8.5 - 10

Problem 6: Gauges and 4-potentials

(a) Write down the Maxwell equations in terms of 3-vectors in a vacuum including sources. These provide eight coupled PDEs for $\vec{E}$ and $\vec{B}$ given the source functions $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$. [1 point]

(b) Introduce the vector potential $\vec{A}(\vec{x},t)$. How is this related to $\vec{B}$ and how does it help solve the Maxwell equations? [2 points]

1-1.5? (c) Plugging $\vec{A}(\vec{x},t)$ into Faraday's equation allows introduction of the scalar potential $\Phi(\vec{x},t)$. How is Φ related to $\vec{A}$ and $\vec{E}$ and how does it help to solve the Maxwell equations? [2 points]

(d) Write the inhomogeneous Maxwell equations in terms of the potentials Φ and $\vec{A}$. What is the Lorenz gauge condition and how does it help to solve the Maxwell equations? [2 points]

(e) If the electro- and magneto-static solutions to Maxwell equations are $\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$ and $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$ what are the corresponding electro*dynamic* (*i.e.* time-dependent) solutions in the Lorenz gauge? What do they have to do with causality? [3 points]

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E+M #6

Gaussian

a) In Gaussian units, the 3-vector form of Maxwell's equations are:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

b) Introducing the vector potential $\vec{A}$, which is related to $\vec{B}$ by:

$$\vec{\nabla}\times\vec{A} = \vec{B}$$

We can use the vector potential to reformulate Maxwell's equations for magneto statics in terms of Poisson's eqn $\left(-\nabla^2\vec{A} = \frac{4\pi}{c}\vec{J}\right)$, which is often times easier to solve than the Biot-Savart Law or Ampere's Law, which only applies to certain geometries

c) Substituting the above into Faraday's equation $\left(\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0\right)$ yields

$$\nabla\times E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times A) = 0$$

$$\nabla\times\left(E + \frac{1}{c}\frac{\partial}{\partial t}A\right) = 0$$

Since the curl of a gradient of a scalar function is idetically 0, we can define Φ as:

$$-\nabla\Phi = \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \iff \vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

Again, this helps us to solve Maxwell's equations by allowing us in electrostatics to reformulate our equation in terms of Poisson's equation $(-\nabla^2\Phi = 4\pi\rho)$, which is easier to solve than Coloumbs Law, and works in more geometries than Gauss' Law.

d) Maxwell's inhomogeneous equations reformulated in terms of the scalar/vector potenials are:

$$\left.\begin{array}{l} -\nabla^2\Phi = 4\pi\rho \\ -\nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} \end{array}\right\} -\Box A^\mu = \frac{4\pi}{c}\vec{J}^\mu$$

#6 (cont.)

d) The Lorenz guage condition specifies that $\partial_\mu A^\mu = 0$. This condition helps to solve Maxwell's equations by allowing use of the Poisson equation in all media and to pair down its solutions to a family of solutions related only by a guage transformation. It is also consistent with time-dependent potentials.

e) In the case of electrodynamics, our potentials become the Lienard-Wiechert potentials, described by:

$$\Phi(t, \vec{r}) = 4\pi \int \frac{\rho\left(t - \frac{|r-r'|}{c}, \vec{r}'\right)}{|r-r'|} d^3r'$$

$$\vec{A}(t, \vec{r}) = \frac{1}{c} \int \frac{J\left(t - \frac{|r-r'|}{c}, \vec{r}'\right)}{|r-r'|} d^3r'$$

These solutions account for the fact that any effects from changes in the fields require time to propogate before being felt by the test particles, since no information can travel faster than the speed of light.