5. A plane-polarized electromagnetic wave traveling in vacuum is observed in the lab frame as

$$\mathbf{E} = \text{Real}\Big\{ E_0 \,\hat{\imath}\, e^{i(kz-\omega t)} \Big\},$$

$$\mathbf{B} = \text{Real}\Big\{ B_0 \,\hat{\jmath}\, e^{i(kz-\omega t)} \Big\},$$

where $k = \omega/c$ and

$$\begin{aligned} B_0 &= E_0/c, && (SI) \\ B_0 &= E_0. && (Gaussian) \end{aligned}$$

A relativistic particle moving in the z-direction ($\mathbf{v} = v_0\hat{k}$, where $\gamma >> 1$) encounters this wave. For this problem you are to find the form of the wave in the rest frame of the particle at the instant the wave is first encountered (before the particle's velocity is changed because of an interaction with the wave).

(a) {2 pts} Start by combining $\mathbf{E}$ and $\mathbf{B}$ into a single 4-tensor $F^{\alpha\beta}(x)$. This includes writing $(kz - \omega t) = \pm k_\alpha x^\alpha$). The sign $\pm$ depends on your choice of Lorentz metrics $(-1, +, +, +)$ or $(1, -1, -1, -1)$. State which you are using.

(b) {2 pts} Give the Lorentz transformation $L^\alpha{}_\beta$ that transforms the lab frame into the particle's rest frame $x'^\alpha = L^\alpha{}_\beta \, x^\beta$.

(c) {2 pts} Apply your Lorentz transformation to $F^{\alpha\beta}(x)$ to find $F'^{\alpha\beta}(x')$, the electromagnetic 4-tensor in the particles rest frame.

(d) {2 pts} From your results in (c) give the 3-dimensional propagation direction of the wave in the particle's frame and the $\mathbf{E}'(x')$ and $\mathbf{B}'(x')$ fields.

(e) {2 pts} Compare the amplitudes and frequency of the wave as seen by the particle in its rest frame with those seen by a lab observer?

Why does Kantowski boost in x-direction?

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E+M #5

a) $E = \hat{i} E_0 e^{i(kz-\omega t)}$ $\quad k = \omega/c$

$B = \hat{j} B_0 e^{i(kz-\omega t)}$

* Use gaussian units so $B_0 = E_0$

* Use $(1,-1,-1,-1)$ metric

$$\Rightarrow kz - \omega t = -k_\alpha x^\alpha$$

$$= -(k_0 ct + k_1 x + k_2 y + k_3 z)$$

$$= -(k^0 ct - k^1 x - k^2 y - k^3 z)$$

$$= -\frac{\omega}{c} ct + k^1 x + k^2 y + k^3 z$$

$$= -\omega t + kz \quad (k^1 = k^2 = 0,\ k^3 = k)$$

$$\Rightarrow F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} E_0 e^{\pm i k_\alpha x^\alpha}$$

b) For a particle moving in the z-direction,

$$L^\alpha{}_\beta = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

#5 (cont.)

c) $F' = L F L^T$

must transform as well

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} E_0 e^{\pm k_\alpha x^\alpha}$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ -\beta\gamma+\gamma & 0 & 0 & \gamma-\beta\gamma \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma+\beta\gamma & 0 & 0 \\ -\beta\gamma+\gamma & 0 & 0 & \gamma-\beta\gamma \\ 0 & 0 & 0 & 0 \\ 0 & \beta\gamma-\gamma & 0 & 0 \end{bmatrix} E_0 e^{\pm k'_\alpha x'^\alpha}$$

$$k' = L^\alpha{}_\beta k_\alpha \qquad\qquad x' = L^\alpha{}_\beta x_\alpha$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} \omega/c \\ 0 \\ 0 \\ k \end{bmatrix} \qquad = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\gamma\omega}{c} - \beta\gamma k \\ 0 \\ 0 \\ -\frac{\omega\beta\gamma}{c} + \gamma k \end{bmatrix} \qquad = \begin{bmatrix} \gamma ct - \beta\gamma z \\ x \\ y \\ -\beta\gamma ct + \gamma z \end{bmatrix}$$

d) $\Rightarrow E'(x') = \hat{x}' E_0 (\gamma - \beta\gamma) e^{i\left(\left(\frac{-\gamma\omega\beta}{c} + \gamma k\right)z - \omega\left(\frac{\gamma\omega}{c} - \beta\gamma k\right)\right)}$

$\vec{B}' = \hat{y}' B_0 (\gamma - \beta\gamma) \exp\left[i\left(\left(\frac{-\omega\beta\gamma}{c} + \gamma k\right)z - \left(\frac{\gamma\omega}{c} - \beta\gamma k\right)t\right)\right]$

e) Amplitudes changed by factor of $(\gamma - \beta\gamma)$

↳ $k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c} \Rightarrow \nu = \frac{ck}{2\pi}$

$\nu =$