Problem 2: Magnetostatics

An infinitely long circular cylinder of radius R (with its axis along the z-direction) carries a magnetization $\vec{M} = ks^2\hat{\phi}$, where k is a constant, s is the distance from the axis, and $\hat{\phi}$ is the azimuthal unit vector.

1. Find the bound currend densities ($\vec{K}_b$ and $\vec{J}_b$). [2 points]
2. Verify that the total bound current in the cylinder is zero. [2 points]
3. Find the magnetic field $\vec{B}$, due to $\vec{M}$, inside and outside the cylinder. [3 points]
4. Verify the boundary conditions for $\vec{B}$ at the interface ($s = R$). [3 points]

Jan 2018 $\underline{E+M \#2}$ Gaussian

a) $\vec{K}_b = \hat{M} \times \hat{n}$ $\qquad \vec{M} = \langle 0, ks^2, 0 \rangle$

$\vec{J}_b = -\nabla \times M$

$$J_b = \langle \tfrac{1}{s}\partial_\varphi M_z - \partial_z M_\varphi,\ \partial_z M_s - \partial_s M_z,\ \tfrac{1}{s}\partial_s(sM_\varphi) - \partial_\varphi M_s \rangle$$

$$= \langle \tfrac{1}{s}\partial_\varphi(0) - \partial_z(ks^2),\ \partial_z(0) - \partial_s(0),\ \tfrac{1}{s}\partial_s(sks^2) - \partial_\varphi(0) \rangle$$

$$= \langle 0, 0, 3ks \rangle$$

$$\vec{K}_B = \begin{vmatrix} \hat{s} & \hat{\varphi} & \hat{z} \\ 0 & ks^2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, 0, -ks^2 \rangle$$

b) $$0 = \oint k_B \cdot d\vec{a} + \int J_B \, dV$$

$$= \int -ks^2 \cdot d\vec{a} + \int_0^s \int_0^{2\pi} \int_0^L 3ks \; s\,ds\,d\varphi\,dz$$

$$= -ks^2 \cdot 2\pi s L + ks^3\Big|_0^s 2\pi L$$

$$= -2\pi L k s^3 + 2\pi L k s^3$$

$$= 0 \checkmark$$

c) $$\nabla \times B = \frac{4\pi}{c} J_f + J_b$$

$$\int B \cdot dl = \frac{4\pi}{c} \int J_b \cdot d\vec{a}$$

$*$ if $s < R'$

$$B \cdot 2\pi s = \frac{4\pi}{c} \int 3ks \; s\,ds\,d\varphi$$

$$2\pi s B = \frac{8\pi^2}{c} 3k \cdot \frac{1}{3} s^3 \Big|_0^s$$

$$2\pi s B = \frac{8\pi^2 k s^3}{c} \quad \Rightarrow \quad B = \frac{4\pi k s^2}{c} \hat{\varphi}$$

#2 (cont)

c) * if $s > R$,

$\vec{B} = 0$ b/c total bound current is 0, as is free current

d) Our boundary conditions are:

$$B_1^{\parallel} - B_2^{\parallel} = \frac{4\pi}{c}\vec{K}$$

$$B_1^{\perp} - B_2^{\perp} = 0$$

* Defining region 1 as inside the cylinder and region 2 as outside

$$B_1^{\perp} - B_2^{\perp} = 0$$

$$0 - 0 = 0 \checkmark$$

$$B_1^{\parallel} - B_2^{\parallel} = \frac{4\pi}{c}\vec{K}$$

$$\frac{4\pi}{c}ks^2 - 0 = \frac{4\pi}{c}ks^2 \checkmark$$