10

Problem 4: Interaction forces and energies

Consider a spherical shell of radius a and charge q uniformly distributed over its surface.

(a) Find the electric field everywhere in space. [1 point]

(b) Calculate the energy of the configuration. [3 points]

(c) Consider now the case in which we add a second spherical shell of radius b ($b > a$) and total charge $-q$ uniformly distributed over its surface. Calculate the energy of the configuration if the two spherical shells are concentric. [3 points]

(d) Does the superposition principle apply to the energy? That is, is the energy of the concentric spherical shells equal to the sum of the energy of two spherical shells taken individually? Justify your answer. [3 points]

Aug 2018

E+M #4

SI

a) For a spherical shell of radius a and total charge q:

$$E = \begin{cases} 0 & r < a \\ \frac{q}{4\pi\epsilon_0 r^2}\hat{r} & r \geq a \end{cases} \quad \text{(from Gauss Law)}$$

$$\int E \cdot da = \frac{q_{enc}}{\epsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{q_{enc}}{\epsilon_0}$$

$$E = \frac{q_{enc}}{4\pi\epsilon_0 r^2}\hat{r}$$

b) The energy stored in the electric field is:

$$W = \frac{\epsilon_0}{2}\int |E|^2 d^3r$$

$$= \frac{\epsilon_0}{2}\int \frac{q^2}{16\pi^2\epsilon_0^2 r^4} r^2 dr d\Omega$$

$$= \frac{q^2}{32\pi^2\epsilon_0}\int \frac{1}{r^2} dr d\Omega$$

$$= \frac{q^2}{8\pi\epsilon_0}\int_a^\infty \frac{1}{r^2} dr$$

$$= \frac{q^2}{8\pi\epsilon_0}\left(\frac{-1}{r}\right)\Big|_a^\infty$$

$$= \frac{q^2}{8\pi\epsilon_0 a}$$

c) If we now add a second shell of charge $-q$ at radius b, $(b > a)$, our total electric field becomes (again by Gauss' Law):

$$\vec{E} = \begin{cases} 0 & r < a & (q_{enc} = 0) \\ \frac{q}{4\pi\epsilon_0 r^2} & a \leq r \leq b & \\ 0 & r \geq b & (q_{enc} = 0) \end{cases}$$

#4 (cont.)

c) We calculate the energy the same way as in part b:

$$W = \frac{\epsilon_0}{2}\int |E|^2 d^3r$$

$$= \frac{\epsilon_0}{2}\int \frac{q^2}{16\pi^2\epsilon_0^2 r^4} r^2 dr\, d\Omega$$

$$= \frac{q^2}{32\pi^2\epsilon_0}\int \frac{1}{r^2} dr\, d\Omega$$

$$= \frac{q^2}{8\pi\epsilon_0}\int_a^b \frac{1}{r^2} dr$$

$$= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

d) In general, the superposition principle does not apply to energy. Since we know the superposition principle applies to fields ($\vec{E}_T = \vec{E}_1 + \vec{E}_2$), we can see:

$$W = \frac{\epsilon_0}{2}\int |E|^2 d^3r$$

$$= \frac{\epsilon_0}{2}\int (E_1 + E_2)^2 d^3r \stackrel{?}{=} \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2)\, d^3r$$

$$= \frac{\epsilon}{2}\int (E_1^2 + 2E_1E_2 + E_2^2)\, d^3r \neq \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2)\, d^3r$$