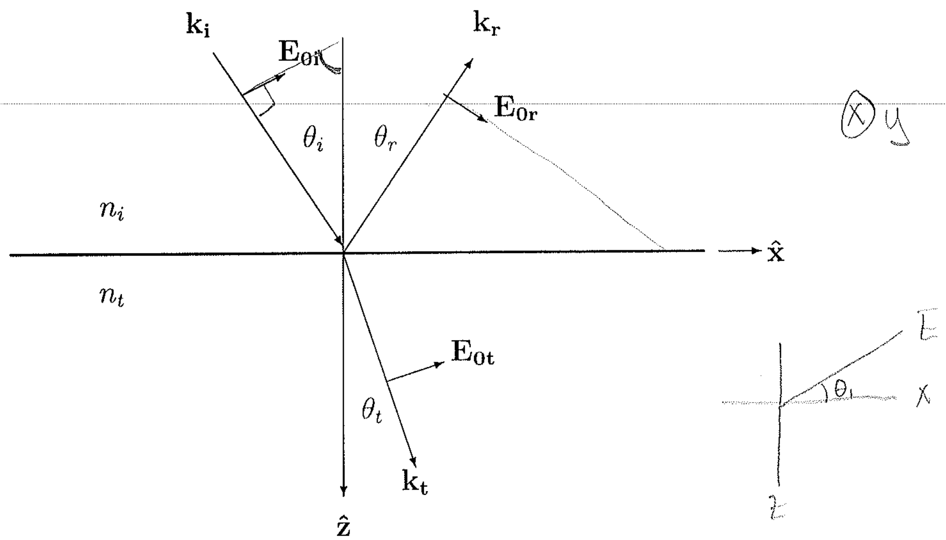

1. A plane monochromatic electromagnetic wave of wave vector $\mathbf{k}_i$, amplitude $\mathbf{E}_{0i}$, and angular frequency ω is incident at the planar interface of two dielectric, non-magnetic ($\mu = \mu_0$ in SI units), non-absorbing media (i.e., have real indices of refraction n_i and n_t). The angle of incidence is equal to θ_i. Part of the incident wave is reflected at an angle $\theta_r = \theta_i$ and part of it is transmitted into the second medium at a transmission angle θ_t. Assume the electric fields of the incident, reflected, and refracted waves lie in the plane of incidence as shown in the figure. Assume coordinates are chosen so that the dielectric interface is the $z = 0$ plane and the polarization is in the x-z plane.

   (a) [2 pts] Use Maxwell's equations to derive an expression for the magnetic induction $\mathbf{B}$ associated with a plane monochromatic electromagnetic wave whose electric field is

   $$\mathbf{E} = \mathbf{E}_0 \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

   traveling in a homogeneous material described by a real index of refraction n. Give the relationship of $|\mathbf{k}|$ to ω.

   (b) [2 pts] From the above figure give $\mathbf{k}_i, \mathbf{k}_r$, and $\mathbf{k}_t$ in terms of their $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components, and evaluate $\mathbf{k} \cdot \mathbf{r}$ in the $z = 0$ plane.

   (c) [1 pts] From the above figure give $\mathbf{E}_{0i}, \mathbf{E}_{0r}$, and $\mathbf{E}_{0t}$ in terms of their $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components.

   (d) [1 pts] State the 4 boundary conditions satisfied by the fields $\mathbf{E}, \mathbf{B}, \mathbf{H}$, and $\mathbf{D}$ at the above $z = 0$ junction.

   (e) [1 pts] Use one of these junction conditions to prove Snell's law, $n_t \sin\theta_t = n_i \sin\theta_i$ (only 2 of the 4 are independent).

   (f) [3 pts] Use two of the junction conditions to determine the ratio of the magnitude of the amplitudes of the reflected and transmitted to the incident electric fields, i.e., evaluate $|\mathbf{E}_{0r}|/|\mathbf{E}_{0i}|$ and $|\mathbf{E}_{0t}|/|\mathbf{E}_{0i}|$ as shown in the figure.

Aug 2013

E+M #1

SI

* Remember:

$\nabla \circ D = \rho_f$ $\qquad$ $\nabla \cdot B = 0$

$\nabla \times E + \frac{\partial B}{\partial t} = 0$ $\qquad$ $\nabla \times H - \frac{\partial D}{\partial t} = \vec{J}_f$

