin(a)}{1 - b^2}.$$

(c) [3 pts] Calculate the radiation part of $\mathbf{B}$ using $\mathbf{B} = \nabla \times \mathbf{A} = (\nabla A^z) \times \hat{z}$ and the radiation part of $\mathbf{E}$ using

$$\mathbf{E}_{\text{rad}} = (c)\mathbf{B}_{\text{rad}} \times \hat{\mathbf{r}}, \qquad (c) \longrightarrow 1 \text{ in Gaussian units.}$$

Remember that the radiation parts are those that are $\propto 1/r$ for large $r \gg \ell$. Also recall that $\nabla\theta \propto 1/r$ and $\nabla(1/r) \propto 1/r^2$

(d) [2 pts] Calculate the time average of the Poynting vector $\langle \mathbf{S} \rangle$ as a function of (r, θ) for $r \gg \ell$ and plot (sketch) $|\langle \mathbf{S} \rangle|$ as a function of θ with $r =$ constant.



6. A green laser pointer (wavelength = 550 nano-meters) has a power of 50 milliwatts with a 2 millimeter beam diameter. Assume the beam can be represented by a plane electromagnetic wave with fields

$$\begin{aligned} E^y &= E_0 \sin(kx - \omega t), \\ B^z &= B_0 \sin(kx - \omega t), \end{aligned}$$

confined to a cylinder of diameter 2 mm. As with any wave the wave number k is related to wavelength λ by $k = 2\pi/\lambda$ and ω is related to the frequency f by $\omega = 2\pi f$.

(a) [2 pts] If the beam travels in vacuum use Maxwell's equations to find the numerical value of f and to relate B_0 to E_0.

(b) [3 pts] Compute the Poynting vector and the energy density at time t and position $\mathbf{r}$ for the above wave as a function of E_0.

(c) [2 pts] What is the time average of the above two quantities at a position within the beam?

(d) [3 pts] Use one or both of the above time-averages and the known 50 milliwatt power of the laser to determine the amplitudes E_0 and B_0.

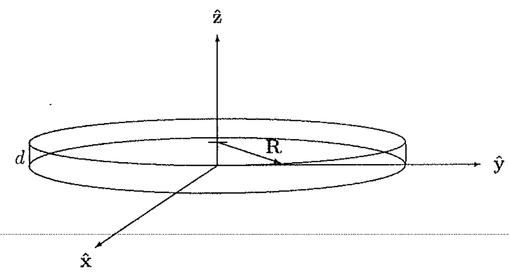
E & M Qualifier

August 15, 2014

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias **(NOT YOUR REAL NAME)** on every page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. number every page as follows
   (a) put the problem number on every page you hand in for that problem,
   (b) starting numbering each problem with page 1,
   (c) when you finish a problem put the total number of pages you used for that problem on every page you hand in for that problem.
6. **DO NOT** staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



1. A large flat thin disk of linear magnetic material of thickness d and radius $R \gg d$ which has magnetic permeability μ is placed in a uniform magnetic field $\mathbf{H} = H_0\hat{\mathbf{z}}$ as shown in the figure. The bottom of the slab is in the x-y plane at $z = 0$ and the top is at $z = d$. Assume the source of the uniform magnetic field is far away and assume the slab is infinite ($R \to \infty$) in the x-y directions. In addition to possessing a linear magnetic susceptibility χ_m related to the materials permeability, the slab also possesses a **uniform permanent magnetization** $M_0\hat{\mathbf{z}}$, producing a total magnetization density

$$\mathbf{M} = \chi_m \mathbf{H} + M_0\hat{\mathbf{z}} \qquad \text{where} \qquad \chi_m^{SI} = 4\pi\chi_m^G.$$

Recall that in SI (mks) and Gaussian (cgs) units

$$\mathbf{B}^{SI} = \mu_0(\mathbf{H}^{SI} + \mathbf{M}^{SI}), \qquad \mathbf{B}^G = \mathbf{H}^G + 4\pi\mathbf{M}^G.$$

(a) [1 pts] In this problem you are to write the magnetic field $\mathbf{H}$ as the gradient of a scalar potential

$$\mathbf{H} = -\boldsymbol{\nabla}\Phi_M.$$

Explain why you can do this.

(b) [3 pts] What is the form of the Poisson equation satisfied by Φ_M inside and outside the slab, i.e.,

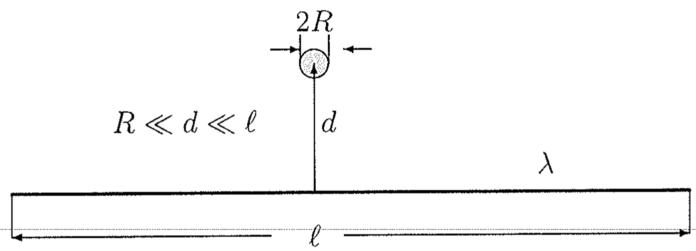
$$\nabla^2\Phi_M = ?$$

Solve this equation for the 3 spatial regions separated by $z \neq 0$ and $z \neq d$. Observe that there is no x or y dependence in this problem. Make sure your Φ_M far above and below the slab produces the uniform magnetic field $\mathbf{H} = H_0\hat{\mathbf{z}}$.

(c) [2 pts] What general boundary conditions are satisfied by $\mathbf{H}$ and $\mathbf{B}$ at the two junctions $z = 0$ and $z = d$. What conditions are placed on Φ_M and its z-derivative by these junction conditions for this particular problem?

(d) [2 pts] Use your solutions from (b) and boundary conditions from (c) to find Φ_M inside and outside the slab.

(e) [2 pts] Calculate $\mathbf{H}$ and $\mathbf{B}$ inside and outside the slab.



2. Consider a tiny sphere of radius R, composed of a linear dielectric material of susceptibility χ_e and permittivity ϵ which is a distance d from a thin but very long ($R << d << \ell$) wire possessing a uniform line charge per unit length λ. Recall that

$$\mathbf{D} = \epsilon \mathbf{E} \quad \text{where} \quad \epsilon^G = \epsilon^{SI}/\epsilon_0 = 1 + \chi_e^{SI} = 1 + 4\pi\chi_e^G$$

$$\mathbf{P} = \chi_e \mathbf{E} \quad \text{where} \quad \chi_e^{SI} = 4\pi\chi_e^G$$

$$\mathbf{D}^{SI} = \epsilon_0(\mathbf{E}^{SI} + \mathbf{P}^{SI}), \qquad \mathbf{D}^G = \mathbf{E}^G + 4\pi\mathbf{P}^G$$

The electrostatic potential for a point dipole at the origin is

$$\Phi^{SI} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3},$$

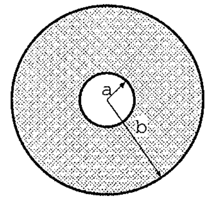
$$\Phi^G = \frac{\mathbf{p}\cdot\mathbf{r}}{r^3},$$

(a) [2 pts] Calculate the magnitude of the electric field E_{wire} **at the center of the sphere** caused by the charge on the wire.

(b) [2 pts] As an approximation, assume the **dielectric sphere** is centered at the origin in a uniform electric field of the form $E_{wire}\,\hat{\mathbf{x}}$. The polarization charge induced on the sphere's surface produces an electric dipole field $\mathbf{E}_{dipole}$ outside the sphere and makes a uniform contribution to the net uniform field $E_0\,\hat{\mathbf{x}}$ that exists inside the sphere. Give an expression for the electric dipole field $\mathbf{E}_{dipole}$ as a function of the sphere's uniform polarization density $\mathbf{P}$ if the dipole is oriented in the $\hat{\mathbf{x}}$ direction, i.e., if $\mathbf{p} = p_0\,\hat{\mathbf{x}} = 4/3\,\pi R^3\,\mathbf{P}$.

(c) [3 pts] What boundary conditions must $\mathbf{E}$ and $\mathbf{D}$ satisfy at the sphere's surface? Use these boundary conditions to calculate the net electric dipole moment $p_0\hat{\mathbf{x}}$ of the sphere?

(d) [3 pts] Compute the force exerted on the sphere by the wire by computing the force on a point dipole in the non-uniform electric field caused by the wire. Is the sphere attracted or repelled by the charged wire?



3. Consider two concentric conducting spherical shells of radii a and b with $b > a$. The space between the two shells is a filled with Ohmic material of constant conductivity σ, permittivity ϵ_0, and permuability μ_0. The system is charged such that at time $t = 0$ the inner conductor has charge $+Q_0$ and the outer conductor has charge $-Q_0$. At times $t > 0$ the charge will flow from the inner shell to the outer shell.

   (a) [2 pts] Use Gauss's law to relate the electric field $\mathbf{E}(t, \mathbf{r})$ between the plates to the charge $Q(t)$ on the inner plate.

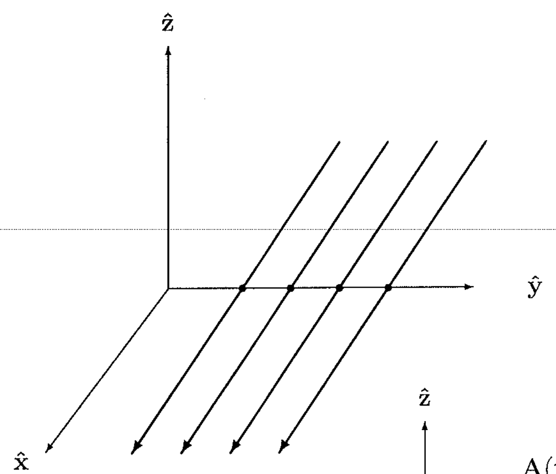
   (b) [4 pts] Use the conservation of charge and

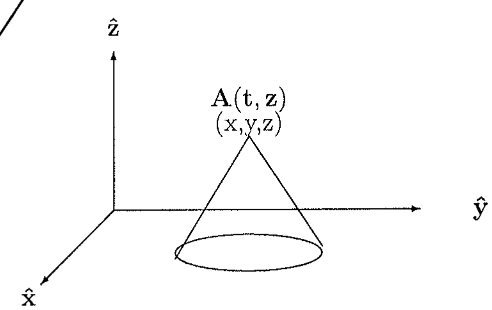
   $$\mathbf{J}(t, \mathbf{r}) = \sigma \, \mathbf{E}(t, \mathbf{r}),$$

   to find $Q(t)$.

   (c) [2 pts] Use Faraday's law and your electric field to show that $\mathbf{B}(t, \mathbf{r}) = 0$.

   (d) [2 pts] Confirm that Ampère's law is satisfied.





4. A uniform sheet of current in the (x, y) plane at $z = 0$ suddenly turns on at $t = 0$ and has a surface current density

$$\begin{aligned} \mathbf{K}(t, \mathbf{r}) &= 0, \qquad t < 0, \\ \mathbf{K}(t, \mathbf{r}) &= K_0\,\hat{\mathbf{x}}, \qquad t \geq 0, \end{aligned} \tag{1}$$

where K_0 has units of current/length. The corresponding volume current density is

$$\mathbf{J}(t, \mathbf{r}) = \mathbf{K}(t, \mathbf{r})\,\delta(z).$$

The retarded vector potential in SI units and in the Lorentz gauge for an arbitrary current source can found by integrating

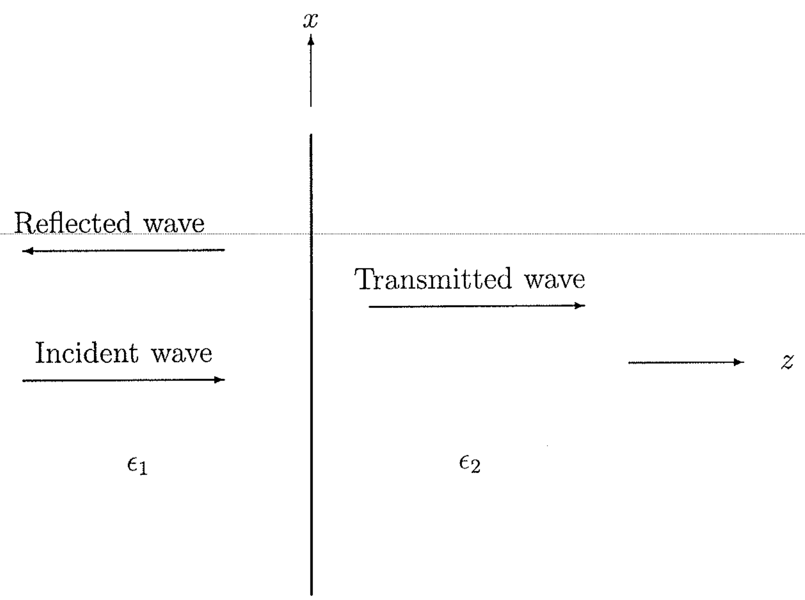
$$\mathbf{A}(t, \mathbf{r}) = \left(\frac{\mu_0}{4\pi}\right) \int \frac{\mathbf{J}(t - |\mathbf{r} - \mathbf{r}'|/c, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'.$$

In Gaussian units the factor $\mu_0/4\pi$ is replaced by $1/c$.

(a) [4 pts] In cylindrical polar coordinates evaluate 2 of the 3 integrals in the above expression for $\mathbf{A}(t, \mathbf{r})$, i.e., integrate over z' and ϕ' leaving $\mathbf{A}(t, \mathbf{r})$ as an integral over the single coordinate ρ'.

(b) [3 pts] Evaluate the ρ' integral giving $\mathbf{A}(t, \mathbf{r})$ as a function of t and z only.

(c) [3 pts] Compute the magnetic induction from your vector potential.



5. A linearly-polarized harmonic ($e^{-i\omega t}$) plane electromagnetic wave traveling to the right in a homogeneous dielectric medium described by a real dielectric constant ϵ_1, strikes a second homogeneous dielectric material described by another real dielectric constant $\epsilon_2 > \epsilon_1$ (see the figure). Assume that both materials have no magnetic susceptibility, $\chi_m = 0$, and that the incidence angle is 0^o (i.e., the wave is traveling perpendicular to the junction). Assume the incoming wave is polarized in the $\hat{x}$ direction and that its electric field amplitude is E_0, i.e., assume the incoming electric field is the real part of

$$\mathbf{E} = E_0\, e^{i(kz - wt)}\, \hat{x}.$$

(a) [2 pts] Give the direction of the magnetic induction $\mathbf{B}$ associated with the above incoming wave and give its amplitude B_0 as a function of E_0. Also give k as a function of ω.

(b) [2 pts] Give similar expressions for $\mathbf{E}$ and $\mathbf{B}$ of the reflected and transmitted waves. Use E_0'' and E_0' for the respective electric field amplitudes of the reflected and transmitted waves.

(c) [3 pts] Apply the boundary conditions at the junction/interface between the dielectrics to the incoming, reflected, and transmitted wave to compute E_0'' and E_0' as functions of E_0 and the two dielectric constants ϵ_1 and ϵ_2.

(d) [3 pts] Evaluate the reflection and transmission coefficients, R and T, for above waves. Recall that R and T are computed from ratios of time averaged Poynting vectors which are defined by

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} \equiv \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

Aug 2014 $\quad$ E+M #5 $\quad$ Gaussian

a) *To find relationship b/w k and ω, we derive wave equation

$$\cancel{\nabla \times \left(\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} \right) = 0}$$

$$\nabla \times (\overset{0}{\nabla \cdot \vec{E}}) - \nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times B) = 0$$

$$-\nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mu H) = 0$$

$$-\nabla^2 E + \frac{\mu}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial D}{\partial t}\right) = 0$$

$$-\nabla^2 E + \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2} E = 0$$

*Given $\vec{E} = E_0 \exp[i(kz - \omega t)]\hat{x}$

$$k^2 E_0 \exp[i(kz-\omega t)] - \frac{\mu\epsilon}{c^2}\omega^2 E_0 \exp[i(kz-\omega t)] = 0$$

$$k^2 - \frac{\mu\epsilon}{c^2}\omega^2 = 0$$

$$\hookrightarrow k = \frac{\sqrt{\mu\epsilon}}{c}\omega$$

$$\vec{B} = c\int \nabla \times E \, dt$$

$$\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0 & 0 & 0 \end{vmatrix} = \langle 0, \partial_z E_0, 0 \rangle = ikE_0 \exp[ikz - \omega t]$$

$$\vec{B} = \int ikc E_0 \exp[i(kz - \omega t)]\,dt\,\hat{y}$$

$$= \frac{+kc}{\omega} E_0 \exp[i(kz - \omega t)]\hat{y}$$

b) * For the reflected wave:

$$\vec{E}_r = E_0'' \exp[i(-k''z - \omega t)]\hat{x} \qquad \vec{B}_r = -\frac{k''c}{\omega} E_0'' \exp[i(-kz - \omega t)]\hat{y}$$

* For the transmitted wave:

$$\vec{E}_T = E_0' \exp[i(k'z - \omega t)]\hat{x} \qquad \vec{B}_T = \frac{k'c}{\omega} E_0' \exp[i(k'z - \omega t)]\hat{y}$$

#5 (cont.)

c) * In general, our boundary conditions are:

① $D_1^{\perp} - D_2^{\perp} = 4\pi \sigma_f$ ③ $B_1^{\perp} - B_2^{\perp} = 0$

② $E_1^{\parallel} - E_2^{\parallel} = 0$ ④ $H_1^{\parallel} - H_2^{\parallel} = \frac{4\pi}{c} \vec{K}_f$

* Using conditions ① and ④ we generate the following system of equations

① $E_0 + E_0'' = E_0'$

② $\frac{ck}{\omega} E_0 - \frac{ck}{\omega} E_0'' = \frac{ck'}{\omega} E_0'$

$* \; k = \frac{\sqrt{\epsilon}}{c} \omega$

$$\sqrt{\epsilon_1} E_0 - \sqrt{\epsilon_1} E_0'' = \sqrt{\epsilon_2} E_0'$$

$$\sqrt{\epsilon_1} E_0 - \sqrt{\epsilon_1}(E_0' - E_0) = \sqrt{\epsilon_2} E_0'$$

$$2\sqrt{\epsilon_1} E_0 - \sqrt{\epsilon_1} E_0' = \sqrt{\epsilon_2} E_0'$$

$$2\sqrt{\epsilon_1} E_0 = (\sqrt{\epsilon_1} + \sqrt{\epsilon_2}) E_0' \Rightarrow E_0' = \frac{2\sqrt{\epsilon_1} E_0}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}}$$

$$\Rightarrow E_0'' = E_0' - E_0$$

$$= E_0 \left(\frac{2\sqrt{\epsilon_1}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} - 1 \right)$$

d) $R = \frac{\langle |S''| \rangle}{\langle |S| \rangle}$ $T = \frac{\langle |S'| \rangle}{\langle |S'| \rangle}$

* in both cases E, H are scaled by factors found in part c, therefore R, T are those ratio's squared

$R = \left(\frac{2\sqrt{\epsilon_1}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} - 1 \right)^2$ $T = \left(\frac{2\sqrt{\epsilon_1}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} \right)^2$

6. In the lab you measure a uniform electric field and a uniform magnetic induction

$$\mathbf{E} = E_0(\cos 45^\circ\,\hat{\mathbf{x}} + \sin 45^\circ\,\hat{\mathbf{y}}),$$

$$\mathbf{B} = B_0\,\hat{\mathbf{x}},$$

where $B_0 = E_0$ in Gaussian units or $B_0 = E_0/c$ in SI units. The goal of this problem is to compute the $\mathbf{E}'$ and $\mathbf{B}'$ fields an observer sees if moving relative to the lab with a velocity $\mathbf{v} = v_0\,\hat{\mathbf{z}}$.

(a) [2 pts] Combine $\mathbf{E}$ and $\mathbf{B}$ into a single 4×4 anti-symmetric electromagnetic field tensor $F^{\alpha\beta}$.

(b) [2 pts] Give the 4×4 Lorentz boost L^α_β that transforms the lab coordinates (ct, x, y, z) into the moving frame's coordinates (ct', x', y', z') i.e., $x'^\alpha = L^\alpha_\beta x^\beta$ where $x^\beta = (ct, x, y, z)$. In matrix notation $x' = L\,x$.

(c) [3 pts] Find $\mathbf{E}'$ and $\mathbf{B}'$ by by boosting the F tensor, i.e., compute $F'^{\alpha\beta} = L^\alpha_\sigma L^\beta_\lambda F^{\sigma\lambda}$ which in matrix notation is $F' = LFL^\top$

(d) [3 pts] For what value of v_0 will $\mathbf{E}'$ and $\mathbf{B}'$ be parallel?

Aug 2014 $\underline{E+M \# 6}$ Gaussian

a) $F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & +B_x & 0 \end{bmatrix}$

$\vec{E} = \langle E_0 \cos(45°), E_0 \sin(45°), 0 \rangle$

$\vec{B} = \langle B_0, 0, 0 \rangle$

$= \begin{bmatrix} 0 & -E_0\frac{\sqrt{2}}{2} & -E_0\frac{\sqrt{2}}{2} & 0 \\ E_0\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ E_0\frac{\sqrt{2}}{2} & 0 & 0 & -B_0 \\ 0 & 0 & B_0 & 0 \end{bmatrix}$

b) $L^{\hat{\alpha}}_{\beta} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix}$

$x'^{\alpha} = \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$

$= \begin{bmatrix} \gamma ct - \gamma\beta z \\ x \\ y \\ -\gamma\beta ct + \gamma z \end{bmatrix}$

c) $F' = L F L^T$

$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -E_0/\sqrt{2} & -E_0/\sqrt{2} & 0 \\ E_0/\sqrt{2} & 0 & 0 & 0 \\ E_0/\sqrt{2} & 0 & 0 & -B_0 \\ 0 & 0 & B_0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$

#6 (cont.)

c) $$F' = \begin{bmatrix} 0 & -\gamma E_0/\sqrt{2} & -\gamma E_0/\sqrt{2} - \gamma\beta B_0 & 0 \\ E_0/\sqrt{2} & 0 & 0 & 0 \\ E_0/\sqrt{2} & 0 & 0 & -B_0 \\ 0 & \beta\gamma E_0/\sqrt{2} & \beta\gamma E_0/\sqrt{2} + \gamma B_0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma E_0/\sqrt{2} & -\gamma E_0/\sqrt{2} - \gamma\beta B_0 & 0 \\ \gamma E_0/\sqrt{2} & 0 & 0 & -\beta\gamma E_0/\sqrt{2} \\ \gamma E_0/\sqrt{2} + \gamma\beta B_0 & 0 & 0 & -\beta\gamma E_0/\sqrt{2} \cdot \gamma B_0 \\ 0 & \beta\gamma E_0/\sqrt{2} & \beta\gamma E_0/\sqrt{2} + \gamma B_0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{E} = \langle \gamma E_0/\sqrt{2},\ \gamma E_0/\sqrt{2} + \gamma\beta B_0,\ 0 \rangle$$

$$\vec{B} = \langle \beta\gamma E_0/\sqrt{2} + \gamma B_0,\ \beta\gamma E_0/\sqrt{2},\ 0 \rangle$$

d) * Because we are in Gaussian units, $E_0 = B_0$

$$\Rightarrow \vec{E} = E_0 \langle \gamma/\sqrt{2},\ \frac{\gamma}{\sqrt{2}} + \gamma\beta,\ 0 \rangle$$

$$\vec{B} = E_0 \langle \beta\gamma/\sqrt{2} + \gamma,\ \beta\gamma/\sqrt{2},\ 0 \rangle$$

$$\frac{\beta\gamma}{\sqrt{2}} + \gamma = \frac{\beta\gamma}{\sqrt{2}}$$

$$\frac{\beta}{\sqrt{2}} + 1 = \frac{\beta}{\sqrt{2}}$$

$$\beta + \sqrt{2} =$$

E & M Qualifier

January 7, 2015

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

1. (a) [2 pts] Write down all four of Maxwell's equations in differential form for the four fields $\mathbf{E}, \mathbf{H}, \mathbf{D}$, and $\mathbf{B}$. Which two are homogeneous equations and which two are non-homogeneous equations? Which one is called Faraday's law of induction, which one is called Ampère's law, and which one is equivalent to Gauss's law?

   (b) [1 pts] Maxwell's equations simplify for fields defined in static, nonconducting, homogeneous, isotropic, and linear materials. Eliminate $\mathbf{H}$ and $\mathbf{D}$ from two of the four equations and simplify these two equations by assuming $\mathbf{B} = \mu\mathbf{H}$, and $\mathbf{D} = \epsilon\mathbf{E}$, where the permittivity ϵ and permeability μ are both real positive constants. (Do not assume the free charge density or the free current density vanishes.)

   (c) [2 pts] Using your Maxwell equations from part (a) explain exactly why $\mathbf{E}$ and $\mathbf{B}$ can be replaced by potentials ϕ and $\mathbf{A}$. What freedom (non-uniqueness) exists in ϕ and $\mathbf{A}$ for a given pair of fields $\mathbf{E}$ and $\mathbf{B}$?

   (d) [2 pts] Replace $\mathbf{E}$ and $\mathbf{B}$ by ϕ and $\mathbf{A}$ in your four Maxwell equations of part (b) and explain how it is possible to solve them for ϕ and $\mathbf{A}$ if these quantities are not unique?

   (e) [3 pts]Given charge and current densities $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ bounded in space (i.e., contained entirely in $r < R$) and static before some early time $t = t_0$, simplify your Maxwell equations from part (d) by using the Coulomb ($\nabla \cdot \mathbf{A} = 0$) gauge constraint. Give the retarded solution for ϕ and $\mathbf{A}$ to your Maxwell equations as 3-d spatial integrals.

2. Consider a capacitor composed of two thin concentric spherical metal shells, the inner one with radius a and the outer one with radius b. The region between the spherical metal shells is filled with a linear dielectric with permittivity $\epsilon = k/r^2$. A charge $+Q$ exists on the inner metallic shell and $-Q$ on the outer metallic shell.

   (a) [2 pts] Find the electric displacement $\mathbf{D}$ everywhere in space.

   (b) [3 pts] Find the capacitance of the configuration.

   (c) [5 pts] Calculate the bound charge densities within the dielectric and on its surfaces, and verify that the total net bound charge is zero.

3. In this problem you will construct the 4-dimensional (4-d) electromagnetic stress-energy-momentum tensor from the 4-d electromagnetic field tensor $F^{\mu\nu}$ in **Gaussian** units. Recall that $F^{\mu\nu}$ is antisymmetric ($F^{\mu\nu} = -F^{\nu\mu}$) and is constructed from components of the electric and magnetic induction fields $\mathbf{E}$ and $\mathbf{B}$ by choosing

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon^{ijk}B^k.$$

Here we use the Einstein convention of summing over repeated indices, where Greek letters run from 0 to 3, while Latin letters run from 1 to 3. The symbol ϵ^{ijk} is the totally anti-symmetric 3-dimensional Levi-Civita symbol and satisfies $\epsilon^{123} = +1$. The time coordinate is given by $x^0 = ct$, where c is the speed of light and the 4-d metric used to raise and lower Greek indices is $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

(a) [3 pts] Define the 4-current J^μ and show that in a region containing no polarizable materials ($\epsilon = \mu = 1$) Maxwell equations are written in 4-d form as

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.$$

(b) [1 pts] From Maxwells equations prove that charge is conserved, i.e., show that

$$\partial_\mu J^\mu = 0.$$

(c) [3 pts] The 4-d stress-energy-momentum tensor is a traceless symmetric second-rank tensor, quadratic in the field strengths defined by

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F_\lambda{}^\nu - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\beta\alpha}\right].$$

Show that the 4 parts of $T^{\mu\nu}$ can be identified with the electromagnetic energy density u by $T^{00} = u$, the momentum density $\mathbf{g}$ and the Poynting vector $\mathbf{S}$ by $T^{0i} = T^{i0} = cg^i = S^i/c$, and the 3-d Maxwell stress tensor $\overleftrightarrow{\mathbf{T}}_M$ by $T^{ij} = -T^{ij}_M$. Be sure to give u, $\mathbf{g} = \mathbf{S}/c^2$, and $\overleftrightarrow{\mathbf{T}}_M$ as functions of $\mathbf{E}$ and $\mathbf{B}$.

(d) [3 pts] Use Maxwell's equations to compute $\partial_\mu T^{\mu\nu}$. Show that $\partial_\mu T^{\mu\nu} = 0$ in a region where $J^\mu = 0$ and that this one 4-d vector equation is equivalent to the local conservation of electromagnetic energy and momentum in 3-d, i.e., that

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = 0,$$

and

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \overleftrightarrow{\mathbf{T}}_M.$$

Jan 2015

E+M #3

a) We define the 4-current J^μ as:

$$J^\mu = \langle c\rho, \vec{j} \rangle$$

* To prove $\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu$, we consider two cases:

$\mu = 0$:

$$\partial_\nu F^{\nu 0} = \frac{4\pi}{c} J^0$$

$$\partial_0 F^{00} + \partial_i F^{i0} = \frac{4\pi}{c} c\rho$$

$$0 + \nabla \cdot E = 4\pi\rho \checkmark$$

$\mu = i$:

$$\partial_0 F^{0i} + \partial_j F^{ij} = \frac{4\pi}{c} j^i$$

$$-\frac{1}{c}\frac{\partial}{\partial t}(\vec{E}) - \partial_j \epsilon_{ijk} B_k = \frac{4\pi}{c} j^i$$

$$-\frac{1}{c}\frac{\partial E_i}{\partial t} + \epsilon_{jik}\frac{\partial}{\partial x^i} B_k = \frac{4\pi}{c} j^i$$

$$-\frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} \checkmark$$

* To prove $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$, we again consider two cases

$\lambda, \mu, \nu \in \{1, 2, 3\}$:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

$$\frac{\partial}{\partial x}(-B_x) + \frac{\partial}{\partial y}(-B_y) + \frac{\partial}{\partial z}(-B_z) = 0$$

$$\nabla \cdot B = 0 \checkmark$$

$\lambda, \mu, \nu \in \{0, i, j\}$:
$i \neq j$

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0$$

$$\partial_0(-\epsilon_{ijk} B_k) - \partial_i(E_j) + \partial_j(E_i)$$

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \nabla \times E = 0 \checkmark$$

#3 (cont.)

b) The equation of charge conservation says $\partial_\mu J^\mu = 0$

$$\partial_\mu J^\mu = \partial_\mu(\partial_\nu F^{\nu\mu})$$

$$= \partial_\nu \partial_\mu F^{\nu\mu} \quad \text{(order of partial differentiation doesn't matter)}$$

$$= -\partial_\nu \partial_\mu F^{\mu\nu} \quad \text{(anti-symmetric exchange} \Rightarrow F^{\nu\mu} = -F^{\mu\nu})$$

$\Rightarrow \partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}$ $\rightarrow$ only 0 is equal to its own negative

$\hookrightarrow \partial_\mu J^\mu = 0$ ✓

c) Given $T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F_\lambda{}^\nu - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\beta\alpha}\right]$

The constituent parts of the equation are:

$$F^{\alpha\beta}F_{\beta\alpha} = 2(|E|^2 - |B|^2)$$

$$F^{0\lambda}F_\lambda{}^0 = |E|^2$$

$$F^{0\lambda}F_\lambda{}^i = (-E^j)(-\epsilon_{ijk}B_k)$$

$$= (E \times B)^i$$

$$F^{i\lambda}F_\lambda{}^j = F^{i0}F_0{}^j + F^{ik}F_k{}^j$$

$$= (E_i)(-E_j) + (-\epsilon^{ikm}B^m)(\epsilon^{kjp}B^p)$$

$$= -E_iE_j - (\delta^j_m \delta^i_p - \delta^j_i \delta^p_m)B^mB^p$$

$$= -E_iE_j - B^iB^j + \delta^{ij}|B|^2$$

#3 (cont)

c) $T^{00} = u$

$$= \frac{1}{4\pi}\left[F^{00}F_0^{\ 0} - \frac{1}{4}g^{00}F^{\alpha\beta}F_{\beta\alpha}\right]$$

$$= \frac{1}{4\pi}\left[|E|^2 - \frac{1}{4}\left(2\left[|E|^2 - |B|^2\right]\right)\right]$$

$$= \frac{1}{4\pi}\left[|E|^2 - \frac{1}{2}|E|^2 - \frac{1}{2}|B|^2\right]$$

$$= \frac{1}{8\pi}\left(|E|^2 + |B|^2\right) \qquad \text{(Energy Density)}$$

$T^{0i} = T^{i0} = cg^i = \frac{1}{c}S^i$

$$= \frac{1}{4\pi}\left[F^{0i}F_i^{\ 0} - \frac{1}{4}g^{01}F^{\alpha\beta}F_{\beta\alpha}\right]$$

$$= \frac{1}{4\pi}(E \times B)^i$$

$$\Rightarrow \quad \frac{S^i}{c} = \frac{1}{4\pi}(E \times B)^i$$

$$S^i = \frac{c}{4\pi}(E \times B)^i \qquad \text{(Poynting vector)}$$

$T^{ij} = -T^{ji}$

$$= \frac{1}{4\pi}\left[F^{ij}F_j^{\ i} - \frac{1}{4}g^{ij}F^{\alpha\beta}F_{\beta\alpha}\right]$$

$$= \frac{1}{4\pi}\left[-E_iE_j - B_iB_j + \delta_{ij}|B|^2 - \frac{1}{4}\delta^{ij}\left(2|E|^2 - 2|B|^2\right)\right]$$

$$= -\frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(|E|^2 - |B|^2\right)\right] \qquad \text{(Maxwell Stress Tensor}$$

#3 (cont.)

d) $$\partial_\mu T^{\mu\nu} = \partial_\mu \left[\frac{1}{4\pi} \left(F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\beta\alpha} \right) \right]$$

$$= \frac{1}{4\pi} \left[(\partial_\mu F^{\mu\lambda}) F_\lambda{}^\nu + F^{\mu\lambda} (\partial_\mu F_\lambda{}^\mu) - \frac{1}{4} g^{\mu\nu} [(\partial_\mu F^{\alpha\beta}) F_{\beta\alpha} + F^{\alpha\beta} (\partial_\mu F_{\beta\alpha})] \right])]$$

$$= \frac{1}{4\pi} \left[-J^\lambda F_\lambda{}^\nu + F^{\alpha\beta} (\partial_\alpha F_\beta{}^\gamma) - \frac{1}{4} g^{\mu\nu} [(\partial_\mu F^{\alpha\beta}) F_{\beta\alpha} + F^{\alpha\beta} (\partial_\alpha F_{\beta\alpha})] \right]$$

4. A solution of dextrose, which is optically active, is characterized by a polarization vector $\mathbf{P} = \gamma \nabla \times \mathbf{E}$ where γ is a real constant that depends on the concentration of dextrose. The solution is non-conducting ($\mathbf{J}_{\text{free}} = 0$) and non-magnetic ($\mathbf{M} = 0$). Consider a plane electromagnetic wave of angular frequency ω propagating along the $+z$-axis in such a solution.

