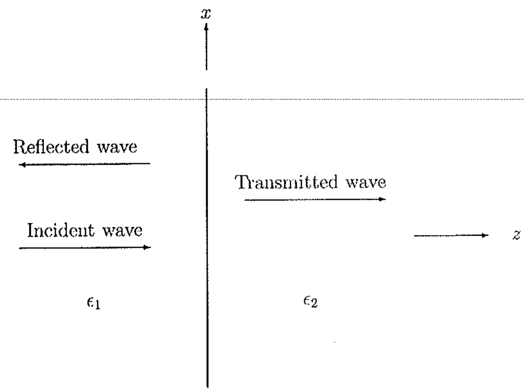


5. A plane-polarized harmonic ($e^{-i\omega t}$) plane electromagnetic wave traveling to the right in a homogeneous dielectric medium described by an dielectric constant ϵ_1, strikes a second homogeneous dielectric material described by dielectric constant $\epsilon_2 > \epsilon_1$ (see the figure). Assume that both materials have the same magnetic permeability μ_0 and that the incidence angle is 0^o (i.e., the wave is traveling perpendicular to the junction). Assume the incoming wave is polarized in the $\hat{x}$ direction and that its electric field amplitude is E_0, i.e., assume the incoming electric field is the real part of

$$\mathbf{E} = E_0\, e^{i(kz-wt)}\, \hat{x}.$$

   (a) {3 pts} Give the magnetic induction $\mathbf{B}$ associated with the above incoming wave. Make sure your wave satisfies Maxwell's equations, e.g., give k as a function of ω, the direction of $\mathbf{B}$, and the amplitude of $\mathbf{B}$ as a function of E_0.

   (b) {1 pts} Give similar expressions for the $\mathbf{E}$ and $\mathbf{B}$ components of the reflected and transmitted waves. Use E_0'' and E_0' for the respective amplitudes of reflected and transmitted waves.

   (c) {2 pts} In general, what conditions must be satisfied at the junction between two materials by the electromagnetic fields $\mathbf{E}, \mathbf{B}, \mathbf{D}$, and $\mathbf{H}$, if Maxwell's equations are to be satisfied?

   (d) {2 pts}Apply these junction conditions to the combined incoming, reflected, and transmitted wave to compute E_0'' and E_0' as functions of E_0 and the two dielectric constants ϵ_1 and ϵ_2.

   (e) {2 pts} Evaluate the time averages of the Poynting vectors of the incident, reflected, and transmitted waves. Recall that

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} \equiv \frac{1}{4\pi}\mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

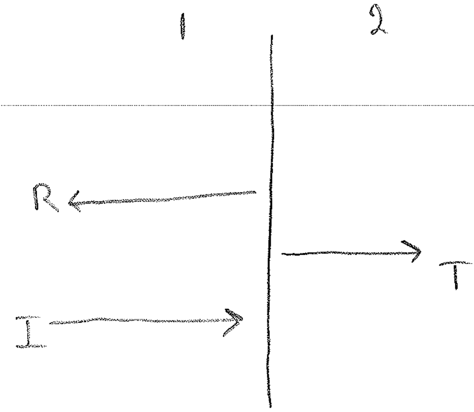
   The sum of the magnitudes of the reflected and transmitted time averaged Poynting vectors should equal the magnitude of the incident wave's time averaged Poynting vector.

Jan 2010

E+M #5

Gaussian

a)



$\vec{E} = E_0 \exp[i(k_1 z - \omega t)]\,\hat{x}$

$\vec{B} = n\hat{k}_1 \times \vec{E}$

$= \sqrt{\epsilon_1}\, E_0 \exp[i(k_1 z - \omega t)]\,\hat{y}$

* The relevant Maxwell equations in Gaussian units are:

① $\nabla \cdot D = 4\pi \rho_f$ ③ $\nabla \cdot B = 0$

② $\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$ ④ $\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$

* Starting w/ equation ③:

$$\nabla \cdot B = \frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z$$

$$= \frac{\partial}{\partial y}\left(-\sqrt{\epsilon_1}\, E_0 \exp[i(k_1 z - \omega t)]\right)$$

$$= 0 \checkmark$$

* Now equation ②:

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$

$$\hookrightarrow \nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = \langle 0, +\partial/\partial z\, E_x, -\tfrac{\partial}{\partial y}E_x \rangle$$

$$= \langle 0, ik_1 \; E_0 \exp[i(k_1 z - \omega t)], 0 \rangle$$

#5 (cont.)

a) $\frac{1}{c}\frac{\partial}{\partial t}B = \frac{1}{c}\frac{\partial}{\partial t}\left(-\sqrt{\epsilon_1}\,E_0 \exp[i(k_1 z - \omega t)]\right)\hat{y}$

$= \frac{-1}{c}\sqrt{\epsilon_1}\,i\omega\,E_0 \exp[i(k_1 z - \omega t)]\,\hat{y}$

$\Rightarrow \nabla\times\vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} \overset{?}{=} 0$

$= -ik \quad E_0 \exp[i(kz-\omega t)] + -i\frac{\omega}{c}\sqrt{\epsilon_1}\,E_0 \exp[i(kz-\omega t)]$

