5. An infinitely long, uniformly charged wire of radius a and total charge per unit length λ, is at rest on the z-axis of the lab frame.

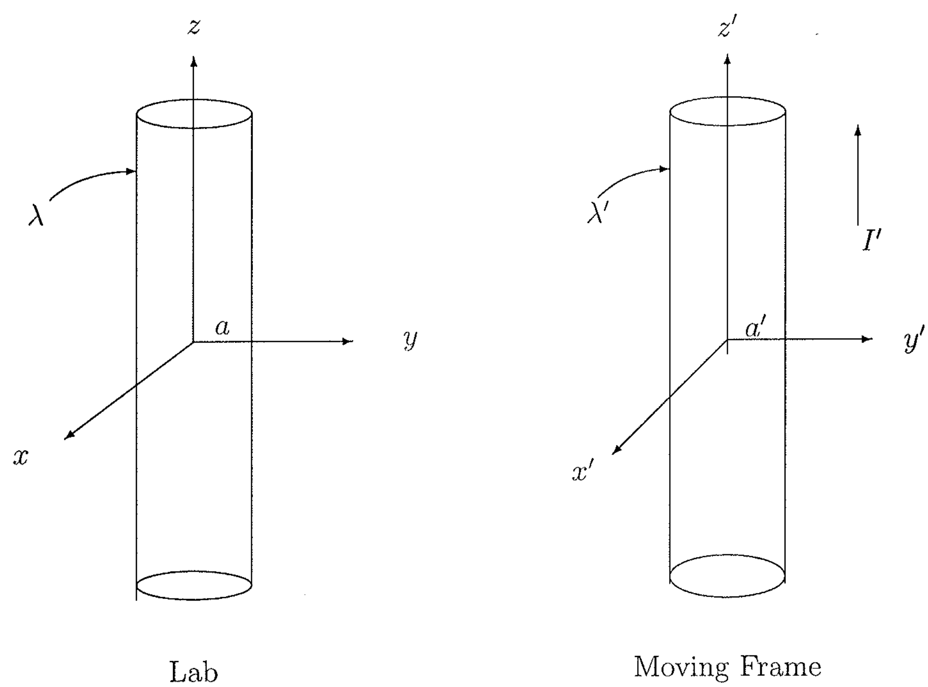
(a) [2 pts] Compute the electric field $\mathbf{E}(x, y, z)$ exterior to the wire in the lab frame by solving Gauss's law in that frame. What is the magnetic induction $\mathbf{B}(x, y, z)$ in this frame?

(b) [2 pts] If you are moving in the lab's negative z direction with speed v how are your spatial and time coordinates related to those of the lab's? To answer this question simply give the Lorentz boost $x'^{\sigma} = L^{\sigma}{}_{\mu} x^{\mu}$ that relates the two sets of coordinates.

(c) [2 pts] In your frame what is the radius a' of the wire? What is the charge/length λ' of the wire and what is the current I' in the wire?

(d) [1 pts] Combine the E and B fields in the lab into a single electromagnetic field tensor $F^{\alpha\beta}$ using $F^{\sigma\mu} = -F^{\mu\sigma}$ and $F^{0i} = -E^i$. In Gaussian units $F^{12} = -B^z$, $F^{23} = -B^x$ and $F^{13} = B^y$, and in SI units $F^{12} = -c\,B^z$, $F^{23} = -c\,B^x$ and $F^{13} = c\,B^y$, .

(e) [3 pts] What electric field $\mathbf{E}\,'(x', y', z')$ and what magnetic induction $\mathbf{B}\,'(x', y', z')$ will you measure exterior to the wire in your frame? To answer this part you can use your answers for part (c) or you can compute $F' = LFL^T$.



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E+M #5

a) From Gauss Law, we know that

$$\int \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$E \int da = \frac{\lambda L}{\epsilon_0}$$

$$E(2\pi r L) = \frac{\lambda L}{\epsilon_0}$$

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r} \hat{r}$$

* However, we most convert this from cylindrical coordinates to cartesian

$$\Rightarrow r = \sqrt{x^2+y^2}$$

$$\hat{r} = \cos\varphi \,\hat{x} + \sin\varphi \,\hat{y}$$

$$\hookrightarrow \vec{E}(x,y,z) = \frac{\lambda}{2\pi\epsilon_0} \frac{\langle \cos\varphi, \sin\varphi, 0 \rangle}{\sqrt{x^2+y^2}}$$

* Since there is no moving charge in the wire, $\vec{B} = 0$

b) In a moving frame, we know coordinates are transformed according to:

$$x'^{\sigma} = L^{\sigma}{}_{\mu} x^{\mu} \iff x' = L_z x$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma ct + \gamma\beta z \\ x \\ y \\ \gamma\beta ct + \gamma z \end{bmatrix} \Rightarrow \begin{aligned} t' &= \frac{\gamma}{c}(ct + \beta z) \\ x' &= x \\ y' &= y \\ z' &= \gamma(ct\beta + z) \end{aligned}$$

#5(cont.)

c) * Since the radius of the wire only depends on x and y, it is unchanged moving from the unprimed frame to the primed frame.

* We can transform the current vector to find λ' and I'

$$\vec{J} = \begin{bmatrix} c\lambda \\ J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} c\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{J}' = L_z J$$

$$= \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} c\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c\lambda\gamma \\ 0 \\ 0 \\ \gamma\beta c\lambda \end{bmatrix} \Rightarrow \begin{array}{l} c\lambda' = c\lambda\gamma \rightarrow \lambda' = \gamma\lambda \\ J_z' = \gamma\beta c\lambda \end{array}$$

$$I' = \oint \vec{J}' \cdot d\vec{a}'$$

$$= J_z \cdot \pi a^2$$

$$= \gamma\beta c\lambda \pi a^2$$

d) In the lab (unprimed) frame, our field tensor is:

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & 0 \\ E_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E_x = \frac{\lambda \cos\varphi}{2\pi\epsilon_0 \sqrt{x^2+y^2}} \qquad E_y = \frac{\lambda \sin\varphi}{2\pi\epsilon_0 \sqrt{x^2+y^2}}$$

#5 (cont.)

e) The field tensor is transformed according to:

$$F' = L F L^T$$

$$= \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & 0 \\ E_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & 0 \\ \gamma E_x & 0 & 0 & \gamma\beta E_x \\ \gamma E_y & 0 & 0 & \gamma\beta E_y \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma E_x & -\gamma E_y & 0 \\ \gamma E_x & 0 & 0 & \gamma\beta E_x \\ \gamma E_y & 0 & 0 & \gamma\beta E_y \\ 0 & -\gamma\beta E_x & -\gamma\beta E_y & 0 \end{bmatrix}$$

$$\Rightarrow \quad E_x' = \gamma E_x \qquad E_z' = 0$$

$$E_y' = \gamma E_y$$

$$B_y' = \frac{\gamma\beta}{c} E_x \qquad B_z' = 0$$

$$B_x' = -\frac{\gamma\beta}{c} E_y$$

$$\Rightarrow \vec{E} = \frac{\gamma\lambda}{2\pi\epsilon_0\sqrt{x^2+y^2}} \langle \cos\varphi\,\hat{x}, \sin\varphi\,\hat{y}, 0 \rangle$$

$$\vec{B} = \frac{\gamma\beta\lambda}{c\,2\pi\epsilon_0\sqrt{x^2+y^2}} \langle \sin\varphi\,\hat{x}, -\cos\varphi\,\hat{y}, 0 \rangle$$