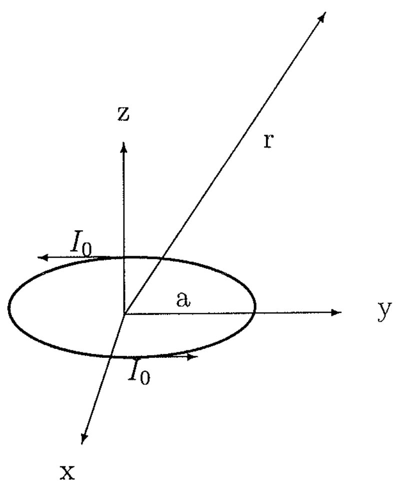
4. (a) [2 pts] Write down any vector potential that produces the uniform magnetic induction

$$\mathbf{B} = B_0 \, \hat{\mathbf{z}}.$$

(b) [4 pts] What is the magnetic induction $\mathbf{B}$ and an associated vector potential $\mathbf{A}$ ($\mathbf{B} = \nabla \times \mathbf{A}$) produced by a very long wire located on the z-axis and carrying a current I_0 in the +z direction?

(c) [4 pts] A small circular loop of wire of radius a, centered at the origin and lying in the $z = 0$ plane, carries a current I_0 as shown in the figure. Derive an approximate expression for the vector potential at large distances ($r >> a$) from the loop. Recall that

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r} \left\{ 1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \mathcal{O}\left(\frac{r'}{r}\right)^2 \right\}$$



Aug 2013

E+M #4

a) We know that $\nabla \times A = B$, therefore if $B_z = B_0 \hat{z}$

$$B_z = \partial_x A_y - \partial_y A_x \qquad \Rightarrow \text{ set } A_z = 0$$

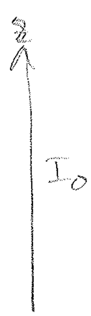
$$= \frac{\partial}{\partial x}\frac{B_0}{2}x - \frac{\partial}{\partial y}\frac{-B_0}{2}y$$

$$= \frac{B_0}{2} + \frac{B_0}{2}$$

$$= B_0 \checkmark$$

$$\Rightarrow \quad A = \left\langle \frac{-B_0}{2}y,\ \frac{B_0}{2}x,\ 0 \right\rangle$$

b) * We first solve for B



$$\int B \cdot dl = \frac{4\pi}{c} I_0$$

$$B \cdot 2\pi r = \frac{4\pi}{c} I_0$$

$$B = \frac{2I_0}{cr}\hat{\varphi}$$

$$= \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{A} = \left\langle \frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z},\ \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho},\ \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(A_\varphi) - \frac{\partial A_\rho}{\partial \varphi}\right) \right\rangle$$

↳ set $A_\varphi = 0$; $A_\rho A_z$ has no φ dependence

$$= \frac{2I_0}{c}\left\langle \frac{1}{\rho}\cancel{\partial_\varphi A_z},\ \partial_z A_\rho - \partial_\rho A_z,\ \frac{1}{\rho}\cancel{\partial_\varphi A_\rho}^{\,0} \right\rangle$$

$$= \frac{2I_0}{c}\langle 0,\ \partial_z A_\rho - \partial_\rho A_z,\ 0 \rangle$$

↳ $\frac{1}{r} = \partial_z A_\rho - \partial_\rho A_z$

*since an arbitrary A works, set $A_\rho = 0$,

$$\int \frac{1}{r} d\rho = \int \partial_\rho A_z \, d\rho$$

$$\ln(r) = A_z$$

$$\Rightarrow \quad A = \frac{2I_0}{c}\langle 0, 0, \ln(r) \rangle$$