6. E&M Waves

A monochromatic, plane polarized, plane electromagnetic wave traveling in the z-direction in the lab (in a vacuum) can be written in the following 3+1 dimensional form:

$$\mathbf{E} = E_0\ \hat{\mathbf{x}}\ e^{i(k\,z-\omega\,t)},$$

$$\mathbf{B} = B_0\ \hat{\mathbf{y}}\ e^{i(k\,z-\omega\,t)}.$$

(a) (3 pts) Combine this $\mathbf{E}$ and $\mathbf{B}$ into a single electromagnetic field tensor $F^{\alpha\beta}$ and use Maxell's equations in the 4-dimensional form

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0,$$

$$\partial_\alpha F^{\alpha\beta} = 0$$

to find all constraints on the 4 constants E_0, B_0, k, and ω (i.e., the above wave won't satisfy Maxwell's equations for arbitrary values of all four of these parameters). Depending on your choice of conventions: $x^\alpha = (x^0, x^1, x^2, x^3)$ with $x^0 = ct$ or $x^\alpha = (x^1, x^2, x^3, x^4)$ with $x^4 = ct$ and $x^1 = x, x^2 = y, x^3 = z$.

(b) (1 pts) What are the values of the invariants $F^{\alpha\beta}F_{\alpha\beta}$ and $\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$ for this wave?

(c) (3 pts) Use a Lorentz boost to find $F'^{\alpha\beta}$ in a frame moving in the $+z$ direction with a speed v. Don't forget to express your answer in terms of the moving coordinates ct' and x', y', z'.

(d) (2 pts) What is the frequency and the wavelength of this wave in the moving frame?

(e) (1 pts) How have the electric and magnetic fields changed in direction and/or magnitude?

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E+M #6

Gaussian

a) $\vec{E} = \langle E_0, 0, 0 \rangle \exp[i(kz-\omega t)]$

$\vec{B} = \langle 0, B_0, 0 \rangle \exp[i(kz-\omega t)]$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} \exp[i(kz-\omega t)]$$

* for $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$, the only non-zero terms are: F_{01}, F_{13}

$$\partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(-B_0 \exp[i(kz-\omega t)]\right) + \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial z}\left(E_0 \exp[i(kz-\omega t)]\right) = 0$$

$$-\frac{i\omega}{c} B_0 \exp[i(kz-\omega t)] + ik E_0 \exp[i(kz-\omega t)] = 0$$

$$k E_0 = \frac{\omega}{c} B_0 \quad ①$$

* for $\partial_\alpha F^{\alpha\beta} = 0$, $\beta = 1$ to get both E_0, B_0

$$\partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(E_0 \exp[i(kz-\omega t)]\right) + 0 + 0 + \frac{\partial}{\partial z}\left(B_0 \exp[i(kz-\omega t)]\right) = 0$$

$$\frac{-i\omega}{c} E_0 \exp[i(kz-\omega t)] + ik B_0 \exp[i(kz-\omega t)] = 0$$

$$\hookrightarrow \frac{\omega}{c} E_0 = k B_0 \quad ②$$

* Combining ① and ② yields

$$\frac{\omega}{c} E_0 = k\left(\frac{ck}{\omega} E_0\right)$$

$$\frac{\omega^2}{c^2} = k^2 \Rightarrow \boxed{k = \pm \frac{\omega}{c}}$$

$$\frac{\omega}{c} E_0 = \pm \frac{\omega}{c} B_0 \Rightarrow \boxed{E_0 = \pm B_0}$$

#6 (cont.)

b) $F^{\alpha\beta} F_{\alpha\beta} = \begin{bmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & E_0 & 0 & 0 \\ -E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} E_0^2 & 0 & 0 & -E_0 B_0 \\ 0 & E_0^2 - B_0^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_0 B_0 & 0 & 0 & -B_0^2 \end{bmatrix}$ * but we want the trace of this matrix

$= 2(E_0^2 - B_0^2)$ *Should be $2(B_0^2 - E_0^2)$

$\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} = 2 F^{\gamma\delta} F_{\gamma\delta}$

$= \begin{bmatrix} 0 & 0 & -B_0 & 0 \\ 0 & 0 & 0 & 0 \\ B_0 & 0 & 0 & E_0 \\ 0 & 0 & -E_0 & 0 \end{bmatrix} \begin{bmatrix} 0 & E_0 & 0 & 0 \\ -E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} 2$

$= 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2B_0E_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= 0$, makes sense b/c $\hat{x} \cdot \hat{y} = 0$ *Should be $-8(B \cdot E)$

#6 (cont.)

c) $$L = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$F' = L^T F L$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma+\beta\gamma & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \beta\gamma-\gamma & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} E_0 \exp[i(kz-\omega t)]$$

$$= \begin{bmatrix} 0 & -\gamma+\beta\gamma & 0 & 0 \\ \gamma-\beta\gamma & 0 & 0 & \gamma-\beta\gamma \\ 0 & 0 & 0 & 0 \\ 0 & -\gamma+\beta\gamma & 0 & 0 \end{bmatrix} E_0 \exp[i(kz-\omega t)]$$

* but we must now transform the coordinates, which we can rewrite according to

$$kz - \omega t = -k_\sigma x^\sigma, \qquad k_\sigma = \frac{\omega}{c}\langle ct, 0, 0, -z\rangle$$
$$= k\langle 1, 0, 0, -1\rangle$$

$$k'_\lambda x'^\lambda = k_\sigma B^{-1\sigma}{}_\lambda$$

$$= [1\ 0\ 0\ -1] \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix}$$

$$= [\gamma(1-\beta) \quad 0 \quad 0 \quad -\gamma(1-\beta)]$$

$$\Rightarrow k' = k\gamma(1-\beta) \quad \Rightarrow \quad \frac{\omega'}{c} = \frac{\omega}{c}\gamma(1-\beta)$$

#6 (cont.)

c) Since wave will have same form in both frames

$$E' = E_0 \gamma(1-\beta) \exp[i(k'z' - \omega' t')]$$

$$B' = B_0 \gamma(1-\beta) \exp[i(k'z' - \omega' t')]$$

d) where $k' = k\gamma(1-\beta)$, $\omega' = \omega\gamma(1-\beta) = \omega\sqrt{\frac{1-\beta}{1+\beta}} \Rightarrow \lambda' = \lambda\sqrt{\frac{1+\beta}{1-\beta}}$

e) Direction is same but magnitude has changed

$$E_0' = E_0 \gamma(1-\beta)$$

$$B_0' = B_0 \gamma(1-\beta)$$

$$= E_0 \gamma(1-\beta)$$