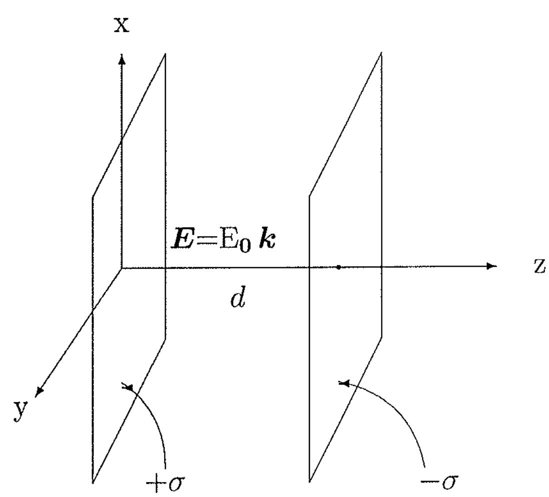
6. A uniform static electric field $\mathbf{E} = E_0\mathbf{k}$ exists between two large thin conducting metal plates. The positive plate is at $z = 0$ and the negative plate is at $z = a$. You can assume the plates are infinitely large in the x-y directions.

(a) {1 pts} Use Maxwell's equations to relate the value of E_0 to the surface charge density $\pm\,\sigma$ on the plates.

(b) {4 pts} Lorentz transform $F^{\alpha\beta}$ to obtain the $\mathbf{E}$ and $\mathbf{B}$ fields seen by an observer moving between the plates with velocity c/2 $\hat{\imath}$?

(c) {2 pts} What are the 4-current densities J^{σ} of the plates in the rest frame and in a frame moving with the observer?

(d) {3 pts} Show that Maxwell's inhomogeneous equations are satisfied by your fields and charge-currents in the moving frame. (Hint: Using $\mathbf{E}$ and $\mathbf{B}$ rather than $F^{\alpha\beta}$ is probably easier.)



Aug 2011

E+M #6

Gaussian

a) Using Gauss Law, we see

$$\int \vec{E} \cdot d\vec{a} = 4\pi q_{enc}$$

$$E \cdot 2A = 4\pi \sigma A$$

$$E = 2\pi\sigma \quad \text{(for one plate)}$$

$$\hookrightarrow \vec{E}_0 = 4\pi\sigma$$

b) $$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* In matrix form $F' = \Lambda F \Lambda^T$

$$F' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & +\beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ \gamma E_z & -\beta\gamma E_z & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{E}' = \gamma \vec{E}$$

$$\vec{B}' = -\beta\gamma E_z \hat{y}$$

* If $\beta = \frac{v}{c} = \frac{1}{2}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{1}{4}}}$

$$= \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2}{\sqrt{3}}$$

#6 (cont.)

c) The 4-current vector J^α is defined as $J^\alpha = \langle c\rho, \vec{J} \rangle$ and transforms according to: $J'^\beta = \Lambda^\beta{}_\alpha J^\alpha$

$$J^\alpha = \langle c[\sigma\delta(z) - \sigma\delta(z-d)], 0 \rangle$$

$$J' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\rho \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J' = \begin{bmatrix} \gamma c\rho \\ -\beta\gamma c\rho \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c\rho' \\ J'_x \\ J'_y \\ J'_z \end{bmatrix} \Rightarrow \begin{array}{l} \rho' = \gamma\sigma(\delta(z) - \delta(z-d)) \\ \vec{J} = \langle -\beta\gamma c\sigma(\delta(z) - \delta(z-d)), 0, 0 \rangle \end{array}$$

d) The inhomogeneous Maxwell equations in Gaussian units are:

$$\nabla \cdot E' = 4\pi\rho'$$

$$\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$$

– To test Gauss Law

$$\nabla \cdot E' = \frac{\partial}{\partial z} \frac{\partial}{\sqrt{3}}(4\pi\sigma)$$

$$= \frac{8\pi}{\sqrt{3}} \frac{\partial}{\partial z}\left(\sigma[\Theta(z) - \Theta(z-d)]\right) \leftarrow$$ Assumption works, not sure why since σ only exists on plates

$$= \frac{8\pi}{\sqrt{3}}\sigma(\delta(z) - \delta(z-d))$$

$$= 4\pi\left(\frac{2}{\sqrt{3}}\sigma[\delta(z - \delta(z-d)]\right)$$

$$= 4\pi\rho' \checkmark$$

#6 (cont)

d) - Evaluating Ampere's Law

$$\nabla \times B' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & -\beta\gamma E_z & 0 \end{vmatrix} = \langle -\partial_z(\beta\gamma E_z),\ 0,\ \partial_x(-\beta\gamma E_z) \rangle$$

$$= \langle -\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{3}} 4\pi\sigma[\Theta(z)-\Theta(z-d)]\right),\ 0,\ \frac{\partial}{\partial x}\left(\frac{-1}{\sqrt{3}} 4\pi\sigma[\Theta(z)-\Theta(z-d)]\right) \rangle$$

$$= \langle -\frac{4\pi\sigma}{\sqrt{3}}(\delta(z)-\delta(z-d)),\ 0,\ 0 \rangle$$

$$= \langle -\frac{4\pi}{c}\left(\frac{c\sigma}{\sqrt{3}}[\delta(z)-\delta(z-d)]\right),\ 0,\ 0 \rangle$$

$$= \langle -\frac{4\pi}{c}(\beta\gamma c E_z),\ 0,\ 0 \rangle$$

$$= -\frac{4\pi}{c} \langle J'_x,\ 0,\ 0 \rangle \ \checkmark$$

(* Note: $\frac{1}{c}\frac{\partial E}{\partial t} = 0$ b/c no time dependence in formula)