Problem 3: Maxwell equations

Consider a medium with nonzero scalar conductivity σ ($\mathbf{J}_f = \sigma\mathbf{E}$ is the current density), permeability μ, permittivity ϵ, and with no free charge ($\rho_f = 0$).

1. Write down the set of four differential Maxwell's equations appropriate for this medium. [2points]

2. Derive the wave equation for $\mathbf{E}$ in this medium. Highlight the additional term arising from the non-zero $\mathbf{J}_f$. [3 points]

3. Consider a monochromatic wave moving in the $+x$ direction with E_y given by

$$E_y = A\, e^{i(kx-\omega t)}$$

Show that this wave has an amplitude A which decreases exponentially. Find the attenuation length Δx, the distance after which the amplitude has decayed by a factor of $1/e$ from its initial value, as a function of σ. Show that your solution correctly predicts $\Delta x = 0$ if $\sigma = 0$. [5 points]

Jan 2017 $\qquad$ E+M #3 $\qquad$ Gaussian

a) In general, maxwell's eqns in media are:

$$\nabla \cdot D = \cancel{4\pi\rho_f}^{\,0} \qquad \nabla \cdot B = 0$$

$$\nabla \times E - \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$$

b) To derive the wave equation for E:

$$\nabla \times \left(\nabla \times E - \frac{1}{c}\frac{\partial B}{\partial t} \right) = 0$$

$$\nabla(\nabla \cdot E) - \nabla^2 E - \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times B) = 0$$

* Note: $D = \epsilon E$, $B = \mu H$

$$\frac{1}{\epsilon}\nabla(\cancel{\nabla \cdot D}^{\,0}) - \nabla^2 E - \frac{\mu}{c}\frac{\partial}{\partial t}(\nabla \times H) = 0$$

$$-\nabla^2 E - \frac{\mu}{c}\frac{\partial}{\partial t}\left(\frac{4\pi}{c}\vec{J}_f + \frac{1}{c}\frac{\partial D}{\partial t} \right) = 0$$

$$-\nabla^2 E - \frac{\mu\partial}{c\partial t}\left(\frac{4\pi}{c}\sigma E + \frac{\epsilon}{c}\frac{\partial E}{\partial t} \right) = 0$$

$$-\nabla^2 E - \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2} - \underbrace{\frac{4\pi\mu}{c^2}\sigma\frac{\partial E}{\partial t}}_{\text{extra non-zero term due to } \vec{J}_f} = 0$$

c) Given $\vec{E} = \langle 0, A\exp[i(kx - \omega t)], 0 \rangle$

$$-\nabla^2 E = -\frac{d^2 E}{dx^2}$$

$$= k^2 A\exp[i(kx - \omega t)]$$

$$\frac{\partial^2 E}{\partial t^2} = \omega^2 A\exp[i(kx - \omega t)]$$

$$\frac{\partial E}{\partial t} = -i\omega A\exp[i(kx - \omega t)]$$

#3 (cont.)

c) $\Rightarrow 0 = k^2 A \exp[i(kx-\omega t)] - \frac{\mu\epsilon}{c^2}\omega^2 A \exp[i(kx-\omega t)] - \frac{4\pi\mu}{c^2}\sigma(-i\omega) A \exp[i(kx-\omega t$

$$0 = k^2 A - \frac{\mu\epsilon}{c^2}\omega^2 A + i\frac{4\pi\mu\sigma}{c^2}\omega A$$

$$= A\left(k^2 + i\omega\frac{4\pi\mu\sigma}{c^2} - \frac{\mu\epsilon}{c^2}\omega^2\right)$$

* Helpful simplifications: $k = \omega v$

$V = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$

$n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$

$= A($