Problem 1: Electrostatics

A non-conducting solid sphere of radius R carries a charge density $\rho(r) = k\,r$ (where k is a constant).

(a) Find the electric field at a distance r such that $r \geq R$ [1 point]

(b) Find the electric field at a distance r such that $r \leq R$ [1 point]

(c) State the boundary conditions on the electric field components on the surface of the sphere, and show that your answers to parts a) and b) are consistent with them.
HINT: The surface charge density of the sphere is zero in this case. [1 point]

(d) Calculate the electric potential for all r using $\lim_{r\to\infty} V(r) = 0$ [2 points]

(e) Find the work required to assemble this charge [2 points]

(f) If the *non-conducting* solid sphere was replaced by a *conducting* solid sphere with the same total charge, how does that change your answer to parts a, b, and c?
Explicitly show that the new field satisfies the new boundary conditions across the surface of the sphere. [3 point]

Aug 2017

E+M #1

Gaussian

a) We use Gauss' Law to find the field at $r \geq R$

$$\int \vec{\nabla} \cdot \vec{E}\, dV = \int 4\pi \rho\, dV$$

$$\oint \vec{E} \cdot d\vec{a} = 4\pi \int kr \cdot r^2 \sin\theta\, dr d\theta d\varphi$$

$$\vec{E} \cdot 4\pi R^2 = 16\pi^2 k \int_0^R r^3 dr$$

$$4\pi r^2 \vec{E} = 4\pi^2 k\, r^4 \Big|_0^R$$

$$4\pi r^2 \vec{E} = 4\pi^2 k R^4$$

$$\vec{E} = \pi k \frac{R^4}{r^2} \hat{r}$$ (know $\hat{r}$ direction due to spherical symmetry of problem)

b) We proceed as above, now with $r \leq R$

$$\int \vec{\nabla} \cdot \vec{E}\, dV = \int 4\pi \rho\, dV$$

$$\oint \vec{E} \cdot d\vec{a} = 4\pi k \int r'^3 \sin\theta\, dr' d\theta d\varphi$$

$$\vec{E} \cdot 4\pi r^2 = 4\pi^2 k\, r'^4 \Big|_0^r$$

$$\vec{E} \cdot 4\pi r^2 = 4\pi^2 k r^4$$

$$\vec{E} = k\pi r^2\, \hat{r}$$

c) The boundary conditions of $\vec{E}$ are: ① $E_1^\perp - E_2^\perp = 4\pi\sigma$

② $E_1^\parallel = E_2^\parallel$

② is automatically satisfied as $\vec{E}$ only points perpendicular to surface in both cases

* Defining region 1 as inside the sphere and region 2 as outside the sphere,

$$E_1^\perp = \vec{E}_1 \cdot \hat{n} \big|_{r=R} \qquad E_2^\perp = \vec{E}_2 \cdot \hat{n} \big|_{r=R}$$

$$= E_1 \cdot \hat{r} \big|_{r=R} \qquad = \vec{E}_1 \cdot \hat{r} \big|_{r=R}$$

$$= k\pi R^2 \qquad = k\pi R^2$$

* Since we know $\sigma = 0$, condition ① is also satisfied

* See Griffiths Ch 2 for origin of formulas

#1 (cont.)

d) We calculate the potential according to:

$$V(\vec{x}) = \int \frac{\rho(x)}{|x-x'|} d^3x' = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

* due to spherical symmetry, it is easier to use the second form and integrate along a line radially pointing inwards from infinity to r

$$V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

* since the field changes at $r = R$, we must proceed with 2 cases

* if $r > R$

$$V(r) = -\int_{\infty}^{r} \frac{\pi k R^4}{r'^2} dr'$$

$$= \frac{\pi k R^4}{r'} \Big|_{\infty}^{r}$$

$$= \frac{\pi k R^4}{r}$$

* if $r < R$

$$V(r) = -\left(\int_{\infty}^{R} \frac{\pi k R^4}{r'} dr' + \int_{R}^{r} k\pi r'^2 dr' \right)$$

$$= \frac{\pi k R^4}{r'} \Big|_{\infty}^{R} - \frac{\pi k r'^3}{3} \Big|_{R}^{r}$$

$$= \pi k R^3 - \left(\tfrac{1}{3} \pi k r^3 - \pi k R^3 \right)$$

$$= \tfrac{4}{3} \pi k R^3 - \tfrac{1}{3} \pi k r^3$$

e) We can find the work necessary to assemble this charge distribution according to

$$W = \frac{1}{8\pi} \int E^2 dV$$

$$= \frac{1}{2} \int_{0}^{\infty} E^2 r'^2 dr'$$

* again we must consider our two regions

#1 (cont.)

e) $W = \frac{1}{2}\left[\int_0^R E_1^2 r'^2 dr' + \int_R^\infty E_2^2 r'^2 dr'\right]$

$$= \frac{1}{2}\left[\int_0^R k^2\pi^2 r'^6 dr' + \int_R^\infty k^2\pi^2 \frac{R^8}{r'^4} r'^2 dr'\right]$$

$$= \frac{1}{2}\left[k^2\pi^2 \frac{1}{7} r'^7 \Big|_0^R + k^2\pi^2 R^8 \left(\frac{-1}{r'}\Big|_R^\infty\right)\right]$$

$$= \frac{k^2\pi^2}{2}\left(\frac{1}{7}R^7 + R^7\right)$$

$$= \frac{4k^2\pi^2 R^7}{7}$$

f) If we replace the non-conducting sphere with a conducting sphere of equal charge

a) Field outside conductor remains unchanged

b) Field goes to 0 inside conductor

c) Boundary conditions are same, except all charge accumulates on surface

$$\sigma = \frac{1}{4\pi R^2}\int \rho \, dV$$

$$= \frac{1}{4\pi R^2} 4\pi k \cdot \frac{1}{4} R^4$$

$$= \frac{1}{4} k R^2$$

$$E_1^\perp - E_2^\perp = 4\pi\sigma$$

$$0 - \frac{\pi k R^4}{r^2} = 4\pi\left(\frac{1}{4}kR^2\right)$$

$$-\pi k R^2 = \pi k R^2 \checkmark$$