6. In the lab you measure a uniform electric field and a uniform magnetic induction

$$\mathbf{E} = E_0(\cos 45^\circ\,\hat{\mathbf{x}} + \sin 45^\circ\,\hat{\mathbf{y}}),$$

$$\mathbf{B} = B_0\,\hat{\mathbf{x}},$$

where $B_0 = E_0$ in Gaussian units or $B_0 = E_0/c$ in SI units. The goal of this problem is to compute the $\mathbf{E}'$ and $\mathbf{B}'$ fields an observer sees if moving relative to the lab with a velocity $\mathbf{v} = v_0\,\hat{\mathbf{z}}$.

(a) [2 pts] Combine $\mathbf{E}$ and $\mathbf{B}$ into a single 4×4 anti-symmetric electromagnetic field tensor $F^{\alpha\beta}$.

(b) [2 pts] Give the 4×4 Lorentz boost L^α_β that transforms the lab coordinates (ct, x, y, z) into the moving frame's coordinates (ct', x', y', z') i.e., $x'^\alpha = L^\alpha_\beta x^\beta$ where $x^\beta = (ct, x, y, z)$. In matrix notation $x' = L\,x$.

(c) [3 pts] Find $\mathbf{E}'$ and $\mathbf{B}'$ by by boosting the F tensor, i.e., compute $F'^{\alpha\beta} = L^\alpha_\sigma L^\beta_\lambda F^{\sigma\lambda}$ which in matrix notation is $F' = LFL^\top$

(d) [3 pts] For what value of v_0 will $\mathbf{E}'$ and $\mathbf{B}'$ be parallel?

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a) $$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & +B_x & 0 \end{bmatrix}$$

$\vec{E} = \langle E_0 \cos(45°), E_0 \sin(45°), 0 \rangle$

$\vec{B} = \langle B_0, 0, 0 \rangle$

$$= \begin{bmatrix} 0 & -E_0\frac{\sqrt{2}}{2} & -E_0\frac{\sqrt{2}}{2} & 0 \\ E_0\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ E_0\frac{\sqrt{2}}{2} & 0 & 0 & -B_0 \\ 0 & 0 & B_0 & 0 \end{bmatrix}$$

b) $$L_{\hat{\beta}} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix}$$

$$x'^{\alpha} = \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \gamma ct - \gamma\beta z \\ x \\ y \\ -\gamma\beta ct + \gamma z \end{bmatrix}$$

c) $F' = L F L^T$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -E_0/\sqrt{2} & -E_0/\sqrt{2} & 0 \\ E_0/\sqrt{2} & 0 & 0 & 0 \\ E_0/\sqrt{2} & 0 & 0 & -B_0 \\ 0 & 0 & B_0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

#6 (cont.)

c) $$F' = \begin{bmatrix} 0 & -\gamma E_0/\sqrt{2} & -\gamma E_0/\sqrt{2} - \gamma\beta B_0 & 0 \\ E_0/\sqrt{2} & 0 & 0 & 0 \\ E_0/\sqrt{2} & 0 & 0 & -B_0 \\ 0 & \beta\gamma E_0/\sqrt{2} & \beta\gamma E_0/\sqrt{2} + \gamma B_0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma E_0/\sqrt{2} & -\gamma E_0/\sqrt{2} - \gamma\beta B_0 & 0 \\ \gamma E_0/\sqrt{2} & 0 & 0 & -\beta\gamma E_0/\sqrt{2} \\ \gamma E_0/\sqrt{2} + \gamma\beta B_0 & 0 & 0 & -\beta\gamma E_0/\sqrt{2} - \gamma B_0 \\ 0 & \beta\gamma E_0/\sqrt{2} & \beta\gamma E_0/\sqrt{2} + \gamma B_0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{E} = \langle \gamma E_0/\sqrt{2},\ \gamma E_0/\sqrt{2} + \gamma\beta B_0,\ 0 \rangle$$

$$\vec{B} = \langle \beta\gamma E_0/\sqrt{2} + \gamma B_0,\ \beta\gamma E_0/\sqrt{2},\ 0 \rangle$$

d) * Because we are in Gaussian units, $E_0 = B_0$

$$\Rightarrow \vec{E} = E_0 \langle \gamma/\sqrt{2},\ \frac{\gamma}{\sqrt{2}} + \gamma\beta,\ 0 \rangle$$

$$\vec{B} = E_0 \langle \beta\gamma/\sqrt{2} + \gamma,\ \beta\gamma/\sqrt{2},\ 0 \rangle$$

$$\frac{\beta\gamma}{\sqrt{2}} + \gamma = \frac{\beta\gamma}{\sqrt{2}}$$

$$\frac{\beta}{\sqrt{2}} + 1 = \frac{\beta}{\sqrt{2}}$$

$$\beta + \sqrt{2} =$$