5. An infinitely long, uniformly charged wire of radius a and total charge per unit length λ, is at rest on the z-axis of the lab frame.

   (a) (2 pts) Compute the electric field $\mathbf{E}(x, y, z)$ interior and exterior to the wire in the lab frame by solving Gauss's law in that frame.

   (b) Complete the next 4 steps to compute $\mathbf{E}'(x', y', z')$ and $\mathbf{B}'(x', y', z')$ in a frame moving in the positive z-direction with speed v.

      i. (2 pts) Give the Lorentz boost $x'^{\sigma} = L^{\sigma}_{\mu} x^{\mu}$ ($\mathbf{x}' = \mathbf{L}\mathbf{x}$) from the Lab to the moving frame (take $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$).

      ii. (2 pts) Construct the electromagnetic field tensor $F^{\alpha\beta}$ from the electric field you found in part (a).

      iii. (2 pts) Use your lorentz boost to compute the electromagnetic field tensor $F'^{\alpha\beta} = L^{\alpha}_{\mu} L^{\beta}_{\nu} F^{\mu\nu}$ ($\mathbf{F}' = \mathbf{L}\mathbf{F}\mathbf{L}^{\mathbf{T}}$) in the moving frame.

      iv. (2 pts) From your $F'^{\alpha\beta}$ give the answer to (b).

Hint: Recall that in both SI and Gaussian units $F^{\sigma\mu} = -F^{\mu\sigma}$ and $F^{0i} = -E^{i}$. In Gaussian units $F^{12} = -B^{z}$, $F^{23} = -B^{x}$ and $F^{13} = B^{y}$, but in SI units $F^{12} = -c\, B^{z}$, $F^{23} = -c\, B^{x}$ and $F^{13} = c\, B^{y}$

E+M #5

a) We can use Gauss' Law to find $\vec{E}$

$$\int \nabla \cdot E = \int 4\pi\rho$$

$$\oint E \cdot d\vec{a} = 4\pi Q_{enc}$$

* If we are at $r > a$:

$$\oint \vec{E} \cdot d\vec{a} = 4\pi \lambda L$$

$$E \cdot 2\pi L r = 4\pi\lambda L$$

$$\vec{E} = \frac{2\lambda}{r}\hat{r}$$

$$= \frac{2\lambda}{\sqrt{x^2+y^2}} \langle \cos\varphi, \sin\varphi, 0 \rangle \quad \text{(in Cartesian)}$$

* If $r < a$:

$$Q_{enc} = \frac{\lambda r^2 L}{a^2} \quad \text{since wire is uniformly charged}$$

$$\hookrightarrow \oint \vec{E} \cdot d\vec{a} = 4\pi \frac{\lambda r^2 L}{a^2}$$

$$E \cdot 2\pi L r = \frac{4\pi L \lambda r^2}{a^2}$$

$$\vec{E} = \frac{2\lambda r}{a^2}\hat{r}$$

$$= \frac{2\lambda\sqrt{x^2+y^2}}{a^2} \langle \cos\varphi, \sin\varphi, 0 \rangle$$

$$\Rightarrow E = \begin{cases} \dfrac{2\lambda\sqrt{x^2+y^2}}{a^2} \langle \cos\varphi, \sin\varphi, 0 \rangle & \text{if } r < a \\ \dfrac{2\lambda}{\sqrt{x^2+y^2}} \langle \cos\varphi, \sin\varphi, 0 \rangle & r > a \end{cases}$$

#5 (cont.)

b) i) For a boost of velocity v in $\hat{z}$ direction:

$$L = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$\Rightarrow x'^{\sigma} = L^{\sigma}{}_{\mu} x^{\mu}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma ct - \beta\gamma z \\ x \\ y \\ -\beta\gamma ct + \gamma z \end{bmatrix}$$

ii) $$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\cos\varphi & -\sin\varphi & 0 \\ \cos\varphi & 0 & 0 & 0 \\ \sin\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{cases} \dfrac{2\lambda\sqrt{x^2+y^2}}{a^2} & r<a \\ \dfrac{2\lambda}{\sqrt{x^2+y^2}} & r>a \end{cases}$$

iii) $$F'^{\alpha\beta} = L^{\alpha}{}_{\mu} L^{\beta}{}_{\nu} F^{\mu\nu}$$

$$= L F L^T$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -\cos\varphi & -\sin\varphi & 0 \\ \cos\varphi & 0 & 0 & 0 \\ \sin\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma\cos\varphi & -\gamma\sin\varphi & 0 \\ \cos\varphi & 0 & 0 & 0 \\ \sin\varphi & 0 & 0 & 0 \\ 0 & \beta\gamma\cos\varphi & \beta\gamma\sin\varphi & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

#5 (cont.)

b) iii)
$$F'^{\alpha\beta} = \begin{bmatrix} 0 & -\gamma\cos\varphi & -\gamma\sin\varphi & 0 \\ \gamma\cos\varphi & 0 & 0 & -\beta\gamma\cos\varphi \\ \gamma\sin\varphi & 0 & 0 & -\beta\gamma\sin\varphi \\ 0 & \beta\gamma\cos\varphi & \beta\gamma\sin\varphi & 0 \end{bmatrix}$$

iv)
$$E' = \begin{cases} \dfrac{\gamma 2\lambda\sqrt{x^2+y^2}}{a^2} \langle \cos\varphi, \sin\varphi, 0 \rangle & r < a \\ \dfrac{\gamma 2\lambda}{\sqrt{x^2+y^2}} \langle \cos\varphi, \sin\varphi, 0 \rangle & r > a \end{cases}$$

$$B' = \begin{cases} \dfrac{\beta\gamma 2\lambda\sqrt{x^2+y^2}}{a^2} \langle \cos\varphi, -\sin\varphi, 0 \rangle & r < a \\ \dfrac{\beta\gamma 2\lambda}{\sqrt{x^2+y^2}} \langle \cos\varphi, -\sin\varphi, 0 \rangle & r > a \end{cases}$$