5. Consider an infinitely long, solid, nonmagnetic conducting rod of radius a centered on the z axis. An infinitely long, hollow, conducting cylinder with inner radius $b > a$ and outer radius d is coaxial with the rod. Let r be the radial distance perpendicular to the axis of the rod and the cylinder. The region between the conducting rod and the conducting cylinder (that is, $a < r < b$) is filled with a nonconducting, linear, isotropic magnetic material with a constant relative permeability $K = \mu/\mu_0$, where μ is the permeability of the material, and μ_0 is the permeability of free space ($\mu_0 = 1$ in Gaussian units).

   The rod carries a current I in the $+z$ direction while the concentric cylinder carries a current I in the $-z$ direction. We assume that the current density $\mathbf{j}$ is uniform and of the same magnitude in both the rod and the cylinder,

$$j = \frac{I}{\pi a^2} = \frac{I}{\pi(d^2 - b^2)}.$$

a) 3 pts. Calculate the magnetic field $H(r)$ for the four regions

$$\text{I: } r \leq a, \quad \text{II: } a \leq r \leq b, \quad \text{III: } b \leq r \leq d, \quad \text{IV: } d \leq r.$$

b) 3 pts. Calculate the magnetic flux (per unit length in the z direction) crossing a half-plane extending from the axis of the coaxial system and extending to infinity, that is, the surface defined by $x > 0$, $y = 0$, $-\infty < z < \infty$. Use this result to find the self-inductance L per unit length of the coaxial conductor.

c) 2 pts. Compute the magnetic energy U per unit length along the z axis stored in the region filled with the linear magnetic material, that is for region II, $a < r < b$.

d) 2 pts. Using the result from part c), show that the contribution to L coming from the region $a \leq r \leq b$, L_{II}, is consistent with the contribution from the same region that you calculated in part b) above. That is, compute $\frac{1}{2}L_{\mathrm{II}}I^2$ and compare with the result of part c).

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E+M #5

Gaussian

a) We can compute the magnetic field via Ampere's Law (from $\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$)

$$\int \nabla \times B \, da = \int \frac{4\pi}{c} \vec{J} \cdot d\vec{a}$$

$$\int B \cdot d\vec{l} = \frac{4\pi}{c} \int \vec{J} \cdot da$$

$H = \vec{B} - 4\pi\vec{M}$

$B = \mu H$, $\mu = 1 + 4\pi\chi_m$

$M = \chi_m H$ * for linear, isotropic material

- We have 4 regions to consider:

Region I: $r \leq a$

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c} \int \left(\frac{I}{\pi a^2}\right) \cdot d\vec{a}$$

$$B \cdot \cancel{2\pi} r = \frac{{}^{2}\cancel{4\pi}}{c} \frac{I}{\cancel{\pi} a^2} \cdot \cancel{\pi} r^2$$

$$\vec{B} = \frac{2Ir}{ca^2}\hat{\varphi} \quad \text{(direction determined via symmetry)}$$

$$\vec{H} = \vec{B} - 4\pi\vec{M}$$

$$\vec{H} = \frac{2Ir}{ca^2}\hat{\varphi} - 0 \quad \text{(non-magnetic material)}$$

$$\vec{H} = \frac{2Ir}{ca^2}\hat{\varphi}$$

Region II: $a \leq r \leq b$

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c} \int \frac{I}{\pi a^2} \cdot d\vec{a}$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \frac{I}{\pi a^2} \cdot \pi a^2$$

$$B = \frac{2I}{cr}\hat{\varphi}$$

* Because our magnetization is non-zero, $M = \chi_m H$, $H = \frac{1}{\mu}B$, $\mu = 1 + 4\pi\chi_m$

$$H = B - 4\pi M$$

$$H = B - 4\pi\chi_m H$$

$$H(1 + 4\pi\chi_m) = B$$

$$H\mu = B$$

$$H = \frac{1}{\mu}B$$

$$\Rightarrow H = \frac{2I}{\mu c r}\hat{\varphi}$$

#5 (cont.)

Region III: $b \le r \le d$

*Must separate $\vec{J}$ - integral to account for multiple currents

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c} \int \vec{J} \cdot da$$

$$B \cdot 2\pi r = \frac{4\pi}{c}\left[\int \vec{J}_{rod} \cdot da_{rod} + \int \vec{J}_{shell} \cdot da_{shell} \right]$$

$$B \cdot 2\pi r = \frac{4\pi}{c}\left[\frac{I}{\pi a^2} \cdot \pi a^2 + \frac{-I}{\pi(d^2-b^2)} \cdot \pi(r^2-b^2) \right]$$

$$B \cdot 2\pi r = \frac{4\pi}{c}\left[I - \frac{I(r^2-b^2)}{d^2-b^2} \right]$$

$$\vec{B} = \frac{2I}{cr}\left[1 - \frac{r^2-b^2}{d^2-b^2} \right] \hat{\varphi}$$

