3. In this problem you will construct the 4-dimensional (4-d) electromagnetic stress-energy-momentum tensor from the 4-d electromagnetic field tensor $F^{\mu\nu}$ in **Gaussian** units. Recall that $F^{\mu\nu}$ is antisymmetric ($F^{\mu\nu} = -F^{\nu\mu}$) and is constructed from components of the electric and magnetic induction fields $\mathbf{E}$ and $\mathbf{B}$ by choosing

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon^{ijk}B^k.$$

Here we use the Einstein convention of summing over repeated indices, where Greek letters run from 0 to 3, while Latin letters run from 1 to 3. The symbol ϵ^{ijk} is the totally anti-symmetric 3-dimensional Levi-Civita symbol and satisfies $\epsilon^{123} = +1$. The time coordinate is given by $x^0 = ct$, where c is the speed of light and the 4-d metric used to raise and lower Greek indices is $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

(a) [3 pts] Define the 4-current J^μ and show that in a region containing no polarizable materials ($\epsilon = \mu = 1$) Maxwell equations are written in 4-d form as

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.$$

(b) [1 pts] From Maxwells equations prove that charge is conserved, i.e., show that

$$\partial_\mu J^\mu = 0.$$

(c) [3 pts] The 4-d stress-energy-momentum tensor is a traceless symmetric second-rank tensor, quadratic in the field strengths defined by

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F_\lambda{}^\nu - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\beta\alpha}\right].$$

Show that the 4 parts of $T^{\mu\nu}$ can be identified with the electromagnetic energy density u by $T^{00} = u$, the momentum density $\mathbf{g}$ and the Poynting vector $\mathbf{S}$ by $T^{0i} = T^{i0} = cg^i = S^i/c$, and the 3-d Maxwell stress tensor $\overleftrightarrow{\mathbf{T}}_M$ by $T^{ij} = -T^{ij}_M$. Be sure to give u, $\mathbf{g} = \mathbf{S}/c^2$, and $\overleftrightarrow{\mathbf{T}}_M$ as functions of $\mathbf{E}$ and $\mathbf{B}$.

(d) [3 pts] Use Maxwell's equations to compute $\partial_\mu T^{\mu\nu}$. Show that $\partial_\mu T^{\mu\nu} = 0$ in a region where $J^\mu = 0$ and that this one 4-d vector equation is equivalent to the local conservation of electromagnetic energy and momentum in 3-d, i.e., that

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = 0,$$

and

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \overleftrightarrow{\mathbf{T}}_M.$$

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E+M #3

a) We define the 4-current J^μ as:

$$J^\mu = \langle c\rho, \vec{j} \rangle$$

* To prove $\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu$, we consider two cases:

$\mu = 0$:

$$\partial_\nu F^{\nu 0} = \frac{4\pi}{c} J^0$$

$$\partial_0 F^{00} + \partial_i F^{i0} = \frac{4\pi}{c} c\rho$$

$$0 + \nabla \cdot E = 4\pi\rho \checkmark$$

$\mu = i$:

$$\partial_0 F^{0i} + \partial_i F^{ij} = \frac{4\pi}{c} \vec{j}^i$$

$$-\frac{1}{c}\frac{\partial}{\partial t}(\vec{E}) - \partial_j \epsilon_{ijk} B_k = \frac{4\pi}{c} j^i$$

$$-\frac{1}{c}\frac{\partial E_i}{\partial t} + \epsilon_{jik}\frac{\partial}{\partial x^i} B_k = \frac{4\pi}{c} j^i$$

$$-\frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} \checkmark$$

* To prove $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$, we again consider two cases

$\lambda, \mu, \nu \in \{1,2,3\}$:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

$$\frac{\partial}{\partial x}(-B_x) + \frac{\partial}{\partial y}(-B_y) + \frac{\partial}{\partial z}(-B_z) = 0$$

$$\nabla \cdot B = 0 \checkmark$$

$\lambda, \mu, \nu \in \{0, i, j\}$:
$i \neq j$

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0$$

$$\partial_0(-\epsilon_{ijk} B_k) \underbrace{- \partial_i(E_j) + \partial_j(E_i)}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \nabla \times E = 0 \checkmark$$

#3 (cont.)

