5-10

Problem 3: Waves

Consider an electromagnetic wave propagating in a vacuum where there are no charges or electric currents, with electric field of $\vec{E}(z,t) = E_0 e^{i(kz-\omega t)}\hat{x}$.

1. Show that three of Maxwell's equations can be combined to give $\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$ for a region with no charges or electric currents. Show that three of Maxwell's equations can also lead to an analogous equation for the magnetic field $\vec{B}$. *Hint:* Start by calculating $\nabla \times (\nabla \times \vec{E})$. [2 points]

2. In the equation for $\vec{E}(z,t)$ how are ω and k related to μ_0 and ϵ_0? What is the equation for $\vec{B}(z,t)$ of the electromagnetic wave? Be explicit about how the amplitude, direction and phase of $\vec{B}$ are related to those of $\vec{E}$. [3 points]

   2.5?

3. Now suppose the wave propagates from vacuum into a dielectric material with permittivity of $\epsilon = \kappa\epsilon_0$, where κ is a positive constant. Assuming a normal angle of incidence at the vacuum/dielectric interface, calculate the amplitude of the electric field in the dielectric material. Express your answer in terms of E_0 and κ. [3 points]

4. What fraction of the incident energy is transmitted across the boundary? [2 points]

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E+M #3

a) In Gaussian units, Maxwell's eqns are:

① $\nabla \cdot E = 4\pi\rho$ ③ $\nabla \cdot B = 0$

② $\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$ ④ $\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$

*If we take $\nabla \times$ ②

$$\nabla \times \nabla \times E + \nabla \times \left(\frac{1}{c}\frac{\partial B}{\partial t}\right) = \nabla \times 0$$

$$\nabla(\nabla \cdot E) - \nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times B) = 0$$

$$\nabla(4\pi\rho) - \nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t}\right) = 0$$

*Since we are in a vacuum, $\rho = 0$, $\vec{J} = 0$

$$-\nabla^2 E + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \checkmark$$ *Note: $c^2 = \frac{1}{\mu_0\epsilon_0}$

b) *From Jackson, $\frac{\omega}{k} = \frac{c}{n} = \frac{1}{\sqrt{\mu\epsilon}}$

↳ $\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ in a vacuum

*To calculate $\vec{B}$, we simply plug E into Maxwell's eqns

↳ Remember, in SI units (forced by problem), Maxwell's eqns are:

$\nabla \cdot E = \frac{\rho}{\epsilon}$ $\nabla \cdot B = 0$

$\nabla \times E + \frac{\partial B}{\partial t} = 0$ $\nabla \times B - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0 \vec{J}$

$$\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = \langle 0, \frac{\partial}{\partial z}E_x, 0 \rangle = \langle 0, ikE_0 \exp[i(kz-\omega t)], 0 \rangle$$

$$\Rightarrow ikE_0 \exp[i(kz-\omega t)]\hat{y} = -\frac{\partial B}{\partial t}$$

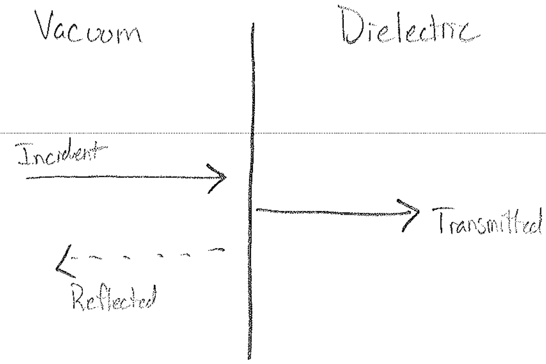
$$\int ikE_0 \exp[i(kz-\omega t)]\,\hat{y}\,dt = -\vec{B}$$

$$-\frac{k}{\omega}E_0 \exp[i(kz-\omega t)]y = -\vec{B}$$

$$\therefore \boxed{\vec{B} = \sqrt{\mu_0\epsilon_0}\,E_0 \exp[i(kz-\omega t)]\,\hat{y}}$$

#3 (cont.)

c)



*Across all boundaries,

$$\epsilon_1 E_{1,\perp} = \epsilon_2 E_{2\perp}$$

$$E_{1,\parallel} = E_{2,\parallel}$$

$$B_{1,\perp} = B_{2,\perp}$$

$$\frac{1}{\mu_1} B_{1,\parallel} = \frac{1}{\mu_2} B_{2,\parallel}$$