2. (a) {3 pts} Use Maxwell's equations to derive the continuity equation (in differential form) relating charge ρ and current density $\mathbf{J}$.

(b) {2 pts} Use the divergence theorem and the results of part (a) to derive the conservation of charge, $\dot{Q} = 0$, for a bounded charge distribution.

(c) {2 pts} Show that the continuity equation can be written in 4-vector form using the 4-current J^{μ}. Define all symbols you use.

(d) {3 pts} Use Maxwell's equations to derive Poynting's theorem, the equation analogous to the continuity of charge equation relating the Poynting vector $\mathbf{S}$ and the energy density u in the $\mathbf{E}$ and $\mathbf{B}$ fields, that represents conservation of electromagnetic energy. Assume the electric and magnetic fields are in vacuum, i.e., no charges, currents, or polarizable materials are present.

Aug 2011

E+M #2

Gaussian

a) We know: $\nabla \cdot D = 4\pi \rho$ $\qquad \nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$

$\nabla \cdot B = 0$ $\qquad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}$

Continuity Eqn: $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$

$$\frac{\partial}{\partial t} \nabla \cdot D = 4\pi \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \left(\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} \right) = \frac{4\pi}{c} \nabla \cdot J$$

$$\nabla \cdot \nabla \times H - \frac{1}{c}\frac{\partial}{\partial t} \nabla \cdot D = \frac{4\pi}{c} \nabla \cdot J$$

$$\cancelto{0}{\nabla \cdot \nabla \times H} - \frac{4\pi}{c}\frac{\partial \rho}{\partial t} = \frac{4\pi}{c} \nabla \cdot J$$

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot J$$

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot J \ \checkmark$$

b) In general, the Divergence theorem says: $\int_V \vec{\nabla} \cdot \vec{A}\, dV = \oint \vec{A} \cdot d\vec{a}$

↳ If we want to show conservation of charge, we define a surface that encloses all of our charge with $J_{bound} = 0$

$$\Rightarrow Q_{enc} = \frac{1}{4\pi} \int \rho \cdot dV$$

$$\dot{Q} = 0 = \frac{1}{4\pi} \frac{\partial}{\partial t} \int \rho \cdot dV$$

$$= \frac{-1}{4\pi} \int \nabla \cdot J \, dV$$

$$= \frac{-1}{4\pi} \int \cancelto{0}{J \cdot d\vec{a}}$$

$$= 0 \ \checkmark$$

c) $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}$

*if $J^0 = c\rho$, $J^i = j_i$, $i \in \{x, y, z\}$

↳ $\frac{\partial}{\partial x^\mu} J^\mu = 0 \ \checkmark$

#2 (cont.)

d) In Gaussian units: Poynting vector: $\vec{S} = \frac{c}{4\pi} E \times H$

Energy density: $u = \frac{1}{4\pi}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$

$$\vec{E}\cdot\left(\nabla\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\right) = \vec{E}\cdot\frac{4\pi}{c}\vec{J}$$

$$\vec{E}\cdot\nabla\times\vec{H} - \frac{1}{c}\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\cancel{\vec{E}\cdot\vec{J}}$$ 0 b/c no current

$$-\nabla\cdot(E\times H) + H\cdot(\nabla\times E) - \frac{1}{c}E\cdot\frac{\partial D}{\partial t} = 0$$

$$-\nabla\cdot(E\times H) + H\cdot\left(-\frac{1}{c}\frac{\partial B}{\partial t}\right) - \frac{1}{c}E\cdot\frac{\partial D}{\partial t} = 0$$

$$-\nabla\cdot(E\times H) - \frac{1}{c}\left[H\cdot\frac{\partial B}{\partial t} + E\cdot\frac{\partial D}{\partial t}\right] = 0$$

$$-\nabla\cdot(E\times H) - \frac{1}{c}\frac{\partial}{\partial t}(E\cdot D + H\cdot B) = 0$$ ← b/c ϵ, μ have no time dependence

$E = \epsilon D$, $H = \frac{1}{\mu}B$

* multiplying by $\frac{-c}{4\pi}$

$$\nabla\cdot\vec{S} + \frac{\partial u}{\partial t} = 0 \checkmark$$

Aug 2011 $\quad$ E + M #2 $\quad$ Gaussian

a) The generalized Maxwell Eqns are:

$$\nabla \cdot D = 4\pi \rho_f \qquad \nabla \cdot B = 0$$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$$

$$\nabla \cdot \left(\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t}\right) = \nabla \cdot \frac{4\pi}{c}\vec{J}_f$$

$$\cancel{\nabla \cdot (\nabla \times H)}^{\,0} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot D) = \frac{4\pi}{c}\nabla \cdot \vec{J}_f$$

$$-\cancel{\frac{4\pi}{c}}\frac{\partial}{\partial t}\rho_f = \cancel{\frac{4\pi}{c}}\nabla \cdot \vec{J}_f$$

$$-\frac{\partial}{\partial t}\rho_f = \nabla \cdot \vec{J}_f \checkmark$$

b) The divergence theorem states: $\int \nabla \cdot A \, dV = \oint A \cdot d\vec{a}$

$$\int \nabla \cdot \vec{J}_f \, dV = -\frac{\partial}{\partial t}\int \rho_f \, dV$$

$$\oint \vec{J}_f \cdot d\vec{a} = -\frac{\partial}{\partial t} Q$$

* if our arbitrary surface totally encloses region of $\vec{J}_f$, the flux is 0

$$0 = \dot{Q} \checkmark$$

c) We define J^μ as follows

$$J^\mu = \begin{bmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{bmatrix} \Rightarrow \quad 0 = \frac{1}{c}\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}J_i$$

$$= \frac{1}{\partial x^\mu} J^\mu \checkmark$$

#2 (cont.)

d) * Remember that $\vec{S} = \frac{c}{4\pi} E \times H$

$U = \frac{1}{4\pi}(E \cdot D + B \cdot H)$

$\Rightarrow \quad E \cdot \left(\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} \right) = E \cdot 0$ (b/c in vaccum)

$$E \cdot \nabla \times H - \frac{1}{c}\frac{\partial}{\partial t} E \cdot D = 0$$

$$-\nabla \cdot (E \times H) + H \cdot (\nabla \times E) - \frac{1}{c}\frac{\partial}{\partial t} E \cdot D = 0$$

$$-\nabla \cdot (E \times H) + H \cdot \left(-\frac{1}{c}\frac{\partial B}{\partial t} \right) - \frac{1}{c}\frac{\partial}{\partial t} E \cdot D = 0$$

$$-\nabla \cdot (E \times H) - \frac{1}{c}\frac{\partial}{\partial t}(E \cdot D + H \cdot B) = 0$$

$$\frac{c}{4\pi}\left(\cdot \nabla \cdot (E \times H) \right) + \frac{1}{4\pi}\frac{\partial}{\partial t}(E \cdot D + H \cdot B) = 0$$

$$\nabla \cdot S + \frac{\partial}{\partial t} U = 0 \checkmark$$