7/10

Problem 1: Magnetostatics

√(a) Use the Biot-Savart law to calculate the magnitude and direction of the magnetic induction , **B**, on the axis of a circular loop of radius R carrying a current I. Let the z axis lie on the axis of the loop with the orgin at the center of the loop. [2 points]

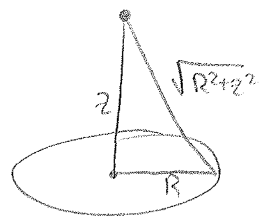
√? (b) Consider a short solenoid with N closely wound coils per meter each carrying current I. Using the result from part (a), find the magnetic induction, **B**, at the center of the solenoid. Assume the solenoid has length L, and consists of circular coils of radius R. [3 points]

√(c) Show that for any point on the axis of the solenoid, $B = \frac{1}{2}\mu_0 IN(cos(\alpha_1) + cos(\alpha_2))$ where α_1 and α_2 are the angles subtended at the point by a radius R at either end of the solenoid. [3 points]

√(d) Write expressions for $cos(\alpha_1)$ and $cos(\alpha_2)$ in terms of R, L, and x where x is the distance of the point from the center of the solenoid. [2 points]

Aug 2018 $\quad$ E+M #1 $\quad$ SI

a)



* We use the Biot-Savart Law to derive the magnetic field, which in SI units reads:

$$B = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}) \times (r-r')}{|r-r'|^3}\, dr'$$

$$\Rightarrow B = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}) \times (r-r')}{|r-r'|^3}\, dr'$$

* note that $J(\vec{r}')dr' = I\,dl\,\hat{\varphi}$

$$(r-r') = -R\hat{r} + z\hat{z}$$

$$\Rightarrow J(\vec{r}) \times (r-r')dr' = \begin{vmatrix} \hat{s} & \hat{\varphi} & \hat{z} \\ 0 & Idl & 0 \\ -R & 0 & z \end{vmatrix}$$

$$= \langle zIdl,\ 0,\ RIdl \rangle$$

* we ignore radial term, as axial symmetry will cancel this term during integration

$$B = \frac{\mu_0 I}{4\pi}\int \frac{Rdl}{(R^2+z^2)^{3/2}}$$

$$= \frac{IR^2\mu_0}{2\cdot(R^2+z^2)^{3/2}}\hat{z}$$

b) * To build up the field of a solenoid from current loops, we get the infinitesimal dB from a current loop of thickness dz

$$\Rightarrow dB = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\, dz'$$

We also note that $z \to z-z'$ since the source point is no longer constrained to be at $z=0$

$$dB = \frac{\mu_0 I R^2}{2(R^2+(z-z')^2)^{3/2}}\, dz'$$

and that the total current $I \to NI$

#1 (cont.)

b) $$dB = \frac{\mu_0 N I R^2}{2(R^2 + (z-z')^2)^{3/2}} dz'$$

* Integrating this from 0 to L in z to get the total field we find

$$B = \frac{\mu_0 N I}{2} \int_0^L \frac{R^2}{(R^2+(z-z')^2)^{3/2}} dz'$$

* According to Schaums: $\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}$

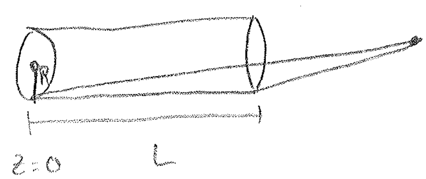
↳ if $a^2 = R^2$

$x^2 = (z-z')^2 \qquad dx = dz'$

$$B = \frac{\mu_0 N I R^2}{2} \left(\frac{z-z'}{R^2(R^2+(z-z')^2)^{1/2}} \right) \Big|_0^L$$

$$= \frac{\mu_0 N I}{2} \left(\frac{z-L}{(R^2+(z-L)^2)^{1/2}} - \frac{z}{(R^2+z^2)^{1/2}} \right) \hat{z}$$

c)



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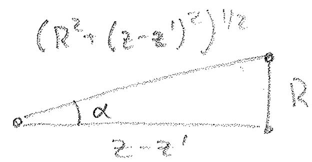

$\cos \alpha = \frac{z-z'}{(R^2+(z-z')^2)}$

* For the point at the base of the solenoid, $z-z' = z$, and for the point at the top of the solenoid, $z-z' = z-L$

$$\Rightarrow B = \frac{\mu_0 I N}{2} \left(\cos(\alpha_2) - \cos(\alpha_1) \right)$$

d) From our work above: $\cos(\alpha_2) = \frac{z-L}{(R^2+(z+L)^2)^{1/2}}$

$$\cos(\alpha_1) = \frac{z}{(R^2+z^2)^{1/2}}$$