* We again must consider magnetization in region II, following above work,

$$H = \frac{1}{\mu} B$$

$$= \frac{2I}{c\mu r}\left[1 - \frac{r^2-b^2}{d^2-b^2} \right] \hat{\varphi}$$

Region IV: $r \ge d$

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c} \int \vec{J} \cdot d\vec{a}$$

$$B \cdot 2\pi r = \frac{4\pi}{c}\left[\int \vec{J}_{rod} \cdot da_{rod} + \int \vec{J}_{shell} \cdot da_{shell} \right]$$

$$B \cdot 2\pi r = \frac{4\pi}{c}\left[\frac{I}{\pi a^2}(\pi a^2) + \frac{-I}{\pi(d^2-b^2)}(d^2-b^2)\pi \right]$$

$$\vec{B} = \frac{4\pi}{c}\left[I - I \right] \cdot \frac{1}{2\pi r}$$

$$\vec{B} = 0$$

$$\vec{H} = \frac{1}{\mu}\vec{B}$$

$$\vec{H} = 0$$

#5 (cont.)

b) *We know that the self-inductance L is related to the flux via: $\Phi = LI$

To get total flux of system, we sum the fluxes from the individual regions

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^\ell dz \int dx\, B$$

Region I: $r \le a$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^\ell dz \int_0^a \frac{2I\sqrt{x^2+y^2}}{ca^2}\,dx$$

$$= \int_0^r \frac{2I}{ca^2}\,x\,dx$$

$$= \frac{I}{ca^2}\,x^2\Big|_0^a$$

$$= \frac{I}{c}$$

Region II: $a \le r \le b$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^\ell dz \int_a^b \frac{2I}{c\sqrt{x^2+y^2}}\,dx$$

$$= \int_a^b \frac{2I}{cx}\,dx$$

$$= \frac{2I}{c}\ln(x)\Big|_a^b$$

$$= \frac{2I}{c}\left(\ln(b/a)\right)$$

Region III: $b \le r \le d$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^\ell dz \int_b^d \frac{2I}{cr}\left[1 - \frac{r^2-b^2}{a^2-b^2}\right]dx$$

$$= \int_b^d \frac{2I}{cx}\left[1 - \frac{x^2-b^2}{d^2-b^2}\right]dx$$

$$= \frac{2I}{c}\int\left[\frac{1}{x} - \frac{x}{d^2-b^2} + \frac{b^2}{x(d^2-b^2)}\right]dx$$

$$= \frac{2I}{c}\left[\ln(x) - \frac{1}{2}\frac{x^2}{d^2-b^2} + \frac{b^2}{d^2-b^2}\ln(x)\right]\Big|_b^d$$

#5 (cont.)

b) $\frac{\Phi}{l} = \frac{2I}{c}\left[\ln\left(\frac{d}{b}\right) - \frac{1}{2} + \frac{b^2}{d^2-b^2}\ln\left(\frac{d}{b}\right)\right]$

$= \frac{2I}{c}\left[\left(1+\frac{b^2}{d^2-b^2}\right)\ln\left(\frac{d}{b}\right) - \frac{1}{2}\right]$

Region IV : $r \geq d$

$B = 0 \Rightarrow \frac{\Phi}{l} = 0$

* Thus our overall flux per unit length is:

$$\frac{\Phi}{l} = \frac{I}{c} + \frac{2I}{c}\left(\ln\left(b/a\right)\right) + \frac{2I}{c}\left[\left(1+\frac{b^2}{d^2-b^2}\left(\ln\left(\frac{d}{b}\right)\right) - \frac{1}{2}\right]$$

$$= \cancel{\frac{I}{c}} + \frac{2I}{c}\ln(b/a) + \frac{2I}{c}\ln\left(\frac{d}{b}\right) + \frac{2I}{c}\frac{b^2}{d^2-b^2}\ln\left(\frac{d}{b}\right) - \cancel{\frac{I}{c}}$$

$$= \frac{2I}{c}\left[\ln(b/a) + \left(1+\frac{b^2}{d^2+b^2}\right)\ln\left(\frac{d}{b}\right)\right]$$

$$\frac{\Phi}{l} = \frac{LI}{l} \Rightarrow \frac{L}{l} = \frac{\Phi}{lI}$$

$$\Rightarrow \frac{L}{l} = \frac{2}{c}\left[\ln(b/a) + \left(1+\frac{b^2}{d^2+b^2}\right)\ln\left(\frac{d}{b}\right)\right]$$

c) $U = \frac{1}{8\pi}B^2$

$$U = \frac{1}{8\pi}\left(\frac{2I}{cr}\hat{\varphi}\right)^2$$

$$= \frac{1}{8\pi}\frac{4I^2}{c^2r^2}$$

$$U = \int \frac{I^2}{4\pi c^2 r^2} \, r\,dr\,d\varphi\,dz$$

$$= \frac{lI^2}{2c^2}\int_a^b \frac{1}{r}\,dr$$

$$= \frac{lI^2}{2c^2}\ln\left(b/a\right)$$

$$\frac{U}{l} = \frac{I^2}{2c^2}\ln(b/a)$$

#5 (cont.)

d) $\frac{1}{2} L_{II} I^2 = \frac{1}{2}\left(\frac{2}{c}\ln(b/a)\right) I^2$

$= \frac{I^2}{c}\ln(b/a)$

* off by factor $\frac{1}{2c}$, likely due to units being Gaussian v SI