   (a) [5 pts] Using Maxwell's equations show that left and right circularly polarized waves travel at 2 distinct speeds ($v_\pm$) in this medium. Calculate the indices of refraction $n_\pm = (ck_\pm)/\omega = c/v_\pm$ as a function of ω and γ for left and right circularly polarized waves. Recall that left ($+$) and right ($-$) circularly polarized waves are of the form

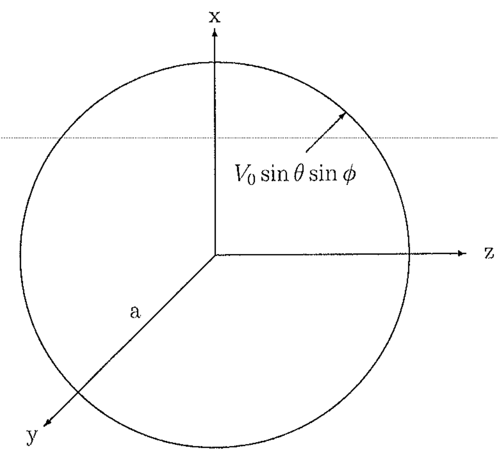
$$\mathbf{E} = E_0 \, (\hat{\mathbf{x}} \pm i \, \hat{\mathbf{y}}) \, e^{i(kz - \omega t)}$$

   (b) [5 pts] Suppose linearly polarized light is incident on the dextrose solution. After traveling a distance L through the solution, the light is still linearly polarized but its direction of polarization rotated by an angle $\Delta\phi$. Calculate $\Delta\phi$ in terms of L, γ, and ω. Hint: Write $k_\pm = \overline{k} \pm \Delta k$ where

$$\overline{k} \equiv \frac{k_+ + k_-}{2} \quad \text{and} \quad \Delta k \equiv \frac{k_+ - k_-}{2}.$$

   Also recall that the amplitude of a wave linearly polarized at an angle ϕ relative to the x-direction can be written as a combination of circularly polarized amplitudes as

$$(\cos\phi \; \hat{\mathbf{x}} + \sin\phi \; \hat{\mathbf{y}}) = e^{i\phi} \left(\frac{\hat{\mathbf{x}} - i \, \hat{\mathbf{y}}}{2} \right) + e^{-i\phi} \left(\frac{\hat{\mathbf{x}} + i \, \hat{\mathbf{y}}}{2} \right).$$



5. Assume that in spherical polar coordinates (r, θ, ϕ), the potential on the surface of a sphere of radius a, centered on the origin, is known to be $V(\theta, \phi)$.

   (a) [2 pts] If the space inside the sphere is empty give an expression for the potential $\Phi(r, \theta, \phi)$ everywhere inside as an expansion in spherical harmonics with arbitrary constants. If you knew the potential $V(\theta, \phi)$ on the surface how would you evaluate the constants in your expansion?

   (b) [2 pts] If the space outside the sphere is empty give an expression for the potential $\Phi(r, \theta, \phi)$ everywhere outside as an expansion in spherical harmonics with arbitrary constants. If you knew the potential $V(\theta, \phi)$ on the surface how would you evaluate the constants in your expansion?

   (c) [6 pts] If $V(\theta, \phi) = V_0 \sin\theta \sin\phi$ give exact expressions for $\Phi(r, \theta, \phi)$ inside and outside the sphere.

   The spherical harmonics are ortho-normal on the sphere and for $\ell = 1$

$$
\begin{aligned}
Y_1^{-1}(\theta, \phi) &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}, \\
Y_1^{0}(\theta, \phi) &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\
Y_1^{1}(\theta, \phi) &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}.
\end{aligned}
$$

Jan 2015

E+M #5

a) The general solution to LaPlace's eqn, expanded in spherical coordinates is:

$$\Phi(r,\theta,\varphi) = \sum_{\ell} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}}\right) Y_{\ell}^{m}(\theta,\varphi)$$

Since we are in empty space inside the sphere, $B_{\ell m}=0$ since $\Phi(0,\theta,\varphi)=0$

$$\hookrightarrow \Phi(r,\theta,\varphi) = \sum_{\ell} \sum_{m=-\ell}^{\ell} A_{\ell m} r^{\ell} Y_{\ell}^{m}(\theta,\varphi)$$

We can then determine the values of $A_{\ell m}$ by the orthogonality of the spherical harmonics:

$$V(\theta,\varphi) = \sum_{\ell} \sum_{m=-\ell}^{\ell} A_{\ell m} a^{\ell} Y_{\ell}^{m}(\theta,\varphi)$$

$$\int_0^{2\pi}\int_0^{\pi} V(\theta,\varphi) Y_{\ell}^{*m}(\theta,\varphi) = A_{\ell m} a^{\ell}$$

$$\hookrightarrow A_{\ell m} = \iint V(\theta,\varphi) Y_{\ell}^{*m}(\theta,\varphi)\, d\Omega \cdot \frac{1}{a^{\ell}}$$

b) We proceed similiarly to part a, except here $A_{\ell m}=0$ b/c $\Phi \to 0$ at $r\to\infty$

$$\hookrightarrow \Phi(r,\theta,\varphi) = \sum_{\ell} \sum_{m=-\ell}^{\ell} B_{\ell m} \frac{1}{r^{\ell+1}} Y_{\ell}^{m}(\theta,\varphi)$$

Again, similior to above, we use orthogonality of spherical harmonics to determine $B_{\ell m}$

$$\hookrightarrow B_{\ell m} = a^{\ell+1} \iint V(\theta,\varphi) Y_{\ell}^{*m}(\theta,\varphi)\, d\Omega$$

c) If we now specify that $\Phi(a,\theta,\varphi) = V_0 \sin\theta \sin\varphi$

* Rewriting Φ in terms of spherical harmonics:

$$\sin\theta\sin\varphi = \sin\theta\left(\frac{1}{2i}e^{i\varphi} - e^{-i\varphi}\right)$$

$$= \sqrt{\frac{2\pi}{3}}\, i\left(Y_1^1 + Y_1^{-1}\right)$$

#5 (cont.)

c) * For $r < a$

$$A_{\ell m} = \iint V(\theta,\varphi) Y_\ell^{*m} \, d\Omega \cdot \frac{1}{a^\ell}$$

$$= \frac{1}{a} V(r) \iint \sqrt{\frac{2\pi}{3}} \, i \left(Y_1^1 + Y_1^{-1} \right) Y_\ell^{*m} \, d\Omega$$

$$= \frac{1}{a} V(r) \sqrt{\frac{2\pi}{3}} \, i \left(Y_1^1 + Y_1^{-1} \right)$$

$$= \frac{1}{a} V(r) \sin\theta \sin\varphi$$

$$\Rightarrow \Phi(r,\theta,\varphi) = \frac{r}{a} V(r) \sin\theta \sin\varphi$$

* For $r \geq a$

$$B_{\ell m} = \iint a^{\ell+1} V(\theta,\varphi) Y_\ell^{*m} \, d\Omega$$

$$= a^2 V(r) \iint \sqrt{\frac{2\pi}{3}} \, i \left(Y_1^1 + Y_1^{-1} \right) Y_\ell^{*m} \, d\Omega$$

$$= a^2 V(r) \sin\theta \sin\varphi$$

$$\Rightarrow \Phi(r,\theta,\varphi) = \left(\frac{a}{r}\right)^2 V(r) \sin\theta \sin\varphi$$

6. In this problem you are to describe properties of waves penetrating into conductors. If the conductor is static, homogeneous, isotropic, linear, and ohmic, you can replace $\mathbf{D}, \mathbf{H}$, and $\mathbf{J}$ in Maxwell's equations using

$$\mathbf{D} = \epsilon \mathbf{E}, \ \mathbf{B} = \mu \mathbf{H}, \ \mathbf{J} = \sigma \mathbf{E},$$

where ϵ, μ, and σ are real positive constants. For simplicity you can also assume the wave is a harmonic plane wave propagating in the z-direction, e.g., its electric field and magnetic induction are of the form

$$\mathbf{E} = E_0 e^{i(kz-\omega t)} \hat{\mathbf{x}},$$

$$\mathbf{B} = B_0 e^{i(kz-\omega t)} \hat{\mathbf{y}}.$$

(a) [4 pts] Use Maxwell's equations to relate k to ω. Explain what the imaginary part of $k = k_{Re} + ik_{Im}$ does to the amplitude of the wave.

(b) [3 pts] Use Maxwell's equations to relate B_0 to E_0. Explain what the phase of $k = |k|e^{i\phi}$ does to the phase of $\mathbf{B}$ as compared to the phase of $\mathbf{E}$.

(c) [3 pts] If the conductor is a "good" conductor, i.e., if for low frequencies, $\epsilon \ll \sigma/\omega$, what does k simplify to, what is the attenuation distance (skin depth) of the wave, and what is the phase delay of the magnetic induction $\mathbf{B}$ relative to the electric field $\mathbf{E}$?

E & M Qualifier

August 20, 2015

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

1. (a) [4 pts] Use δ-functions to give volume charge densities ρ_f for each of the following:

   i. Give $\rho_f(\rho, \phi, z)$ in cylindrical-polar coordinates for a cylindrical shell of charge of a radius $\rho = b$, centered on the z-axis, which has a surface charge density $\sigma_f(\phi, z)$.

   ii. Give $\rho_f(\rho, \phi, z)$ in cylindrical-polar coordinates for a line of charge located at $\rho = b, \phi = \beta$ which has a charge/length $=$ $\lambda_f(z)$.

   iii. Give $\rho_f(r, \theta, \phi)$ in spherical-polar coordinates for a spherical shell of charge of radius $r = a$, centered on the origin, which has a surface density $\sigma_f(\theta, \phi)$.

   (b) [2 pts] Use Gauss's law to compute the electric field caused by the cylindrically symmetric charge density

$$\rho_f(\rho) = \frac{\lambda_0}{\pi b^2} e^{-\rho^2/b^2}.$$

   (c) [2 pts] What charge density produces an electrostatic potential

$$\Phi(z) = V_0 e^{-z^2/a^2}.$$

   (d) [2 pts] What charge density produces an electrostatic potential

$$\Phi(z) = -E_0 \, |z|.$$

Aug 2015

E + M #1

Gaussian

a) i) $\rho_f(\rho, \varphi, z) = \sigma_f(\varphi, z)\,\delta(\rho - b)$

ii) $\rho_f(\rho, \varphi, z) = \lambda_f(z)\,\delta(\rho - b)\,\delta(\varphi - \beta) \cdot \frac{1}{b}$

iii) $\rho_f(r, \theta, \varphi) = \sigma_f(\varphi, z)\,\delta(r - a)$

b) The integral form of Gauss' Law states

$$\int \vec{\nabla} \cdot \vec{E} = \int 4\pi\rho$$

$$\int \vec{E} \cdot d\vec{a} = 4\pi \int \frac{\lambda_0}{\pi b^2} \exp\left[-\frac{\rho^2}{b^2}\right] dV$$

$$E \cdot 2\pi\rho L = 4\pi \int \frac{\lambda_0}{\pi b^2} \exp\left[-\frac{\rho^2}{b^2}\right] \rho \, d\rho \, d\varphi \, dz$$

$$E \cdot 2\pi\rho L = 4\lambda_0 L \int \frac{\rho}{b^2} \exp\left[-\frac{\rho^2}{b^2}\right] d\rho \, d\varphi$$

$$E \cdot 2\pi\rho L = 8\pi\lambda_0 L \int \frac{\rho}{b^2} \exp\left[-\frac{\rho^2}{b^2}\right] d\rho$$

$$E \cdot 2\pi\rho L = 8\pi\lambda_0 L \left(1 - \exp\left[-\frac{\rho^2}{b^2}\right]\right)$$

$$E = \frac{4\lambda_0}{\rho}\left(1 - \exp\left[-\frac{\rho^2}{b^2}\right]\right) \hat{\rho}$$

c) $-\nabla^2 \Phi = 4\pi\rho$

* In Cartesian Coordinates $\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$

$$-\nabla^2\left(V_0 \exp\left[-z^2/a^2\right]\right) = 4\pi\rho$$

$$-\frac{d^2}{dz^2}\left(V_0 \exp\left[-z^2/a^2\right]\right) = 4\pi\rho$$

$$-\frac{d}{dz}\left(V_0\left[\frac{-2z}{a^2} \exp\left[z^2/a^2\right]\right)\right) = 4\pi\rho$$

$$-V_0\left(-\frac{2}{a^2} \exp\left[\frac{z^2}{a^2}\right] + \frac{4z^2}{a^4} \exp\left[z^2/a^2\right]\right) = 4\pi\rho$$

$$\hookrightarrow \rho = \frac{-V_0}{4\pi} \exp\left[z^2/a^2\right]\left(\frac{z^2 4}{a^4} - \frac{1}{a^2}\right)$$

#1 (cont.)

d) * Similiarly to part c

$$-\nabla^2 \Phi = 4\pi \rho$$

$$-\frac{d^2}{dz^2}(E_0|z|) = 4\pi\rho$$

↳ if $z \neq 0$, $\Phi = 0$

* Due to the discontinuity at $z=0$, we proceed via Gauss Law

$$-\nabla \Phi = E$$

↳ $E = \begin{cases} E_0 z & z > 0 \\ -E_0 z & z < 0 \end{cases}$

$$\int E \cdot d\vec{a} = 4\pi\rho$$

$$2E_0 A = 4\pi\rho$$

$$= 4\pi\sigma A$$

↳ $\sigma = \frac{E_0}{2\pi}$

↳ $\rho = \frac{E_0}{2\pi}\delta(z)$

2. The magnetic field of a plane wave in vacuum is

$$\mathbf{B} = B_0 \sin(kx - \omega t)\,\hat{\mathbf{y}},$$

where $\hat{\mathbf{y}}$ is a unit vector pointing in the positive y-direction.

(a) [1 pts] Give the wavelength λ of this wave as a function of k, ω and/or c (the speed of light).

(b) [2 pts] Write an expression for the wave part of the electric field $\mathbf{E}$ associated with the above magnetic field.

(c) [1 pts] What is the direction and magnitude of the Poynting vector $\mathbf{S}$ associated with this wave?

(d) [1 pts] Assume this plane wave is totally reflected by a thin conducting sheet lying in the y-z plane at x=0. What is the resulting time averaged radiation pressure on the sheet? Recall that the momentum density $\mathbf{g}$ and the Poynting vector of the incoming and reflected waves are related by $\mathbf{g} = \mathbf{S}/c^2$

(e) [2 pts] The component of an electric field parallel to the surface of an ideal conductor must be zero on the surface. Using this fact, find expressions for the reflected electric and magnetic fields. Recall that the electric and magnetic fields vanish within an ideal conductor.

(f) [3 pts] An oscillating surface current $\mathbf{K}$ flows in the thin conducting sheet as a result of this reflection. Along which axis does $\mathbf{K}$ point and what is its amplitude? Hint: To find $\mathbf{K}$ use an Amperian loop with one side just inside the conducting sheet and one side just outside the sheet.

Aug 2015

E+M #2

Gaussian

a) $c = \frac{\omega}{k}$ $\quad k = 2\pi/\lambda$

$\hookrightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega}$

b) $\vec{B} = B_0 \sin(kx - \omega t)\, \hat{y}$

* To determine $\vec{E}$, we use $\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \vec{J}$

$$\nabla \times B = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & B_0 \sin(kx-\omega t) & 0 \end{bmatrix}$$

$$= \langle 0,\ 0,\ \frac{\partial}{\partial x} B_0 \sin(kx - \omega t) \rangle$$

$$= k B_0 \cos(kx - \omega t)\, \hat{z}$$

* Since we are in a vacuum, $\vec{J} = 0$

$\hookrightarrow k B_0 \cos(kx - \omega t)\hat{z} = \frac{1}{c}\frac{\partial E}{\partial t}$

$$ck B_0 \cos(kx - \omega t)\hat{z} = \frac{\partial E}{\partial t}$$

$$-\frac{ck}{\omega} B_0 \sin(kx - \omega t)\, \hat{z} = E$$

$\hookrightarrow E = -B_0 \sin(kx - \omega t)\, \hat{z}$

c) * In Gaussian units $\vec{S} = \frac{c}{4\pi}(E \times H)$, $\quad H = \vec{B} - 4\pi \vec{M}$

$\hookrightarrow \vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B})$

$$= \frac{c}{4\pi} B_0^2 \sin^2(kx - \omega t)\, \hat{x}$$

#2 (cont.)

d) Radiation pressure is related to $\vec{g}$ via: $P = 2\langle cg \rangle$

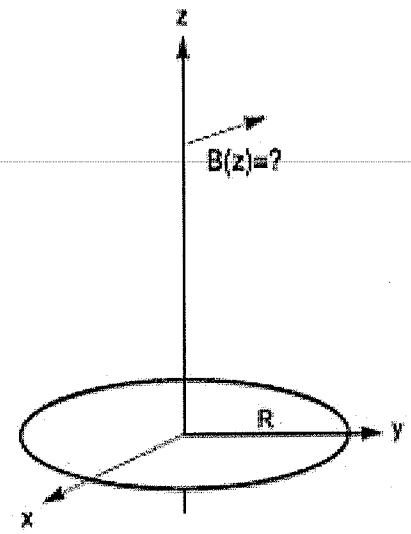
$$\hookrightarrow P = 2\langle c\vec{g} \rangle$$

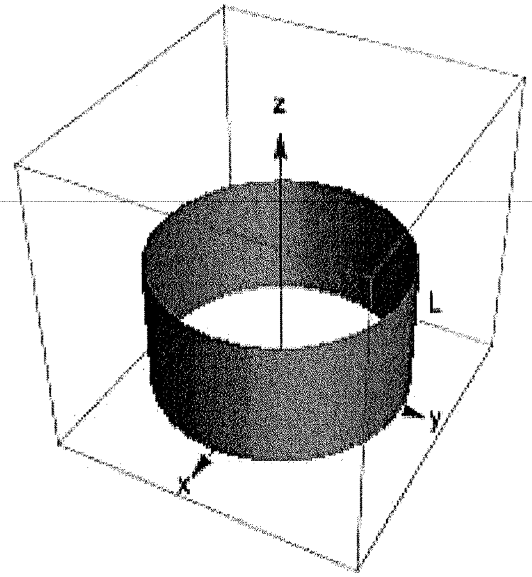
$$= \frac{2}{c}\langle \vec{S} \rangle$$

$$= \frac{2}{c}\frac{B_0^2}{4\pi}\hat{x}$$

e)

3.





(a) [5 pts] A circular loop of wire of radius R carries a current I as shown in the first figure. Find the magnitude and direction of the magnetic induction $\mathbf{B}(z)$ **on the axis** of the loop as a function of z.

(b) [5 pts] Use the result of part (a) to find $\mathbf{B}(z)$ along the axis of a solenoid of radius R and length L wound with n turns per unit length (total turns $N = n \times L$).

Hint:

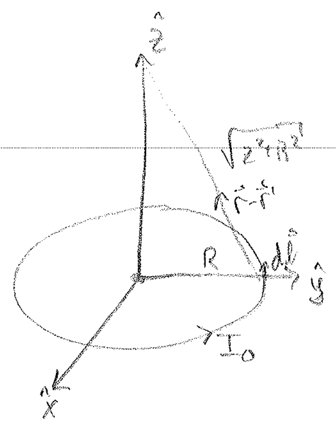
$$\int \frac{dx}{[a^2 + x^2]^{3/2}} = \frac{x}{a^2[a^2 + x^2]^{1/2}} + \text{constant}.$$

Aug 2015

E+M #3

Gaussian

a)



$$|r-r'| = \sqrt{z^2+R^2}$$

* We can find $\vec{B}$ via the Biot-Savart Law

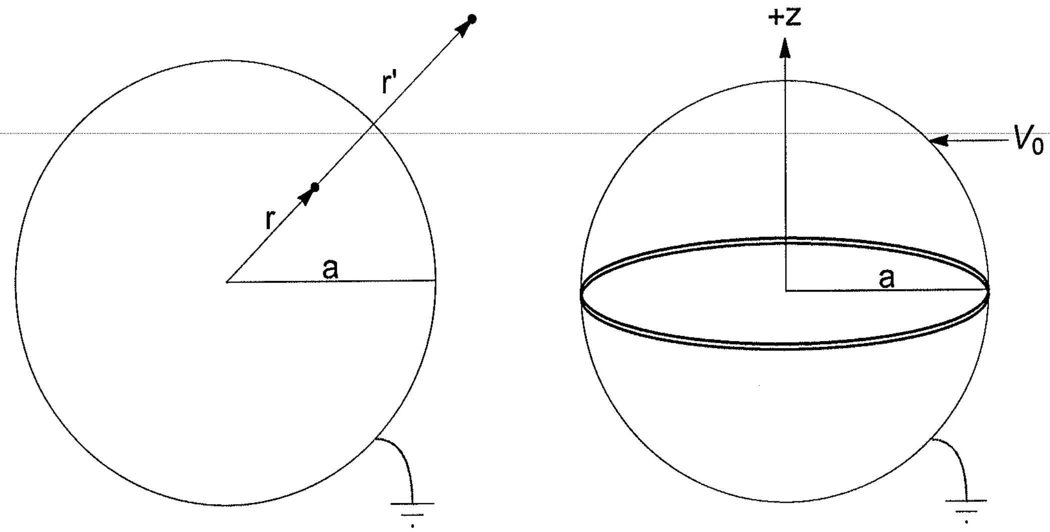
$$\Rightarrow \vec{B} = \frac{1}{c}\int \frac{\vec{J}\times(\vec{r}-\vec{r}')}{|r-r'|^3}\, d^3r'$$

$$= \frac{I_0}{c}\int \frac{d\vec{l}'\times(r-r')}{|r-r'|^3}$$

$$= \frac{I_0}{c}\int \frac{dl\, R}{(z^2+R^2)^{3/2}}$$

$$= \frac{2\pi R^2 I_0}{c(z^2+R^2)^{3/2}}\,\hat{z}$$ (any non $\hat{z}$ component cancels due to circular symmetry about axis)

b)



4. (a) [3 pts] What is the Dirichlet Green's function $G^D(\mathbf{r}, \mathbf{r}')$ for the Laplace operator for the 3-dimensional volume interior to a sphere of radius $r = a$?

   { Hint: The Dirichlet Green's function vanishes on $r = a$ and the method of images gives the Green's function as the sum of potentials of two point charges, a positive unit charge located inside the sphere and a larger negative point charge located outside. }

   (b) [2 pts] If a grounded conducting sphere of radius $r = a$ contains a positive charge q at $z = a/2$ and a negative charge $-q$ at $z = -a/2$, what will the electrostatic potential be inside the sphere?

   (c) [5 pts] If a conducting sphere of radius $r = a$, centered at the origin, is cut in two halves along the $z = 0$ plane and the top half ($z > 0$) is held at potential V_0 and the bottom half ($z < 0$) is grounded, what is the potential **on the** $+z$**-axis, inside** the sphere $r \leq a$.

   { Hint: Use spherical polar coordinates and recall that the Dirichlet Green's function can be used to calculate the electrostatic potential via:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') G^D(\mathbf{r}, \mathbf{r}') d^3r' - \frac{1}{4\pi} \int_{\partial V} \Phi(\mathbf{r}') \, \frac{\partial G^D(\mathbf{r}, \mathbf{r}')}{\partial n'} da'.$$

   In this case $\frac{\partial}{\partial n'} = \frac{\partial}{\partial r'}$ and $da' = a^2 \sin\theta' d\theta' d\phi'$.}

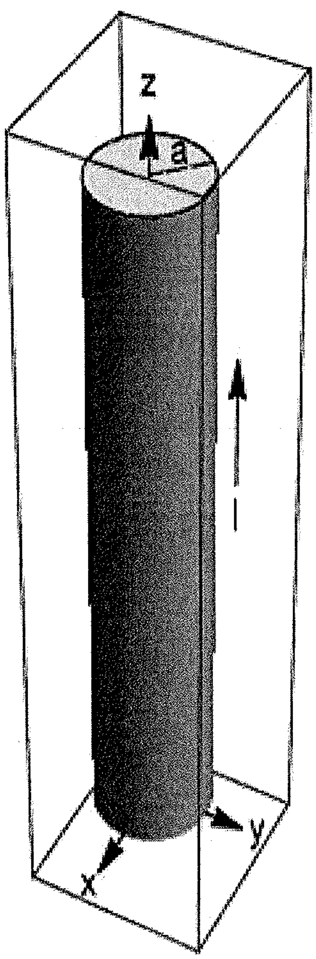
5. A very-very long (∞-length) wire of radius a is centered along the z-axis. The wire, at rest in the lab, is uncharged and carries a **uniform** current I in the +z direction.

(a) [2 pts] Compute the magnetic induction $\mathbf{B}$ as a function of (x, y) for all $\sqrt{x^2 + y^2} \geq a$ **(outside the wire)**.

(b) [2 pts] Give the electro-magnetic field tensor $F^{\alpha\beta}(x, y)$ in the lab frame for $\sqrt{x^2 + y^2} \geq a$**(outside the wire)**.

(c) [3 pts]
An observer is moving with constant velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$ in the lab's +z direction. Compute $F'^{\alpha\beta}(x', y')$, **outside the wire** in the frame of the moving observer. Assume the moving frame coordinates are simply a boost of the lab coordinate in the z direction. Don't forget to write your answer as functions of the moving frame's coordinates.

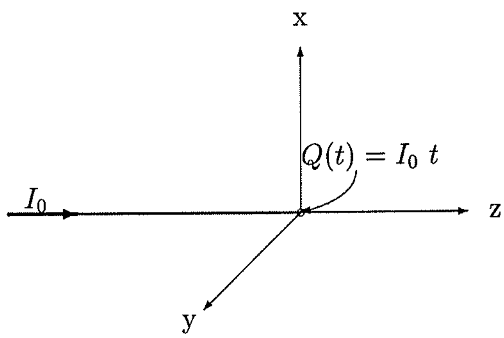
(d) [3 pts]
Give the electric and magnetic fields $\mathbf{E}'$ and $\mathbf{B}'$, **outside the wire** in the moving frame.



6. In the absence of polarizable and/or magnetizable material (i.e., only free charges and currents present) Maxwell's equations, in the Lorenz gauge, reduce to the inhomogeneous wave equation:

$$\Box \begin{Bmatrix} \Phi \\ A^x \\ A^y \\ A^z \end{Bmatrix} = \frac{4\pi}{c} \begin{Bmatrix} c\rho \\ J^x \\ J^y \\ J^z \end{Bmatrix}, \quad \text{where } \Box \equiv \left(\frac{\partial}{c\partial t}\right)^2 - \nabla^2.$$

A time dependent charge $Q(t) = I_0\, t,\ t \geq 0$ is fixed at the origin



of a cylindrical polar coordinate system (ρ, ϕ, z) The charge increases linearly with time because a constant current I_0 flows in along a thin wire attached to the charge on its left, see the figure. Assume the wire carries no current for $t < 0$, however, at $t = 0$ a current I_0 abruptly starts flowing in the +z direction and remains constant for $t \geq 0$. Assume the wire remains neutral except for the charge that grows at the origin. Find the following quantities at time t for points (ρ, ϕ, z):

(a) [4 pts] The retarded scalar potential $\Phi(t, \rho, \phi, z)$, for all t and at all points in space.

(b) [6 pts] The retarded vector potential $\mathbf{A}(t, \rho, \phi, z)$ for all t at all points $z > 0$.

Recall that the retarded solution to $\Box F(t, \mathbf{r}) = S(t, \mathbf{r})$ is

$$F(t, \mathbf{r}) = \frac{1}{4\pi} \int \frac{S(t - |\mathbf{r} - \mathbf{r}'|/c, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'.$$

You might need the indefinite integral

$$\int \frac{dX}{\sqrt{X^2 + a^2}} = \ln(\sqrt{X^2 + a^2} + X).$$

E & M Qualifier

January 14, 2016

To insure that the your work is graded correctly you MUST:

1. use only the reference material supplied (Schaum's Guides),
2. use only the blank answer paper provided,
3. write only on one side of the page,
4. put your alias **(NOT YOUR REAL NAME)** on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
   (d) try to answer every problem, but if you don't please include a single numbered page stating that you have skipped that problem.
7. **DO NOT** staple your exam when done. Paper clips will be provided.

1. Consider a Lorentz frame K containing no polarizable materials in which there is a magnetic induction $\mathbf{B} = B^x \hat{\mathbf{x}} + B^y \hat{\mathbf{y}} + B^z \hat{\mathbf{z}}$ but no electric field.

   (a) [1 pt] For the above magnetic induction, write down the 4-dimensional electromagnetic field tensor $F^{\alpha\beta}$ in frame K as a matrix.

   (b) [1 pt] Write down a homogeneous Lorentz boost $\Lambda^{\alpha}{}_{\beta}$ in the y-direction from frame K to another frame K′ which is moving with velocity $\mathbf{v} = v_0 \hat{\mathbf{y}}$ as seen by observers that are at rest in frame K.

   (c) [2 pt] Apply the boost $\Lambda^{\alpha}{}_{\beta}$ to $F^{\alpha\beta}$ to find $F'^{\alpha\beta}$, the field strength tensor as seen in the moving frame K′ .

   (d) [2 pt] What are the electric field components E'^x, E'^y, and E'^z and the magnetic induction components B'^x, B'^y, and B'^z in frame K′?

   (e) [4 pt] Consider explicitly a **B** field in the K frame caused by an **uncharged** infinitely long and thin wire centered on the y-axis $(x, z) = (0, 0)$ which carries a current I in the $+y$ direction. Assume that no polarizable materials are present, i.e., assume $\epsilon_r = 1$ and $\mu_r = 1$. What are $\mathbf{B}'(x', y', z')$ and $\mathbf{E}'(x', y', z')$ in the K′ frame, written as functions of the K′-coordinates? Where does $\mathbf{E}'$ point?

Jan 2016 # E+M #1 Gaussian

a) Given $E=0$, $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad \text{(in general)}$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B_z & B_y \\ 0 & B_z & 0 & -B_x \\ 0 & -B_y & B_x & 0 \end{bmatrix}$$

b) For a Lorentz boost in y-direction

$$\Lambda^{\alpha}{}_{\beta} = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) In matrix form: $F' = \Lambda F \Lambda^T$

$$F' = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B_z & B_y \\ 0 & +B_z & 0 & -B_x \\ 0 & -B_y & +B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\beta\gamma B_z & 0 & \beta\gamma B_x \\ 0 & 0 & -B_z & B_y \\ 0 & \gamma B_z & 0 & -\gamma B_x \\ 0 & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

#1 (cont.)

c) $$F'^{\alpha\beta} = \begin{bmatrix} 0 & -\beta\gamma B_z & 0 & \beta\gamma B_x \\ \beta\gamma B_z & 0 & -\gamma B_z & B_y \\ 0 & \gamma B_z & 0 & -\gamma B_x \\ -\beta\gamma B_x & -B_y & \gamma B_x & 0 \end{bmatrix}$$

d) $\vec{E}' = \beta\gamma(B_z\hat{x} - B_x\hat{z}) \Rightarrow E'_x = \beta\gamma B_z \quad E'_y = 0 \quad E'_z = -\beta\gamma B_x$

$\vec{B}' = \gamma B_x\hat{x} + B_y\hat{y} + \gamma B_z\hat{z} \Rightarrow B'_x = \gamma B_x \quad B'_y = B_y \quad B'_z = \gamma B_z$

e) We find $\vec{B}$ according to Ampere's law

$$\oint B \cdot dl = \frac{4\pi}{c} I_{enc}$$

$$B \cdot 2\pi r = \frac{4\pi}{c} I$$

$$B = \frac{2I}{cr} \langle \cos\varphi\,\hat{x} + \sin\varphi\,\hat{z} \rangle$$

* This is best done in cylindrical coordinates, but professors are stupid question writers. φ measured ccw from $+x$ axis in x-z plane

$$\Rightarrow \vec{E}' = \langle \beta\gamma\sin\varphi \frac{2I}{cr},\ 0,\ -\beta\gamma\cos\varphi\frac{2I}{cr} \rangle$$

$$\vec{B}' = \langle \gamma\sin\varphi\frac{2I}{cr},\ 0,\ \gamma\cos\varphi\frac{2I}{cr} \rangle$$

2. Consider a very long hollow cylinder made of iron that is placed with its axis perpendicular to a uniform external magnetic induction $\mathbf{B}_0 = B_0\hat{\mathbf{x}}$. Assume the inner radius of the hollow cylinder is a and the outer radius is b. Also assume the permeability μ of the iron is a constant. The goal of this problem is to calculate the magnetic induction $\mathbf{B}$ inside the hollow region $(0 \leq \rho \equiv \sqrt{x^2+y^2} < a)$.

