6. Maxwell's equations in 4 dimensions

(a) {2 pts} Write the Maxwell equations in the absence of polarizable materials using 4-vector notation, making use of the field strength tensor $F_{\mu\nu}$.

(b) {4 pts} Show that the equations of part (a) reduce to the usual form of Maxwell's equations in 3-vector notation.

(c) {2 pts} The Lagrangian density of the EM field is given by

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}, \qquad (SI)$$

or

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}. \qquad (Gaussian)$$

Recall that all repeated Greek indices are summed over 4-dimensions (1 time and 3 space). Show that the Lagrangian density is invariant under a gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha(x)$, where α is an arbitrary function of spacetime $x \equiv (ct, \vec{x})$.

(d) {2 pts} If we add an interaction term $\mathcal{L} \to \mathcal{L} + \Delta\mathcal{L}$ where

$$\Delta\mathcal{L} = j^\mu A_\mu, \qquad (SI)$$

or

$$\Delta\mathcal{L} = \frac{1}{c} j^\mu A_\mu, \qquad (Gaussian)$$

to the Lagrangian– where j^μ is some spatially bounded and conserved 4-current density– how does the action $I \equiv \int \mathcal{L} d^4 r$ change under a gauge transformation and do the resulting equations of motion change?

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E+M #6

Gaussian

a) In free space, 4-D Maxwell equations in tensor form are:

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \qquad (\text{Dual Tensor } (B \to E,\ E \to B))$$

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

b) To show these reduce to the 3 vector form:

$$\nabla \cdot E = 4\pi\rho \qquad \nabla \cdot B = 0$$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$$

It is easiest to show with the inhomogeneous equations first

$$J^\beta = \begin{bmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{bmatrix} \qquad F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

* If $\beta = 0$:

$$\partial_\alpha F^{\alpha 0} = \frac{4\pi}{c} J^0$$

$$\partial_0 F^{00} + \partial_i F^{i0} = \frac{4\pi}{c}(c\rho)$$

$$0 + \nabla \cdot E = 4\pi\rho \checkmark$$

If $\beta = i$:
($i \in \{1,2,3\}$)

$$\partial_\alpha F^{\alpha i} = \frac{4\pi}{c} J^i$$

$$\partial_0 F^{0i} + \partial_j F^{ji} = \frac{4\pi}{c} J^i$$

$$\frac{1}{c}\frac{\partial E}{\partial t} + \nabla \times B = \frac{4\pi}{c}\vec{J} \checkmark$$

* Proceeding to the homogeneous eqns:

$$\mathcal{F}^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_x & E_y \\ B_y & E_x & 0 & E_z \\ B_z & E_y & E_z & 0 \end{bmatrix}$$

#6 (cont.)

b) * If $\beta = 0$: $\partial_\alpha F^{\alpha 0} = 0$

$$\partial_0 F^{00} + \partial_i F^{i0} = 0$$

$$0 + \nabla \cdot B = 0 \checkmark$$

* if $\beta = i,\ i \in \{1,2,3\}$: $\partial_\alpha F^{\alpha i} = 0$

$$\partial_0 F^{0i} + \partial_j F^{ji} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{B} - \nabla \times E = 0$$

$$-\frac{1}{c}\frac{\partial}{\partial t} B + \nabla \times E = 0 \checkmark$$

c) * Remember that $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $F^{\alpha\beta} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$

$$\Rightarrow \mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

* if $A_\mu \to A_\mu + \partial_\mu \alpha(x)$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} \left(\partial_\mu [A_\nu - \partial_\nu \alpha(x)] - \partial_\nu [A_\mu - \partial_\mu \alpha(x)]\right)\left(\partial_\mu [A_\nu - \partial_\nu \alpha(x)] - \partial_\nu [A_\mu - \partial_\mu \alpha(x)]\right)$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} \left(\partial_\mu A_\nu - \cancel{\partial_\mu \partial_\nu \alpha(x)} - \partial_\nu A_\mu + \cancel{\partial_\nu \partial_\mu \alpha(x)}\right)\left(\partial_\mu A_\nu + \cancel{\partial_\mu \partial_\nu \alpha(x)} - \partial_\nu A_\mu + \cancel{\partial_\nu \partial_\mu \alpha(x)}\right)$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\mu\nu} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \checkmark$$