$D = \epsilon E$ $\qquad$ $n = \sqrt{\mu_r \epsilon_r} = c/v$

$B = \mu H$ $\qquad$ $\mu = \mu_0 \Rightarrow \mu_r = 1$

a) To determine the relationship b/w $|k|$ and ω, we derive the wave equation

$$\nabla \times \left(\nabla \times E + \frac{\partial B}{\partial t} \right) = 0$$

$$\nabla \times \nabla \times E + \frac{\partial}{\partial t} \nabla \times B = 0$$

$$-\nabla^2 E + \nabla(\nabla \circ E) + \frac{\partial}{\partial t} \mu (\nabla \times H) = 0$$

$$-\nabla^2 E + \frac{\partial}{\partial t} \mu \left(\frac{\partial}{\partial t} D \right) = 0 \qquad \text{(no free charges / currents)}$$

$$-\nabla^2 E + \mu \epsilon \frac{\partial^2}{\partial t^2} E = 0$$

* Given $\vec{E} = E_0 \exp[i(\vec{k} \circ \vec{r} - \omega t)]$

$$\frac{\partial^2}{\partial t^2} E = -\omega^2 E_0 \exp[i(\vec{k} \circ \vec{r} - \omega t)]$$

$$-\nabla^2 E = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2} \right) E_0 \exp[i(\vec{k} \circ \vec{r} - \omega t)]$$

$$= |k|^2 \exp[i(k \cdot r - \omega t)]$$

$$|k|^2 - n^2 \omega^2 = 0 \Rightarrow |k| = n\omega$$

b) $\vec{k}_i = \langle \sin\theta_i, 0, \cos\theta_i \rangle |k_i|$

$\vec{k}_r = \langle \sin\theta_r, 0, -\cos\theta_r \rangle |k_r|$

$\vec{k}_t = \langle \sin\theta_t, 0, \cos\theta_t \rangle |k_t|$

#1 (cont.)

b) $k_i \cdot r = k_i x \sin\theta_i$

$k_r \cdot r = k_r x \sin\theta_r$ $\Bigg\}$ $r = \langle x, y, 0\rangle$ b/c in $z=0$ plane

$k_t \cdot r = k_t x \sin\theta_t$

c) $\vec{E}$ must be perpendicular to $\vec{k}$

$\vec{E}_i = E_{0,i} \langle \cos\theta_i, 0, -\sin\theta_i \rangle$

$\vec{E}_r = E_{0,r} \langle \cos\theta_r, 0, \sin\theta_r \rangle$

$\vec{E}_t = E_{0t} \langle \cos\theta_t, 0, -\sin\theta_t \rangle$

d) Our boundary conditions are:

① $D_1^{\perp} - D_2^{\perp} = \sigma_f$ ③ $B_1^{\perp} - B_2^{\perp} = 0$

② $E_1^{\parallel} - E_2^{\parallel} = 0$ ④ $H_1^{\parallel} - H_2^{\parallel} = k_f \cdot \frac{1}{\mu_0}$

* However, since there are no free charges

$D \cdot \hat{n}$, $B \cdot \hat{n}$, $E \times \hat{n}$, $H \times \hat{n}$ are all continuous

e) Using boundary condition ①:

$\epsilon_1 E_1 \cdot \hat{z} = \epsilon_2 E_2 \cdot \hat{z}$

$\epsilon_1 (E_i + E_r) \cdot \hat{z} = \epsilon_2 (E_t \cdot \hat{z})$

$\epsilon_1 (-E_{0,i} \sin\theta_1 + E_{0,r} \sin\theta_r) \exp[i(\underline{x k_i \sin\theta_i} - \omega t)] = -\epsilon_2 E_t \sin\theta_t \exp[i(\underline{x k_t \sin\theta_t} - \omega t)]$

$x k_i \sin\theta_i = x k_t \sin\theta_t$

$n_i \sin\theta_i = n_t \sin\theta_t$

#1 (cont.)

f) * From part e we know

$$\epsilon_1 (E_r - E_i) \sin\theta_i = \epsilon_2 E_t \sin\theta_t$$

$$n_i^2 \cancel{\epsilon_0} (E_r - E_i) \cancel{\sin\theta_i} = n_t^2 \cancel{\epsilon_0} E_t \cancel{\sin\theta_t}$$

* using our $E \times \hat{n}$ boundary condition:

$$(E_i + E_r)\cos\theta_i = E_t \cos\theta_t$$

* Thus:

① $n_i (E_r - E_i) = n_t E_t$

② $(E_i + E_r)\cos\theta_i = E_t \cos\theta_t$

* Solving for E_r:

$$(E_i + E_r)\cos\theta_i = \frac{n_i}{n_t}(E_r - E_i)\cos\theta_t$$

$$E_i\left(\cos\theta_i + \frac{n_i}{n_t}\cos\theta_t\right) = E_r\left(\frac{n_i}{n_t}\cos\theta_t - \cos\theta_i\right)$$

$$\frac{E_r}{E_i} = \frac{n_i/n_t \cos\theta_t - \cos\theta_i}{\cos\theta_i - n_i/n_t \cos\theta_t} = \frac{n_i\cos\theta_t - n_t\cos\theta_i}{n_t\cos\theta_i - n_i\cos\theta_t}$$

* Solving for E_t

$$E_r = \frac{n_t}{n_i}E_t + E_i$$

$$\left(2E_i + \frac{n_t}{n_i}E_t\right)\cos\theta_i = E_t\cos\theta_t$$

$$2E_i\cos\theta_i = E_t\left(\cos\theta_t - \frac{n_t}{n_i}\cos\theta_i\right)$$

$$\frac{E_t}{E_i} = \frac{\cos\theta_t - \frac{n_t}{n_i}\cos\theta_i}{2\cos\theta_i}$$