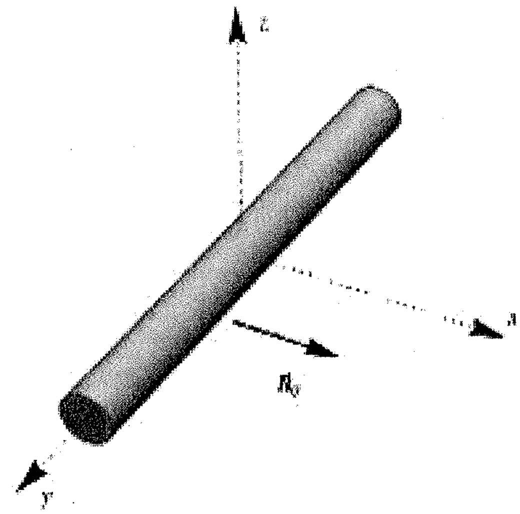
   (a) [3 pt] Starting with Maxwell's equations for static $\mathbf{B}$ and $\mathbf{H}$ fields and assuming that there is no free current density, $\mathbf{J}_f = 0$, prove that the field $\mathbf{H}$ can be written as the negative gradient of a magnetic scalar potential Φ_M that satisfies the Poisson equation with an appropriate source term. For this particular problem the Poisson equation reduces to the Laplace equation except at the cylinder's boundaries.

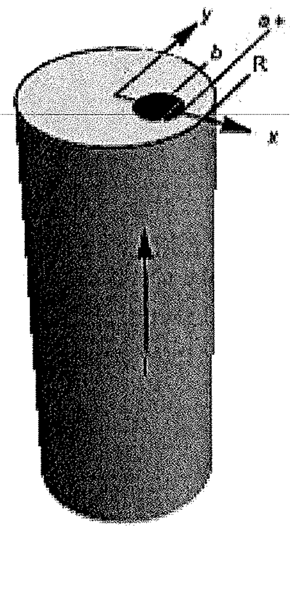
   (b) [3 pt] Derive the appropriate boundary conditions to be satisfied by the scalar potential Φ_M and the magnetic field $\mathbf{H}$ at $\rho = a$ and $\rho = b$.

   (c) [4 pt] Solve for the $\mathbf{H}$ field in the interior region $\rho < a$. Hint: solve the Laplace equation for Φ_M in the three regions $0 \leq r < a$, $a < r < b$, and $b < r < \infty$, and appropriately match these solutions at the cylinder's boundaries. Show that for large μ, (i.e., when $\mu \to \infty$) the iron provides complete shielding from the magnetic field, i.e., $\mathbf{H} \to 0$ for $\rho < a$.

Hint:

$$\nabla^2\Phi_M = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_M}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi_M}{\partial\phi^2} + \frac{\partial^2\Phi_M}{\partial z^2}.$$



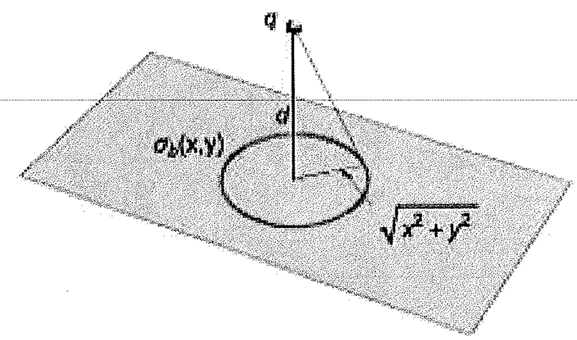


3. A very long straight conductor has a circular cross section of radius R and carries a current I. Inside the conductor, there is a cylindrical hole of radius a whose axis is parallel to the axis of the conductor and a distance b from it ($a + b < R$). The goal of this problem is to show that the magnetic induction $\mathbf{B}(x, y)$ inside the hole is uniform and to calculate its value. Assume the wire of radius R is centered on the z axis, i.e., at $(x, y) = (0, 0)$ and the cylindrical hole of radius a is centered at $(x, y) = (b, 0)$. Assume the current I is uniformly distributed in the conducting material.

   (a) [3 pts]
   Ignoring the hole, use Amperés Law to find the magnetic induction, $\mathbf{B}_R(x, y)$, inside a homogeneous cylindrical wire of radius R that carries a uniform current density $J_R = I_R/\pi R^2$ in the $+z$ direction.

   (b) [4 pts] Ignoring the current in the wire of radius R assume an imaginary wire of radius a located at $(x, y) = (b, 0)$ carries a current density $J_a = I_a/\pi a^2$ in the $-z$ direction. Use Amperes Law to find the magnetic induction, $\mathbf{B}_a(x, y)$, inside the imaginary wire of radius a caused by J_a.

   (c) [3 pts]
   By adjusting the two current densities to have the same magnitude, and superimposing the two magnetic inductions, find the resultant $\mathbf{B}(x, y)$ field inside the hole in the original conductor that carries a current I described at the beginning of this problem.



4. Consider a large flat interface at $z = 0$ between a dielectric and free space. The region where $z < 0$ is filled with a uniform linear dielectric material with a relative permittivity ϵ_r (equivalently a dielectric constant ϵ_r). If the only free charge present is a point charge $q > 0$ situated a distance d from the origin at $\mathbf{r}_q = (0, 0, d)$, where $d > 0$, answer the following 5 questions.

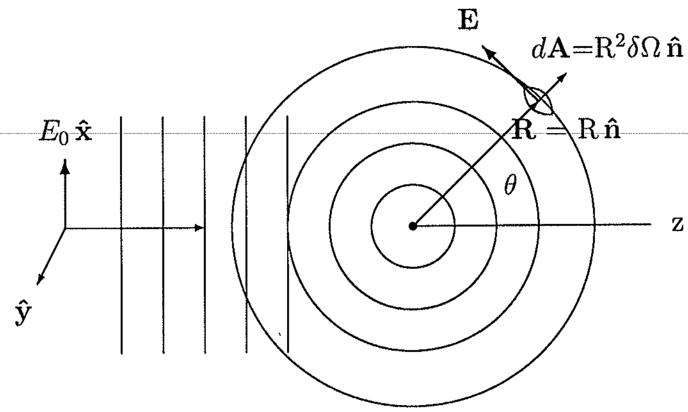
   To answer them you should look at the electric field as a sum of two fields, a coulomb part $\mathbf{E}_q$ caused by the point charge q and a second part $\mathbf{E}_b$ caused by the bound surface charge $\sigma_b(x, y)$ located on the $z = 0$ interface.

   (a) [2 pts] Write two expressions for the z component of the total electric field $E^z = E^z_q + E^z_b$, one just above the dielectric's surface and one just below the dielectric's surface. The E^z_b part is directly related to σ_b by Gauss's law.

   (b) [3 pts] Use the two electric fields from part (a) and the continuity of the normal part of the displacement vector ϵE^z to solve for $\sigma_b(x, y)$ as a function of the known coulomb field $E^z_q(x, y, 0)$.

   (c) [3 pts] Calculate the electric field at the position of the charge q caused by the bound surface charge σ_b. You simply have to integrate a superposition of coulomb fields. From symmetry the resultant field points in the $\pm z$ direction.

   (d) [2 pts] Show that this resultant bound charge field at $(0, 0, d)$ can be interpreted as the field of a single image charge q' located at point $\mathbf{r}_{q'} = (0, 0, -d)$. What is the value of q'?



5. In this question a monochromatic linearly polarized plane wave is scattered by a free electron. If the initial speed of the particle is non-relativistic (i.e., $\beta \ll 1$) and the frequency of the plane wave satisfies $h\nu \ll m_e c^2$, then the electron is accelerated by the plane wave's electric field in accord with Newton's 2^{nd} law, but its speed remains non-relativistic. Due to its acceleration, the electron emits radiation in all directions thus scattering the original plane wave. See the figure.

   (a) [2 pts] Assume the plane wave travels in the z-direction and is polarized in the x-direction as shown in the figure. Compute the acceleration, $\dot{\boldsymbol{\beta}}(t) = \dot{\mathbf{v}}(t)/c$, of the electron caused by the plane wave's electric field.

   (b) [3 pts] Compute the electric field $\mathbf{E}$, the magnetic induction $\mathbf{B}$, and the Poynting vector $\mathbf{S}$ of the radiated wave. { Hint: In Gaussian units $\mathbf{E}_G = q[\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\boldsymbol{\beta}})]/(cR)|_{ret}$, $\mathbf{B}_G = \hat{\mathbf{n}} \times \mathbf{E}$, and $\mathbf{S}_G = (c/4\pi)\mathbf{E} \times \mathbf{H}$. In SI units $\mathbf{E}_{SI} = (1/4\pi\epsilon_0)\mathbf{E}_G$, $\mathbf{B}_{SI} = (1/c)\mathbf{B}_G$, and $\mathbf{S}_{SI} = \mathbf{E} \times \mathbf{H}$. }

   (c) [3 pts] Use your results to compute the differential scattering cross section

   $$\frac{d\sigma(\theta,\phi)}{d\Omega} = \frac{< \mathbf{S} \cdot d\mathbf{A} >}{| < \mathbf{S}_0 > | \delta\Omega}.$$

   In the above $<>$ stands for a time average and $| < \mathbf{S}_0 > |$ is the magnitude of the time averaged Poynting vector of the incoming plane wave. The detector area element $d\mathbf{A}$ subtends a solid angle $\delta\Omega$ at the radiating electron and is typically of the form

   $$d\mathbf{A} = R^2 \delta\Omega\, \hat{\mathbf{n}}.$$

   (d) [2 pts] Integrate your differential cross section over all (θ, ϕ) directions to obtain the total Thompson cross section σ_T.

6. (a) [2 pts] In a homogeneous, linear and isotropic conducting material whose electromagnetic properties (at low frequencies) are described by constant (and real) values of the permittivity, permeability, and conductivity respectively ϵ, μ, and σ, show that Maxwell's equations require that the electric field satisfy the telegraph equation

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0, \qquad \text{(SI)}$$

$$\nabla^2 \mathbf{E} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \mathbf{E}}{\partial t} = 0. \qquad \text{(Gaussian)}$$

(b) [3 pts] Given a linearly polarized plane wave of angular frequency ω whose electric field is of the form

$$\mathbf{E}(z,t) = \text{Real}\left\{E_0\, e^{i(kz-\omega t)}\right\} \hat{\mathbf{x}},$$

evaluate k^2 as a function of ϵ, μ, σ, and ω.

(c) [2 pts] Find the real and imaginary parts of k assuming $\sigma >> \omega\epsilon$.

(d) [3 pts] Using your results from (c) find the skin depth δ of the conductor. The skin depth is defined by the depth at which the wave's amplitude decreases by e^{-1}, i.e.,

$$\frac{|\mathbf{E}(z+\delta,t)|}{|\mathbf{E}(z,t)|} = \frac{1}{e}$$

E & M Qualifier

January 11, 2017

To insure that the your work is graded correctly you MUST:

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2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
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6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
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   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

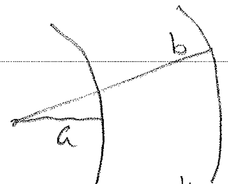
Problem 1: Electrostatics

Consider a capacitor composed of two concentric spherical metal shells, the inner one with radius a and the outer one with radius b. The region between the spherical metal shells is filled with a linear dielectric with permittivity $\epsilon = \frac{k}{r^2}$. Place charge $+Q$ on the inner metallic shell and $-Q$ on the outer metallic shell.

1. Find the electric displacement $\vec{D}$ everywhere in space. [3 points]
2. Find the capacitance of the configuration. [3 points]
3. Calculate the bound charge densities in the linear dielectric and verify that the total net bound charge is zero. [4 points]

Jan 2017 $\quad$ **E+M #1** $\quad$ SI

a)



$\epsilon = \frac{k}{r^2}$

$$\nabla \cdot D = \frac{\rho_f}{\epsilon}$$

* if $r < a$:

$$\nabla \cdot D = \frac{\rho_f}{\epsilon}$$

$\hookrightarrow \vec{D} = 0 \quad$ b/c $\rho_f = 0$

* if $a < r < b$:

$$\int D \cdot d\vec{a} = \int \frac{\rho_f}{\epsilon} dV$$

$$D \cdot 4\pi r^2 = \frac{Q}{\epsilon}$$

$$D = \frac{Q}{4\pi \epsilon r^2} \hat{r}$$

* if $r > b$:

$$\int D \cdot d\vec{a} = \int \frac{\rho_f}{\epsilon} dV$$

$\vec{D} = 0 \quad$ (total enclosed charge is 0)

b) $C = \frac{Q}{\Delta V} \quad$ or $\quad W = \frac{\epsilon_r}{2} C V^2 \qquad (D = \epsilon E)$

$$W = \frac{1}{2} \int D \cdot E \, dV$$

$$= \frac{1}{2} \int \epsilon E^2 \, dV$$

$$= \frac{1}{2} \int \frac{Q^2}{16\pi^2 r^4} r^2 dr d\theta d\varphi$$

$$= \frac{Q^2}{32\pi^2} \int \frac{1}{r^2} dr d\theta d\varphi$$

$$= \frac{Q^2}{8\pi} \frac{1}{r}\Big|_a^b = \frac{Q^2}{8\pi}\left(\frac{1}{b} - \frac{1}{a}\right)$$

#1 (cont.)

b) $\frac{Q^2}{8\pi}\left(\frac{1}{b}-\frac{1}{a}\right) = \frac{\epsilon_r}{2} C V^2$

???

c) $\rho_b = -\nabla \cdot \vec{P}$ $\sigma_b = \vec{P} \cdot \hat{n}$ $\vec{P} = \frac{\chi_e}{\epsilon_r} D$, $\epsilon_r = \frac{\epsilon}{\epsilon_0}$

$$\rho_b = -\nabla \cdot \frac{\chi_e}{\epsilon_r} D$$
$$= -\frac{\chi_e}{\epsilon_r} Q$$
$$= \frac{-\chi_e Q r^2}{k}$$

$$\sigma_b = \frac{\chi_e}{\epsilon_r}$$
$$= \frac{\chi_e r^2}{k} D \cdot \hat{n}$$
$$= \frac{\chi_e r^2}{k} \frac{Q}{4\pi\epsilon_r r^2}$$
$$= \frac{\chi_e Q}{4\pi\epsilon k}$$

$$Q_{b,tot} = \int \rho_b \, dV + \oint \sigma_b \, d\vec{a}$$
$$= -\frac{\chi_e}{k} Q \int r^4 \, dr \, d\varphi \, d\theta + \int \frac{\chi_e Q}{4\pi\epsilon k} \, da$$
$$= \frac{-4\pi\chi_e Q}{k} \left. \frac{1}{5} r^5 \right|_a^b$$

*Parts b+c mostly wrong

Problem 2: Magnetostatics

Consider a sphere of radius R composed of magnetic material with a magnetization given by $\vec{M} = M_0\hat{z}$.

1. Starting with the Maxwell equations for a static magnetic field $\vec{H}$ and a static magnetic induction $\vec{B}$, prove that the magnetic field $\vec{H}$ can be written in terms of a scalar magnetic potential. From this derive the Poisson equation that solves the potential. In addition, derive expressions for the magnetic volume charge density and the bound current density from the Maxwell equations. [2 points]

2. Derive the boundary conditions on $\vec{H}$ and $\vec{B}$. Be sure to clearly define the effective surface magnetic charge density and the surface magnetic current density. [2 points]

3. Derive the fields inside and outside the sphere. *You can assume that $\vec{B}$ and $\vec{M}$ are parallel.* [6 points]

Jan 2017

E+M #2

Gaussian

a) Maxwell's equations are

$$\nabla \cdot B = 0$$

$$\nabla \times H = \frac{4\pi}{c} \vec{J}_f \qquad \left(\nabla \times B = \frac{4\pi}{c} \left(\vec{J}_f + \vec{J}_b\right) \right)$$

Problem 3: Maxwell equations

Consider a medium with nonzero scalar conductivity σ ($\mathbf{J}_f = \sigma\mathbf{E}$ is the current density), permeability μ, permittivity ϵ, and with no free charge ($\rho_f = 0$).

1. Write down the set of four differential Maxwell's equations appropriate for this medium. [2points]

2. Derive the wave equation for $\mathbf{E}$ in this medium. Highlight the additional term arising from the non-zero $\mathbf{J}_f$. [3 points]

3. Consider a monochromatic wave moving in the $+x$ direction with E_y given by

$$E_y = A\, e^{i(kx-\omega t)}$$

   Show that this wave has an amplitude A which decreases exponentially. Find the attenuation length Δx, the distance after which the amplitude has decayed by a factor of $1/e$ from its initial value, as a function of σ. Show that your solution correctly predicts $\Delta x = 0$ if $\sigma = 0$. [5 points]

Jan 2017 E+M #3 Gaussian

a) In general, Maxwell's eqns in media are:

$$\nabla \cdot D = \cancel{4\pi \rho_f}^{\,0} \qquad \nabla \cdot B = 0$$

$$\nabla \times E - \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$$

b) To derive the wave equation for E:

$$\nabla \times \left(\nabla \times E - \frac{1}{c}\frac{\partial B}{\partial t} \right) = 0$$

$$\nabla(\nabla \cdot E) - \nabla^2 E - \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times B) = 0$$

* Note: $D = \epsilon E, \quad B = \mu H$

$$\frac{1}{\epsilon}\nabla(\cancel{\nabla \cdot D}^{\,0}) - \nabla^2 E - \frac{\mu}{c}\frac{\partial}{\partial t}(\nabla \times H) = 0$$

$$-\nabla^2 E - \frac{\mu}{c}\frac{\partial}{\partial t}\left(\frac{4\pi}{c}\vec{J}_f + \frac{1}{c}\frac{\partial D}{\partial t} \right) = 0$$

$$-\nabla^2 E - \frac{\mu \partial}{c \partial t}\left(\frac{4\pi}{c}\sigma E + \frac{\epsilon}{c}\frac{\partial E}{\partial t} \right) = 0$$

$$-\nabla^2 E - \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2} - \underbrace{\frac{4\pi\mu}{c^2}\sigma\frac{\partial E}{\partial t}}_{\text{extra non-zero term due to } \vec{J}_f} = 0$$

c) Given $\vec{E} = \langle 0, A\exp[i(kx - \omega t)], 0 \rangle$

$$-\nabla^2 E = -\frac{d^2 E}{dx^2}$$

$$= k^2 A \exp[i(kx - \omega t)]$$

$$\frac{\partial^2 E}{\partial t^2} = \omega^2 A \exp[i(kx - \omega t)]$$

$$\frac{\partial E}{\partial t} = -i\omega A \exp[i(kx - \omega t)]$$

#3 (cont.)

c) $\Rightarrow 0 = k^2 A\exp[i(kx-\omega t)] - \frac{\mu\epsilon}{c^2}\omega^2 A\exp[i(kx-\omega t)] - \frac{4\pi\mu}{c^2}\sigma(-i\omega)A\exp[i(kx-\omega t$

$$0 = k^2 A - \frac{\mu\epsilon}{c^2}\omega^2 A + i\frac{4\pi\mu\sigma}{c^2}\omega A$$

$$= A\left(k^2 + i\omega\frac{4\pi\mu\sigma}{c^2} - \frac{\mu\epsilon}{c^2}\omega^2\right)$$

* Helpful simplifications: $k = \omega v$

$V = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$

$n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$

$= A($

Problem 4: EM radiation

A particle of mass m and charge q is attached to a spring with force constant k, which is hanging from the ceiling. The particle's equilibrium position is a distance h above the floor. Suppose the particle is pulled down a distance d below its equilibrium position and released at time $t = 0$. Useful information: When the wavelength is much greater than the spatial amplitude, the electric field from an oscillating dipole is

$$E = -\frac{\mu_o p_o \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos(\omega t - [\omega r/c]) \tag{1}$$

1. Calculate the intensity of the radiation hitting the floor, as a function of the distance R from the point directly below the point particle. Assume $d \ll \lambda \ll h$ and neglect radiative damping of the oscillator. [3 points]

2. At what R is the radiation most intense? [2 points]

3. Assume the floor is of infinite extent. Calculate the average energy per unit time striking the entire floor. How does this compare to the total power radiated by the oscillating charge? [3 points]

4. Because energy is lost in the form of radiation, the amplitude of the oscillation will gradually decrease. At what time τ has the oscillator's energy been reduced to $1/e$ of its initial value? Assume the fraction of the total energy lost in one cycle is very small. [2 points]

Problem 5: Special relativity

Consider two frames K and K' with a uniform relative velocity. Observers at rest in K' are moving along the positive x axis of K with a velocity v. $\beta = v/c$, $\gamma = (1-\beta^2)^{-1/2}$, and c is the speed of light.

1. Let x^ν be the four-dimensional space-time vector in the K frame with the components: $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$, and x'^μ be the corresponding vector in the K' frame with the Lorentz transformation of $x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$, where Einstein's summation rule is implied. What are the components of $\Lambda^\mu_{\ \nu}$? [2 points]

2. An object is moving with a three-dimensional velocity $\vec{u}$ in K, and the velocity is measured to be $\vec{u'}$ in K'. What are the components of the object's four-velocity in K' in terms of u_x, u_y and u_z? [4 points]

3. Let θ be the angle between $\vec{u}$ and x in K, and θ' be the angle between $\vec{u'}$ and x' in K'. Show that
$$\tan\theta = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)}. \tag{2}$$
[2 point]

4. A source is emitting isotropically in its rest-frame and moves with an ultra-relativistic velocity in K with $\gamma \gg 1$. Show that in K half of the radiation power is concentrated in a cone with a half open angle of $1/\gamma$. [2 points]

Jan 2017

E+M #5

a) Given K' is moving relative to K with $\vec{v} = v_x \hat{x}$, in matrix form,

$$\Lambda^\mu_{\ \nu} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) To transform $\vec{v}$ to $\vec{v}'$, $\vec{v}' = \Lambda v$

$$v' = \begin{bmatrix} ct' \\ v_x' \\ v_y' \\ v_z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\begin{bmatrix} ct' \\ v_x' \\ v_y' \\ v_z' \end{bmatrix} = \begin{bmatrix} \gamma ct - \beta\gamma v_x \\ -\beta\gamma ct + \gamma v_x \\ v_y \\ v_z \end{bmatrix}$$

c)

Problem 6: Relativistic electrodynamics

The Lagrangian for the EM field generated from a 4-current j_μ is given (in SI units) by

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - A_\mu j^\mu \tag{3}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, $A_\mu = (\phi/c, -\vec{A})$ and $j_\mu = (c\rho, -\vec{j})$ and where $c^2 = 1/\epsilon_0\mu_0$.

1. Show that $\mathcal{L}$ is invariant under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda(t, \vec{x})$. [2 points]
2. Derive the covariant form of Maxwell's equations from the Euler-Lagrange equations using $\mathcal{L}(A_\mu, \partial_\nu A_\mu)$. [4 points]
3. Show that these reduce to the usual form of Maxwell's equations in 3-vector notation. [4 points]

Jan 2017

E+M #6

a) $\mathcal{L} = \frac{-1}{4\mu_0} F^{\mu\nu}F_{\mu\nu} - A_\mu j^\mu$

$* \text{ if } F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$= \frac{-1}{4\mu_0}\left(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right)F_{\mu\nu} - A_\mu j^\mu$$

$$= \frac{-1}{8\mu_0}\epsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial_\mu A_\nu - \partial_\nu A_\mu) - A_\mu j^\mu$$

$* \text{ if } A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(t,\vec{x})$

$$= \frac{-1}{8\mu_0}\epsilon^{\mu\nu\rho\sigma}\big(\partial_\rho(A_\sigma + \partial_\sigma\Lambda) - \partial_\sigma(A_\rho + \partial_\rho\Lambda)\big)\big(\partial_\mu(A_\nu + \partial_\nu\Lambda) - \partial_\nu(A_\mu + \partial_\mu\Lambda)\big)$$

$$- (A_\mu + \partial_\mu\Lambda)j^\mu$$

$$= \frac{-1}{8\mu_0}\epsilon^{\mu\nu\rho\sigma}\big(\partial_\rho A_\sigma + \cancel{\partial_\rho\partial_\sigma\Lambda} - \partial_\sigma A_\rho - \cancel{\partial_\sigma\partial_\rho\Lambda}\big)\big(\partial_\mu A_\nu + \cancel{\partial_\mu\partial_\nu\Lambda} - \partial_\nu A_\mu - \cancel{\partial_\nu\partial_\mu\Lambda}\big)$$

$$- A_\mu j^\mu - \partial_\mu \Lambda j^\mu$$

$$= \frac{-1}{8\mu_0}\epsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial_\mu A_\nu - \partial_\nu A_\mu) - A_\mu j^\mu - \overset{0}{\cancel{\partial_\mu \Lambda j^\mu}}$$

$$= \frac{-1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} - A_\mu j^\mu$$

E & M Qualifier

August 16, 2017

To insure that the your work is graded correctly you MUST:

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7. **DO NOT** staple your exam when done.

Problem 1: Electrostatics

A non-conducting solid sphere of radius R carries a charge density $\rho(r) = k\,r$ (where k is a constant).

(a) Find the electric field at a distance r such that $r \geq R$ [1 point]

(b) Find the electric field at a distance r such that $r \leq R$ [1 point]

(c) State the boundary conditions on the electric field components on the surface of the sphere, and show that your answers to parts a) and b) are consistent with them.
HINT: The surface charge density of the sphere is zero in this case. [1 point]

(d) Calculate the electric potential for all r using $\lim_{r\to\infty} V(r) = 0$ [2 points]

(e) Find the work required to assemble this charge [2 points]

(f) If the *non-conducting* solid sphere was replaced by a *conducting* solid sphere with the same total charge, how does that change your answer to parts a, b, and c?
Explicitly show that the new field satisfies the new boundary conditions across the surface of the sphere. [3 point]

Aug 2017

E+M #1

Gaussian

a) We use Gauss' Law to find the field at $r \geq R$

$$\int \vec{\nabla} \cdot \vec{E}\, dV = \int 4\pi \rho\, dV$$

$$\oint \vec{E} \cdot d\vec{a} = 4\pi \int kr \cdot r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$\vec{E} \cdot 4\pi R^2 = 16\pi^2 k \int_0^R r^3 dr$$

$$4\pi r^2 \vec{E} = 4\pi^2 k\, r^4 \Big|_0^R$$

$$4\pi r^2 \vec{E} = 4\pi^2 k R^4$$

$$\vec{E} = \pi k \frac{R^4}{r^2} \hat{r}$$ (know $\hat{r}$ direction due to spherical symmetry of problem)

b) We proceed as above, now with $r \leq R$

$$\int \vec{\nabla} \cdot \vec{E}\, dV = \int 4\pi \rho\, dV$$

$$\oint \vec{E} \cdot d\vec{a} = 4\pi k \int r'^3 \sin\theta\, dr'\, d\theta\, d\varphi$$

$$\vec{E} \cdot 4\pi r^2 = 4\pi^2 k\, r'^4 \Big|_0^r$$

$$\vec{E} \cdot 4\pi r^2 = 4\pi^2 k r^4$$

$$\vec{E} = k\pi r^2\, \hat{r}$$

c) The boundary conditions of $\vec{E}$ are: ① $E_1^{\perp} - E_2^{\perp} = 4\pi\sigma$

② $E_1^{\parallel} = E_2^{\parallel}$

② is automatically satisfied as $\vec{E}$ only points perpendicular to surface in both cases

* Defining region 1 as inside the sphere and region 2 as outside the sphere,

$$E_1^{\perp} = \vec{E}_1 \cdot \hat{n}\big|_{r=R} = E_1 \cdot \hat{r}\big|_{r=R} = k\pi R^2$$

$$E_2^{\perp} = \vec{E}_2 \cdot \hat{n}\big|_{r=R} = \vec{E}_1 \cdot \hat{r}\big|_{r=R} = k\pi R^2$$

* Since we know $\sigma = 0$, condition ① is also satisfied

#1 (cont.)

* See Griffiths Ch 2 for origin of formulas

d) We calculate the potential according to:

$$V(\vec{x}) = \int \frac{\rho(x)}{|x-x'|} d^3x' = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

* due to spherical symmetry, it is easier to use the second form and integrate along a line radially pointing inwards from infinity to r

$$V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

* since the field changes at $r = R$, we must proceed with 2 cases

* if $r > R$

$$V(r) = -\int_{\infty}^{r} \frac{\pi k R^4}{r'^2} dr'$$

$$= \frac{\pi k R^4}{r'} \Big|_{\infty}^{r}$$

$$= \frac{\pi k R^4}{r}$$

* if $r < R$

$$V(r) = -\left(\int_{\infty}^{R} \frac{\pi k R^4}{r'} dr' + \int_{R}^{r} k\pi r'^2 dr' \right)$$

$$= \frac{\pi k R^4}{r'} \Big|_{\infty}^{R} - \frac{\pi k r'^3}{3} \Big|_{R}^{r}$$

$$= \pi k R^3 - \left(\tfrac{1}{3}\pi k r^3 - \pi k R^3 \right)$$

$$= \tfrac{4}{3}\pi k R^3 - \tfrac{1}{3}\pi k r^3$$

e) We can find the work necessary to assemble this charge distribution according to

$$W = \frac{1}{8\pi} \int E^2 \, dV$$

$$= \frac{1}{2} \int_{0}^{\infty} E^2 r'^2 dr'$$

* again we must consider our two regions

#1 (cont.)

e) $W = \frac{1}{2}\left[\int_0^R E_1^2 r'^2 dr' + \int_R^\infty E_2^2 r'^2 dr' \right]$

$= \frac{1}{2}\left[\int_0^R k^2\pi^2 r'^6 dr' + \int_R^\infty k^2\pi^2 \frac{R^8}{r'^4} r'^2 dr' \right]$

$= \frac{1}{2}\left[k^2\pi^2 \frac{1}{7} r'^7 \Big|_0^R + k^2\pi^2 R^8 \left(\frac{-1}{r'} \Big|_R^\infty \right) \right]$

$= \frac{k^2\pi^2}{2}\left(\frac{1}{7} R^7 + R^7 \right)$

$= \frac{4k^2\pi^2 R^7}{7}$

f) If we replace the non-conducting sphere with a conducting sphere of equal charge

a) Field outside conductor remains unchanged

b) Field goes to 0 inside conductor

c) Boundary conditions are same, except all charge accumulates on surface

$\sigma = \frac{1}{4\pi R^2} \int \rho \, dV$

$= \frac{1}{4\pi R^2} 4\pi k \cdot \frac{1}{4} R^4$

$= \frac{1}{4} k R^2$

$E_1^\perp - E_2^\perp = 4\pi\sigma$

$0 - \frac{\pi k R^4}{r^2} = 4\pi \left(\frac{1}{4} k R^2 \right)$

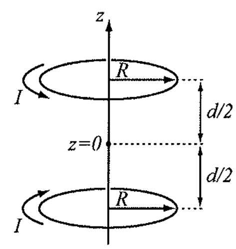
$-\pi k R^2 = \pi k R^2$ ✓

Problem 2: Magnetostatics

Consider a circular loop of radius R which carries a steady current I.

