Problem 1: Electrostatics

Consider a capacitor composed of two concentric spherical metal shells, the inner one with radius a and the outer one with radius b. The region between the spherical metal shells is filled with a linear dielectric with permittivity $\epsilon = \frac{k}{r^2}$. Place charge $+Q$ on the inner metallic shell and $-Q$ on the outer metallic shell.

1. Find the electric displacement $\vec{D}$ everywhere in space. [3 points]
2. Find the capacitance of the configuration. [3 points]
3. Calculate the bound charge densities in the linear dielectric and verify that the total net bound charge is zero. [4 points]

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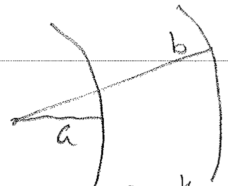
a)



$\nabla \cdot D = \frac{\rho_f}{\epsilon}$

$\epsilon = \frac{k}{r^2}$

* if $r < a$:

$\nabla \cdot D = \frac{\rho_f}{\epsilon}$

$\hookrightarrow \vec{D} = 0$ b/c $\rho_f = 0$

* if $a < r < b$:

$\int D \cdot d\vec{a} = \int \frac{\rho_f}{\epsilon} dV$

$D \cdot 4\pi r^2 = \frac{Q}{\epsilon}$

$D = \frac{Q}{4\pi \epsilon r^2} \hat{r}$

* if $r > b$:

$\int D \cdot d\vec{a} = \int \frac{\rho_f}{\epsilon} dV$

$\vec{D} = 0$ (total enclosed charge is 0)

b) $C = \frac{Q}{\Delta V}$ or $W = \frac{\epsilon_r}{2} CV^2$ $(D = \epsilon E)$

$W = \frac{1}{2} \int D \cdot E \, dV$

$= \frac{1}{2} \int \epsilon E^2 dV$

$= \frac{1}{2} \int \frac{Q^2}{16\pi^2 r^4} r^2 dr d\theta d\varphi$

$= \frac{Q^2}{32\pi^2} \int \frac{1}{r^2} dr d\theta d\varphi$

$= \frac{Q^2}{8\pi} \left. \frac{1}{r} \right|_a^b = \frac{Q^2}{8\pi} \left(\frac{1}{b} - \frac{1}{a} \right)$

#1 (cont.)

b) $\frac{Q^2}{8\pi}\left(\frac{1}{b}-\frac{1}{a}\right) = \frac{\epsilon_r}{2} C V^2$

???

c) $\rho_b = -\nabla \cdot \vec{P}$ $\quad \sigma_b = \vec{P} \cdot \hat{n}$ $\quad \vec{P} = \frac{\chi_e}{\epsilon_r} D$, $\epsilon_r = \frac{\epsilon}{\epsilon_0}$

$$\rho_b = -\nabla \cdot \frac{\chi_e}{\epsilon_r} D = -\frac{\chi_e}{\epsilon_r} Q = \frac{-\chi_e Q r^2}{k}$$

$$\sigma_b = \frac{\chi_e}{\epsilon_r} = \frac{\chi_e r^2}{k} D \cdot \hat{n} = \frac{\chi_e r^2}{k} \frac{Q}{4\pi\epsilon_r r^2} = \frac{\chi_e Q}{4\pi\epsilon k}$$

$$Q_{b,\,tot} = \int \rho_b \, dV + \oint \sigma_b \, d\vec{a} = -\frac{\chi_e}{k} Q \int r^4 \, dr \, d\varphi \, d\theta + \int \frac{\chi_e Q}{4\pi\epsilon k} \, da = \frac{-4\pi\chi_e Q}{k} \frac{1}{5} r^5 \Big|_a^b$$

* Parts b+c mostly wrong