1. Consider a Lorentz frame K containing no polarizable materials in which there is a magnetic induction $\mathbf{B} = B^x\,\hat{\mathbf{x}} + B^y\,\hat{\mathbf{y}} + B^z\,\hat{\mathbf{z}}$ but no electric field.

   (a) [1 pt] For the above magnetic induction, write down the 4-dimensional electromagnetic field tensor $F^{\alpha\beta}$ in frame K as a matrix.

   (b) [1 pt] Write down a homogeneous Lorentz boost $\Lambda^{\alpha}{}_{\beta}$ in the y-direction from frame K to another frame K′ which is moving with velocity $\mathbf{v} = v_0\,\hat{\mathbf{y}}$ as seen by observers that are at rest in frame K.

   (c) [2 pt] Apply the boost $\Lambda^{\alpha}{}_{\beta}$ to $F^{\alpha\beta}$ to find $F'^{\alpha\beta}$, the field strength tensor as seen in the moving frame K′ .

   (d) [2 pt] What are the electric field components E'^x, E'^y, and E'^z and the magnetic induction components B'^x, B'^y, and B'^z in frame K′?

   (e) [4 pt] Consider explicitly a **B** field in the K frame caused by an **uncharged** infinitely long and thin wire centered on the y-axis $(x, z) = (0, 0)$ which carries a current I in the $+y$ direction. Assume that no polarizable materials are present, i.e., assume $\epsilon_r = 1$ and $\mu_r = 1$. What are $\mathbf{B}'(x', y', z')$ and $\mathbf{E}'(x', y', z')$ in the K′ frame, written as functions of the K′-coordinates? Where does $\mathbf{E}'$ point?

Jan 2016 # $E+M$ #1 Gaussian

a) Given $E=0$, $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$ (in general)

$$F^{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B_z & B_y \\ 0 & B_z & 0 & -B_x \\ 0 & -B_y & B_x & 0 \end{bmatrix}$$

b) For a Lorentz boost in y-direction

$$\Lambda^{\alpha}_{\ \beta} = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) In matrix form: $F' = \Lambda F \Lambda^T$

$$F' = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B_z & B_y \\ 0 & +B_z & 0 & -B_x \\ 0 & -B_y & +B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\beta\gamma B_z & 0 & \beta\gamma B_x \\ 0 & 0 & -B_z & B_y \\ 0 & \gamma B_z & 0 & -\gamma B_x \\ 0 & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

#1 (cont.)

c) $$F'^{\alpha\beta} = \begin{bmatrix} 0 & -\beta\gamma B_z & 0 & \beta\gamma B_x \\ \beta\gamma B_z & 0 & -\gamma B_z & B_y \\ 0 & \gamma B_z & 0 & -\gamma B_x \\ -\beta\gamma B_x & -B_y & \gamma B_x & 0 \end{bmatrix}$$

d) $\vec{E}' = \beta\gamma(B_z\hat{x} - B_x\hat{z}) \Rightarrow E'_x = \beta\gamma B_z \quad E'_y = 0 \quad E'_z = -\beta\gamma B_x$

$\vec{B}' = \gamma B_x\hat{x} + B_y\hat{y} + \gamma B_z\hat{z} \Rightarrow B'_x = \gamma B_x \quad B'_y = B_y \quad B'_z = \gamma B_z$

e) We find $\vec{B}$ according to Ampere's Law

$$\oint B \cdot dl = \frac{4\pi}{c} I_{enc}$$

$$B \cdot 2\pi r = \frac{4\pi}{c} I$$

$$B = \frac{2I}{cr} \langle \cos\varphi\,\hat{x} + \sin\varphi\,\hat{z} \rangle$$

* This is best done in cylindrical coordinates, but professors are stupid question writers. φ measured ccw from $+x$ axis in x-z plane

$$\Rightarrow \vec{E}' = \langle \beta\gamma\sin\varphi \frac{2I}{cr},\ 0,\ -\beta\gamma\cos\varphi\frac{2I}{cr} \rangle$$

$$\vec{B}' = \langle \gamma\sin\varphi\frac{2I}{cr},\ 0,\ \gamma\cos\varphi\frac{2I}{cr} \rangle$$