2. (a) (3 pts) From Maxwell's Equations, derive the wave equation for $\mathbf{E}$ with no sources ($\rho = 0, \mathbf{J} = 0$) in a homogeneous, isotropic, linear medium of permittivity ϵ and permeability μ.

(b) (1 pts) Show that if $\mathbf{E} = E(t, z)\,\hat{\mathbf{y}}$, the wave equation reduces to

$$\begin{aligned} \frac{\partial^2 E}{\partial z^2} &= \epsilon\mu \frac{\partial^2 E}{\partial t^2}, && \text{in SI units} \\ \frac{\partial^2 E}{\partial z^2} &= \frac{\epsilon\mu}{c^2}\frac{\partial^2 E}{\partial t^2}. && \text{in Gaussian units} \end{aligned}$$

(c) (4 pts) By introducing the change of variables

$$\begin{aligned} \xi &= t + \sqrt{\epsilon\mu}\, z, && \text{in SI units} \\ \xi &= ct + \sqrt{\epsilon\mu}\, z, && \text{in Gaussian units} \\ \eta &= t - \sqrt{\epsilon\mu}\, z, && \text{in SI units} \\ \eta &= ct - \sqrt{\epsilon\mu}\, z, && \text{in Gaussian units} \end{aligned}$$

show that the wave equation assumes a form that is easily integrated.

(d) (2 pts) Integrate the equation to obtain

$$E(z, t) = E_1(\xi) + E_2(\eta),$$

where E_l and E_2 are arbitrary functions.

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E+M #2

Gaussian

a) Maxwells eqns: $\nabla \cdot D = \cancel{4\pi\rho}^{\,0}$ $\quad \nabla \cdot B = 0$

$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \quad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \cancel{\vec{J}}^{\,0}$

To derive wave eqn $\left(-\nabla^2 E = \frac{-\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}E \right)$

$$\nabla \times (\nabla \times E) + \frac{1}{c}\nabla \times \frac{\partial B}{\partial t} = 0$$

$$\nabla(\cancel{\nabla \cdot E}^{\,0}) - \nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mu H) = 0$$

$$-\nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mu}{c}\frac{\partial D}{\partial t} \right) = 0$$

$$-\nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mu\epsilon}{c}\frac{\partial E}{\partial t} \right)$$

$$-\nabla^2 E + \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t} = 0 \checkmark$$

b) If $\vec{E} = E(t,z)\hat{y}$

$$-\nabla^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

$$\hookrightarrow -\nabla^2 E = \frac{\partial^2}{\partial z^2}\vec{E}$$

$$-\frac{\partial^2 E}{\partial z^2} + \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \Rightarrow \frac{\partial^2 E}{\partial z^2} = \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2}$$

c) Introducing the following variable substitutions: $\zeta = ct + \sqrt{\epsilon\mu}\,z$

$\eta = ct - \sqrt{\mu\epsilon}\,z$

$$\frac{\partial}{\partial z} = \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta}$$

$$= \sqrt{\mu\epsilon}\frac{\partial}{\partial \zeta} - \sqrt{\mu\epsilon}\frac{\partial}{\partial \eta} = \sqrt{\mu\epsilon}\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \eta}\right)$$

$$\frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial \zeta}{\partial t}\frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}$$

$$= c\frac{\partial}{\partial \zeta} + c\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta}$$

#2 (cont.)

c) $\Rightarrow \; -\frac{\partial^2}{\partial z^2} = \mu\epsilon\left(-\frac{\partial^2}{\partial \xi} + 2\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial^2}{\partial \eta^2}\right)$

$\frac{\mu\epsilon}{c^2}\frac{\partial}{\partial t^2} = \mu\epsilon\left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial^2}{\partial \eta^2}\right)$

$\Rightarrow \; -\frac{\partial^2 E}{\partial z^2} + \frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2} = 4\mu\epsilon\frac{\partial^2}{\partial \xi \partial \eta} E = 0$

d)