(a) Calculate the magnetic field a distance z above the center of the loop. [3 points]

(b) Now consider a configuration composed of two circular loops a distance d apart and with currents flowing in opposite directions, as shown in the figure. This configuration is know as anti-Helmholtz coils. Calculate the magnetic field along the z-axis as a function of z. [2 points]



(c) For what value of z will the the magnetic field due to the anti-Helmholtz coils be equal to zero? Give a physical explanation for your result. [2 points]

(d) Calculate the magnetic dipole moment of the configuration. [3 points]

Aug 2017

E+M #2

a) We calculate the magnetic field according to the Biot-Savart Law

$$\vec{B} = \frac{4\pi}{c}\int \frac{\vec{J}\times(\vec{r}-\vec{r}')}{|r-r'|^3}dr'$$

$$= \frac{4\pi}{c} I_0 \int \frac{dl \times (\vec{r}-\vec{r}')}{|r-r'|^3}$$

$$= \frac{4\pi}{c} I_0 \int \frac{R\,dl}{\sqrt{(R^2+z^2)^3}}\hat{z}$$ (from evaluation of cross product + symmetry of problem, leaving only z-component non-zero)

$$= \frac{4\pi}{c} I_0 \frac{2\pi R^2}{(R^2+z^2)^{3/2}}\hat{z}$$

$$= \frac{8\pi^2 I_0 R^2}{c(R^2+z^2)^{3/2}}\hat{z}$$

b) In our current configuration, we call the loop with the CCW current loop 1, and the loop with CW current loop 2

$-I_0$ b/c CW current
$\pm d/2$ added to reflect shift from origin

$$\Rightarrow B_t = B_1 + B_2$$

$$= \frac{8\pi^2 I_0 R^2}{c(R^2+[z-d/2]^2)^{3/2}}\hat{z} + \frac{8\pi^2(-I_0)R^2}{c(R^2+[z+d/2]^2)^{3/2}}\hat{z}$$

$$= \frac{8\pi^2 I_0 R^2}{c}\left[\frac{1}{(R^2+[z-d/2]^2)^{3/2}} - \frac{1}{(R^2+[z+d/2]^2)^{3/2}}\right]\hat{z}$$

c) $\vec{B}=0$ at $z=0$. This is due to the fields generated by each coil pointing in the opposite direction

d) The formula for the magnetic dipole moment is:

$$\vec{m} = I\int d\vec{a}$$

$$\hookrightarrow \vec{m}_{tot} = \vec{m}_1 + \vec{m}_2$$

$$= I_0\pi R^2\hat{z} - I_0\pi R^2\hat{z}$$

$= 0$? or $2I_0\pi R^2$ (depending on if we have positive or negative normal on $\vec{m}_2$ integral)

Problem 3: Waves

An electromagnetic wave with an angular frequency of ω passes from medium 1, through a slab of medium 2 (with thickness d), and into medium 3. All three media are linear and homogeneous, and have a permeability of μ_o. The index of refraction is n_1, n_2, and n_3 for medium 1, 2, and 3, respectively.

(a) In medium 1, there is an incident plane wave (electric field of amplitude E_I) and a reflected plane wave (electric field of amplitude E_R). In medium 3, there is a transmitted plane wave (electric field of amplitude E_T). In medium 2, there is a plane wave going toward medium 3 (electric field of amplitude E_r) and a plane wave going toward medium 1 (electric field of amplitude E_l). Write down expressions that describe the electric and magnetic fields in medium 1, medium 2, and medium 3. [4 points]

(b) Apply the boundary conditions for the electric and magnetic fields at the interface between medium 1 and medium 2. Express E_I in terms of E_r, E_l, n_1, and n_2. [1.5 points]

(c) Apply the boundary conditions for the electric and magnetic fields at the interface between medium 2 and medium 3. Express E_r in terms of E_T, n_2, n_3 and d. Also express E_l in terms of E_T, n_2, n_3 and d. [1.5 points]

(d) Combine your answers for part b and c to get an expression for E_T/E_I in terms of n_1, n_2, n_3, and d. What percentage of the incident wave is transmitted into medium 3? Express your answer in terms of n_1, n_2, n_3 and d. [1 points]

(e) Suppose medium 1 is water ($n_1 = 4/3$), medium 2 is glass ($n_2 = 3/2$), and medium 3 is air ($n_3 = 1$). Will the thickness of the glass make much difference in how well you (in medium 3) can see the fish (in medium 1)? How would your answer change if instead you were in the water and the fish was in the air? [2 points]

Problem 4: ED in media

In this problem we consider the propagation of an electromagnetic wave through an optically active media; a media that causes the direction of polarization to rotate about the direction of propagation. Such a media can be describe using the susceptibility tensor

$$\boldsymbol{\chi} = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix}.$$

Furthermore, recall that the polarization of the media and the electric field are related by $\vec{P} = \epsilon_0 \boldsymbol{\chi} \vec{E}$ in SI units. *You can assume a non-magnetic linear dielectric material with no free charges or currents.*

(a) Starting from the Maxwell equations, the expressions given in the statement of the problem, and assuming a transverse electromagnetic wave propagating in the z direction, derive the magnitude of the wave vector k for the two possible polarizations in terms of the components of $\boldsymbol{\chi}$. [4 points]

(b) Prove that the allowed wave modes correspond to circularly polarized waves. [2 points]

(c) Under the assumption that $\chi_{12} \ll \chi_{11}$, derive an expression for the amount that the polarization rotates over a distance ℓ. The approximation $\sqrt{1-\epsilon} \approx 1 + \epsilon/2$ might be useful. [4 points]

Problem 5: Radiation

Two oscillating dipole moments $\vec{d_1}$ and $\vec{d_2}$ are oriented parallel to each other in the direction of the y-axis and are separated by a distance L. They oscillate in phase at the same frequency ω. For an observer at a distance r with $r \gg L$ located at an angle θ with respect to the y-axis in the plane of the two dipoles.

(a) Show that [9 points]

$$\frac{dP}{d\Omega} = \frac{\omega^4 \sin\theta^2}{8\pi c^3}(d_1^2 + 2d_1 d_2 \cos\delta + d_2^2), \tag{1}$$

where

$$\delta = \frac{\omega L \sin\theta}{c} \tag{2}$$

and where P is the time-averaged power. The units are Gaussian.

(b) Show that when $L \ll \lambda$, the radiation can be approximated as from a single oscillating dipole of amplitude $d_1 + d_2$. [1 point]

Problem 6: Relativity

(a) Write down both the homogeneous and inhomogeneous Maxwell's equations in manifestly Lorentz covariant form using the 2nd rank field strength tensor $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. (State which system of units you are using) [2 points]

(b) Write down the components (in matrix form) of $F_{\mu\nu}$. [2 points]

(c) Lorentz transformations on a four-vector are given by $x'_\mu = \Lambda^\nu_\mu x_\nu$. Write down the form of Λ^μ_ν for a boost to velocity $\beta_x = v_x/c$ along the x-direction. [2 points]

(d) Using Λ^μ_ν, calculate the Lorentz transformation relations for $\vec{E}$ and $\vec{B}$ for a boost along the x-direction. [2 points]

(e) Depict the lines of electric field $\vec{E}$ from a point charge a) at rest and b) moving with some large velocity β_x. [2 points]

Aug 2017

E+M #6

a) The manifestly covariant form of Maxwell's eqns are (in SI units)

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

$$\partial_\alpha \left(\tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}\right) = 0$$

* Rest of problem in Gaussian

b) $$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

c) $$\Lambda^\mu_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d) $F'^{\alpha\beta} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F_{\mu\nu} \iff F' = \Lambda F \Lambda^T$ in matrix form

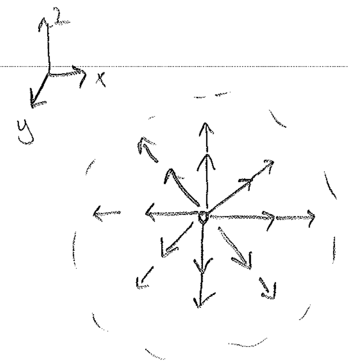
$$F' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \beta\gamma E_x & \gamma E_x & \gamma E_y + \beta\gamma B_z & \gamma E_z - \beta\gamma B_y \\ -\gamma E_x & -\beta\gamma E_x & -\beta\gamma E_y - \gamma B_z & -\beta\gamma E_z + \gamma B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

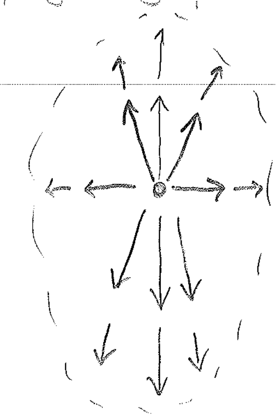
$$= \begin{bmatrix} 0 & (\underbrace{-\beta^2\gamma^2 + \gamma^2}_{1})E_x & \gamma E_y + \beta\gamma B_z & \gamma E_z - \beta\gamma B_y \\ (\underbrace{-\gamma^2 + \beta^2\gamma^2}_{-1})E_x & 0 & -\beta\gamma E_y - \gamma B_z & -\beta\gamma E_z + \gamma B_y \\ -\gamma E_y - \beta\gamma B_z & \beta\gamma E_y + \gamma B_z & 0 & -B_x \\ -\gamma E_z + \beta\gamma B_y & \beta\gamma E_z - \gamma B_y & B_x & 0 \end{bmatrix}$$

#6 (cont.)

e) For a point charge at rest



For a rapidly moving point charge



E & M Qualifier

January 10, 2018

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

G:

$\nabla \cdot E = 4\pi \rho$ $\qquad$ $\nabla \cdot B = 0$

$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$ $\qquad$ $\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c} j$

$\nabla \cdot D = 4\pi \rho_f$ $\qquad$ $\nabla \cdot B = 0$

$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t}$ $\qquad$ $\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$

$D = E + 4\pi P$

$H = B + 4\pi M$

Boundary's

$E_{\parallel} = 0 \qquad D_{\perp} = \sigma_b$

$B_{\perp} = 0 \qquad H_{\parallel} = \frac{4\pi}{c} K \times \hat{n}$

$S = E \times B$

$= E \times H$

Problem 1: Electrostatics

A wire of radius R_1 is insulated with a dielectric of outer radius R_2 that is itself enclosed in a grounded conducting sheath. Let the charge per unit length on the wire be λ.

1. Find an expression for the electric field, $\vec{E}$, on the wire at a radius ρ from the center of the wire. [3 points]

2. Find the voltage, V, between the inner and outer conductors. [2 points]

3. Calculate the force per unit volume on the insulating material in the coaxial cable. [3 points]

4. Estimate the size of the force for $R_1 = 1$ mm, $R_2 = 5$ mm, $\epsilon_r = 2.5$, and $V = 25,000$ volts. Is this force larger than the force of gravity if the dielectric has the same density as water (10^3 *kilograms/meter*^{3})? [2 points]

[Hint: The force per unit volume on a dielectric is given by $\frac{1}{2}(\epsilon - \epsilon_0)\nabla E^2$; also, $\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/(\mathrm{Nm}^2)$.]

$h_1 = 1 \quad h_2 = s \quad h_3 = 1$

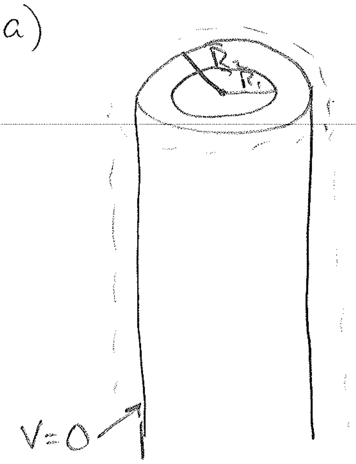
$$\nabla \Phi = \left\langle \frac{1}{h_1}\frac{\partial \Phi}{\partial s}, \frac{1}{h_2}\frac{\partial \Phi}{\partial \varphi}, \frac{1}{h_3}\frac{\partial \Phi}{\partial z} \right\rangle$$

Jan 2018

E + M #1

Gaussian

a)



$\nabla \cdot D = 4\pi \rho_f$

* if $s < R_1$:

$$\int \nabla \cdot E \, dV = \oint E \cdot d\vec{a} = \int 4\pi \rho_f$$

$$E \cdot 2\pi s L = 4\pi \lambda L \frac{s^2}{R_1^2}$$

$$\vec{E} = 2\lambda \frac{s}{R_1^2} \hat{\varphi}$$

* if $R_1 < s < R_2$:

$$\oint \vec{D} \cdot d\vec{a} = \int 4\pi \rho_f$$

$$D \cdot 2\pi s L = 4\pi \lambda L$$

$$\vec{D} = \frac{2\lambda}{s} \hat{\varphi}$$

* Assuming dielectric is linear, $D = \epsilon E$

$\hookrightarrow E = \frac{2\lambda}{\epsilon s} \hat{\varphi}$

* if $s > R_2$:

$\vec{E} = 0$ (identically true for conductors)

#1 (cont.)

b) $\Delta E = -\int_a^b \vec{E} \cdot dl$

$= -\int_{R_2}^{R_1} \vec{E} \cdot dl$

$= -\int_{R_2}^{R_1} \frac{2\lambda}{\epsilon s}\, ds$

$= -\frac{2\lambda}{\epsilon} \ln(s) \Big|_{R_2}^{R_1}$

$= \frac{2\lambda}{c} \ln(R_1/R_2)$

c) $F = qE \Rightarrow \frac{F}{V} = \frac{qE}{V}$

$\Rightarrow \frac{F}{V} = \frac{\lambda E}{\pi(R_2^2 - R_1^2)}$

$\frac{F}{V} = \frac{2\lambda^2}{\pi \epsilon s (R_2^2 - R_1^2)}$

Problem 2: Magnetostatics

An infinitely long circular cylinder of radius R (with its axis along the z-direction) carries a magnetization $\vec{M} = ks^2\hat{\phi}$, where k is a constant, s is the distance from the axis, and $\hat{\phi}$ is the azimuthal unit vector.

1. Find the bound currend densities ($\vec{K}_b$ and $\vec{J}_b$). [2 points]
2. Verify that the total bound current in the cylinder is zero. [2 points]
3. Find the magnetic field $\vec{B}$, due to $\vec{M}$, inside and outside the cylinder. [3 points]
4. Verify the boundary conditions for $\vec{B}$ at the interface ($s = R$). [3 points]

Jan 2018 $\underline{E+M \ \#2}$ Gaussian

a) $\vec{K}_b = \hat{M} \times \hat{n}$ $\quad \vec{M} = \langle 0, ks^2, 0 \rangle$

$\vec{J}_b = -\nabla \times M$

$$J_b = \langle \tfrac{1}{s}\partial_\varphi M_z - \partial_z M_\varphi,\ \partial_z M_s - \partial_s M_z,\ \tfrac{1}{s}\partial_s(sM_\varphi) - \partial_\varphi M_s \rangle$$

$$= \langle \tfrac{1}{s}\partial_\varphi(0) - \partial_z(ks^2),\ \partial_z(0) - \partial_s(0),\ \tfrac{1}{s}\partial_s(sks^2) - \partial_\varphi(0) \rangle$$

$$= \langle 0, 0, 3ks \rangle$$

$$\vec{K}_B = \begin{vmatrix} \hat{s} & \hat{\varphi} & \hat{z} \\ 0 & ks^2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, 0, -ks^2 \rangle$$

b) $0 = \oint K_B \cdot d\vec{a} + \int J_B \, dV$

$$= \int -ks^2 \cdot d\vec{a} + \int_0^s \int_0^{2\pi} \int_0^L 3ks \; s\,ds\,d\varphi\,dz$$

$$= -ks^2 \cdot 2\pi sL + ks^3\Big|_0^s 2\pi L$$

$$= -2\pi L k s^3 + 2\pi L k s^3$$

$$= 0 \checkmark$$

c) $\nabla \times B = \frac{4\pi}{c} J_f + J_b$

$$\int B \cdot dl = \frac{4\pi}{c} \int J_b \cdot d\vec{a}$$

*if $s < R$

$$B \cdot 2\pi s = \frac{4\pi}{c} \int 3ks \; s\,ds\,d\varphi$$

$$2\pi s B = \frac{8\pi^2}{c} 3k \cdot \tfrac{1}{3} s^3 \Big|_0^s$$

$$2\pi s B = \frac{8\pi^2 k s^3}{c} \Rightarrow B = \frac{4\pi k s^2}{c} \hat{\varphi}$$

#2 (cont)

c) * if $s > R$,

$\vec{B} = 0$ b/c total bound current is 0, as is free current

d) Our boundary conditions are:

$$B_1^{\parallel} - B_2^{\parallel} = \frac{4\pi}{c}\vec{K}$$

$$B_1^{\perp} - B_2^{\perp} = 0$$

* Defining region 1 as inside the cylinder and region 2 as outside

$$B_1^{\perp} - B_2^{\perp} = 0$$

$$0 - 0 = 0 \checkmark$$

$$B_1^{\parallel} - B_2^{\parallel} = \frac{4\pi}{c}\vec{K}$$

$$\frac{4\pi}{c}ks^2 - 0 = \frac{4\pi}{c}ks^2 \checkmark$$

5-10

Problem 3: Waves

Consider an electromagnetic wave propagating in a vacuum where there are no charges or electric currents, with electric field of $\vec{E}(z,t) = E_0 e^{i(kz-\omega t)}\hat{x}$.

1. Show that three of Maxwell's equations can be combined to give $\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$ for a region with no charges or electric currents. Show that three of Maxwell's equations can also lead to an analogous equation for the magnetic field $\vec{B}$. *Hint:* Start by calculating $\nabla \times (\nabla \times \vec{E})$. [2 points]

2. In the equation for $\vec{E}(z,t)$ how are ω and k related to μ_0 and ϵ_0? What is the equation for $\vec{B}(z,t)$ of the electromagnetic wave? Be explicit about how the amplitude, direction and phase of $\vec{B}$ are related to those of $\vec{E}$. [3 points]

   2.5?

3. Now suppose the wave propagates from vacuum into a dielectric material with permittivity of $\epsilon = \kappa\epsilon_0$, where κ is a positive constant. Assuming a normal angle of incidence at the vacuum/dielectric interface, calculate the amplitude of the electric field in the dielectric material. Express your answer in terms of E_0 and κ. [3 points]

4. What fraction of the incident energy is transmitted across the boundary? [2 points]

Jan 2018

E+M #3

a) In Gaussian units, Maxwell's eqns are:

① $\nabla \cdot E = 4\pi \rho$ ③ $\nabla \cdot B = 0$

② $\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$ ④ $\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$

*If we take $\nabla \times$ ②

$$\nabla \times \nabla \times E + \nabla \times \left(\frac{1}{c}\frac{\partial B}{\partial t}\right) = \nabla \times 0$$

$$\nabla(\nabla \cdot E) - \nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times B) = 0$$

$$\nabla(4\pi\rho) - \nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t}\right) = 0$$

*Since we are in a vacuum, $\rho = 0$, $\vec{J} = 0$

$$-\nabla^2 E + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \checkmark$$

*Note: $c^2 = \frac{1}{\mu_0 \epsilon_0}$

b) * From Jackson, $\frac{\omega}{k} = \frac{c}{n} = \frac{1}{\sqrt{\mu\epsilon}}$

↳ $\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ in a vacuum

*To calculate $\vec{B}$, we simply plug E into Maxwell's eqns

↳ Remember, in SI units (forced by problem), Maxwell's eqns are:

$\nabla \cdot E = \frac{\rho}{\epsilon}$ $\nabla \cdot B = 0$

$\nabla \times E + \frac{\partial B}{\partial t} = 0$ $\nabla \times B - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0 \vec{J}$

$$\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = \langle 0, \tfrac{\partial}{\partial z}E_x, 0 \rangle = \langle 0, ikE_0 \exp[i(kz-\omega t)], 0 \rangle$$

$$\Rightarrow ikE_0 \exp[i(kz-\omega t)]\hat{y} = -\frac{\partial B}{\partial t}$$

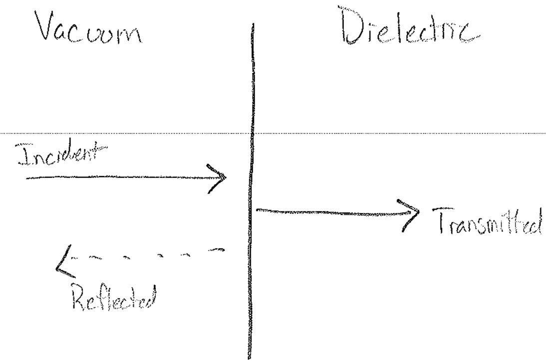
$$\int ikE_0 \exp[i(kz-\omega t)]\,\hat{y}\,dt = -\vec{B}$$

$$-\frac{k}{\omega}E_0 \exp[i(kz-\omega t)]y = -\vec{B}$$

$$\therefore \boxed{\vec{B} = \sqrt{\mu_0\epsilon_0}\,E_0 \exp[i(kz-\omega t)]\,\hat{y}}$$

#3 (cont.)

c)



*Across all boundaries,

$$\epsilon_1 E_{1,\perp} = \epsilon_2 E_{2\perp}$$

$$E_{1,\parallel} = E_{2,\parallel}$$

$$B_{1,\perp} = B_{2,\perp}$$

$$\frac{1}{\mu_1} B_{1,\parallel} = \frac{1}{\mu_2} B_{2,\parallel}$$

2+7

Problem 4: Maxwell Eq'n in 4-d

For this problem, use the following metric: $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$, and $g_{ij} = 0$ for $i \neq j$ and consider Maxwell's equations in four dimensions (4D)

$$\sum_\nu \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$$
$$\sum_\nu \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

1. ✓ (0.5 pt) Write down the field strength tensor $F^{\mu\nu}$, its dual tensor $G^{\mu\nu}$, and the 4-vector charge-current density J^μ in vector form in terms of $\mathbf{E}, \mathbf{B}, \rho, \mathbf{j}$, and c.
2. ✓ (0.5 pt) Explicitly derive the equation $\sum_\mu \partial J^\mu / \partial x^\mu = 0$ from Maxwell's equations in 4D. What is the physical meaning of this equation?
3. ✓ (1.0 pt) Show that the field tensor can be written as $F^{\mu\nu} = \partial A^\nu / \partial x_\mu - \partial A^\mu / \partial x_\nu$ by introducing a 4-vector potential A^μ.
4. ? (1.0 pt) Show that with the introduction of the 4-vector potential A^μ the 4D Maxwell equation involving $G^{\mu\nu}$ is automatically satisfied.
5. (1.0 pt) Impose the Lorentz gauge on A^μ and show that, with this gauge, the 4D Maxwell's equations reduce to the inhomogeneous 4D wave equation for the 4-vector potential.
6. ? (1.0 pt) Consider the Minkowski force acting on a charge q, $K^\mu = q\eta_\nu F^{\mu\nu}$, where η_ν is the proper velocity. Find the $\mu = 1, 2, 3$ components of K^μ in terms of $\mathbf{E}, \mathbf{B}$, q and c.
7. ? (0.5 pt) What is the physical meaning of the Minkowski force expression for $\mu = 1, 2, 3$?
8. ? (1.0 pt) Find the $\mu = 0$ component of the Minkoswki force in terms of $\mathbf{E}$ and $\mathbf{B}$, q and c.
9. ? (0.5 pt) What is the physical meaning of the Minkowski force expression for $\mu = 0$?
10. (3.0 pt) Consider a particle starting from rest at the origin under the influence of a constant Minkowski force in the x-direction. Find an implicit relativistic expression for the particle velocity v. Leave your answer in implicit form (t as a function of v).

Problem 5: Radiation

A current source $\vec{J}(\vec{x},t)$ is localized within a sphere of radius a near the origin of a co-ordinate system and oscillates with harmonic time dependence $e^{-i\omega t}$. Would an oscillating charge density all by itself ($\vec{J}=0$) contribute to the power radiated into the radiation zone? Why or why not? [1 point] ?

The vector potential (in SI units) is given by

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(x',t')}{|\vec{x}-\vec{x}'|}\delta(t'-(t-\frac{|\vec{x}-\vec{x}'|}{c}))d^3x'dt' \tag{1}$$

1. Integrate out the time dependence to find $\vec{A}(\vec{x})$. [1 point]

2. In the radiation zone, $|\vec{x}-\vec{x}'| \simeq r - \hat{n}\cdot\vec{x}'$. What is the vector potential in the radiation zone? [2 points]

3. What approximation must be made to gain the electric dipole contribution to $\vec{A}$? [2 points]

4. The electric dipole moment is given by $\vec{p}=\int \vec{x}'\rho(\vec{x}')d^3x'$. What is the vector potential in the radiation zone in terms of the EDP moment $\vec{p}$? [2 points]
   (*Hint:* $\partial_i(x_k J_i) = \delta_{ik}J_i + x_k\partial_i J_i$ and the equation of continuity for harmonic time dependence is given by $\nabla\cdot\vec{J}=i\omega\rho$.)

5. For a dipole $\vec{p}$ oriented along the z-axis, what angular distribution do you expect in the power radiated from EDP radiation? (You need not actually do the calculation.) [2 points]

Problem 6: Stress tensor

The manifestly covariant form of the electromagnetic field Lagrangian is given by $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$ in Gaussian units where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

? (a) For $\mathcal{L} = \mathcal{L}(A^\nu, \partial^\mu A^\nu)$, write down the Euler-Lagrange equations. [1 point]

(b) Apply these to derive the covariant form of the inhomogeneous Maxwell equations. [1 point]

$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$

✓ (c) From the Maxwell equations, show that the equation of continuity $\partial_\mu J^\mu = 0$ is satisfied. [2 points]

(d) List at least three important steps in deriving the symmetrized electromagnetic stress-energy tensor $\Theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} f_{\mu\nu} F^{\mu\nu}\right)$. [2 points]

(e) Express $\Theta^{\alpha\beta}$ in matrix form in terms of the EM energy density u, momentum density $c\vec{g}$ and the Maxwell stress tensor $T^M_{ij} = \frac{1}{4\pi}\left(E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(\vec{E}^2 + \vec{B}^2)\right)$. [2 points]

(f) Express the zeroth component of the conservation equation $\partial_\mu \Theta^{\mu\nu} = 0$ in terms of u and the Poynting vector $\vec{S} = c^2 \vec{g}$. What is the significance of this equation? [2 points]

$c^2 \, E \times B$

$\partial_\mu \sum_\nu \frac{\partial E^{\mu\nu}}{\partial x_\nu} = \partial_\mu \mu_0 J^\mu$

Jan 2018

E+M #6

a) General Euler-Lagrang eqn: $\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$

$$\Rightarrow \quad \frac{\partial \mathcal{L}(A^\nu, \partial^\mu A^\nu)}{\partial A^\nu} = \frac{d}{dt}\frac{\partial \mathcal{L}(A^\nu, \partial^\mu A^\nu)}{\partial(\partial^\mu A^\nu)}$$

b) The covariant form of the inhomogeneous Maxwell Eqns is: $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$

Given: $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

November 14, 2015

PHYS 5583 (E & M II)

Exam 2

Average = 43.3/100

Some useful formulas.

In class we have shown that the power radiated per unit solid angle into direction $\hat{\mathbf{n}}$ by an accelerating point particle is

$$\frac{\delta P}{\delta \Omega} = \frac{c\,q^2}{4\pi c^2} \frac{|\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^5},$$

and that it becomes

$$\begin{aligned} \frac{\delta P}{\delta \Omega} &= \frac{q^2}{4\pi c(1-\beta\cos\theta)^5} \Big\{ \sin^2\theta\,\dot{\beta}_{||}^2 \\ &+ \left[(1-\beta\cos\theta)^2 - \gamma^{-2}\sin^2\theta\cos^2\phi\right]\dot{\beta}_{\perp}^2 \\ &+ 2(\beta - \cos\theta)\sin\theta\cos\phi\dot{\beta}_{\perp}\dot{\beta}_{||} \Big\}, \end{aligned}$$

when the particle's velocity is in the $\hat{\mathbf{z}}$ direction and its acceleration is in the $\hat{\mathbf{z}}$-$\hat{\mathbf{x}}$ plane

$$\begin{aligned} \boldsymbol{\beta} &= \beta\,\hat{\mathbf{z}} \\ \dot{\boldsymbol{\beta}} &= \dot{\beta}_{||}\,\hat{\mathbf{z}} + \dot{\beta}_{\perp}\hat{\mathbf{x}}, \\ \hat{\mathbf{n}} &= \sin\theta\,[\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}] + \cos\theta\,\hat{\mathbf{z}}. \end{aligned}$$

We have also shown that when $\delta P/\delta\Omega$ is integrated over all directions the following expressions for the total power radiated result

$$\begin{aligned} P(t) \equiv \int \frac{\delta P}{\delta \Omega}\delta\Omega &= -\frac{2}{3}\frac{q^2}{c^3}a^\sigma a_\sigma, \\ &= \frac{2}{3}\frac{q^2}{c}\gamma^4\left[|\dot{\boldsymbol{\beta}}|^2 + \gamma^2(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\right], \\ &= \frac{2}{3}\frac{q^2}{c}\gamma^6\left[|\dot{\boldsymbol{\beta}}|^2 - |\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}|^2\right]. \end{aligned}$$

1 eV=1.6×10^{-12} ergs, e=4.8×10^{-10} statcoul, 1 erg = 10^{-7} Joules.

1. {20%, Ave=6.9/20}

An electron (Mc^2=0.5 MeV) is accelerated from rest ($\boldsymbol{\beta} = 0$) between the plates of a parallel plate capacitor. Assume it starts at $t = 0$ at $\mathbf{r} = 0$ on the grounded plate ($\Phi = 0$ Volts) located in the $z = 0$ plane and travel to the positive plate whose potential is $\Phi = V_0 = 5 \times 10^5$ Volts located at the $z = d = 10$ cm plane. The goal of this problem is to give two expressions for the power radiated per unit solid angle by the electron as a function of its spherical polar angle θ, one for when it leaves $z = 0$ and the second just as it reaches the positive plate at $z = d$. To obtain this result carry out the following steps:

(a) Calculate γ, β, and $\dot{\beta}$ of the electron as it moves between the capacitor's plates as a function of z. Hint: The Lorentz force equations for a charge moving in a uniform electric field $E_0 = V_0/d$ are

$$\frac{d}{dt}(m\gamma c^2) = mc^2\dot{\gamma} = qE^z\dot{z} = eE_0 c\beta,$$

$$\frac{d}{dt}(m\gamma\beta c) = mc(\dot{\gamma}\beta + \gamma\dot{\beta}) = eE_0.$$

(b) What are the starting ($z = 0$) numerical values and the finishing ($z = d$) numerical values for γ, β, and $\dot{\beta}$

(c) Give an expression for

$$\frac{dP}{d\Omega}(\theta, z),$$

as a function of β and $\dot{\beta}$, appropriate for this type of acceleration.

(d) Give the power radiated

$$\frac{dP}{d\Omega}(\theta, 0),$$

just after the electron leaves the grounded plate at $z = 0$ and sketch its shape as a function of θ.

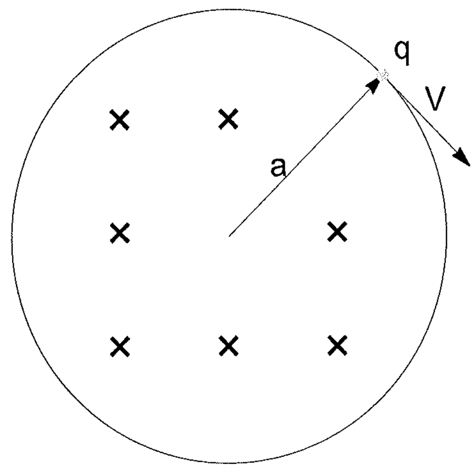
(e) Give the power radiated

$$\frac{dP}{d\Omega}(\theta, d),$$

just before the electron reaches the positive plate at $z = d$ and sketch its shape as a function of θ. For this part you will have to expand $\beta \approx 1 - \gamma^{-2}/2$ and $\cos(\theta) \approx 1 - \theta^2/2$. What is the angular opening of the radiation cone?

2. {20%, Ave=7.1/20}

A fast moving electron whose total energy is 10 MeV enters a uniform magnetic induction $\vec{B}_0 = B_0\hat{k}$ in the lab and moves in a circular orbit of radius $a = 10$ cm, orthogonal to the magnetic induction.



(a) What are the electron's γ and β values?

(b) What is the value of B_0?

(c) Find the time the electron takes to move around a complete circle as seen by a lab observer.

(d) How much total energy does the electron loose as radiation during one complete revolution?

(e) What fraction of the electron's total energy is lost to radiation during that one revolution?

$m_e c^2 = 0.5MeV$, 1 eV=1.6×10^{-12} ergs, e=4.8×10^{-10} statcoul.

3. {20%, Ave=13.8/20}

The Lienard-Weichert potentials for a point charge q moving on a world line $(t, \mathbf{r}_q(t))$ is

$$\begin{aligned} \phi(t,\mathbf{r}) &= \frac{q}{R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})} = \frac{q}{(R-\mathbf{R}\cdot\boldsymbol{\beta})}, \\ \mathbf{A}(t,\mathbf{r}) &= \frac{q\,\boldsymbol{\beta}}{R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})} = \frac{q\,\boldsymbol{\beta}}{(R-\mathbf{R}\cdot\boldsymbol{\beta})}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{R} &\equiv \mathbf{r} - \mathbf{r}_q(t_{ret}) = R\,\hat{\mathbf{n}}, \\ R &\equiv |\mathbf{r} - \mathbf{r}_q(t_{ret})|, \\ \boldsymbol{\beta} &\equiv \dot{\mathbf{r}}_q(t_{ret})/c. \end{aligned}$$

The retarded time $t_{ret}(t,\mathbf{r})$ is a function of the space-time point $(t,\mathbf{r})$ at which you are computing $\phi(t,\mathbf{r})$ and $\mathbf{A}(t,\mathbf{r})$ and is defined by the constraint equation

$$t_{ret} = t - \frac{|\mathbf{r} - \mathbf{r}_q(t_{ret})|}{c}.$$

In this problem your task is to compute the coulomb part of the electric field $\mathbf{E}_{coul}(t,\mathbf{r})$, i.e., the part of the electric field that falls off as $1/R^2$. To obtain this goal use the following two results (**do not** derive them):

$$\begin{aligned} \frac{\partial\, t_{ret}(t,\mathbf{r})}{\partial t} &= \frac{1}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})}, \\ c\,\nabla t_{ret}(t,\mathbf{r}) &= -\frac{\hat{\mathbf{n}}}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})}, \end{aligned}$$

and treat $\boldsymbol{\beta}$ as a constant.

