Problem 6: Relativistic electrodynamics

The Lagrangian for the EM field generated from a 4-current j_μ is given (in SI units) by

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - A_\mu j^\mu \tag{3}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, $A_\mu = (\phi/c, -\vec{A})$ and $j_\mu = (c\rho, -\vec{j})$ and where $c^2 = 1/\epsilon_0\mu_0$.

1. Show that $\mathcal{L}$ is invariant under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda(t, \vec{x})$. [2 points]
2. Derive the covariant form of Maxwell's equations from the Euler-Lagrange equations using $\mathcal{L}(A_\mu, \partial_\nu A_\mu)$. [4 points]
3. Show that these reduce to the usual form of Maxwell's equations in 3-vector notation. [4 points]

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E+M #6

a) $\mathcal{L} = \frac{-1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - A_\mu j^\mu$

$* \text{if } F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$= \frac{-1}{4\mu_0} \left(\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \right) F_{\mu\nu} - A_\mu j^\mu$$

$$= \frac{-1}{8\mu_0} \epsilon^{\mu\nu\rho\sigma} \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho \right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) - A_\mu j^\mu$$

$* \text{if } A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(t, \vec{x})$

$$= \frac{-1}{8\mu_0} \epsilon^{\mu\nu\rho\sigma} \left(\partial_\rho (A_\sigma + \partial_\sigma \Lambda) - \partial_\sigma (A_\rho + \partial_\rho \Lambda) \right)\left(\partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda) \right)$$

$$- (A_\mu + \partial_\mu \Lambda) j^\mu$$

$$= \frac{-1}{8\mu_0} \epsilon^{\mu\nu\rho\sigma} \left(\partial_\rho A_\sigma + \cancel{\partial_\rho \partial_\sigma \Lambda} - \partial_\sigma A_\rho - \cancel{\partial_\sigma \partial_\rho \Lambda} \right)\left(\partial_\mu A_\nu + \cancel{\partial_\mu \partial_\nu \Lambda} - \partial_\nu A_\mu - \cancel{\partial_\nu \partial_\mu \Lambda} \right)$$

$$- A_\mu j^\mu - \partial_\mu \Lambda j^\mu$$

$$= \frac{-1}{8\mu_0} \epsilon^{\mu\nu\rho\sigma} \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho \right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) - A_\mu j^\mu - \cancelto{0}{\partial_\mu \Lambda j^\mu}$$

$$= \frac{-1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - A_\mu j^\mu$$