2. Consider an infinite slab of thickness $d$, carrying uniform charge density $\rho$, centered on the origin and extending in the $x$-$y$ plane. Assume both the electric permittivity $\varepsilon$ and the magnetic permeability $\mu$ have their vacuum values.

a) 2 pt. Find the electric field vector, $\mathbf{E}$, and magnetic flux density (magnetic induction), $\mathbf{B}$, everywhere. Do not just write the answer down, but in all cases clearly articulate your arguments and solution to receive credit.

*For parts (b)–(e) consider an observer moving at velocity* $\mathbf{v} = v_0\hat{\mathbf{x}}$. *Do not assume that* $v_0 \ll c$.

b) 1 pt. What is the current density, $\mathbf{J}'$, in the observer's frame of reference? [Hint: How does the charge density transform?]

c) 3 pt. Find the electric field vector, $\mathbf{E}'$, and magnetic flux density, $\mathbf{B}'$, everywhere in the observer's frame of reference. Do this by solving Ampére's and Gauss' law in the observer's frame.

d) 2 pt. Alternatively, obtain the same result by performing a Lorentz transformation on the fields found in part (a).

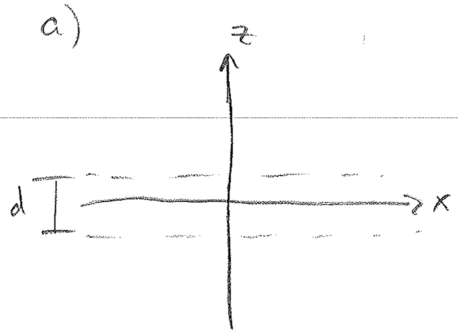
e) 2 pt. Show explicitly that $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ have the same value in both the rest frame of the slab and the observer's frame. Why is that? Is it possible to find a frame where $\mathbf{E} = \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$?

Jan 2008

# E+M #2

Gaussian Units

a)




* Inside and outside slab must be considered seperately

* $\vec{B} = 0$ everywhere since there are no moving charges

Inside slab

Integral form of Gauss law states:

$$\int \vec{E} \cdot d\vec{a} = 4\pi Q_{enc}$$

$$E \cdot \int d\vec{a} = 4\pi \rho \cdot A \cdot (2|z|) \quad \Leftarrow \text{Gaussian prism centered on origin of height } 2z \text{ and bottom area } A$$

$$E \cdot 2A = 4\pi \rho A \cdot 2z \, \hat{z}$$

$$E = 4\pi \rho z \, \hat{z} \quad \Leftarrow +\hat{z} \text{ direction if } z > 0, \; -\hat{z} \text{ direction } z < 0$$

Outside slab

$$\oint E \cdot d\vec{a} = 4\pi Q_{enc}$$

$$E \int d\vec{a} = 4\pi d A \rho \hat{z}$$

$$E \cdot 2A = 4\pi d A \rho \hat{z}$$

$$E = 2\pi d \rho \hat{z} \quad (\text{follows same } \pm \hat{z} \text{ as above})$$

b) $v = v_0 \hat{x}$

$J' = L J$ where

$$L = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} c\rho \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# 2 (cont.)

b) $$J' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\rho \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma c\rho \\ -\beta\gamma c\rho \\ 0 \\ 0 \end{bmatrix} \Rightarrow J' = -\beta\gamma c\rho\, \hat{x}$$

c) We proceed by transforming our coordinate system

$$x' = Lx$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \gamma ct - \beta\gamma x \\ \gamma x - \beta\gamma ct \\ y \\ z \end{bmatrix}$$

* From our transformations in parts b and c, we see

$z' = z$ and $\rho' = \gamma\rho$ $\Rightarrow$ previous results will be correct w/ coordinate substitutions

$$\Rightarrow \vec{E} = \begin{cases} 4\pi\gamma\rho'|z'|\,\hat{z}' & z < d \\ 2\pi d\gamma\rho'\hat{z} & z > d \end{cases}$$

#2 (cont.)

d) $$F^{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -E_z & 0 & 0 & 0 \end{bmatrix}$$

$$F' = LFL^T$$

$$= \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ \gamma E_z & -\beta\gamma E_z & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} E_z' &= \gamma \vec{E}_z \\ B_y' &= -\beta\gamma \vec{E}_z \end{aligned}$$

e) $E \cdot B$: * In rest frame

$$\vec{E} \cdot \vec{B} = 0 \quad (\vec{B} = \langle 0, 0, 0 \rangle)$$

* In moving frame

$$\vec{E} \cdot \vec{B} = 0 \quad (E_x, E_y, B_x, B_z = 0)$$

$E^2 - B^2$: * In rest frame

$$E^2 - B^2 = E^2 = E_z^2 \checkmark$$

* In moving frame

$$E^2 - B^2 = \gamma^2 E_z^2 - \beta^2\gamma^2 E_z^2$$

$$= E_z^2 \checkmark$$