(a) Keep $\boldsymbol{\beta}$ constant and evaluate the following (use these but **do not** derive them)

$$\nabla R = \hat{\mathbf{n}} - (\hat{\mathbf{n}}\cdot\boldsymbol{\beta})\,c\,\nabla t_{ret}$$
$$\nabla(\mathbf{R}\cdot\boldsymbol{\beta}) = \boldsymbol{\beta} - \beta^2\,c\,\nabla t_{ret}.$$

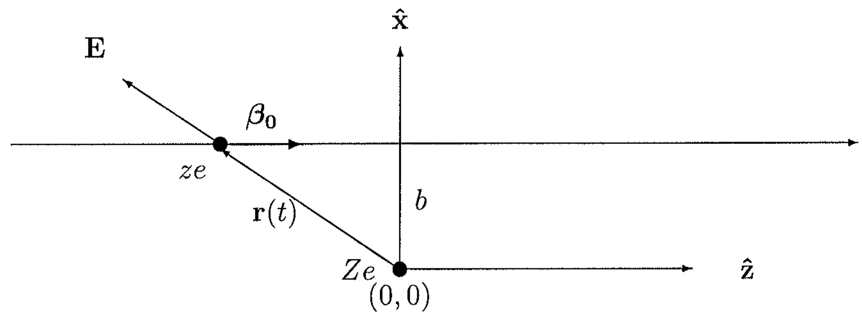
(b) Next keep $\boldsymbol{\beta}$ constant and evaluate the following (use these but **do not** derive them)

$$\frac{\partial R}{\partial t} = -(\hat{\mathbf{n}}\cdot\boldsymbol{\beta})\,c\,\frac{\partial t_{ret}}{\partial t},$$
$$\frac{\partial(\mathbf{R}\cdot\boldsymbol{\beta})}{\partial t} = -\beta^2\,c\,\frac{\partial t_{ret}}{\partial t}.$$

(c) Use your results from (a) and (b) to evaluate the coulomb part of

$$\mathbf{E}(t,\mathbf{r}) = -\nabla\phi(t,\mathbf{r}) - \frac{1}{c}\frac{\partial\mathbf{A}(t,\mathbf{r})}{\partial t}.$$

4. {20%, Ave=7.0/20}



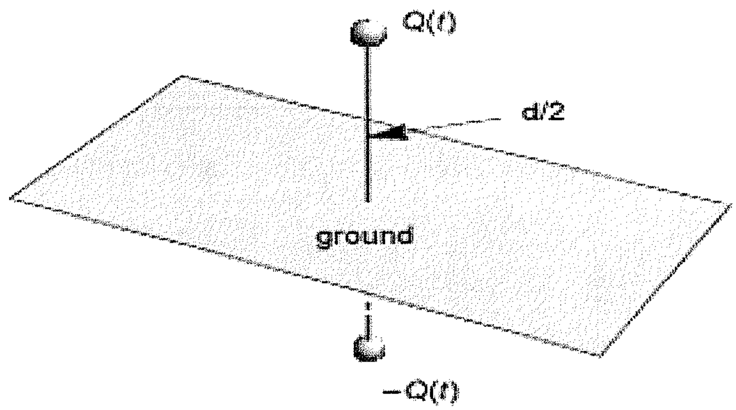
A swiftly moving particle of charge ze and mass m passes a fixed point charge Ze approximately in a straight line path at impact parameter b and nearly constant constant speed v_0. Show that the total energy radiated in the encounter is

$$\int_{-\infty}^{+\infty} P(t)\,dt = \Delta W = \frac{\pi z^4 Z^2 e^6}{4m^2 c^4 \beta_0}\left(\gamma^2 + \frac{1}{3}\right)\frac{1}{b^3}.$$

Hint:

$$\int_0^{\infty} \frac{1}{(\xi^2+1)^3}\,d\xi = \frac{3\pi}{16}$$

$$\int_0^{\infty} \frac{\xi^2}{(\xi^2+1)^3}\,d\xi = \frac{\pi}{16}$$



5. {20%, Ave=7.9/20} A dipole antenna is constructed by attaching a large metal sphere at the top of an antenna tower (height = $d/2$) by a long wire to an ac voltage source $\propto \cos(\omega t)$ at the antenna's base. The ac voltage has the effect of giving the sphere a time dependent charge $Q(t) = Q_0 \cos(\omega t)$. The earth acts as a grounded conductor producing an image charge $-Q_0 \cos(\omega t)$ at a depth of $-d/2$ below the surface.

   (a) Treat this antenna as a point time-dependent electric dipole and compute the dipole's current density

   $$\mathbf{J}(t, \mathbf{r}) = \dot{\mathbf{p}}(t)\, \delta^3(\mathbf{r}).$$

   (b) Calculate the vector potential for this current density

   $$\mathbf{A}(t, \mathbf{r}) = \frac{1}{c} \int \frac{\mathbf{J}(t - |\mathbf{r} - \mathbf{r}'|/c, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'.$$

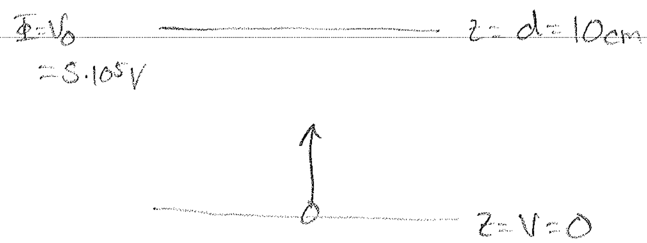
   (c) Calculate **only** the radiation part of the magnetic induction $\mathbf{B}_{rad} \propto 1/r$.

   (d) Calculate **only** the radiation part of the electric field using $\mathbf{E}_{rad} = \mathbf{B}_{rad} \times \hat{\mathbf{r}}$.

   (e) Give the time average radiation pattern $\delta P/\delta\Omega$ for this antenna.

Exam 2 2015

#1



a) $F^\alpha = \frac{dp^\alpha}{d\tau} = \frac{dp^\alpha}{\gamma^{-1}dt}$ $\qquad p^\alpha = \begin{bmatrix} mc^2\gamma \\ mc\gamma\vec{\beta} \end{bmatrix} \Rightarrow \begin{bmatrix} d\mathcal{E}/dt \cdot \frac{1}{c} \\ d\vec{p}/dt \end{bmatrix}$

$$\frac{d\mathcal{E}}{c\,dt} = qE\cdot v$$

$$= -qE_0\dot{z}$$

$$= \frac{d}{dt} mc^2\gamma$$

$$\Rightarrow mc^2\dot{\gamma} = -qE_0$$

Exam 2 2015

#2

a) $\gamma = \frac{E_{tot}}{E_{rest}}$

$= \frac{10\text{ MeV}}{0.5\text{ MeV}}$

$= 20$

$\gamma^2 = \frac{1}{1-\beta^2}$

$\gamma^2(1-\beta^2) = 1$

$\beta^2 = \gamma^{-2} - 1$

$\beta = \sqrt{\gamma^{2}-1}$

$\beta = \sqrt{(\frac{1}{20})^2 - 1}$

$= \sqrt{\frac{1}{400} - 1}$

$= \sqrt{399}$

b) $F^\alpha = m^\alpha = \frac{dp^a}{d\tau} = \frac{dp^a}{\gamma^{-1}dt}$

$= \gamma \begin{bmatrix} \frac{1}{c}\frac{dE}{dt} \\ \frac{dp}{dt} \end{bmatrix}$

$= \gamma_q \begin{bmatrix} \frac{1}{c}\cancel{\vec{E}}^{0}\cdot\vec{v} \\ \cancel{E}^{0} + \beta \times B \end{bmatrix}$

$F = \frac{mv^2}{a} = \gamma q \vec{\beta} \times B$

$\frac{mv^2}{a} \quad q\beta\gamma B_0$

$\sin\theta = 1 \quad b/c \quad \theta = \pi/2$

$\Rightarrow B_0 = \frac{mv^2}{qa\beta\gamma}$

$= \frac{mc^2\beta}{q\gamma a}$

Exam 2 2015

#2 (cont.)

c) $T = \frac{2\pi a}{c\beta}$

d) $P = \frac{2}{3}\frac{q^2}{c}\gamma^4\left[|\dot{\vec{\beta}}|^2 + \gamma^2(\vec{\beta}\cdot\dot{\vec{\beta}})\right.$

Exam 2 2015

#3

$$\varphi(t,\vec r) = \frac{q}{R(1-\hat n\cdot\beta)} = \frac{q}{(R-\vec R\cdot\vec\beta)} \qquad \frac{\partial t_{ret}}{\partial t} = \frac{1}{(1-\hat n\cdot\beta)}$$

$$A(t,\vec r) = \frac{q\beta}{R(1-\hat n\cdot\beta)} = \frac{q\beta}{(R-\vec R\cdot\vec\beta)} \qquad c\nabla t_{ret} = \frac{-\hat n}{(1-\hat n\cdot\beta)}$$

a) $$\nabla R = \hat n - (\hat n\cdot\beta)\, c\nabla t_{ret}$$
$$= \hat n - (\hat n\cdot\beta)\left(\frac{-\hat n}{1-\hat n\cdot\beta}\right)$$

$$\nabla(R\cdot\beta) = \beta - \beta^2 c\nabla t_{ret}$$
$$= \beta - \beta^2\left(\frac{-\hat n}{1-\hat n\cdot\beta}\right)$$

b) $$\frac{\partial R}{\partial t} = (-\hat n\cdot\beta)\, c\,\frac{\partial t_{ret}}{\partial t}$$
$$= (-\hat n\cdot\beta)\, c\,\frac{1}{(1-\hat n\cdot\beta)}$$

$$\frac{\partial(R\cdot\beta)}{\partial t} = -\beta^2 c\,\frac{\partial t_{ret}}{\partial t}$$
$$= \frac{-\beta^2 c}{(1-\hat n\cdot\beta)}$$

c) $$E(t,\vec r) = -\nabla\varphi(t,r) - \frac{1}{c}\frac{\partial A}{\partial t}$$

$$= -\nabla\left(\frac{q}{R-\vec R\cdot\vec\beta}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{q\beta}{R-R\cdot\beta}\right)$$

$$= +\frac{q\,\nabla(R-R\cdot\beta)}{(R-R\cdot\beta)^2} + \frac{1}{c}\,\frac{q\beta\,\frac{\partial}{\partial t}(R-R\cdot\beta)}{(R-R\cdot\beta)^2}$$

$$= \frac{q\left[\left(\hat n-(\hat n\cdot\beta)\left(\frac{\hat n}{1-\hat n\cdot\beta}\right)\right)-\left(\beta+\beta^2\left(\frac{-\hat n}{(1-\hat n\cdot\beta)}\right)\right)\right]}{(R-R\cdot\beta)^2} + \frac{1}{c}\,\frac{q\beta}{(R-R\cdot\beta)^2}\left[c\left(\frac{-(\hat n\cdot\beta)}{1-\hat n\cdot\beta}\right)+\frac{-\beta^2 c}{(1-\hat n\cdot\beta)}\right]$$

$$= \frac{q}{(R-R\cdot\beta)^2}\left[\hat n - \frac{\hat n(\hat n\cdot\vec\beta)}{1-\hat n\cdot\beta} - \beta + \frac{-\hat n\beta^2}{1-\hat n\cdot\beta} + \frac{-\beta(\hat n\cdot\beta)}{1-\hat n\cdot\beta} + \frac{\beta^2\vec\beta}{(1-\hat n\cdot\beta)}\right]$$

$$= \frac{q}{R^2(1-\hat n\cdot\beta)^3}\left[(\hat n-\beta)(1-\hat n\cdot\beta) - \hat n(\hat n\cdot\beta) - \hat n\beta^2 - \beta(\hat n\cdot\beta) + \beta^2\beta\right]$$

Want: $\frac{q(\hat{n}-\beta)}{R^2\gamma^2(1-\hat{n}\cdot\beta)^3}$

#3 (cont.)

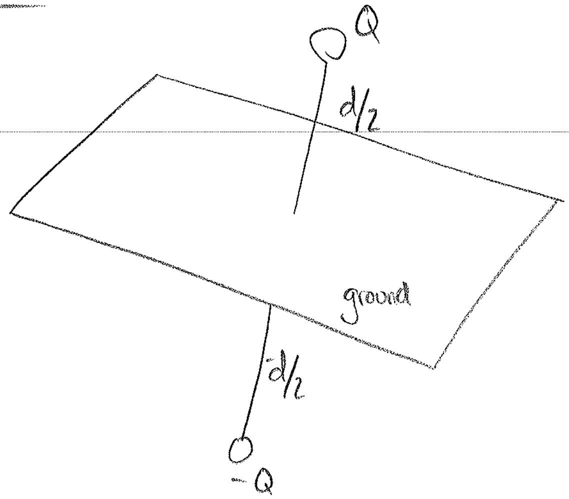
$$= \frac{q}{R^2(1-\hat{n}\cdot\beta)^3}\left[(\hat{n}-\beta)(1-\hat{n}\cdot\beta) - (\hat{n}+\beta)(\hat{n}\cdot\beta) + (\hat{n}+\beta)\beta^2\right]$$

$$= \frac{q}{R^2(1-\hat{n}\cdot\beta)^3}\left[(\hat{n}-\beta)(1-\hat{n}\cdot\beta) - (\hat{n}-\beta)(\hat{n}\cdot\beta-\beta^2)\right]$$

$$= \frac{q(\hat{n}-\beta)}{R^2\gamma^2(1-\hat{n}\cdot\beta)^3}$$

2015 Exam 2

#5



a) $\vec{J}(\vec{r},t) = \dot{\vec{p}}(t)\,\delta^3(\vec{r})$

$$\vec{p} = \sum_a e_a r_a$$

$$= Q_0\cos(\omega t)\frac{d}{2} - Q_0\cos(\omega t)\left(\frac{-d}{2}\right)$$

$$= Q_0 d\cos(\omega t)$$

$$\dot{\vec{p}} = -Q_0 d\omega\sin(\omega t)$$

$$\Rightarrow \vec{J}(\vec{r},t) = -Q_0 d\omega\sin(\omega t)\,\delta^3(\vec{r})$$

b) $\vec{A}(t,\vec{r}) = \frac{1}{c}\int \frac{1}{|\vec{r}-\vec{r}'|}\vec{J}\left(t-\frac{|\vec{r}-\vec{r}'|}{c},\vec{r}'\right)d^3\vec{r}'$

$$= \frac{1}{c}\int \frac{1}{|\vec{r}-\vec{r}'|} - Q_0 d\omega\sin\left(\omega\left[t-\frac{|\vec{r}-\vec{r}'|}{c}\right]\right)\delta^3(\vec{r}')\,d^3\vec{r}'$$

$$= \frac{-Q_0 d\omega}{c}\frac{1}{r}\sin\left(\omega\left[t-\frac{r}{c}\right]\right)$$

c) $\vec{B} = \vec{\nabla}\times\vec{A}$

$$= \nabla\times\left(\frac{-Q_0 d\omega}{c}\cdot\frac{1}{r}\sin\left(\omega\left[t-\frac{r}{c}\right]\right)\right)$$

$$= \frac{-Q_0 d\omega}{cr}\vec{\nabla}\times\sin\left(\omega\left[t-\frac{r}{c}\right]\right)$$

* but $r = r\hat{n}$

$$= \frac{-Q_0 d\omega}{cr}\left[-\frac{\hat{n}}{c}\times\omega\cos\left(\omega\left[t-\frac{r}{c}\right]\right)\right]$$

$$= \frac{Q_0 d\omega^2}{c^2 r}\left[\hat{n}\times\cos\left(\omega\left[t-\frac{r}{c}\right]\right)\right]$$

#5 (cont.)

d) $E_{rad} = B_{rad} \times \hat{n}$

$$= \frac{dQ_0 \omega^2}{c^2 r} \left(\hat{n} \times \cos(\alpha)\right) \times \hat{n}$$

$$= -\frac{dQ_0 \omega^2}{c^2 r} \hat{n} \times \left(\hat{n} \times \cos(\alpha)\right)$$

$$= \frac{-dQ_0 \omega^2}{c^2 r} \left[\hat{n}\left(\hat{n} \cdot \cos(\alpha)\right) - \cos(\alpha)\left(\hat{n} \cdot \hat{n}\right)\right]$$

$$= \frac{-dQ_0 \omega^2}{c^2 r} \cos(\alpha) \left[\hat{n}\left(\hat{n} \cdot \hat{z}\right) - \hat{z}\right]$$

$$= \frac{-dQ_0 \omega^2}{c^2 r^2} \cos\left(\omega\left[t - \frac{r}{c}\right]\right) \left[\cos\theta \, \hat{n} - \hat{z}\right]$$

2015 Exam II

Problem 1

a)

$\uparrow \hat{z}$ $\quad \vec{\beta} \propto \dot{\vec{\beta}} \propto \hat{z}$

$\vec{E} = E_0 \hat{z}$

$q = -e$

a) $F^\alpha = m a^\alpha = \dfrac{dp^\alpha}{d\tau} = \gamma \dfrac{dp^\alpha}{dt} = \gamma \begin{pmatrix} \frac{1}{c}\frac{d\epsilon}{dt} \\ \frac{d\vec{p}}{dt} \end{pmatrix} =$

(1) $\boxed{\dfrac{d\epsilon}{dt} = q\vec{E}\cdot\vec{v}} = -qE_0\dot{z} = \dfrac{d}{dt}(\gamma m c^2)$

$-qE_0\dot{z} = mc^2\dot{\gamma}$

$-qE_0\dfrac{dz}{dt} = mc^2\dfrac{d\gamma}{dt}$

$-qE_0(z - \underbrace{z_0}_{=0}) = mc^2(\gamma - \gamma_0)$ ← from rest $\Rightarrow \gamma_0 = 1$

$\therefore \boxed{\gamma = 1 + \dfrac{\cancel{q}\,e\,E_0 z}{mc^2}} \leftarrow q = -e$

$\beta = \sqrt{1 - \gamma^{-2}}$

$= \sqrt{1 - \dfrac{1}{\left(1 + \frac{qE_0 z}{mc^2}\right)^2}}$

2015
1-3

c)

$$\frac{dP}{d\Omega}(\theta, z)$$

we know $\dot\beta_\perp = 0$

$$\therefore \boxed{\frac{dP}{d\Omega} = \frac{q^2}{4\pi c(1-\beta\cos\theta)^5}\sin^2\theta\, \dot\beta_\parallel^2}$$

$\beta(z)$ from a) $\quad |\dot\beta|^2$

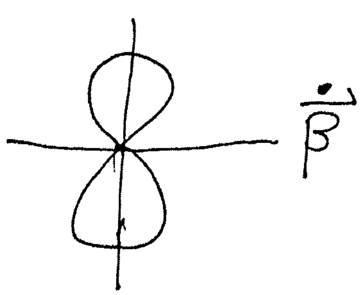
d) $z = 0 \Rightarrow \gamma = 1,\ \beta = 0,\ \dot\beta = \frac{eE_0}{mc}$

$$\cancel{\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\sin^2\theta\left(\frac{eE_0}{mc}\right)}$$

$\beta = 0$ non relativistic

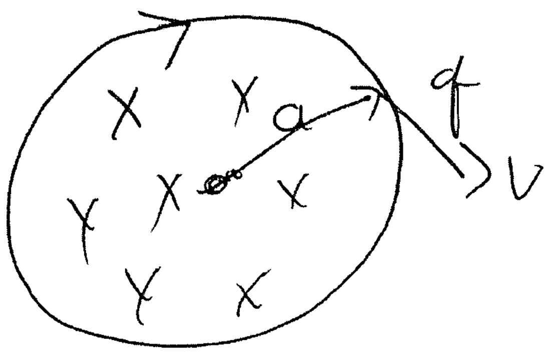
$\therefore$



2015 Exam II

Problem 2

$\vec{E} = 0$



$$\frac{d\epsilon}{dt} = q\,\vec{E}\cdot\vec{v} = 0$$

$$\frac{d}{dt}(m\gamma c^2) = 0 \qquad \boxed{\gamma = \text{const}}$$

a) γ, β

$$\vec{r}(t) = a\cos\omega_0 t\,\hat{x} + a\sin\omega_0 t\,\hat{y}$$

$$\vec{\beta} = \frac{\dot{\vec{r}}}{c} = -\frac{a\omega_0}{c}\sin\omega_0 t\,\hat{x} + \frac{a\omega_0}{c}\cos\omega_0 t\,\hat{y}$$

$$\dot{\vec{\beta}}(t) = -\frac{a\omega_0^2}{c}\cos(\omega_0 t)\hat{x} + \frac{a\omega_0^2}{c}\sin(\omega_0 t)\hat{y}$$

$$\gamma = \frac{\gamma mc^2}{mc^2} = \frac{10\,\text{MeV}}{0.5\,\text{MeV}} = 20 \qquad \boxed{\gamma = 20}$$

$$\beta = \sqrt{1 - \frac{1}{20^2}} \Rightarrow \boxed{\beta = 0.9987}$$

b) B_0

$$F^\alpha = ma^\alpha = \frac{dp^\alpha}{d\tau} = \frac{\gamma\, dp^\alpha}{dt} = \gamma\begin{pmatrix}\frac{1}{c}\frac{d\epsilon}{dt} \\ \frac{d\vec{p}}{dt}\end{pmatrix} = \gamma q\begin{pmatrix}\frac{1}{c}\cancel{\vec{E}\cdot\vec{v}} \\ \cancel{\vec{E}} + \vec{\beta}\times\vec{B}\end{pmatrix} \qquad \vec{E}=0$$

$$\vec{F} = \frac{mv^2}{a} = \gamma q\,\vec{\beta}\times\vec{B}$$

3-1

2015

2015 Exam II

Problem 3

a) $\nabla R = \hat{n} - (\hat{n}\cdot\vec{\beta})\, c\nabla t_{ret}$

$$= \hat{n} - (\hat{n}\cdot\vec{\beta})\left(-\frac{\hat{n}}{1-\hat{n}\cdot\vec{\beta}}\right)$$

$$= \frac{(1-\hat{n}\cdot\vec{\beta})\,\hat{n} + (\hat{n}\cdot\vec{\beta})\,\hat{n}}{1-\hat{n}\cdot\vec{\beta}}$$

$$= \frac{\hat{n} - (\hat{n}\cdot\vec{\beta})\,\hat{n} + (\hat{n}\cdot\vec{\beta})\,\hat{n}}{1-\hat{n}\cdot\vec{\beta}}$$

$$\boxed{\nabla R = \frac{\hat{n}}{1-\hat{n}\cdot\vec{\beta}}}$$

$$\nabla(\vec{R}\cdot\vec{\beta}) = \vec{\beta} - \beta^2\left(\frac{-\hat{n}}{1-\hat{n}\cdot\vec{\beta}}\right)$$

$$\boxed{\nabla(\vec{R}\cdot\vec{\beta}) = \frac{\vec{\beta}(1-\hat{n}\cdot\vec{\beta}) + \beta^2\hat{n}}{1-\hat{n}\cdot\vec{\beta}}}$$

b) $$\boxed{\frac{dR}{dt} = -c\,\frac{\hat{n}\cdot\vec{\beta}}{(1-\hat{n}\cdot\vec{\beta})}}$$

$$\boxed{\frac{d(\vec{R}\cdot\vec{\beta})}{dt} = \frac{-\beta^2 c}{(1-\hat{n}\cdot\vec{\beta})}}$$

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$$= \frac{q\vec{\beta}}{c} \frac{\cancel{2}}{\cancel{R}} \frac{1}{R^2(1-\hat{n}\cdot\vec{\beta})^2}\left[\frac{-c\,\hat{n}\cdot\vec{\beta}}{1-\hat{n}\cdot\vec{\beta}} + \frac{\beta^2 c}{(1-\hat{n}\cdot\vec{\beta})}\right]$$

$$\left.= \frac{q\vec{\beta}\,[\beta^2-\hat{n}\cdot\vec{\beta}]}{R^2(1-\hat{n}\cdot\vec{\beta})^3}\right\}$$

$$\therefore \vec{E}_{cal} = \frac{q}{R^2(1-\hat{n}\cdot\vec{\beta})^3}\left[\gamma^{-2}\hat{n} - \vec{\beta}(1-\hat{n}\cdot\vec{\beta}) + (\beta^2-\hat{n}\cdot\vec{\beta})\vec{\beta}\right]$$

$$= \frac{q}{R^2(1-\hat{n}\cdot\vec{\beta})^3}\left[\gamma^{-2}\hat{n} - \vec{\beta} + \cancel{\vec{\beta}(\hat{n}\cdot\vec{\beta})} + \beta^2\vec{\beta} - \cancel{\vec{\beta}(\hat{n}\cdot\vec{\beta})}\right]$$

$$= \frac{q}{R^2(1-\hat{n}\cdot\vec{\beta})^3}\left[\gamma^{-2}\hat{n} - \underbrace{(1-\beta^2)}_{\gamma^{-2}}\vec{\beta}\right]$$

$$\boxed{\vec{E}_{cal} = \frac{q\,\gamma^{-2}}{R^2(1-\hat{n}\cdot\vec{\beta})^3}\left[\hat{n}-\vec{\beta}\right]}$$

2015 4-2

$$\vec{E} = \frac{(Ze)}{r^2}\hat{r} = \frac{(Ze)\,\vec{r}}{r^3}$$

$$\vec{r} = (V_0 t)\hat{z} + b\hat{x} \qquad \vec{\beta} = \frac{V_0}{c}\hat{z}$$

$$r^2 = (V_0 t)^2 + b^2$$

$$(\vec{E})^2 = (\vec{E}\cdot\vec{E}) = \frac{(Ze)^2\left[(V_0 t)^2 + b^2\right]}{\left[(V_0 t)^2 + b^2\right]^3} = \frac{(Ze)^2}{\left[(V_0 t)^2 + b^2\right]^2}$$

$$(\vec{E}\cdot\vec{\beta})^2 = \left(\frac{Ze}{r^3}\,\frac{V_0^2 t}{c}\right)^2 = \frac{(Ze)^2}{r^6}\,\frac{V_0^4 t^2}{c^2} = \frac{(Ze)^2\beta^2 V_0^2 t^2}{\left[(V_0 t)^2 + b^2\right]^3}$$

$$\therefore P(t) = \frac{2}{3}\frac{\gamma^2 q^4}{m^2 c^3}\left\{\frac{(Ze)^2}{\left[(V_0 t)^2 + b^2\right]^2} - \frac{(Ze)^2\beta^2 V_0^2 t^2}{\left[(V_0 t)^2 + b^2\right]^3}\right\}$$

$$= \underbrace{\frac{2}{3}\frac{(ze)^4 (Ze)^2\gamma^2}{m^2 c^3}}_{K}\left\{\frac{1}{\left[(V_0 t)^2 + b^2\right]^2} - \frac{\beta^2 V_0^2 t^2}{\left[(V_0 t)^2 + b^2\right]^3}\right\}$$

$$\Delta W = \int_{-\infty}^{\infty} P\,dt = K\int_{-\infty}^{\infty} \frac{1}{\left[\left(\frac{V_0 t}{b}\right)^2 + 1\right]^2 b^4} - \frac{\beta^2 V_0^2 t^2}{\left[\left(\frac{V_0 t}{b}\right)^2 + 1\right]^3 b^6}\,dt$$

$$\text{let } \frac{Vt}{b} = \xi \Rightarrow t = \frac{b\xi}{V} \Rightarrow dt = \frac{b}{V}\,d\xi$$

October 31, 2014

PHYS 5583 (E & M II)

Exam 2

Some useful formulas.

In class we have shown that the power radiated per unit solid angle into direction $\hat{\mathbf{n}}$ by an accelerating point particle is

$$\frac{\delta P}{\delta \Omega} = \frac{c\, q^2}{4\pi c^2} \frac{|\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^5},$$

and that it becomes

$$\begin{aligned} \frac{\delta P}{\delta \Omega} &= \frac{q^2}{4\pi c (1 - \beta \cos\theta)^5} \Big\{ \sin^2\theta \, \dot{\beta}_{||}^2 \\ &+ \left[(1 - \beta\cos\theta)^2 - \gamma^{-2} \sin^2\theta \cos^2\phi \right] \dot{\beta}_{\perp}^2 \\ &+ 2(\beta - \cos\theta) \sin\theta \cos\phi \dot{\beta}_{\perp} \dot{\beta}_{||} \Big\}, \end{aligned}$$

when the particle's velocity is in the $\hat{\mathbf{z}}$ direction and its acceleration is in the $\hat{\mathbf{z}}$-$\hat{\mathbf{x}}$ plane

$$\begin{aligned} \boldsymbol{\beta} &= \beta\, \hat{\mathbf{z}} \\ \dot{\boldsymbol{\beta}} &= \dot{\beta}_{||}\, \hat{\mathbf{z}} + \dot{\beta}_{\perp} \hat{\mathbf{x}}, \\ \hat{\mathbf{n}} &= \sin\theta \left[\cos\phi\, \hat{\mathbf{x}} + \sin\phi\, \hat{\mathbf{y}} \right] + \cos\theta\, \hat{\mathbf{z}}. \end{aligned}$$

We have also shown that when $\delta P / \delta \Omega$ is integrated over all directions the following expressions for the total power radiated result

$$\begin{aligned} P(t) \equiv \int \frac{\delta P}{\delta \Omega} \delta \Omega &= -\frac{2}{3} \frac{q^2}{c^3} a^\sigma a_\sigma, \\ &= \frac{2}{3} \frac{q^2}{c} \gamma^4 \left[|\dot{\boldsymbol{\beta}}|^2 + \gamma^2 (\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})^2 \right], \\ &= \frac{2}{3} \frac{q^2}{c} \gamma^6 \left[|\dot{\boldsymbol{\beta}}|^2 - |\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}|^2 \right]. \end{aligned}$$

1. {20%}

A point charge q moves along the z-axis according to $z(t) = a\cos(\omega_0 t)$.

(a) Using spherical polar angles, compute the power radiated by the point particle into the θ, ϕ direction , $\delta P/\delta\Omega$, as a function of time t.

(b) Compute the total power radiated $P(t)$ by this particle as a function of time t.

(c) Compute the average power radiated per oscillation into the θ, ϕ direction , i.e., compute

$$\frac{1}{(2\pi/\omega_0)}\int_0^{(2\pi/\omega_0)} \frac{\delta P}{\delta\Omega}\, dt. \tag{1}$$

Hint:

$$\int_0^{2\pi} \frac{\cos^2\alpha}{(1+A\sin\alpha)^5}\, d\alpha = \frac{\pi}{4}\frac{(4+A^2)}{(1-A^2)^{7/2}}. \tag{2}$$

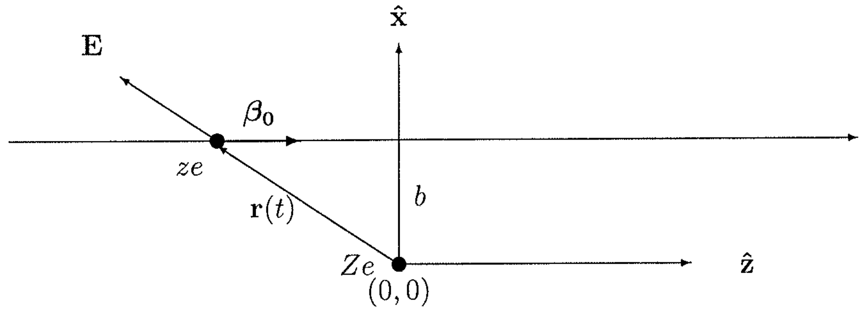
(d) Compute the average total power radiated per oscillation into the θ, ϕ direction , i.e., compute

$$\frac{1}{(2\pi/\omega_0)}\int_0^{(2\pi/\omega_0)} P(t)\, dt. \tag{3}$$

Hint:

$$\int_0^{2\pi} \frac{\cos^2\alpha}{(1-A\sin^2\alpha)^3}\, d\alpha = \frac{\pi}{4}\frac{(4-3A)}{(1-A)^{3/2}}. \tag{4}$$

2. {20%}



A swiftly moving particle of charge ze and mass m passes a fixed point charge Ze approximately in a straight line path at impact parameter b and nearly constant constant speed v_0. Show that the total energy radiated in the encounter is

$$\int_{-\infty}^{+\infty} P(t)\,dt = \Delta W = \frac{\pi z^4 Z^2 e^6}{4m^2c^4\beta_0}\left(\gamma^2 + \frac{1}{3}\right)\frac{1}{b^3}.$$

Hint:

$$\int_0^{\infty} \frac{1}{(\xi^2+1)^3}\,d\xi = \frac{3\pi}{16}$$
$$\int_0^{\infty} \frac{\xi^2}{(\xi^2+1)^3}\,d\xi = \frac{\pi}{16}$$

3. {20%}

A time dependent electric dipole, located at the origin has 4-current

$$J^{\alpha}(t,\mathbf{r}) = \Big[-c\,\mathbf{p(t)}\cdot\vec{\nabla}\delta^3(\mathbf{r}),\ \dot{\mathbf{p}}(\mathbf{t})\delta^3(\mathbf{r})\Big],$$

and a retarded 4-potential (in Gaussian units)

$$A^{\alpha}(t,\mathbf{r}) = \left[-\nabla\cdot\left\{\frac{\mathbf{p}(t-r/c)}{r}\right\},\ \frac{\dot{\mathbf{p}}(t-r/c)}{cr}\right].$$

(a) Compute the radiation part $\mathbf{B}_{rad}$ (i.e., the $\sim 1/r$ part) of the magnetic induction for the above dipole. {Hint: The radiation part will be the part that depends on second time derivative of the dipole moment.}

(b) Compute the radiation part (i.e., the $\sim 1/r$ part) of the electric field by computing $\mathbf{E}_{rad} = -\hat{\mathbf{r}}\times\mathbf{B}_{rad}$.

