Problem 5: Special relativity

Consider two frames K and K' with a uniform relative velocity. Observers at rest in K' are moving along the positive x axis of K with a velocity v. $\beta = v/c$, $\gamma = (1-\beta^2)^{-1/2}$, and c is the speed of light.

1. Let x^ν be the four-dimensional space-time vector in the K frame with the components: $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$, and x'^μ be the corresponding vector in the K' frame with the Lorentz transformation of $x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$, where Einstein's summation rule is implied. What are the components of $\Lambda^\mu_{\ \nu}$? [2 points]

2. An object is moving with a three-dimensional velocity $\vec{u}$ in K, and the velocity is measured to be $\vec{u'}$ in K'. What are the components of the object's four-velocity in K' in terms of u_x, u_y and u_z? [4 points]

3. Let θ be the angle between $\vec{u}$ and x in K, and θ' be the angle between $\vec{u'}$ and x' in K'. Show that
$$\tan\theta = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)}. \tag{2}$$
[2 point]

4. A source is emitting isotropically in its rest-frame and moves with an ultra-relativistic velocity in K with $\gamma \gg 1$. Show that in K half of the radiation power is concentrated in a cone with a half open angle of $1/\gamma$. [2 points]

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E+M #5

a) Given K' is moving relative to K with $\vec{V} = v_x \hat{x}$, in matrix form,

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) To transform $\vec{U}$ to $\vec{U}'$, $\vec{U}' = \Lambda U$

$$U' = \begin{bmatrix} ct' \\ U_x' \\ U_y' \\ U_z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ U_x \\ U_y \\ U_z \end{bmatrix}$$

$$\begin{bmatrix} ct' \\ U_x' \\ U_y' \\ U_z' \end{bmatrix} = \begin{bmatrix} \gamma ct - \beta\gamma U_x \\ -\beta\gamma ct + \gamma U_x \\ U_y \\ U_z \end{bmatrix}$$

c)