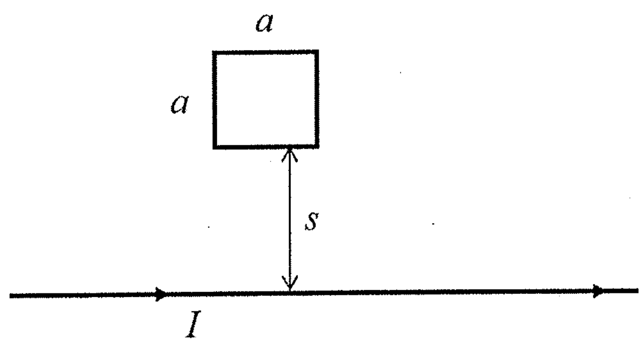
2-3

Problem 2: Maxwell equations

Consider a square loop of side a and with resistance R. As shown below, the loop is a distance s from an infinite straight wire that carries a current I.



(a) Find the magnetic field as a function of distance from the wire. [1 point]

(b) Find the magnetic flux through the square loop. [1 point]

Griffiths 7.18

(c) Suppose the infinite wire is cut, so its current drops to zero. Explain why this will cause a current to flow through the loop. In what direction will the current flow through the loop? [1 points]

(d) For the situation described in part c), what total charge passes a given point in the loop during the time the current flows? [2 points]

An alternating current $I = I_o \cos \omega t$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a.

Griffiths 7.16

(e) Does the induced electric field point in the radial, circumferential, or longitudinal direction? Assuming that the field goes to zero as s goes to infinity, find $E(s,t)$. [2 points]

(f) Find the displacement current density. Integrate your answer to get the total displacement current. [3 points]

Aug 2018

E+M #2

Gaussian

a) We can determine the magnetic field from Ampere's Law

$$\int(\nabla\times B)\cdot d\vec{a} = \frac{4\pi}{c}\int J(\vec{x})\,da$$

$$\oint B\cdot dl = \frac{4\pi}{c} I_{enc}$$

$$B\cdot 2\pi s = \frac{4\pi}{c} I$$

$$\vec{B} = \frac{2I}{cs}\hat{\varphi} \quad \text{(direction determined by RHR)}$$

b) Φ_B, the magnetic flux thru the loop is:

$$\Phi_B = \int \vec{B}\cdot d\vec{a}$$

$$= \int \frac{2I}{cs}\,ds\,dz$$

$$= \frac{2Ia}{c}\int \frac{1}{s}ds$$

$$= \frac{2Ia}{c}\ln\left(\frac{s+a}{s}\right)$$

c) Nature abhors a change in flux. As such, the induced current will flow in the direction necessary to counteract the change. Since the RHR says our initial field points out of the page, so will the induced field. Thus the current flows counter clockwise within our loop.

d) The total charge in the loop is determined by:

$$Q = \frac{1}{R}\nabla\Phi$$

$$= \frac{1}{R}\cdot\frac{2Ia}{c}\ln\left(\frac{s+a}{s}\right)$$

#2 (cont.)

e) * We can use our samework from part a, only $I \to I_0 \cos(\omega t)$ where $I_{enc} \neq 0$. Thus the magnetic field for this configuration is:

$$\vec{B} = \begin{cases} \frac{2 I_0 \cos(\omega t)}{cs} \hat{\varphi} & a < s < b \\ 0 & \text{elsewhere} \end{cases}$$

* From Lenz's law, we know $\vec{E}$ points in the $\hat{z}$ direction. The $\hat{\varphi}$ component must be 0 since $\oint E \cdot dl = 0$ for an amperian loop, and $E(\hat{r}) = 0$ b/c current reversal is equivalent to flipping the ends of the wire in our problem, which will have no impact on the radial symmetry.

* To find $\vec{E}$:

$$\int \nabla \times E da = -\int \frac{1}{c} \frac{\partial B}{\partial t} da$$

$$\oint E \cdot dl = \frac{-1}{c} \int \frac{\partial B}{\partial t} da$$

$$E \cdot l = \frac{-1}{c} \int \frac{-2 I_0 \omega \sin(\omega t)}{cs} da$$

$$E \cdot l = \frac{2 I_0 \omega \sin(\omega t)}{c^2} \int \frac{1}{s} ds dz$$

$$E = \frac{2 I_0 \omega \sin(\omega t)}{c^2} \ln\left(\frac{a}{s}\right)$$

$$\Rightarrow E = \begin{cases} \frac{2 I_0 \omega \sin(\omega t)}{c^2} \ln\left(\frac{a}{s}\right) & 0 < s < a \\ 0 & s > a \end{cases}$$

#2 (cont.)

f) * The displacement current is defined as:

$$I_d = \frac{1}{4\pi}\frac{\partial E}{\partial t}$$

$$= \frac{1}{4\pi}\frac{2I_0 w^2\cos(wt)}{c^2}\ln\left(\frac{a}{s}\right)$$

$$= \frac{I_0 w^2\cos(wt)}{2\pi c^2}\ln\left(\frac{a}{s}\right)$$

* Integrating over a volume of length L, we get

$$\vec{J}_d = \int \frac{I_0 w^2\cos(wt)}{2\pi c^2}\ln\left(\frac{a}{s}\right) s\,ds\,d\varphi\,dz$$

$$= \frac{I_0 w^2\cos(wt)L}{c^2}\int \ln\left(\frac{a}{s}\right) s\,ds$$

$$= \frac{I_0 w^2\cos(wt)L}{c^2}\left[\frac{1}{2}x^2\ln\left(\frac{a}{x}\right)+\frac{x^2}{4}\right]\Big|_s^a$$

$$= \frac{I_0 w^2\cos(wt)L}{c^2}\left[\frac{a^2}{4}-\frac{1}{2}s^2\ln\left(\frac{a}{s}\right)-\frac{s^2}{4}\right]$$