(c) Use the Poynting vector to compute the time averaged power radiated into a unit solid angle, as a function of the spherical polar angle θ for a dipole with

$$\mathbf{p}(t) = p_0[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}].$$

Recall that

$$\left\langle\frac{dP(\theta)}{d\Omega}\right\rangle = \left\langle\frac{\mathbf{S}\cdot d\mathbf{A}}{d\Omega}\right\rangle$$

where

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H},$$

in Gaussian.

4. {20%}

In this problem you are to derive the **coulomb** part of the **electric** and **magnetic** fields of a moving particle. The electric field is

$$\mathbf{E}_{coul} = \frac{q(\hat{\mathbf{n}} - \boldsymbol{\beta})}{R^2\gamma^2(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3}.$$

Start with the Liénard-Wiechert 4-potrential

$$A^\beta = \frac{q\,U^\beta}{R \cdot U},$$

where R^σ is the retarded null vector from the charge at $x^\sigma_{ret}(x)$ to the field point x^σ

$$R^\sigma = x^\sigma - x^\sigma_{ret}(x) = x^\sigma - x^\sigma_q(t_{ret}(x)) = R(1, \hat{\mathbf{n}})\,,$$

and

$$R \cdot U = R^\sigma U_\sigma = c\,R\,\gamma\,(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta}).$$

To obtain the coulomb part of the fields you treat the retarded U as a constant, i.e., simply do not differentiate it. For example simply take

$$\frac{\partial}{\partial x^\alpha}\left(\frac{1}{R \cdot U}\right) = -\left(\frac{1}{R \cdot U}\right)^2 \left(U_\alpha - \frac{c^2 R_\alpha}{R \cdot U}\right),$$

and compute $E^i = F_{0i}$ from

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha\,.$$

The magnetic field is obtained from F_{ij}.

5. {20%}

Two identical charges Q are located on the z-axis at $z = +b$ and $z = -b$.

(a) Compute the electric field $\mathbf{E}(x, y)$ on the x-y plane at $z = 0$.

(b) Compute the Maxwell stress tensor T_M^{ij} on the x-y plane at $z = 0$, recall that for linear and isotropic materials in Gaussian units

$$T_M^{ij} = \frac{1}{4\pi}\left[D^i E^j + B^i H^j - \frac{1}{2}\delta^{ij}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H})\right].$$

(c) Evaluate the surface integral

$$F^i = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} T_M^{ij}\,dA^j,$$

over the x-y plane at $z = 0$.

Hint:

$$\int_0^\infty \frac{\rho^3 d\rho}{(\rho^2 + b^2)^3} = \frac{1}{(2b)^2}.$$

November 8, 2013

PHYS 5583 (E & M II)

Exam 2

1. {25%}

A stationary point magnetic dipole at $\mathbf{r} = 0$ with a time dependant dipole-moment $\mathbf{m}(t)$ has a 4-current

$$J^\alpha = [0, -c\,\mathbf{m}(t) \times \boldsymbol{\nabla}\delta^3(\mathbf{r})].$$

and a 4-potential

$$A^\alpha = \left[0, \boldsymbol{\nabla} \times \left(\frac{\mathbf{m}(t - r/c)}{r}\right)\right].$$

(a) Compute the **radiation** parts of the $\mathbf{E}$ and $\mathbf{B}$ fields, (i.e., the parts of the fields $\propto 1/r$.

Hints:

$$\mathbf{B}_{rad} = \hat{\mathbf{r}} \times \mathbf{E}_{rad},$$

$$\boldsymbol{\nabla} \times \dot{\mathbf{m}}(t - r/c) = -\frac{1}{c}\,\hat{\mathbf{r}} \times \ddot{\mathbf{m}}.$$

(b) For the particular dipole moment

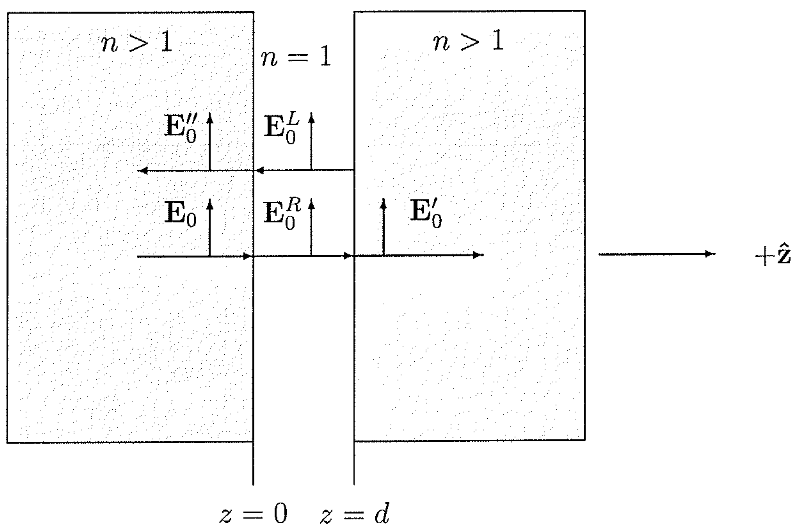
$$\begin{aligned} \mathbf{m}(t) &= m_0\left[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}\right] \\ &= m_0\,\mathrm{Real}\left[(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{-i\omega t}\right], \end{aligned}$$

evaluate the time-average of the radiation part of the Poynting vector ($\mathbf{S} = c/4\pi\mathbf{E} \times \mathbf{H}$) as a function of spherical polar coordinates (r, θ, ϕ)?

(c) What is $dP/d\Omega$, the power radiated into a unit solid angle, as a function of (θ, ϕ), and what is the total power radiated into all directions $P = \int(dP/d\Omega)\,d\Omega$?

2. {25%}

Two plane semi-infinite slabs of the same uniform, isotropic, nonpermeable ($\mu_r = 1$), non-conducting, dielectric with index of refraction $n > 1$ are parallel and separated by an air gap ($n = 1$) of width d. As shown in the figure a plane wave of frequency ω is incident **normally** (i.e., $i = 0$) on the gap from the left slab.



(a) Calculate the magnetic field $\mathbf{B}(t, z)$ assuming the electric fields in the left slab, the center gap, and the right slab, are respectively

$$\begin{aligned}
\mathbf{E}(t,z) &= \mathbf{E}_0\, e^{i(nkz-wt)} + \mathbf{E}_0''\, e^{i(-nkz-wt)}, && z \le 0,\\
\mathbf{E}(t,z) &= \mathbf{E}_0^R\, e^{i(kz-wt)} + \mathbf{E}_0^L\, e^{i(-kz-wt)}, && 0 \le z \le d,\\
\mathbf{E}(t,z) &= \mathbf{E}_0'\, e^{i(nkz-wt)}, && z \ge d,
\end{aligned}$$

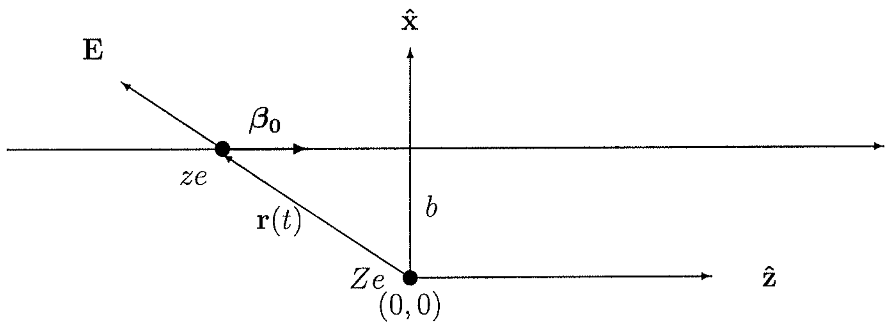
where $\omega/k = c$ and $\mathbf{E}_0 \cdot \hat{\mathbf{z}} = 0$, $\mathbf{E}_0'' \cdot \hat{\mathbf{z}} = 0$, $\mathbf{E}_0^R \cdot \hat{\mathbf{z}} = 0$, $\mathbf{E}_0^L \cdot \hat{\mathbf{z}} = 0$, and $\mathbf{E}_0' \cdot \hat{\mathbf{z}} = 0$.

(b) Match the tangential components of the $\mathbf{E}$ and $\mathbf{H}$ fields at $z = 0$ and $z = d$ to obtain 4 equations relating the 5 amplitudes $\mathbf{E}_0, \mathbf{E}_0'', \mathbf{E}_0^R, \mathbf{E}_0^L$, and $\mathbf{E}_0'$. Use these 4 equations to solve for $\mathbf{E}_0'$ in terms of $\mathbf{E}_0$.

(c) Using your result from (b) calculate the transmission coefficient T, the ratio of the power transmitted into the right slab to the incident power.

3. {25%} A relativistic particle ($\gamma > 1$) of mass m and charge ze passes a nuclear target of charge Ze fixed at the origin ($\mathbf{r} = 0$). Because of the particles high speed it travels **approximately** on a straight line with constant velocity:

$$\mathbf{r}(t) = c\beta_0 t\ \hat{\mathbf{z}} + b\ \hat{\mathbf{x}}.$$



(a) Compute the magnitude of the passing particle's 4-acceleration ($a^\alpha a_\alpha$) as a function of time t caused by the nucleon's Coulomb electric field using the particle's approximate position $\mathbf{r}(t)$.

(b) Use the generalized Larmor formula

$$P(t) = -\frac{2}{3}\left(\frac{q^2}{c^3}\right) a^\alpha a_\alpha$$

to calculate the total energy lost by the passing charge ze,

$$\Delta W = \int_{-\infty}^{+\infty} P(t)\, dt.$$

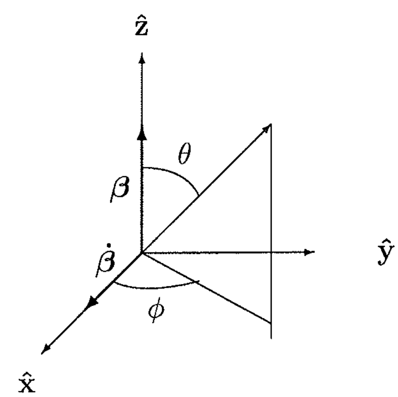
Hint:

$$\int_{-\infty}^{+\infty} \frac{dx}{(b^2+x^2)^2} = \frac{\pi}{2b^3}; \qquad \int_{-\infty}^{+\infty} \frac{dx}{(b^2+x^2)^3} = \frac{3\pi}{8b^5}.$$

4. {25%}

A 10 MeV electron ($m\gamma c^2 = 10$ MeV, $mc^2 = 0.5$ MeV) is traveling in a circle perpendicular to a magnetic induction of strength $B_0 = 50,000$ gauss. Your goal is to analyze the angular dependence of the instantaneous power being radiated by this electron. Assume that at the instant of interest $\boldsymbol{\beta}= \beta\,\hat{\mathbf{z}}$ and $\dot{\boldsymbol{\beta}}=|\dot{\boldsymbol{\beta}}|\,\hat{\mathbf{x}}$. In class we derived the following expression (using spherical polar coordinates) for the power being radiated by a charge whose acceleration is orthogonal to its velocity

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\frac{|\dot{\boldsymbol{\beta}}|^2}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right].$$



(a) Use the Lorentz force in Newton's 2nd law to find numerical values for γ, β, and $|\dot{\boldsymbol{\beta}}|$.

(b) In what direction is $dP/d\Omega$ a maximum and what is its maximum (numerical) value?

(c) Give an approximate form for $dP/d\Omega$ above, valid for large γ and small θ, i.e., assume

$$\begin{aligned}\cos\theta &\approx 1 - \frac{\theta^2}{2},\\ \sin\theta &\approx \theta\\ \beta &\approx 1 - \frac{\gamma^{-2}}{2},\end{aligned}$$

and give $dP/d\Omega$ as a function of $x \equiv (\theta\gamma)$ and ϕ.

$$\begin{aligned} e &= 1.6 \times 10^{-19}\text{coulombs} = 4.8 \times 10^{-10}\text{stat} - \text{coulombs}, \quad \text{(also called esu)}\\ 1\,\text{erg} &= 10^{-7}\text{Joules} = 6.24 \times 10^{5}\text{MeV}\\ 1\,\text{MeV} &= 1.60 \times 10^{-13}\text{Joules} = 1.60 \times 10^{-6}\text{ergs}\end{aligned}$$

April 6, 2012

PHYS 5583 (E & M II)

Exam 2

1. {25%}

The Lienard-Weichert 4-potential for a point charge q moving on a world line $x_q^\alpha(t) = (ct, \mathbf{r}_q(t))$ is

$$A^\alpha(x) = \frac{q\,u^\alpha}{u_\sigma R^\sigma}, \tag{1}$$

where

$$\begin{aligned} R^\sigma &\equiv x^\sigma - x_q^\sigma(t_{ret}) = R(1, \hat{\mathbf{n}}), \\ R &\equiv |\mathbf{r} - \mathbf{r}_q(t_{ret})|, \end{aligned} \tag{2}$$

is the relative-retarded 4-d position vector and

$$u^\sigma = \gamma(t_{ret})c\Big(1, \vec{\beta}(t_{ret})\Big),$$

is the retarded 4-velocity of the point particle.

The retarded time $t_{ret}(t, \mathbf{r})$ is a function of the space-time point $x^\alpha = (t, \mathbf{r})$ at which you are computing $A^\alpha(x)$ and is defined by the constraint equation

$$t_{ret} = t - \frac{|\mathbf{r} - \mathbf{r}_q(t_{ret})|}{c}.$$

(a) Draw a space-time diagram showing the observation point x^α, the retarded point x_{ret}^α, the backward light cone of x^α, the vector R^α, and the retarded velocity vector u^α.

(b) From equation (2) it follows that

$$\frac{\partial R^\alpha}{\partial x^\beta} = \delta_\beta^\alpha - \frac{u^\alpha R_\beta}{u^\sigma R_\sigma}.$$

Do not derive this result but use it and equation (1) to derive the **coulomb** part of the electromagnetic field tensor $F_{\alpha\beta}$. **Do not** compute the radiation part.

(c) From your coulomb $F_{\alpha\beta}$ give the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ in as simple 3-d vector forms as you can.

2. {25%}

An electron ($mc^2 = 0.5$MeV) is accelerated from rest by a uniform electric field $E_0\,\hat{\mathbf{z}}$ through a potential of 10^7 Volts. Assume the electric field is caused by 2 large parallel conducting plates separated by a distance $D = 0.5$ meters (in SI units $E_0 D = 10^7$ Volts). Without radiation the electron should acquire a kinetic energy $T \equiv m\gamma c^2 - mc^2 = 10$ MeV. The goal of this problem is to find out what percentage of this 10 MeV is lost to radiation. The power radiated (integrated over all angles) by an accelerating particle is given by the relativistic Larmor formula:

$$P(t) = -\frac{2}{3}\frac{e^2}{c^3} a^\alpha a_\alpha, \qquad\qquad \text{Gaussian units}$$

where the 4-acceleration is defined by

$$a^\alpha \equiv \frac{du^\alpha}{d\tau}.$$

To find the net energy radiated you can carry out the following steps:

(a) Write down the equations of motion for constant acceleration in an uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$.

(b) Solve the equations of motion to find $a^\alpha a_\alpha$ as well as how long it takes the electron to go the distance $D = 0.5$ m.

(c) Integrate the Larmor formula to obtain the total energy radiated away. What percentage of the electron's kinetic energy was radiated away.

$$\begin{aligned} e &= 1.6 \times 10^{-19}\text{coulombs} = 4.8 \times 10^{-10}\text{stat} - \text{coulombs}, \quad \text{(also called esu)} \\ 1\,\text{erg} &= 10^{-7}\text{Joules} = 6.24 \times 10^{5}\text{MeV} \\ 1\,\text{MeV} &= 1.60 \times 10^{-13}\text{Joules} = 1.60 \times 10^{-6}\text{ergs} \end{aligned}$$

3. {25%}

A 10 MeV electron ($m\gamma c^2 = 10$ MeV, $mc^2 = 0.5$ MeV) is traveling in a circle perpendicular to a magnetic induction of strength $B_0 = 50,000$ gauss. Your goal is to analyze the angular dependence of the instantaneous power being radiated by this electron. Assume that at the instant of interest $\vec{\beta} = \beta\hat{\mathbf{z}}$ and $\dot{\vec{\beta}} = |\dot{\vec{\beta}}|\hat{\mathbf{x}}$. In class we derived the following expression (using spherical polar coordinates) for the power being radiated by a charge whose acceleration is orthogonal to its velocity

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} \frac{|\dot{\vec{\beta}}|^2}{(1 - \beta\cos\theta)^3} \left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - \beta\cos\theta)^2}\right].$$

(a) Use the Lorentz force in Newton's 2nd law to find numerical values for γ, β, and $|\dot{\vec{\beta}}|$.

(b) In what direction is $dP/d\Omega$ a maximum and what is its maximum (numerical) value?

(c) Give an approximate form for $dP/d\Omega$ above, valid for large γ and small θ, i.e., assume

$$\begin{aligned} \cos\theta &\approx 1 - \frac{\theta^2}{2}, \\ \sin\theta &\approx \theta \\ \beta &\approx 1 - \frac{\gamma^{-2}}{2}, \end{aligned}$$

and give $dP/d\Omega$ as a function of $x \equiv (\theta\gamma)$ and ϕ.

(d) For $\phi = 0$ plot (approximate sketch only) $dP/d\Omega$ as a function of x . For what small values of θ (numerical) does $dP/d\Omega$ vanish?

(e) For $\phi = \pi/2$ plot (approximate sketch only) $dP/d\Omega$ as a function of x. For what small values of θ (numerical) has the beam intensity decreased to 1/2 of its maximum value?

$$\begin{aligned} e &= 1.6 \times 10^{-19}\text{coulombs} = 4.8 \times 10^{-10}\text{stat} - \text{coulombs}, \quad \text{(also called esu)} \\ 1\,\text{erg} &= 10^{-7}\text{Joules} = 6.24 \times 10^{5}\text{MeV} \\ 1\,\text{MeV} &= 1.60 \times 10^{-13}\text{Joules} = 1.60 \times 10^{-6}\text{ergs} \end{aligned}$$

4. {25%}

A point electric dipole is fixed in space but has a time dependent dipole moment $\mathbf{p}(t)$. The current density of this dipole is

$$\mathbf{J}(t, \mathbf{r}) = \dot{\mathbf{p}}(t)\delta^3(\mathbf{r} - \mathbf{r}_0).$$

(a) Compute the vector potential $\mathbf{A}(t, \mathbf{r})$ for this current density.

(b) From your vector potential $\mathbf{A}(t, \mathbf{r})$ compute the radiation part of the $\mathbf{B}$ field (the part that falls off as $1/|\mathbf{r} - \mathbf{r}_0|$).

(c) Compute the radiation part of $\mathbf{E}$ by assuming

$$\mathbf{E} = \mathbf{B} \times \hat{\mathbf{R}}$$

where

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{R} \equiv \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|}.$$

(d) If

$$\mathbf{p}(t) = p_0\Big(\hat{\mathbf{x}}\cos(\omega t) + \hat{\mathbf{y}}\sin(\omega t)\Big),$$

compute the time average of the Poynting vector as a function of spherical polar coordinates centered on the dipole (R, θ, ϕ).

April 16, 2010

PHYS 5583 (E & M II)

Exam 2

1. 33%

A time dependent electric dipole, located at the origin has 4-current

$$J^\alpha(t, \mathbf{r}) = \left[-c\,\mathbf{p}(\mathbf{t}) \cdot \vec{\nabla}\delta^3(\mathbf{r}),\ \dot{\mathbf{p}}(\mathbf{t})\delta^3(\mathbf{r})\right],$$

and a retarded 4-potential (in Gaussian units)

$$A^\alpha(t, \mathbf{r}) = \left[-\nabla \cdot \left\{\frac{\mathbf{p}(t - r/c)}{r}\right\},\ \frac{\dot{\mathbf{p}}(t - r/c)}{cr}\right].$$

(a) Compute the radiation part (i.e., the $\sim 1/r$ part) of the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$ for the above dipole. {Hints: The radiation parts depend on second derivatives of the dipole moment and you should find that $\mathbf{B}_{rad} = \hat{r} \times \mathbf{E}_{rad}$.}

(b) What type of polarization, plane or circular, will be seen by a distant observer coming from a dipole with

$$\mathbf{p}(t) = p_0 \cos(\omega t)\hat{k}?$$

(c) Use the Poynting vector to compute

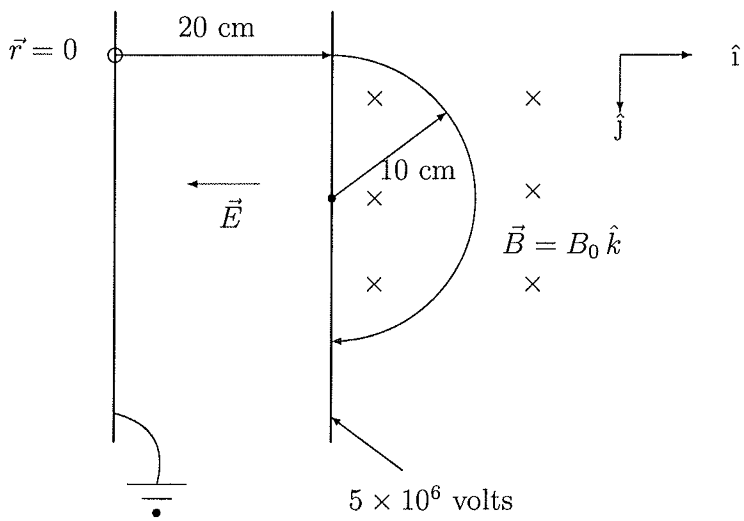
$$\left\langle \frac{dP(\theta)}{d\Omega} \right\rangle,$$

the time averaged power radiated into a unit solid angle, as a function of the spherical polar angle θ for a dipole with

$$\mathbf{p}(t) = p_0[\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath}].$$

2. 33%

An electron starts from $\vec{r} = 0$ at rest on the negative plate of a capacitor and is accelerated 20 cm to the positive plate through a potential difference of 5×10^6 Volts (see the figure). The electron then (through a pin hole in the positive plate) enters a uniform magnetic field $\vec{B}_0 = B_0\hat{k}$ which turns the electron in a semi-circle orbit of radius 10 cm.



(a) Find the (lab) time the electron takes to reach the positive plate and the (lab) time it takes to move around the 1/2 circle.

(b) How much total energy does the electron loose as radiation during its linear acceleration in the capacitor and during its 1/2-circle orbit in the magnetic field? What fraction of the electron's total energy is lost to radiation?

(c)

{Hints: The Larmor formula is

$$P(t) = -\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) = \frac{2}{3}\frac{q^2}{c}\gamma^6, [(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2]$$

and the acceleration is caused by the Lorentz force

$$\frac{dp^\mu}{d\tau} = \frac{q}{c}F^{\mu\lambda}u_\lambda,$$

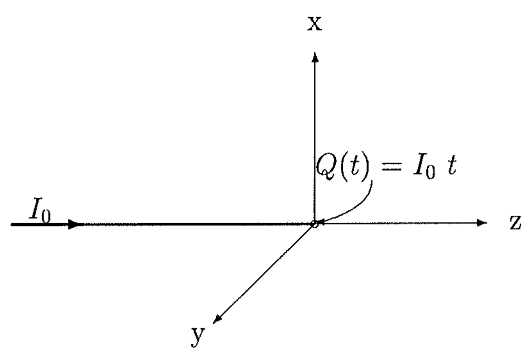
$m_ec^2 = 0.5MeV$, 1 eV=1.6×10^{-12} ergs, e=4.8×10^{-10} statcoul. }

3. 33%

In the absence of polarizable and/or magnetizable material (i.e., only free charges and currents present) Maxwell's equations, in the Lorentz gauge, reduce to the inhomogeneous wave equation:

$$\Box \left\{ \begin{matrix} \Phi \\ A^x \\ A^y \\ A^z \end{matrix} \right\} = \frac{4\pi}{c} \left\{ \begin{matrix} c\rho \\ J^x \\ J^y \\ J^z \end{matrix} \right\}, \text{ where } \Box \equiv \left(\frac{\partial}{c\partial t} \right)^2 - \nabla^2.$$

A time dependent charge $Q(t) = I_0\ t,\ t \geq 0$ is fixed at the origin



of a cylindrical polar coordinate system (ρ, ϕ, z) The charge increases linearly with time because a constant current I_0 flows in along a thin wire attached to the charge on its left, see the figure. Assume the wire carries no current for $t < 0$, however, at $t = 0$ a current I_0 abruptly starts flowing in the +z direction and remains constant for $t \geq 0$. Assume the wire remains neutral as the charge at the origin grows. Find the following quantities at time t for points (ρ, ϕ, z):

(a) The charge density $\rho(t, \rho, \phi, z)$,

(b) The current density $\mathbf{J}(t, \rho, \phi, z)$,

(c) The retarded scalar potential $\Phi(t, \rho, \phi, z)$,

(d) The retarded vector potential $\mathbf{A}(t, \rho, \phi, z)$.

Hints: The retarded Green's function for the $\Box$ operator is:

$$G^{ret}(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta\left(t - t' - |\mathbf{r} - \mathbf{r}'|/c\right)}{4\pi\,|\mathbf{r} - \mathbf{r}'|}.$$

You might need the integral

$$\int \frac{dX}{\sqrt{X^2 + a^2}} = \ln(\sqrt{X^2 + a^2} + X).$$

March 4, 2009

PHYS 5583 (E & M II)

Exam 1

1. (25%)

   A nonrelativistic particle of charge ze, mass m, and initial speed $v_0 << c$ is incident on a fixed charge Ze at an impact parameter b that is large enough to ensure that the particles deflection in the course of the collision is very small. Using the Larmor power formula

   $$P = \frac{2}{3}\frac{q^2}{c}|\dot{\boldsymbol{\beta}}|^2,$$

   and Newtons second law, calculate (approximately) the total energy radiated. Approximate the particles trajectory as a straight line with constant speed but use Newton's second law to compute the acceleration that produces the instantaneous radiation.

2. (25%)

A particle of mass m, charge q, moves in a plane perpendicular to a uniform, static, magnetic induction $\mathbf{B}$.

(a) Calculate the total energy radiated per unit time, expressing it in terms of the above constants and the ratio γ of the particleless total energy to its rest energy.

(b) If at time $t = 0$ the particle has a total energy $E_0 = \gamma_0 mc^2$, show that it will have energy $E = \gamma mc^2 < E_0$ at a time t, where

$$t \approx \frac{3m^3c^5}{2q^2B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right).$$

3. (50%)

An electron (Mc^2=0.5 MeV) is accelerated along the z-axis from rest at the origin ($t = 0$ at $\mathbf{r} = 0$) to a total energy of 10 MeV ($Mc^2\gamma_{max} =$ 10 MeV) at $t = t_{max}$ and $\mathbf{r} = z_{max}\hat{k}$.

(a) Derive an expression for $dP/d\Omega$, the power radiated per unit solid angle by the electron in the $\mathbf{n}$ direction as a function of $\dot{\beta}$, β, and the spherical polar angle θ.

Hints:

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\mathbf{n}\,R^2(1-\mathbf{n}\cdot\boldsymbol{\beta}),$$

$$\mathbf{S} = \frac{c}{4\pi}|\mathbf{E}|^2\,\mathbf{n},$$

$$\mathbf{E} = \frac{e}{c}\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{R\,(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}.$$

(b) If the electron's acceleration is caused by a constant electric field $\mathbf{E} = -10$ MeV/cm $\hat{k}$ it moves with "constant" acceleration in the z-direction and reaches an energy of 10 MeV at $z_{max} = 1$ cm. Recall that the Lorentz force equations are

$$\frac{d}{dt}(m\gamma c) = q\beta E,$$

$$\frac{d}{dt}(m\gamma\beta c) = qE.$$

Solve these equations for $\dot{\beta}$ as a function of the electron's β and then express $dP/d\Omega$ as a function of β and θ.

(c) The TOTAL energy radiated into a unit solid angle $dE/d\Omega$ is an integral of $dE/d\Omega$ over the time up to t_{max} (the time it takes the electron to reach 10 MeV),

$$\frac{dE}{d\Omega} = \int_0^{t_{max}} \frac{dP}{d\Omega}dt. \tag{1}$$

Change the integration variable from t to β, i.e., use

$$\frac{dE}{d\Omega} = \int_0^{\beta_{max}} \frac{dP}{d\Omega}\frac{d\beta}{\dot{\beta}}, \tag{2}$$

and obtain a somewhat simpler integral than what would be obtained if equation (1) were used. Find β_{max} from γ_{max}.

(d) The above β integral can be evaluated; however, it is complicated. Use your results from part (a) to approximate the value of (2) by assuming $\dot{\beta}$ is constant ($= \beta_{max}/t_{max}$). To calculate t_{max} assume the electron moves 1cm and reaches an energy of 10 MeV.

(e) From your result in part (d) does the electron radiate away a large or small fraction of its energy as it accelerates? Recall that the classical radius of the electron is

$$r_e = \frac{e^2}{mc^2} \approx 2.82 \times 10^{-13} cm.$$

April 8, 2005

PHYS 5583 (E & M II)

Exam 2

1. (25 points) A relativistic particle of charge q and mass m moves in a uniform magnetic induction $\mathbf{B} = B_0\hat{\mathbf{k}}$

   (a) Find a vector potential $\mathbf{A}$ for this uniform $\mathbf{B}$.

   (b) Write the Lagrangian for this point particle moving in the uniform $\mathbf{B}$ field.

   (c) Solve the Lagrangian equations of motion for $\mathbf{r}(t)$.

Hint: $\int L(\mathbf{r}, \dot{\mathbf{r}}, t)dt = \int -\left(mc^2 + \frac{q}{c}A_\alpha U^\alpha\right) d\tau$

2. (25 points) A 50 MeV electron ($m\gamma c^2 = 50$ MeV, $mc^2 = 0.5$ MeV) moving along the z-axis is decelerated and brought to a stop after traveling 10 cm in a uniform electric field $E_0\hat{\mathbf{k}}$.

   (a) Compute $\gamma(t)$ assuming the electron starts its deceleration at t=0.

   (b) How long does it take the electron to stop?

   (c) Show that $(\dot{\gamma\beta}) = \gamma^3\dot{\beta}$ for arbitrary motion.

   (d) Compute the total energy radiated by the electron during the 10 cm stopping process. What fraction of the electrons initial energy was lost to radiation?

   Hint:

$$P(t) = \frac{2}{3}\frac{q^2}{c}\gamma^6[(\dot{\vec{\beta}})^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2]$$

3. (25 points) An uncharged, hollow, and conducting sphere of radius a is placed in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{k}}$. Assume the sphere is centered at the origin and is cut into two halves along the $z = 0$ plane.

   (a) Compute the total electric field just outside the sphere by first computing the total electrostatic potential (assume $\Phi = 0$ on the conductor).

   (b) Integrate Maxwell's stress tensor over the $z > 0$ hemispherical surface to find the force exerted on that half of the sphere.

   (c) Are the two halves pressed together by the electrostatic force or do the hemispheres fly apart?

Recall:

$$T_M^{ij} = \frac{1}{4\pi}\left[E^iE^j + B^iB^j - \frac{1}{2}\delta^{ij}(E^2 + B^2)\right]$$

4. (25 points)

(a) Compute the charge-current 4-vector $J^\alpha(t, \mathbf{r})$ for a point electric dipole. Assume the dipole moment is $\mathbf{p}(t)$ and is located at the position $\mathbf{r}_p(t)$. (Hint: think of the dipole as two very close particles of opposite charge and add their 4-currents.)

(b) If the dipole's position is **fixed** at the origin, i.e., $\mathbf{r}_p = 0$, and its dipole moment oscillates $\mathbf{p}(t) = p_0 \cos(\omega t)\hat{\mathbf{k}}$, use the retarded Green's function to compute the **radiation** part of the $\mathbf{E}$ and $\mathbf{B}$ fields.