b) The equation of charge conservation says $\partial_\mu J^\mu = 0$

$$\partial_\mu J^\mu = \partial_\mu (\partial_\nu F^{\nu\mu})$$

$$= \partial_\nu \partial_\mu F^{\nu\mu}$$ (order of partial differentiation doesn't matter)

$$= -\partial_\nu \partial_\mu F^{\mu\nu}$$ (anti-symmetric exchange $\Rightarrow F^{\nu\mu} = -F^{\mu\nu}$)

$\Rightarrow \partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}$ $\rightarrow$ only 0 is equal to its own negative

$\hookrightarrow \partial_\mu J^\mu = 0$ ✓

c) Given $T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\beta\alpha} \right]$

The constituent parts of the equation are:

$$F^{\alpha\beta} F_{\beta\alpha} = 2(|E|^2 - |B|^2)$$

$$F^{0\lambda} F_\lambda{}^0 = |E|^2$$

$$F^{0\lambda} F_\lambda{}^i = (-E^j)(-\epsilon_{ijk} B_k)$$

$$= (E \times B)^i$$

$$F^{i\lambda} F_\lambda{}^j = F^{i0} F_0{}^j + F^{ik} F_k{}^j$$

$$= (E_i)(-E_j) + (-\epsilon^{ikm} B^m)(\epsilon^{kjp} B^p)$$

$$= -E_i E_j - (\delta^j_m \delta^i_p - \delta^j_i \delta^p_m) B^m B^p$$

$$= -E_i E_j - B^i B^j + \delta^{ij} |B|^2$$

#3 (cont)

c) $T^{00} = u$

$$= \frac{1}{4\pi}\left[F^{00}F_0^{\ 0} - \frac{1}{4}g^{00}F^{\alpha\beta}F_{\beta\alpha}\right]$$

$$= \frac{1}{4\pi}\left[|E|^2 - \frac{1}{4}(2[|E|^2 - |B|^2])\right]$$

$$= \frac{1}{4\pi}\left[|E|^2 - \frac{1}{2}|E|^2 - \frac{1}{2}|B|^2\right]$$

$$= \frac{1}{8\pi}(|E|^2 + |B|^2) \qquad \text{(Energy Density)}$$

$$T^{0i} = T^{i0} = cg^i = \frac{1}{c}S^i$$

$$= \frac{1}{4\pi}\left[F^{0i}F_i^{\ 0} - \frac{1}{4}g^{0i}F^{\alpha\beta}F_{\beta\alpha}\right]$$

$$= \frac{1}{4\pi}(E\times B)^i$$

$$\Rightarrow \quad \frac{S^i}{c} = \frac{1}{4\pi}(E\times B)^i$$

$$S^i = \frac{c}{4\pi}(E\times B)^i \qquad \text{(Poynting vector)}$$

$$T^{ij} = -T^{ji}$$

$$= \frac{1}{4\pi}\left[F^{ij}F_j^{\ i} - \frac{1}{4}g^{ij}F^{\alpha\beta}F_{\beta\alpha}\right]$$

$$= \frac{1}{4\pi}\left[-E_iE_j - B_iB_j + \delta_{ij}|B|^2 - \frac{1}{4}\delta^{ij}(2|E|^2 - 2|B|^2)\right]$$

$$= -\frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}(|E|^2 - |B|^2)\right] \qquad \text{(Maxwell Stress Tensor}$$

#3 (cont.)

d) $\partial_\mu T^{\mu\nu} = \partial_\mu \left[\frac{1}{4\pi} \left(F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\beta\alpha} \right) \right]$

$= \frac{1}{4\pi} \left[(\partial_\mu F^{\mu\lambda}) F_\lambda{}^\nu + F^{\mu\lambda} (\partial_\mu F_\lambda{}^\nu) - \frac{1}{4} g^{\mu\nu} \left[(\partial_\mu F^{\alpha\beta}) F_{\beta\alpha} + F^{\alpha\beta} (\partial_\mu F_{\beta\alpha}) \right] \right) \Big]$

$= \frac{1}{4\pi} \left[-J^\lambda F_\lambda{}^\nu + F^{\alpha\beta} (\partial_\alpha F_\beta{}^\nu) - \frac{1}{4} g^{\mu\nu} \left[(\partial_\mu F^{\alpha\beta}) F_{\beta\alpha} + F^{\alpha\beta} (\partial_\mu F_{\beta\alpha}) \right] \right]$