$= -iE_0 \exp[i(k_1 z - \omega t)]\left(-k + \frac{\omega}{c}\sqrt{\epsilon_1}\right)$

$\hookrightarrow$ Only true if $k = \frac{\omega}{c}\sqrt{\epsilon_1}$

* Finally equation (4)

$\nabla\times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$

$\vec{J}_f = 0$

$\nabla\times H = \nabla\times\frac{1}{\mu_0}\vec{B}$

$$= \frac{1}{\mu_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & B_y & 0 \end{vmatrix} = \langle -\partial_z B_y,\ 0,\ \partial_x B_y\rangle$$

$= \frac{1}{\mu_0}\frac{\partial}{\partial z}\left(\sqrt{\epsilon_1}\,E_0 \exp[i(k_1 z - \omega t)]\right)\hat{x}$

$= \frac{-\sqrt{\epsilon_1}}{\mu_0}E_0(ik_1)\exp[i(k_1 z - \omega t)]\,\hat{x}$

$-\frac{1}{c}\frac{\partial D}{\partial t} = \frac{-1}{c}\frac{\partial}{\partial t}\epsilon_1\vec{E}$

$= -\frac{\epsilon_1}{c}(-i\omega)E_0 \exp[i(k_1 z - \omega t)]\,\hat{x}$

$\hookrightarrow \nabla\times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{i\omega\epsilon_1}{c}E_0 \exp[i(k_1 z - \omega t)]\hat{x} + -i\frac{\sqrt{\epsilon_1}k}{\mu_0}E_0 \exp[i(k_1 z - \omega t)]$

$= 0$ if $\frac{\omega\epsilon_1}{c} - \frac{\sqrt{\epsilon_1}k}{\mu_0} = 0 \Rightarrow k = \frac{\omega\sqrt{\epsilon_1}}{c}$ as before ✓

#5 (cont.)

b) * For the transmitted wave:

$$\vec{E}' = E_0' \exp[i(k_2 z - \omega t)]\hat{x}$$

$$\vec{B}' = \sqrt{\epsilon_2}\, E_0' \exp[i(k_2 z - \omega t)]\hat{y}$$

$$k_2 = \frac{\omega\sqrt{\epsilon_2}}{c}$$

* For the reflected waves

$$\vec{E}'' = E_0'' \exp[i(k_1(-z)\omega t)]\hat{x}$$

$$\vec{B}' = +\sqrt{\epsilon_1}\, E_0'' \exp[i(-k_1 z - \omega t)]\hat{y}$$

$$k_1 = \frac{\omega\sqrt{\epsilon_1}}{c}$$

c) The general boundary conditions are:

$$D_1^{\perp} - D_2^{\perp} = 4\pi\sigma_f \qquad B_1^{\perp} - B_2^{\perp} = 0$$

$$E_1^{\parallel} = E_2^{\parallel} \qquad H_1^{\parallel} - H_2^{\parallel} = 4\pi \vec{K}_f$$

d) Applying these boundary conditions to our situation

$$E_1^{\parallel} - E_2^{\parallel} = 0$$

$$(\vec{E}_I^{\parallel} + \vec{E}_R^{\parallel}) - \vec{E}_T^{\parallel} = 0$$

$$[(E_I + E_R) - E_T] \times \hat{z} = 0 \quad \Rightarrow \text{picks out } \hat{y} \text{ components}$$

$$-E_0 \exp[i(k_1 z - \omega t)] + E''\exp[i(-k_1 z - \omega t)] - E'\exp[i(k_2 z - \omega t)] = 0$$

* if we set our boundry to be $z = 0$

$$-E_0 + E_0'' - E_0' = 0$$

#5 (cont.)

d) $H_1^{\parallel} - H_2^{\parallel} = \cancel{4\pi K_f} \to 0$

$\frac{1}{\mu_0}(B_1^{\parallel} - B_2^{\parallel}) = 0 \Rightarrow B_1^{\parallel} - B_2^{\parallel} = 0$

$\hookrightarrow B_1^{\parallel} - B_2^{\parallel} = 0$

$B_I^{\parallel} + B_R^{\parallel} - B_T^{\parallel} = 0$

$[(B_I + B_R) - B_T] \times \hat{z} = 0 \Rightarrow$ picks off x-component

$-E_0\sqrt{\epsilon_1}\exp[i(k_1 z - \omega t)] + E_0''\sqrt{\epsilon_1}\exp[i(-k_1 z - \omega t)] - E_0'\sqrt{\epsilon_2}\exp[i(k_2 z - \omega t)]$

$(-E_0 + E_0'')\sqrt{\epsilon_1} - \sqrt{\epsilon_2}E_0' = 0$

$\hookrightarrow E_0' = \sqrt{\frac{\epsilon_1}{\epsilon_2}}(-E_0 + E_0'')$

*Using our equations to solve for E_0', E_0'' in terms of E_0:

$E_0' = \sqrt{\frac{\epsilon_1}{\epsilon_2}}(-E_0 + E_0'')$

$= \sqrt{\frac{\epsilon_1}{\epsilon_2}}(-E_0 + (E_0' + E_0))$