Problem 6: Stress tensor

The manifestly covariant form of the electromagnetic field Lagrangian is given by $\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}J_\mu A^\mu$ in Gaussian units where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

? (a) For $\mathcal{L} = \mathcal{L}(A^\nu, \partial^\mu A^\nu)$, write down the Euler-Lagrange equations. [1 point]

(b) Apply these to derive the covariant form of the inhomogeneous Maxwell equations. [1 point] $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$

√ (c) From the Maxwell equations, show that the equation of continuity $\partial_\mu J^\mu = 0$ is satisfied. [2 points]

(d) List at least three important steps in deriving the symmetrized electromagnetic stress-energy tensor $\Theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}f_{\mu\nu}F^{\mu\nu}\right)$. [2 points]

(e) Express $\Theta^{\alpha\beta}$ in matrix form in terms of the EM energy density u, momentum density $c\vec{g}$ and the Maxwell stress tensor $T^M_{ij} = \frac{1}{4\pi}\left(E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}(\vec{E}^2 + \vec{B}^2)\right)$. [2 points]

(f) Express the zeroth component of the conservation equation $\partial_\mu \Theta^{\mu\nu} = 0$ in terms of u and the Poynting vector $\vec{S} = c^2\vec{g}$. What is the significance of this equation? [2 points]

$c^2 E \times B$

$\partial_\mu \sum_\nu \frac{\partial F^{\mu\nu}}{\partial x_\nu} = \partial_\mu \mu_0 J^\mu$

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E+M #6

a) General Euler-Lagrang eqn: $\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$

$$\Rightarrow \quad \frac{\partial \mathcal{L}(A^\nu, \partial^\mu A^\nu)}{\partial A^\nu} = \frac{d}{dt}\frac{\partial \mathcal{L}(A^\nu, \partial^\mu A^\nu)}{\partial(\partial^\mu A^\nu)}$$

b) The covariant form of the inhomogeneous Maxwell Eqns is: $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$

Given: $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$