(c) Evaluate the Poynting vector and describe the angular dependence of the radiation pattern.

December 15, 2015

PHYS 5583 (E & M II)

Final

Work only 4 of the following problems!

All formulas are in Gaussian units. If you convert to SI you need to recall that $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 and $\sqrt{\epsilon_0\mu_0} = 1/c$.

1. (25%) An **unpolarized** plane wave travelling in air ($n = 1$) is incident on a flat glass plate ($n = 4/3$) at incidence angle $i = 30°$. Assume $\mu_r = 1$ for air and glass. The goal of this problem is to analyze the amount of linear polarization that exists in the reflected wave. You can think of the unpolarized wave as a superposition of two independent linearly polarized waves of equal intensities, one polarized perpindicular to the plane of incidence and one polarized in the plane of incidence. Because the reflection coefficients for these two polarizations are different the reflected wave will be partially polarized. To answer this problem carry out the following steps:

   (a) Seperately for the two polarization modes match the tangential electric fields **E** and magnetic fields **H** at the air-glass interface to determine the (numerical) amplitudes of the reflected waves E_0'' as functions of the amplitudes of the incident waves E_0.

   (b) Calculate the numerical values for the two reflection coefficients $R_\perp$ and $R_{||}$ for the two polarization modes by evaluating

   $$R = \frac{|E_0''|^2}{|E_0|^2},$$

   for each polarization.

   (c) Evaluate the degree of linear polarization by evaluating

   $$\left|\frac{R_\perp - R_{||}}{R_\perp + R_{||}}\right|.$$

2. (25%) In Gaussian units a wave traveling in a conductor is of the form

$$\mathbf{E} = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

$$\mathbf{B} = \frac{c\,k}{\omega}\left(\hat{\mathbf{k}} \times \mathbf{E}_0\right)\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

where $\mathbf{J} = \sigma\mathbf{E}$ and

$$k = \frac{\omega}{c}\sqrt{\mu_r\left(\epsilon_r + i\frac{4\pi\sigma}{\omega}\right)}.$$

Recall that for good conductors and frequencies below some maximum value, σ is the real valued static conductivity and $\epsilon_r << 4\pi\sigma/\omega$. To answer the following you can assume $\mu_r = 1$.

(a) If the skin depth of silver is 8.29 mm at 60 Hz what is the static conductivity σ?

(b) What will the skin depth of silver be at 10^9 Hz?

To relate the above conductivity σ to the physical properties of the conducting material the Drude model can be used. In this model a cube of material is thought of as a box of free electrons that move in response to the transiting plane electromagnetic wave

$$\mathbf{E} = E_0\,\hat{\mathbf{x}}\, e^{i(kz-\omega t)},$$

according to

$$m\,\ddot{x} = -m\gamma_0\,\dot{x} - eE_0\, e^{i(kz-\omega t)}.$$

The viscosity parameter γ_0 accounts for the energy transfer from the moving electrons to the lattice of atoms. The resulting dipole moment density due to the displaced electrons is

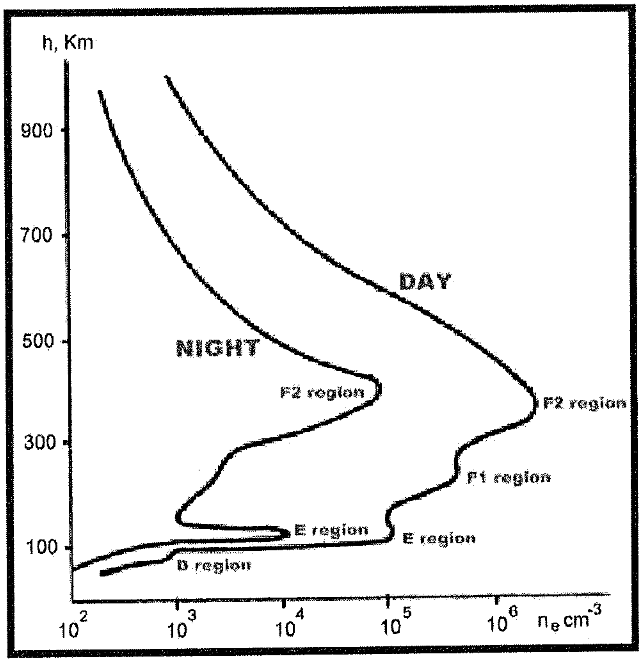
$$\mathbf{P} = N(-e)\,x = i\,N\frac{e^2}{m\,\omega(\gamma_0 - i\omega)}\,\mathbf{E},$$

where N is the density of conducting electrons. Recall that in Gaussian units $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$. Use $e^2/m = 2.5 \times 10^8\ \mathrm{cm^3/s^2}$.

(c) Apply the Drude model to silver to relate N and γ_0 to the static conductivity σ.

(d) If Silver has an electron density of $N = 5.8 \times 10^{22}/\mathrm{cm}^3$ estimate γ_0, the frequency above which this model will fail to correctly give the skin depth for silver.

3. {25%} The ionosphere is a region of the earth's atmosphere containing



a partially ionized plasma. The free electron density $N(h)$ varies with height h above the earth's surface and varies significantly between night and day. The attached figure gives $N(h) = n_e$ for night and day. As shown in the figure the ionosphere is divided into layered regions D, E, F1, and F2. Because a plasma reflects all radio waves of frequency $\omega < \omega_p$ the altitude of each of these regions can be estimated by recording the round trip travel time for a pulse of radio waves **sent straight up in the air** at the appropriate frequencies. You can assume the ionosphere has an index of refraction given by

$$n(h, \omega) = \sqrt{1 - \frac{\omega_p^2(h)}{\omega^2}},$$

where in Gaussian units $e^2/m = 2.5 \times 10^8\ \mathrm{cm}^3/\mathrm{s}^2$ and

$$\omega_p^2 = 4\pi N_e(h)\frac{e^2}{m}.$$

(a) Assuming the attached figure is correct select a frequency ω_E that could be used to measure the height of the bottom of the E layer during the day or night. Choose a value of ω_E that will not be reflected below the E region but will be reflected from the bottom of it.

(b) From the figure select two frequency ω_F(day) and ω_F(night) that could be used to measure the respective heights of the central part of the F2 region during the day and night. For these you can select (respectively day or night) ω_F values slightly less than $\omega_p(F2)$ where $\omega_p(F2)$ is the peak value of ω_p in the central F2 region.

(c) By taking the approximate height $h = h_{F2}$ of the central F2 layer during the day from the figure, give an integral expression for the round trip travel time of a pulsed signal that it reflects. Don't forget to use the group velocity in your integral (not the phase velocity) and don't forget that the index of refraction varies with h,

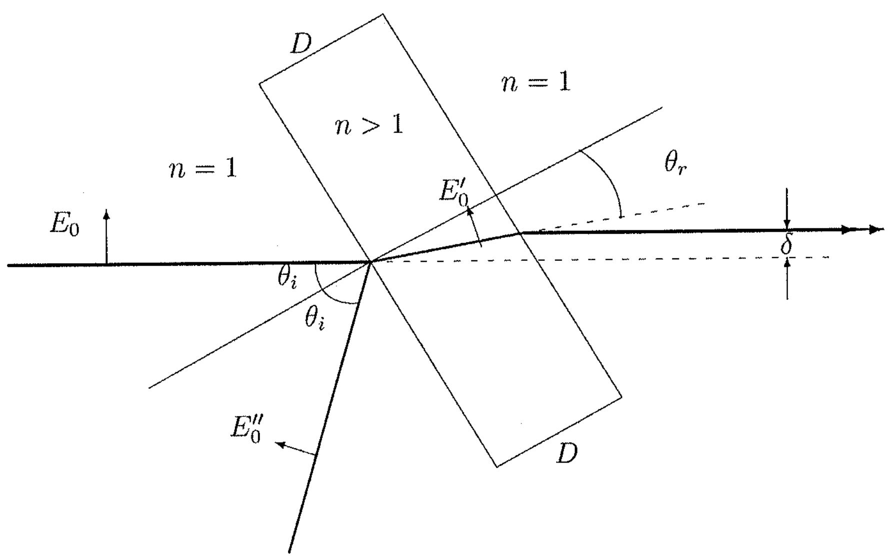
$$v_g(h) = \frac{c}{n + \omega\frac{dn}{d\omega}}.$$

(d) Approximate your day-time travel time integral by assuming the atmosphere everywhere below the F2 layer has a constant electron density equal to the electron density at one half the height of the central F2 region, $N = N(h_{F2}/2)$.

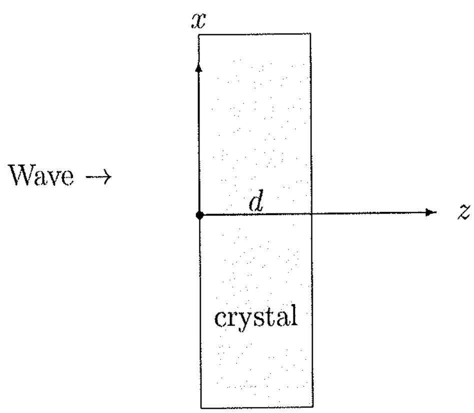
4. {25%}

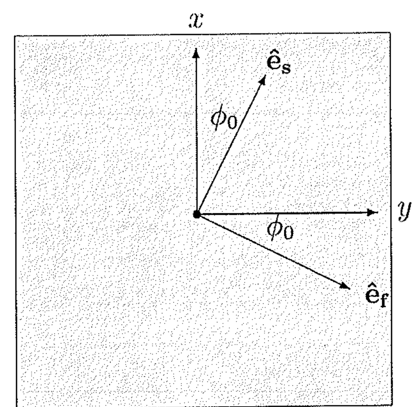
(a) When swimming under water much of the air surface above you looks like a mirror (if the water is clear and calm). The reason for this is that light traveling under water ($\mu_r = 1, n = 1.33$), will be completely reflected if it strikes the air surface at an incidence angle $i \geq i_0$. Derive an expression for i_0 as a function of n and give its value for water?

(b) A narrow columnated monochromatic beam of light from a laser hits a flat piece of ordinary glass of thickness D in air at angle θ_i as shown in the figure. Assume that the index of refraction of air is 1 and of the glass is 1.5. Also assume a relative permeability $\mu_r = 1$ for both the air and glass. Ignore the reflected beam that occurs when the beam exits the glass on the right of the figure. After the transmitted beam exits the glass and returns to air **show that it will be traveling parallel to its original direction but displaced a distance δ as shown in the figure. At Brewster's angle what is the value of δ?** Recall that Brewster's angle i_B, is the angle at which the amplitude of reflected light vanishes if polarized **in** the plane of incidence.



5. {25%} A plane polarized monochromatic light wave traveling in the





$+z$ direction enters a large flat slab of transparent crystal of thickness d, located between $z = 0$ and $z = d$. This crystal has the property that the index of refraction depends on the direction of polarization as follows: Plane waves traveling in the z direction but polarized in the direction

$$\hat{\mathbf{e}}_s = \cos\phi_0 \hat{\mathbf{x}} + \sin\phi_0 \hat{\mathbf{y}},$$

travel with speed $v_s = c/n_s < c$ but those polarized in the orthogonal direction

$$\hat{\mathbf{e}}_f = -\sin\phi_0 \hat{\mathbf{x}} + \cos\phi_0 \hat{\mathbf{y}},$$

travel with the faster speed $v_f = c/n_f < c$ where $n_s = n_f + \Delta n$.

Assume the wave, just after entering the crystal (i.e., for very small z, i.e., $z << \lambda < d$), is polarized in the y direction and hence has the form

$$\mathbf{E}(z \approx 0, t) = E_0 \, \hat{\mathbf{y}} \, e^{-i\omega t}.$$

(a) Prove that when the plane wave reaches $z = d$

$$\mathbf{E}(z = d, t) = [E_x \, \hat{\mathbf{x}} + E_y \, \hat{\mathbf{y}}] \; e^{i(\bar{k}d - \omega t)},$$

where $E_x = iE_0 \sin 2\phi_0 \sin\delta$ and $E_y = E_0(\cos\delta - i\cos 2\phi_0 \sin\delta)$, with $c\bar{k} \equiv \omega(n_s + n_f)/2$ and $\delta \equiv \omega d \, \Delta n/(2c)$.

(b) For what values of δ and ϕ_0 will the wave emerge from the crystal as a circularly polarized wave? (i.e., when will $E_x/E_y = \pm i$).

(c) For what minimum crystal thicknesses $d = d_{min}$ will the wave emerge as a plane polarized wave (i.e., when will $E_x/E_y =$ real) and what is its polarization direction?

December 15, 2015

PHYS 5583 (E & M II)

Final

Work only 4 of the following problems!

1. (25%) An **unpolarized** plane wave travelling in air ($n = 1$) is incident on a flat glass plate ($n = 4/3$) at incidence angle $i = 30^o$. Assume $\mu_r = 1$ for air and glass. The goal of this problem is to analyze the amount of plane polarization that exists in the reflected wave. You can think of the unpolarized wave as a superposition of two independent linearly polarized waves of equal intensities, one polarized perpindicular to the plane of incidence and one polarized in the plane of incidence. To answer this problem carry out the following steps:

   (a) Seperately for the two polarization modes match the tangential electric fields **E** and magnetic fields **H** at the air-glass interface to determine the amplitudes of the reflected waves E_0'' as functions of the amplitudes of the incident waves E_0.

   (b) Calculate the two reflection coefficients $R_\perp$ and $R_{||}$ for the two polarization modes by evaluating

   $$R = \frac{|E_0''|^2}{|E_0|^2},$$

   for each polarization.

   (c) Evaluate the degree of linear polarization by evaluating

   $$\left|\frac{R_\perp - R_{||}}{R_\perp + R_{||}}\right|.$$

$$\Rightarrow \frac{E_0''}{E_0} = \frac{\frac{16}{\sqrt{165}} - 1}{\frac{16}{\sqrt{165}} + 1} = \frac{1 - \frac{\sqrt{165}}{16}}{1 + \frac{\sqrt{165}}{16}} = \frac{0.197}{1.803} = 0.109$$

(a) ∥ $\quad$ for $\vec{E}_{\underline{\parallel}}$

+10

$$\Rightarrow \boxed{R_{\parallel}} = (0.109)^2 = \boxed{0.0119}$$

(b)

(c) $$\left| \frac{R_{\perp} - R_{\parallel}}{R_{\perp} + R_{\parallel}} \right| = \frac{0.031 - 0.012}{0.031 + 0.012} = \frac{0.0189}{0.0429} = 0.44$$

2. (25%) In Gaussian units a wave traveling in a conductor is of the form

$$\mathbf{E} = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

$$\mathbf{B} = \frac{c\,k}{\omega}\left(\hat{\mathbf{k}} \times \mathbf{E}_0\right)\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

where $\mathbf{J} = \sigma\mathbf{E}$ and

$$k = \frac{\omega}{c}\sqrt{\mu_r\left(\epsilon_r + i\frac{4\pi\sigma}{\omega}\right)}.$$

Recall that for good conductors and frequencies below some maximum value, $\epsilon_r << 4\pi\sigma/\omega$.

(a) If the skin depth of silver is 8.29 mm at 60 Hz what is the static conductivity σ?

(b) What will the skin depth of silver be at 10^9 Hz?

To relate the above conductivity σ to the physical properties of the conducting material the Drude model can be used. In this model a cube of material is thought of as a box of free electrons that move in response to the transiting plane electromagnetic wave

$$\mathbf{E} = E_0\,\hat{\mathbf{x}}\, e^{i(kz-\omega t)},$$

according to

$$m\,\ddot{x} = -m\gamma_0\,\dot{x} - eE_0\, e^{i(kz-\omega t)}.$$

The viscosity parameter γ_0 accounts for the energy transfer from the moving electrons to the lattice of atoms. The resulting dipole moment density due to the displaced electrons is

$$\mathbf{P} = \mathcal{N}(-e)\,x = i\,\mathcal{N}\frac{e^2}{m\,\omega(\gamma_0 - i\omega)}\mathbf{E},$$

where $\mathcal{N}$ is the density of conducting electrons. Recall that in Gaussian units $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$. To answer the following you can assume $\mu_r = 1$ for silver and $e^2/m = 2.5 \times 10^8$ cm^3/s^2.

(c) Apply the Drude model to silver to relate $\mathcal{N}$ and γ_0 to the static conductivity σ.

(d) If Silver has an electron density of $\mathcal{N} = 5.8 \times 10^{22}/\text{cm}^3$ estimate γ_0, the frequency above which this model will fail to correctly give the skin depth for silver.

$$\gamma_0 = \frac{(5.8\times10^{22}/cm^3)(2.5\times10^{8}\,cm^3/s^2)}{5.52\times10^{17}\,s^{-1}}$$

$$\gamma_0 = \frac{(5.8)(2.5)}{5.52}\times10^{+13}\,s^{-1} = \boxed{2.6\times10^{13}\,s^{-1}}$$

(d) The skin depth should become complex for ω approaching γ_0 according to the Drude model making the predicted values incorrect.

+5

part of the F2 region during the day and night. For these you can choose a ω_F such that $\omega_F > \omega_p(F2)$ for heights below the central F2 region and $\omega_F < \omega_p(F2)$ for heights above the central F2 region. Here $\omega_p(F2)$ stands for the plasma frequency of the central part of the F2 region (respectively day or night).

(c) By taking the approximate height $h = h_{F2}$ of the central F2 layer during the day from the figure, give an integral expression for the round trip travel time of a pulsed signal that it reflects. Don't forget to use the group velocity in your integral (not the phase velocity) and don't forget that the index of refraction varies with h,

$$v_g(h) = \frac{c}{n + \omega \frac{dn}{d\omega}}.$$

(d) Approximate your travel time integral by assuming the atmosphere everywhere below the F2 layer has a constant electron density equal to the electron density at one half the height of the central F2 region, $N = N(h_{F2}/2)$.

$$\Delta t = \frac{2}{c}\int_0^{h_{F_2}} \frac{dz}{\sqrt{1-\frac{4\pi Me(z)e^2/m}{\omega^2}}}$$

(d)

$$\Delta t \cong \frac{2}{c}\, \frac{h_{F_2}}{\sqrt{1-\frac{4\pi\, Me(\frac{h}{2})\, e^2/m}{\omega^2}}}$$

$$= \frac{2}{3\times10^5\ \frac{km}{s}} \times \frac{350\ km}{\sqrt{1-\frac{4\pi(1\times10^5)(2.5\times10^8)}{(7.9\times10^7)^2}}}$$

1 ↔ 1.5

$$= \frac{\frac{700}{3}\times10^{-5}\ s}{\sqrt{1-\frac{\pi\times10^{14}}{(7.9)^2\times10^{14}}}} = \frac{700\times10^{-5}}{3\times(0.97)}\ s$$

$$\boxed{\Delta t \cong 2.4\times10^{-3}\ s}$$

+5

#4 cont Brewster angle $\quad n = 3/2 \Rightarrow i_B = 56.3°$

$$\boxed{\tan i_B = n \rightarrow \sin i = \frac{n}{\sqrt{n^2+1}} \quad \cos i = \frac{1}{\sqrt{n^2+1}}}$$

$$\Rightarrow \delta = D\left[1 - \frac{1}{\sqrt{n^2+1}\sqrt{n^2 - \frac{n^2}{n^2+1}}}\right]\frac{n}{\sqrt{n^2+1}}$$

$$\boxed{\delta = D\frac{n}{\sqrt{1+n^2}}\left(1-\frac{1}{n^2}\right)} \quad n = 3/2$$

$$= D\frac{3/2}{\sqrt{13/2}}\left(1-\frac{4}{9}\right) = \boxed{D\frac{3 \cdot 5}{9 \cdot \sqrt{13}}} = \boxed{0.46 D}$$

To derive Brewster's angle look at

$\bar{E}_{\parallel}$ for problem #1.

$$E_0 - E_0'' = \frac{\cos r}{\cos i}E_0' \Rightarrow E_0 - E_0'' = \frac{\cos r}{n\cos i}(E_0 + E_0'')$$

$$E_0 + E_0'' = nE_0' \qquad \Rightarrow \frac{E_0''}{E_0} = \frac{1 - \frac{\cos r}{n\cos i}}{1 + \frac{\cos r}{n\cos i}} = 0 \text{ when?}$$

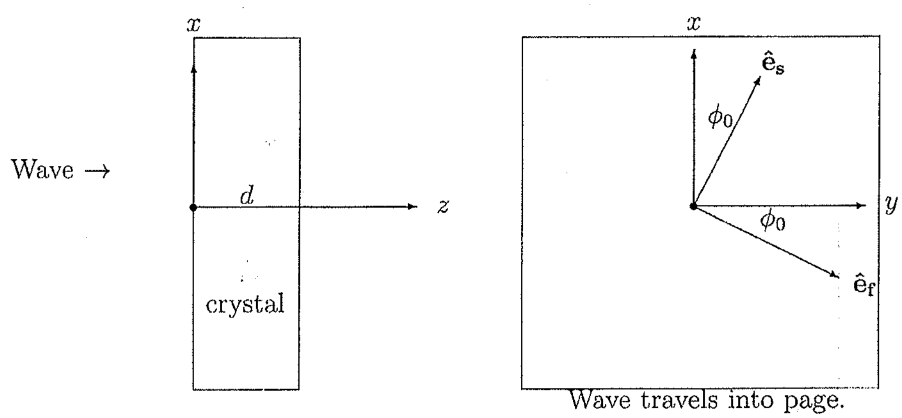
$$n\cos i_b = \cos r = \sqrt{1-\left(\frac{\sin i_b}{n}\right)^2}$$

$$n^2\cos^2 i_b = 1 - \frac{(\sin i_b)^2}{n^2} \quad \cancel{\Rightarrow} \; n^2(1-\sin^2 i_b)$$

$$1 - n^2 = \left(\frac{1}{n^2} - n^2\right)\sin^2 i_b \Rightarrow \sin^2 i_b = \frac{n^2(1-n^2)}{(1-n^4)} = \frac{n^2}{1+n^2}$$

$$\therefore \sin i_b = \frac{n}{\sqrt{1+n^2}} \quad + \cos i_b = \frac{1}{\sqrt{1+n^2}} \quad + \tan i_b = n$$

5. {25%} A plane polarized monochromatic light wave traveling in the



$+z$ direction enters a large flat slab of transparent crystal of thickness d, located between $z = 0$ and $z = d$. This crystal has the property that the index of refraction depends on the direction of polarization as follows: Plane waves traveling in the z direction but polarized in the direction

$$\hat{\mathbf{e}}_s = \cos\phi_0 \hat{\mathbf{x}} + \sin\phi_0 \hat{\mathbf{y}},$$

travel with speed $v_s = c/n_s < c$ but those polarized in the orthogonal direction

$$\hat{\mathbf{e}}_f = -\sin\phi_0 \hat{\mathbf{x}} + \cos\phi_0 \hat{\mathbf{y}},$$

travel with the faster speed $v_f = c/n_f < c$ where $n_s = n_f + \Delta n$.

Assume the wave, just after entering the crystal (i.e., for very small $z << \lambda < d$), is polarized in the y direction and hence has the form

$$\mathbf{E}(z \approx 0, t) = E_0\, \hat{\mathbf{y}}\, e^{-i\omega t}.$$

(a) Prove that when the plane wave reaches $z = d$

$$\mathbf{E}(z = d, t) = [E_x\, \hat{\mathbf{x}} + E_y\, \hat{\mathbf{y}}]\; e^{i(\bar{k}d - \omega t)},$$

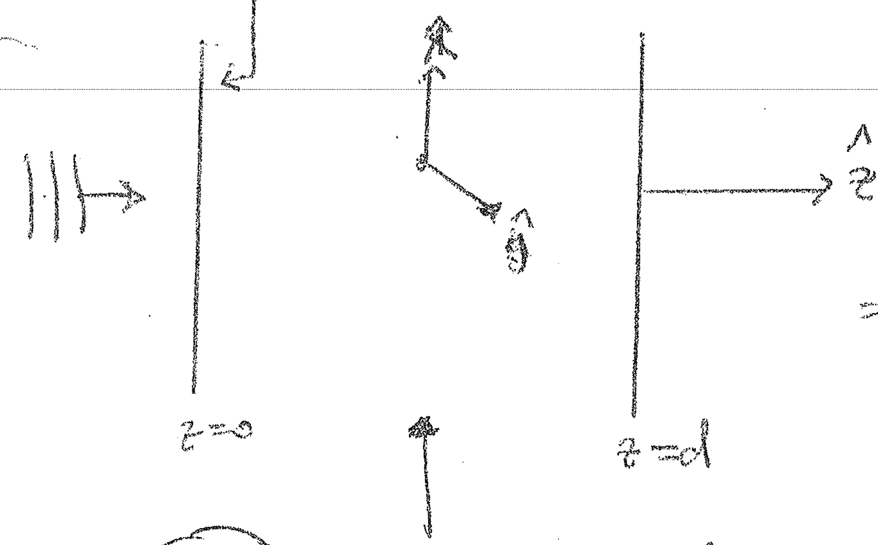
where $E_x = iE_0 \sin 2\phi_0 \sin\delta$ and $E_y = E_0(\cos\delta - i\cos 2\phi_0 \sin\delta)$, with $c\bar{k} \equiv \omega(n_s + n_f)/2$ and $\delta \equiv \omega d\, \Delta n/(2c)$.

(b) For what values of δ and θ_0 will the wave emerge from the crystal as a circularly polarized wave? (i.e., when will $E_x/E_y = \pm i$).

(c) For what minimum crystal thicknesses $d = d_{min}$ will the wave emerge as a plane polarized wave (i.e., when will $E_x/E_y =$ real) and what will its polarization direction be?

P5

$\vec{E} = E_0 \hat{y}\, e^{-i\omega t}$ at $z=0$!



$\hat{e}_s = \cos\phi_0 \hat{x} + \sin\phi_0 \hat{y}$

$\hat{e}_f = -\sin\phi_0 \hat{x} + \cos\phi_0 \hat{y}$

$\Rightarrow \hat{y} = \sin\phi_0 \hat{e}_s + \cos\phi_0 \hat{e}_f$

$\therefore$ ($z>0$)

$$\vec{E} = E_0 \sin\phi_0 \hat{e}_s e^{i(k_s z - \omega t)} + E_0 \cos\phi_0 \hat{e}_f e^{i(k_f z - \omega t}$$

where $k_s = \frac{\omega}{c} n_s \quad k_f = \frac{\omega}{c} n_f \quad n_s = n_f + \delta n$

at $z=0$

$$\vec{E}(d,t) = E_0\left(\sin\phi_0 \hat{e}_s e^{ik_s d} + \cos\phi_0 e^{ik_f d}\right)e^{-i\omega t}$$

similar but $-\delta$

$$e^{i\left(\frac{n_s+n_f}{2}\right)\frac{\omega}{c}d + i\underbrace{\left(\frac{n_s-n_f}{2}\right)\frac{\omega}{c}d}_{\text{defines } \delta}}$$

defines $\bar{n}$

$$\delta = \frac{\delta n}{2}\frac{\omega}{c}d$$

$$= E_0\left(\sin\phi_0 e^{i\delta}\hat{e}_s + \cos\phi_0 e^{-i\delta}\hat{e}_f\right)e^{i(\bar{n}\frac{\omega}{c}d - \omega t)}$$

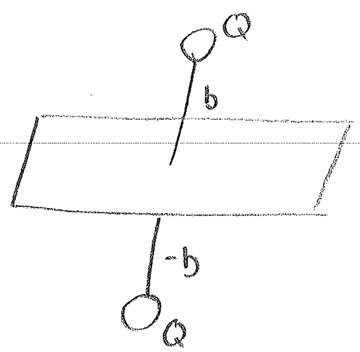
replace

defines $\bar{k}$

$\bar{k}d$

$$\vec{E}(d,t) = E_0\left(\left[\sin\phi_0\cos\phi_0 e^{i\delta} - \cos\phi_0\sin\phi_0 e^{-i\delta}\right]\hat{x} + \left[\sin^2\phi_0 e^{i\delta} + \cos^2\phi_0 e^{-i\delta}\right]\hat{y}\right)e^{i(\bar{k}d - \omega t)}$$

HW #9-1



a) $\vec{E} = \sum_c \frac{Q_a(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

$$= \frac{Q(x\hat{x}+y\hat{y}-b\hat{z})}{\left(\sqrt{x^2+y^2+(z-b)^2}\right)^3} + \frac{Q(x\hat{x}+y\hat{y}+b\hat{z})}{(x^2+y^2+(z+b)^2)^{3/2}}$$

$$= \frac{2Q(x\hat{x}+y\hat{y})}{(x^2+y^2+b^2)^{3/2}}$$ when evaluated in x-y plane

b) $T^{ij} = \frac{1}{4\pi}\left[D^i E^j + B^i H^j - \frac{1}{2}\delta^{ij}(D\cdot E + B\cdot H)\right]$

* Note: $B = H = 0$ b/c no currents

$D = E$ b/c linear, isotropic material

$$\Rightarrow T^{11} = \frac{1}{4\pi}\left(E^1E^1 - \frac{1}{2}E^2\right)$$

$$= \frac{1}{4\pi}\left(\frac{4Q^2x^2}{[x^2+y^2+b^2]^3} - \frac{1}{2}\,\frac{4Q^2(x^2+y^2)}{[x^2+y^2+b^2]^3}\right)$$

$$= \frac{Q^2(x^2-y^2)}{2\pi(x^2+y^2+b^2)^3}$$

$$T^{22} = \frac{Q^2(y^2-x^2)}{2\pi(x^2+y^2+b^2)^3}$$ by similar logic

$$T^{33} = \frac{-1}{2\pi}\,\frac{(x^2+y^2)}{(x^2+y^2+b^2)^3}$$

$$T^{12} = T^{21} = \frac{1}{\pi}\,\frac{xy}{(x^2+y^2+b^2)^3}$$

HW #9-1 (cont.)

c) $F^i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T^{ij}_m \, dA^j$ * Assume normal in positive z-direction

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T^{33}\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-(x^2+y^2)Q^2}{2\pi(x^2+y^2+b^2)}\, dx\, dy$$

$$= \frac{-Q^2}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{r^2}{r^2+b^2}\, dx\, dy$$

$$= \frac{-Q^2}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} \frac{r^3\, dr}{r^2+b^2}$$

* let $u = r^2+b^2$

$du = 2r\,dr$

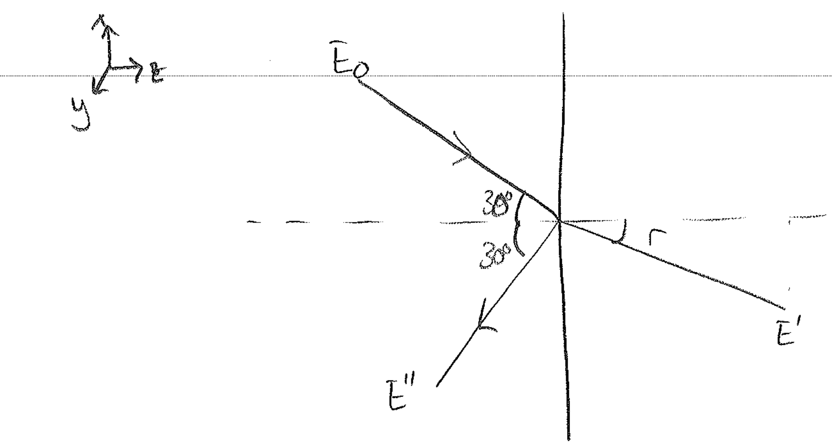
$$= \frac{-Q^2}{2\pi}\int_0^{2\pi} d\theta \int_{b^2}^{\infty} \frac{u^2-b^2}{u^3}\, du$$

$$= \frac{-Q^2}{2\cdot 2\pi}\cdot 2\pi \int_{b^2}^{\infty}\left[\frac{du}{u} - \frac{b^2}{u^3}\right] du$$

$$= \frac{-Q^2}{2}$$

2015 Final

#1



$$k = k\left(\frac{1}{2}\hat{z} + \frac{\sqrt{3}}{2}\hat{x}\right)$$

$$k'' = k\left(-\frac{1}{2}\hat{z} - \frac{\sqrt{3}}{2}\hat{x}\right)$$

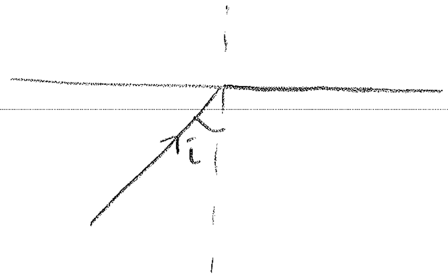
$$k' = k'(\cos r\, \hat{z} - \sin r\, \hat{x})$$

* Our wave equations are:

2015 Final

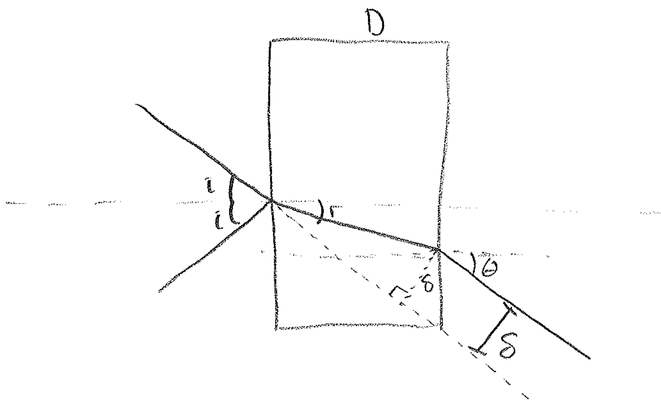
#4

a)



$$n\sin(i) = 1\sin 90$$
$$\sin(i) = \frac{1}{n}$$
$$i = \sin^{-1}\left(\frac{3}{4}\right)$$
$$= 48.59$$

b)



* Our boundary conditions yield: @ $z=0$

$$E_0\cos i + E''\cos i = E_+\cos r$$

$$H_0 - H'' = H_+$$

* but $H = \frac{B}{\mu} = nE$

$$E_0 - E'' = nH_+$$

@ $z = D$

$$E_+\cos(r)e^{ik_+D} = E'\cos i\, e^{ikD}$$

$$nE_+e^{ik_+D} = E'e^{ikD}$$

* Note: Repeated applications of Snell's law shows us $\theta = i$

at 1st boundary

$$\sin(i) = n\sin(r)$$

at 2nd boundary:

$$n\sin(r) = \sin(\theta)$$

$$\Rightarrow \sin(i) = \sin(\theta)$$
$$\hookrightarrow \theta = i$$

2015 Final

#4 (cont.)

