6 - 8

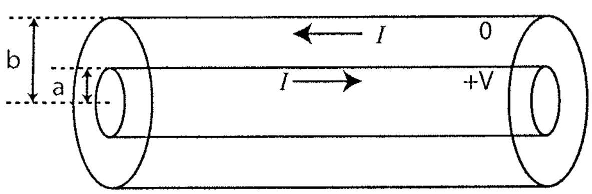
Problem 3: E/p in EM field

? (a) The work required to assemble n point charges q_i is $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$. What is the physical meaning of the factor $\frac{1}{2}$ that appears in this expression? [0.5 pts]

√(b) Generalizing to a volume charge density ρ, the work becomes $W = \frac{1}{2}\int \rho V d^3r$. Use Gauss' law and integration by parts to show from this that the energy stored in the electric field is $W = \frac{\epsilon_0}{2}\int E^2 d^3r$. [1 point]

√(c) Using the expression for W from part 2, find the energy of a uniformly charged spherical shell of total charge Q and radius R. [1 point]

√(d) The work required to set up a current density $\mathbf{J}$ is $W = \frac{1}{2}\int \mathbf{A}\cdot\mathbf{J}\, d^3r$, where $\mathbf{A}$ is the vector potential due to the current, and the integral is over all space. Show from this expression, via integration by parts, that the energy stored in the magnetic field is $W = \frac{1}{2\mu_0}\int B^2\, d^3r$. [1 point]



√(e) For this and the following parts of this problem, consider a long coaxial cable that carries current I along the surface of the inner cylinder, radius a, flowing down in one direction, and the same current I along the outer cylinder surface, radius b, flowing back up in the opposite direction, as shown in the above figure. Find the energy stored in the *magnetic* field of a section of length l of this cable, as a function of I, l, b and a. [2 points]

\5? √(f) What is the energy flux density (magnitude of Poynting vector $\mathbf{S}$) in the cable if the inner conductor is held at a positive potential V with respect to the outer conductor, and current I flows down one conductor and back up the other conductor as shown. Express $|\mathbf{S}|$ as a function of linear charge density λ, radial distance s, I, a and b. [2 points]

$S = \frac{c}{4\pi} E \times H = \frac{1}{\mu_0} E \times B$:

√(g) Which direction does the Poynting vector point? [0.5 points]

? (h) What is the power (energy per unit time) transported down the cable, as a function of linear charge density λ, b, and a? [1 point]

? √(i) Express the power transported in the cable in terms of the current I and potential difference V. [1 point]

$P = IV^2$

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E+M #3

SI

a) The physical meaning of the ½ in the formula $W = \frac{1}{2} \sum_i^{n}{}' q_i V(\vec{r}_i)$ is that it avoids double counting when $V(\vec{r}_i) = \sum_j{}' \frac{q_j}{4\pi\epsilon_0 r_{ij}}$ as $W_{12} = W_{21}$

b)
$$W = \frac{1}{2} \int \rho V d^3r$$
$$= \frac{1}{2} \int \epsilon_0 \frac{\rho}{\epsilon_0} V d^3r$$
$$= \frac{\epsilon_0}{2} \int (\nabla \cdot E) V d^3r$$
$$= \frac{\epsilon_0}{2} \left[\int E \cdot \nabla V d^3r - \oint E V d^3r \right]$$
0 if integrated over all space ($V=0$ at ∞)
$$= \frac{\epsilon_0}{2} \int E \cdot E \, d^3r$$
$$= \frac{\epsilon_0}{2} \int |E|^2 d^3r$$

c) From Gauss Law, we know the field of a uniformly charged spherical shell is:
$$\vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} & r \geq R \\ 0 & r < R \end{cases}$$
$$\Rightarrow W = \frac{\epsilon_0}{2} \int E^2 d^3r$$
$$= \frac{\epsilon_0}{2} \int \frac{Q^2}{16\pi^2 \epsilon_0^2 r^4} r^2 dr \, d\Omega$$
$$= \frac{Q^2}{8\pi\epsilon_0} \left(\frac{-1}{r} \right) \Big|_R^\infty$$
$$= \frac{Q^2}{8\pi\epsilon_0 r}$$

#3 (cont.)

d) $W = \frac{1}{2}\int A \cdot J \, d^3r$

$= \frac{1}{2\mu_0}\int A \cdot (\nabla \times B)\, d^3r$

$= \frac{1}{2\mu_0}\left[-\cancelto{0}{\oint AB \, d^3r} + \int (\nabla \times A) \cdot B \, d^3r \right]$ ($A = 0$ at ∞)

$= \frac{1}{2\mu_0}\int |B|^2 \, d^3r$

e) We determine the magnetic field using Ampere's Law

$\int \vec{\nabla} \times \vec{H} \, da = I_{enc}$

$\oint \vec{H} \, dl = I_{enc}$

$\vec{H} \, 2\pi s = I$

$\vec{H} = \frac{I}{2\pi s} \hat{\varphi}$ (direction determined by symmetry / RHR)

*Applying our general formula above to our specific situation where $B = \mu H$.

- If $s < a$, $\vec{B} = 0$ b/c $I_{enc} = 0$
- If $a \le s \le b$, $\vec{B} = \frac{\mu I}{2\pi s} \hat{\varphi}$
- If $s \ge b$ $\vec{B} = 0$ b/c $I_{enc} = 0$

$$\Rightarrow \vec{B} = \begin{cases} \frac{\mu I}{2\pi s} \hat{\varphi} & a \le s \le b \\ 0 & \text{elsewhere} \end{cases}$$

$W = \frac{1}{2\mu_0}\int B^2 \, d^3r$

$= \frac{1}{2\mu_0}\int \frac{\mu^2 I^2}{4\pi^2 s^2} \, s\,ds\,d\varphi\,dz$

$= \frac{\mu^2 I^2 L}{4\pi \mu} \ln(b/a)$

#3 (cont.)

f) * Assuming a steady current, the amount of charge Q within our region of interest is constant.

* Again because we know nothing about our medium, we assume $\vec{D} = \epsilon \vec{E}$

Using Gauss Law, we find

$$\int \nabla \cdot D \, dV = \int' \rho_f \, dV$$

$$\oint D \cdot d\vec{a} = Q$$

$$D \cdot 2\pi r L = Q$$

$$\vec{D} = \frac{Q}{2\pi r L} \hat{r} \quad \text{(Direction from RHR)}$$

$$\Rightarrow \vec{E} = \begin{cases} \frac{Q}{2\pi s L \epsilon} \hat{r} & a < s < b \\ 0 & \text{elsewhere} \end{cases}$$

$$\vec{S} = \frac{1}{\mu_0} E \times B$$

$$= \frac{1}{\mu_0} |E||B| \hat{z}$$

$$= \frac{1}{\mu_0} \frac{Q}{2\pi s L \epsilon} \cdot \frac{\mu I}{2\pi s} \hat{z}$$

$$= \frac{\mu Q I}{\mu_0 4\pi^2 s^2 L \epsilon} \hat{z}$$

$$= \frac{\mu Q \lambda}{4\pi^2 \mu_0 \epsilon s^2} \hat{z}$$

g) $\hat{z}$ direction

h) $$P = \int S \cdot d\vec{a}$$

$$= \frac{\mu Q \lambda}{4\pi^2 \mu_0 \epsilon} \int \frac{1}{s^2} s \, ds \, d\varphi$$

$$= \frac{\mu Q \lambda}{2\pi \mu_0 \epsilon} \ln\left(\frac{b}{a}\right)$$

j) $$P = \frac{\mu I}{2\pi L} \cdot \frac{V}{b-a}$$