$\Rightarrow$

$\cos(r) = \dfrac{D}{h} \rightarrow h = \dfrac{D}{\cos r}$

$\sin(i-r) = \dfrac{\delta}{h} = \dfrac{\delta \cos r}{D}$

$\Rightarrow D[\sin(i)\cos(r) - \sin(r)\cos(i)] = \delta \cos(r)$

$\delta = D[\sin(i) - \tan(r)\cos(i)]$

* At Brewsters angle

$\delta = D[\sin(i_B) - \tan(r)\cos(i_B)]$

December 10, 2014

PHYS 5583 (E & M II)

Final

Work only 4 of the following problems!

1. (25%)

   (a) When swimming under water much of the air surface above you looks like a mirror (if the water is clear and calm). The reason for this is that light traveling under water ($\mu_r = 1, n = 1.33$), will be completely reflected if it strikes the air surface at an incidence angle $i \geq i_0$. Derive an expression for i_0 as a function of n and give its value for water?

   (b) Light traveling in air ($\mu_r = \epsilon_r = 1$) when reflected from a smooth glass surface ($\mu_r = 1, n = 1.5$) is completely polarized in a direction **perpendicular** to the plane of incidence if the incidence angle is i_B, Brewster's angle. Derive an expression for i_B by computing the angle at which the amplitude of light polarized **in** the plane of incidence vanishes. Give its value for glass?

2. (25%) A plane polarized wave traveling in a vacuum ($n = 1$) in the $+z$ direction of the form

$$\mathbf{E} = E_0 \, e^{i(kz-wt)} \, \hat{\mathbf{x}}$$

is incident normally on a flat homogeneous conducting surface that is located at $z = 0$. Assume the conductor is described by a conductivity σ with $\mu_r \sim 1$ and $Re[\epsilon_r] \sim 1$. Recall that σ is defined by $\mathbf{J} = \sigma \mathbf{E}$ and that in Gaussian units the wave inside the conductor can be found by simply defining a complex permittivity $\epsilon_r = 1 + i4\pi\sigma/\omega$.

(a) Calculate the amplitudes of the reflected (E_0'') and transmitted (E_0') waves as functions of the conductors complex index of refraction n. (Leave them as functions of n and its complex conjugate $\bar{n}$).

(b) Calculate the time averaged Poynting vectors for the incoming and reflected waves and use them to evaluate the reflection coefficient R as a function of the conductor's complex index of refraction n.

(c) For a "good conductor", i.e., when $4\pi\sigma/\omega >> 1$ simplify R and give it's approximate value as as a function of σ and ω.

(d) At higher frequencies the conductor becomes a "poor conductor", i.e., when $4\pi\sigma/\omega << 1$. Simplify R for these higher frequencies and give it's approximate value as as a function of σ and ω.

(e) For an arbitrary conductor show that the transmitted wave's amplitude exponentially decays as it penetrates the conductor according to

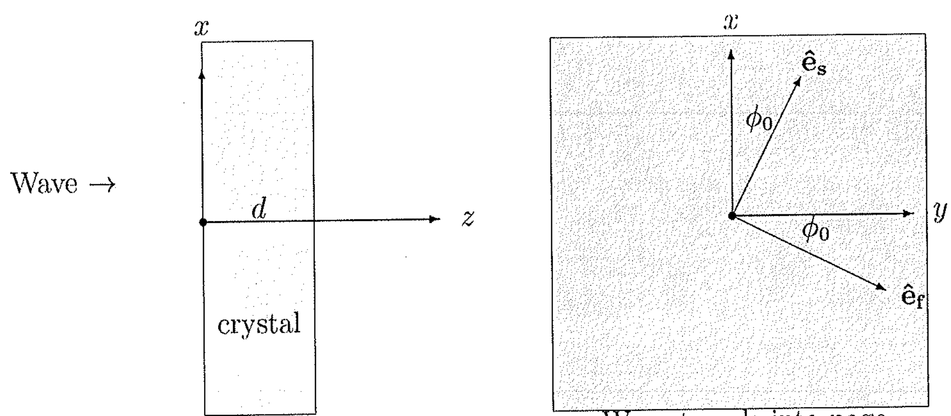
$$|\mathbf{E}'(\mathbf{z})| = |\mathbf{E}'(\mathbf{0})| e^{-z/\delta},$$

and give δ as a function of the conductors complex index of refraction n.

(f) For a "good conductor", i.e., when $4\pi\sigma/\omega >> 1$ simplify δ and give it's approximate value as as a function of σ and ω.

(g) At higher frequencies the conductor becomes a "poor conductor", i.e., when $4\pi\sigma/\omega << 1$. Simplify δ for these higher frequencies and give it's approximate value as a function of σ and ω.

3. {25%} A plane polarized monochromatic light wave traveling in the



Wave travels into page.

$+z$ direction enters a large flat slab of transparent crystal of thickness d, located between $z = 0$ and $z = d$. This crystal has the property that the index of refraction depends on the direction of polarization as follows: Plane waves traveling in the z direction but polarized in the direction

$$\hat{\mathbf{e}}_s = \cos\phi_0 \hat{\mathbf{x}} + \sin\phi_0 \hat{\mathbf{y}},$$

travel with speed $v_s = c/n_s < c$ but those polarized in the orthogonal direction

$$\hat{\mathbf{e}}_f = -\sin\phi_0 \hat{\mathbf{x}} + \cos\phi_0 \hat{\mathbf{y}},$$

travel with the faster speed $v_f = c/n_f < c$ where $n_s = n_f + \Delta n$.

Assume the wave, just after entering the crystal (i.e., for very small $z << \lambda < d$), is polarized in the y direction and hence has the form

$$\mathbf{E}(z \approx 0, t) = E_0\, \hat{\mathbf{y}}\, e^{-i\omega t}.$$

(a) Prove that when the plane wave reaches $z = d$

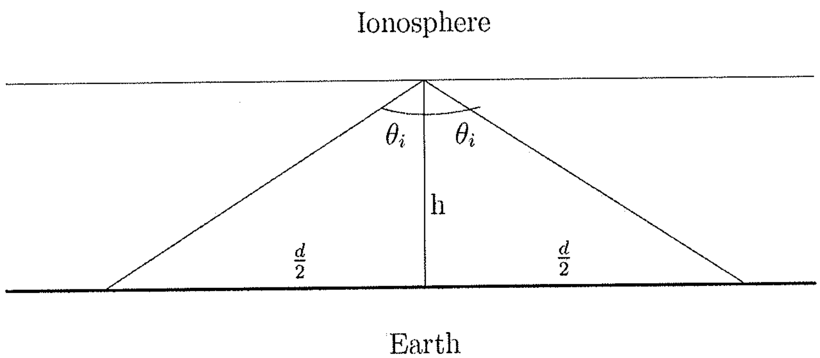
$$\mathbf{E}(z = d, t) = [E_x\, \hat{\mathbf{x}} + E_y\, \hat{\mathbf{y}}]\, e^{i(\bar{k}d - \omega t)},$$

where $E_x = iE_0 \sin 2\phi_0 \sin\delta$ and $E_y = E_0(\cos\delta - i\cos 2\phi_0 \sin\delta)$, with $c\bar{k} \equiv \omega(n_s + n_f)/2$ and $\delta \equiv \omega d\, \Delta n/(2c)$.

(b) For what values of δ and θ_0 will the wave emerge from the crystal as a circularly polarized wave? (i.e., when will $E_x/E_y = \pm i$).

(c) For what minimum crystal thicknesses $d = d_{min}$ will the wave emerge as a plane polarized wave (i.e., when will $E_x/E_y =$ real) and what will its polarization direction be?

4. {25%}



The ionosphere reflects radio waves of frequency ω back to earth if the wave strikes the ionosphere at an incidence angle θ_i greater than $\theta_0(\omega)$ (see the figure). Assume that the ionosphere is a very thick flat layer beginning abruptly at altitude h above the earth.

(a) Assume the ionosphere is a plasma whose relative permittivity is approximated by

$$\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2} = 1 - \frac{\lambda^2}{\lambda_p^2},$$

and whose relative permeability is $\mu_r = 1$. Assume the atmosphere below the ionosphere has an index of refraction $n = 1$ and use snell's law to find θ_0 as a function of ω.

(b) An amateur radio operator finds that when operating at $\lambda = 14$ meter wavelengths in the early evening she can receive stations more distant than 1500 km away. However operating at $\lambda = 20$ meters she can receive stations more distant than 1000 km.

Use the above data and your expression for $\theta_0(\omega)$ to find both the height of the ionosphere h and the plasma frequency ω_p.

(c) The plasma frequency ω_p from part (b) can be used to estimate the free electron number density N_e in the ionosphere by assuming the relative permittivity given in (a) is a result of free electrons being displaced by the wave's electric field

$$m\ddot{\mathbf{r}} = -e\mathbf{E}_0 e^{-i\omega t}.$$

In Gaussian units the resulting expression for the plasma frequency is:

$$\omega_p^2 = 4\pi N_e \frac{e^2}{m}.$$

If $e^2/mc^2 = 2.82 \times 10^{-13}$ cm what is the electron density in the ionosphere?

5. {25%}

(a) Calculate the phase and group velocities of a plane harmonic wave of frequency ω traveling in a homogeneous tenuous plasma. Recall that for such a plasma $\mu_r = 1$ and $\epsilon_r = 1 - \omega_p^2/\omega^2$. Also recall that the phase and group velocities are defined by $v_p = \omega/k$ and $v_g = d\omega/dk$.

(b) For frequencies way above the plasma frequency, i.e., for $\omega_p << \omega$, simplify your answers for v_p and v_g and keep only the lowest order terms in (ω_p/ω).

(c) In the absence of a magnetic field all waves travel at the same speeds independent of their polarization. However, if a magnetic field exists in the plasma left and right circularly polarized waves travel at different speeds. Assume a uniform magnetic induction $B_{||}$ parallel to the wave's propagation direction exists in the plasma. What are the phase and group velocities for left and right circularly polarized waves? Recall that

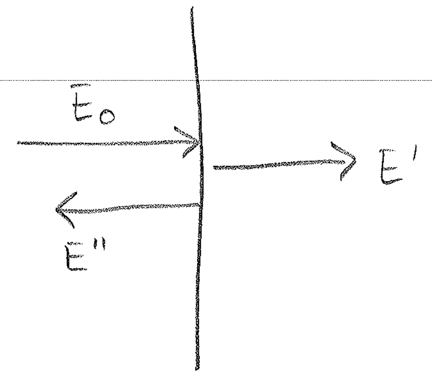
$$\epsilon_r = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)},$$

where the upper sign is for l.c.p. waves and the lower is for r.c.p. waves.

(d) For frequencies way above the plasma frequency, i.e., for $\omega_p << \omega$, and frequencies not too close to ω_B, simplify your answers for v_p and v_g keeping only the lowest order terms in (ω_p/ω) for l.c.p. and r.c.p. waves. Also give differences in v_p and v_g values for l.c.p. and r.c.p. waves. Which one is faster?

2014 Final

#2



* In air:

$$\vec{E}_0 = E_0 e^{i(kz-\omega t)}$$

$$\vec{E}'' = E'' e^{i(-kz-\omega t)}$$

* In conductor

$$\vec{E}' = E' e^{i(k'z-\omega t)}$$

* Applying our boundary conditions:

$$E_0 + E'' = E'$$

$$H_0 - H'' = H'$$

* but $H = \frac{nE}{\mu}$

$$E_0 - E'' = nE'$$

$$\frac{E''}{E^0} = \frac{1-n}{n+1}$$

$$\frac{E'}{E_0} = \frac{2}{n+1}$$

b) $S = \frac{c}{4\pi} Re[E] \times Re[H]$

$= \frac{c}{4\pi} \left(\frac{E+E^*}{2}\right) \times \left(\frac{H+H^*}{2}\right)$

$= \frac{c}{4\pi}\left[E\times H + E\times H^* + E^*\times H + E^*\times H^*\right]$

$\langle S\rangle = \frac{c}{4\pi} \frac{Re[E\times H^*]}{2}$

$\langle S\rangle = \frac{c}{8\pi} Re[E \times n^* E^*]$

$\langle S\rangle = \frac{c\,Re[n^*]}{8\pi} |E|^2$

$\Rightarrow \langle S_0\rangle = \frac{c\,Re[n^*]}{8\pi} |E_0|^2$

$\langle S''\rangle = \frac{c\,Re[n^*]}{8\pi} |E''|^2$

$\Rightarrow R = \frac{\langle S''\rangle}{\langle S\rangle}$

$= \frac{|E''|^2}{|E_0|^2}$

$= \frac{|1-n|^2}{|n+1|^2}$

#2 (cont)

c) $n = \sqrt{\epsilon}$

$$= \sqrt{1 + i\frac{4\pi\sigma}{\omega}}$$

$$\approx \sqrt{i\frac{4\pi\sigma}{\omega}}$$

$$= \frac{(1+i)}{2}\sqrt{\frac{4\pi\sigma}{\omega}}$$

$$\Rightarrow \mathrm{Re}[n] = \sqrt{\frac{2\pi\sigma}{\omega}}$$

$$\Rightarrow R \approx \frac{\left(1 - \left[(1+i)\sqrt{\frac{2\pi\sigma}{\omega}}\right]\right)\left(1 - (1-i)\sqrt{\frac{2\pi\sigma}{\omega}}\right)}{\left(1 + \left[(1+i)\sqrt{\frac{2\pi\sigma}{\omega}}\right]\right)\left(1 + (1-i)\sqrt{\frac{2\pi\sigma}{\omega}}\right)}$$

d) $n \approx 1$

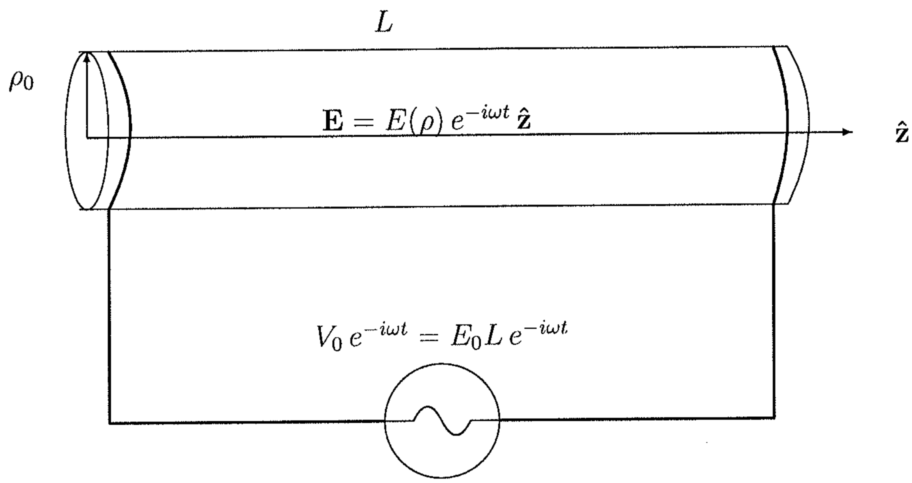
$\hookrightarrow R = 0$ ✓

December 9, 2013

PHYS 5583 (E & M II)

Final

1. {33%}



The a-c current flowing in a very long cylindrical conducting wire of radius ρ_0 and conductivity σ (shown in the figure) can be written as an integral over the wire's cross section as

$$I = \int_0^{\rho_0} \sigma E(\rho) e^{-i\omega t} 2\pi\rho d\rho.$$

The harmonic electric field inside the wire is of the form

$$\mathbf{E} = E(\rho)\,\hat{\mathbf{z}}\, e^{-i\omega t},$$

and can be written as a function of $J_0(x)$, the $m = 0$ Bessel function as

$$E(\rho) = E_0 \frac{J_0(k\rho)}{J_0(k\rho_0)},$$

where $k = (1+i)/\delta$ and $\delta^{-1} = \sqrt{2\pi\mu\sigma\omega}/c$ is the skin depth for a good conductor. Because $J_0(x)$ satisfies the $m = 0$ Bessel's equation

$$\frac{(xJ_0'(x))'}{x} = -J_0(x),$$

the above current can be written as

$$I = \frac{2\pi\sigma\rho_0 E_0}{k}\left(-\frac{J_0'(k\rho_0)}{J_0(k\rho_0)}\right) e^{-i\omega t}.$$

(a) Use Maxwell's equations to compute the Magnetic induction $\mathbf{B}$ at the surface of the wire from $\mathbf{E}$ given above. Your answer will involve a derivative of $J_0(x)$. (Recall $\nabla \times [f(\rho)\hat{\mathbf{z}}] = \nabla f(\rho) \times \hat{\mathbf{z}}$).

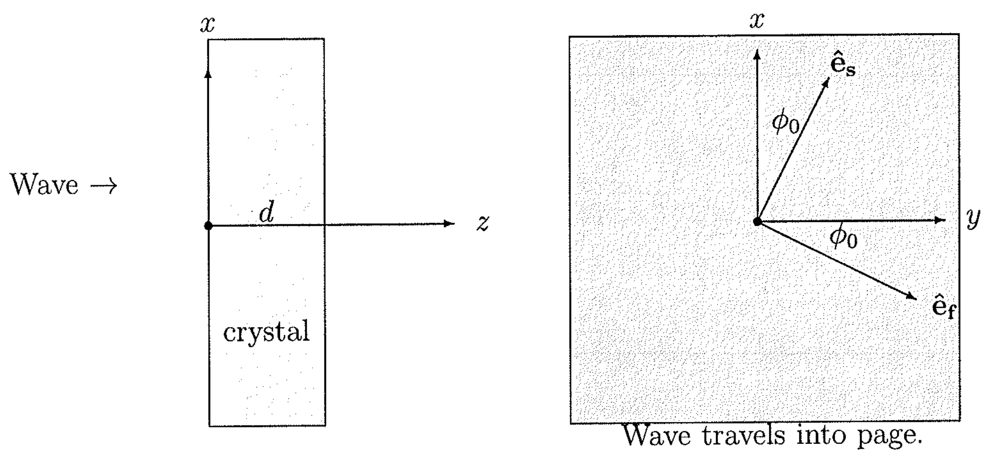
(b) Calculate the time average of the Poynting vector at the surface of the wire (assume $\mu = 1$).

(c) Use the Poynting vector to calculate the average rate energy is flowing into the wire's surface per unit length.

(d) Calculate the heat loss (the I^2R loss) per unit length in the wire. HINT: Simply time-average $Real(I) * Real(V)$ for a 1 cm length of wire to get the heat loss per unit length.

(e) Does the rate energy is lost to heat agree with the energy flow rate calculated in (c)? If not, why not?

2. {33%} A plane polarized monochromatic light wave traveling in the



$+z$ direction enters a large flat slab of transparent crystal of thickness d, located between $z = 0$ and $z = d$. This crystal has the property that the index of refraction depends on the direction of polarization as follows: Plane waves traveling in the z direction but polarized in the direction

$$\hat{\mathbf{e}}_s = \cos\phi_0 \hat{\mathbf{x}} + \sin\phi_0 \hat{\mathbf{y}},$$

travel with speed $v_s = c/n_s < c$ but those polarized in the orthogonal direction

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travel with the faster speed $v_f = c/n_f < c$ where $n_s = n_f + \Delta n$.

Assume the wave, just after entering the crystal (i.e., for very small $z << \lambda < d$), is polarized in the y direction and hence has the form

$$\mathbf{E}(z \approx 0, t) = E_0 \, \hat{\mathbf{y}} \, e^{-i\omega t}.$$

(a) Prove that in general the initial plane wave becomes elliptically polarized when it reaches $z = d$ by deriving the following expression

$$\mathbf{E}(z = d, t) = [E_x \, \hat{\mathbf{x}} + E_y \, \hat{\mathbf{y}}] \, e^{i(\bar{k}d - \omega t)},$$

where

$$\bar{k} \equiv \frac{\omega}{c}\left(\frac{n_s + n_f}{2}\right),$$

and

$$E_x = iE_0 \sin 2\phi_0 \sin\delta,$$
$$E_y = E_0(\cos\delta - i\cos 2\phi_0 \sin\delta),$$

with

$$\delta \equiv \frac{\omega d}{2c}\Delta n.$$

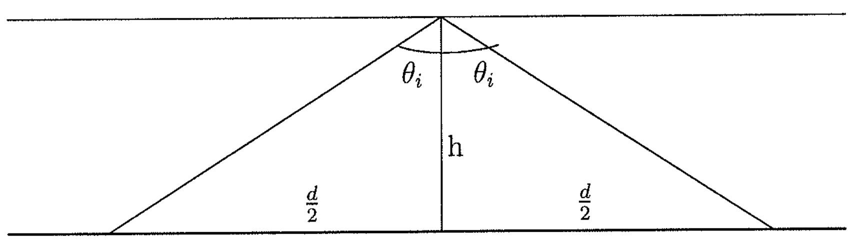
Hint: Write the wave at z=0 as a combination of slow and fast plane polarized parts using $\hat{\mathbf{y}} = \sin\phi_0 \hat{\mathbf{e}}_s + \cos\phi_0 \hat{\mathbf{e}}_f$.

(b) For what values of δ and θ_0 will the wave emerge from the crystal as a circularly polarized wave? ($E_x/E_y = \pm i$).

(c) For what minimum crystal thicknesses $d = d_{min}$ will the wave emerge as a plane polarized wave (E_x/E_y = real) and what will its polarization direction be?

3. {33%}

Ionosphere



Earth

The ionosphere reflects radio waves of frequency ω back to earth if the wave strikes the ionosphere at an incidence angle θ_i greater than $\theta_0(\omega)$ (see the figure). Assume that the ionosphere is a very thick flat layer beginning abruptly at altitude h above the earth.

(a) Assume the ionosphere is a plasma whose relative permittivity is approximated by

$$\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2} = 1 - \frac{\lambda^2}{\lambda_p^2},$$

and whose relative permeability is $\mu_r = 1$. Assume the atmosphere below the ionosphere has an index of refraction $n = 1$ and use snell's law to find θ_0 as a function of ω.

(b) An amateur radio operator finds that when operating at $\lambda = 14$ meter wavelengths in the early evening she can receive stations more distant than 1500 km away. However operating at $\lambda = 20$ meters she can receive stations more distant than 1000 km.

Use the above data and your expression for $\theta_0(\omega)$ to find both the height of the ionosphere h and the plasma frequency ω_p.

(c) The plasma frequency ω_p from part (b) can be used to estimate the free electron number density N_e in the ionosphere by assuming the relative permittivity given in (a) is a result of free electrons being displaced by the wave's electric field

$$m\ddot{\mathbf{r}} = -e\mathbf{E}_0 e^{-i\omega t}.$$

If $e^2/mc^2 = 2.82 \times 10^{-13}$ cm what is the electron density in the ionosphere?

May 8, 2012

PHYS 5583 (E & M II)

Final

1. {25%}

   (a) In a homogeneous, linear and isotropic conducting material whose electromagnetic properties (at low frequencies) are described by constant (and real) values of the permittivity, permeability, and conductivity respectively ϵ, μ, and σ, show that Maxwell's equations require that the electric field satisfy the telegraph equation

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0, \qquad (SI)$$

$$\nabla^2 \mathbf{E} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \mathbf{E}}{\partial t} = 0. \qquad (Gaussian)$$

   (b) Given a plane polarized plane wave of angular frequency ω whose electric field is of the form

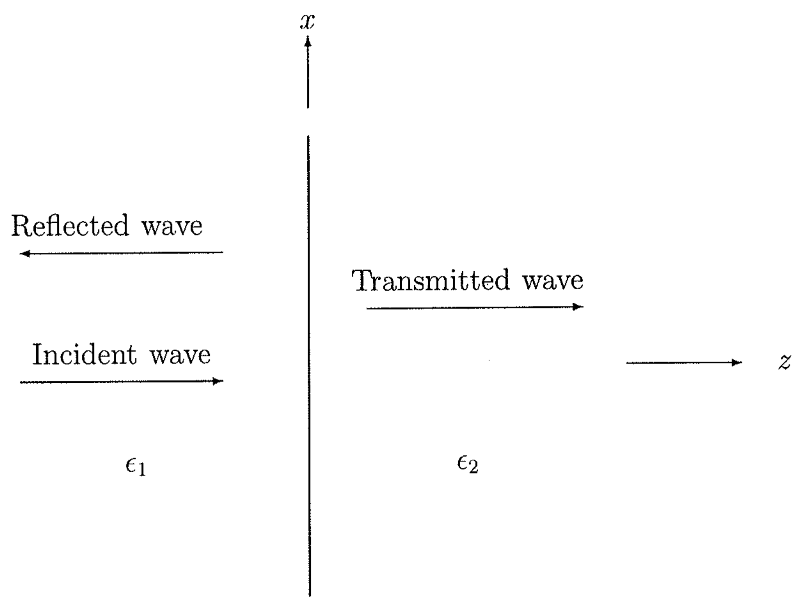
$$\mathbf{E}(z,t) = \text{Real}\left\{\hat{\mathbf{x}} E_0 e^{i(kz-\omega t)}\right\},$$

   evaluate k^2 as a function of ϵ, μ, σ, and ω.

   (c) Find the real and imaginary parts of k assuming $\sigma >> \omega\epsilon$.

   (d) Using your results from (c) find the skin depth δ of the conductor. The skin depth is defined by the depth at which the wave's amplitude decreases by e^{-1}, i.e.,

$$\frac{|\mathbf{E}(z+\delta,t)|}{|\mathbf{E}(z,t)|} = \frac{1}{e}$$



2. {25%} A linearly-polarized harmonic ($e^{-i\omega t}$) plane electromagnetic wave traveling to the right in a homogeneous dielectric medium described by a real dielectric constant ϵ_1, strikes a second homogeneous dielectric material described by another real dielectric constant $\epsilon_2 > \epsilon_1$ (see the figure). Assume that both materials have the same magnetic permeability μ_0 and that the incidence angle is 0^o (i.e., the wave is traveling perpendicular to the junction). Assume the incoming wave is polarized in the $\hat{x}$ direction and that its electric field amplitude is E_0, i.e., assume the incoming electric field is the real part of

$$\mathbf{E} = E_0\, e^{i(kz-wt)}\, \hat{x}.$$

(a) Give the magnetic induction $\mathbf{B}$ associated with the above incoming wave, k as a function of ω, the direction of $\mathbf{B}$, and the amplitude of $\mathbf{B}$ as a function of E_0.

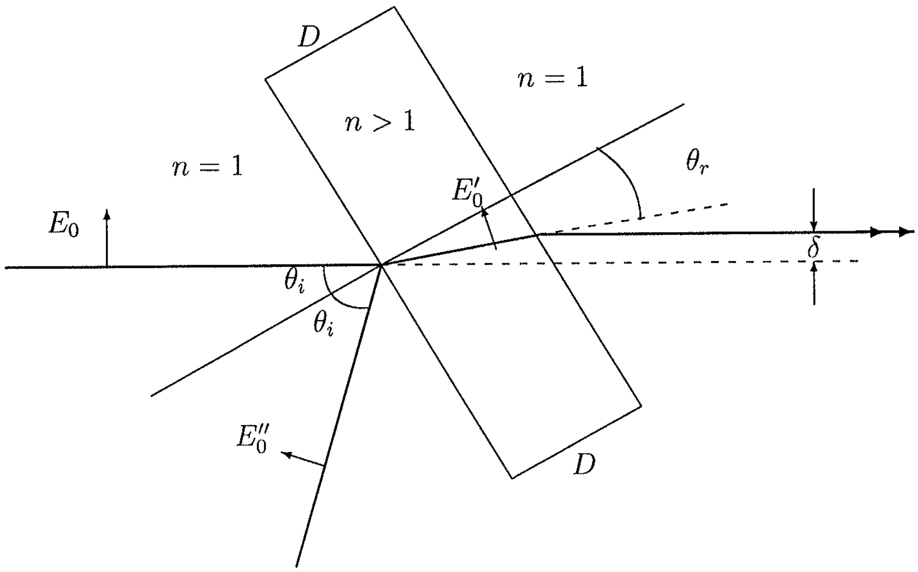
(b) Give similar expressions for the $\mathbf{E}$ and $\mathbf{B}$ components of the reflected and transmitted waves. Use E_0'' and E_0' for the respective amplitudes of reflected and transmitted waves.

(c) Apply the junction conditions to the incoming, reflected, and transmitted wave to compute E_0'' and E_0' as functions of E_0 and the two dielectric constants ϵ_1 and ϵ_2.

(d) Evaluate the reflection and transmission coefficients, R and T, for above waves. Recall that the Poynting vector is defined by

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} \equiv \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$



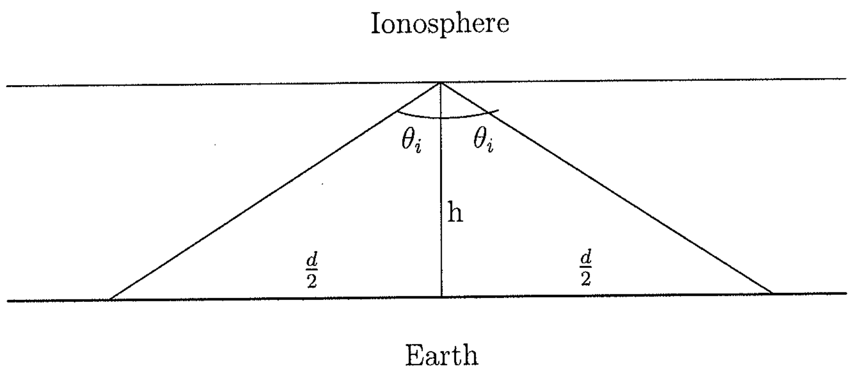
3. {25%} A narrow columnated monochromatic beam of light from a laser hits a flat piece of ordinary glass of thickness D in air at angle θ_i as shown. The incident, reflected, and refracted light beams are linearly polarized in the incidence plane as shown and have respective amplitudes E_0, E''_0, and E'_0 also as shown. Assume that the index of refraction of air is 1 and of the glass is n. Also assume a relative permeability $\mu_r = 1$ for both the air and glass. Ignore the reflected beam that occurs when the beam exits the glass on the right of the figure.

   (a) Use the junction conditions on the $\mathbf{E}, \mathbf{D}, \mathbf{B}$ and $\mathbf{H}$ and Snell's law to derive the Fresnel formulas for E''_0/E_0 and E'_0/E_0 as a function of θ_i.

   (b) Use your results to find the incidence angle θ_B (Brewster's angle) at which the reflected amplitude E''_0 vanishes.

   (c) After the transmitted beam exits the glass and returns to air it will be traveling parallel to its original direction but displaced a distance δ as shown in the figure. At Brewster's angle what will this displacement be?

4. {25%}



An amateur radio operator finds that when operating at $\lambda = 16$ meter wavelengths in the early evening she can receive stations more distant than 1700 km away. However operating at $\lambda = 20$ meters she can receive stations more distant than 1200 km.

Assume, as was done in homework problem # 7.13 that the ionosphere is responsible for reflecting the radio signals and that it is a very thick flat layer beginning abruptly at altitude h above the earth. By assuming the ionosphere is a plasma whose relative permittivity is modeled by

$$\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2} = 1 - \frac{\lambda^2}{\lambda_p^2}$$

and whose relative permeability $\mu_r = 1$, problem # 7.13 concluded that when the incidence angle satisfied

$$\sin^2\theta_i \geq 1 - \frac{\omega_p^2}{\omega^2} = 1 - \frac{\lambda^2}{\lambda_p^2},$$

the reflection was 100%.

Use the above data and the conclusions of problem # 7.13 to find both the height of the ionosphere h and the plasma wavelength λ_p.

Derivation of R and T

* In case where E ∥ to plane of incidence: