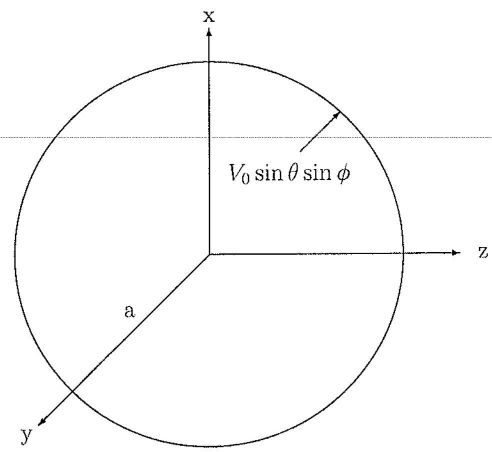

5. Assume that in spherical polar coordinates (r, θ, ϕ), the potential on the surface of a sphere of radius a, centered on the origin, is known to be $V(\theta, \phi)$.

   (a) [2 pts] If the space inside the sphere is empty give an expression for the potential $\Phi(r, \theta, \phi)$ everywhere inside as an expansion in spherical harmonics with arbitrary constants. If you knew the potential $V(\theta, \phi)$ on the surface how would you evaluate the constants in your expansion?

   (b) [2 pts] If the space outside the sphere is empty give an expression for the potential $\Phi(r, \theta, \phi)$ everywhere outside as an expansion in spherical harmonics with arbitrary constants. If you knew the potential $V(\theta, \phi)$ on the surface how would you evaluate the constants in your expansion?

   (c) [6 pts] If $V(\theta, \phi) = V_0 \sin\theta \sin\phi$ give exact expressions for $\Phi(r, \theta, \phi)$ inside and outside the sphere.

   The spherical harmonics are ortho-normal on the sphere and for $\ell = 1$

$$
\begin{aligned}
Y_1^{-1}(\theta, \phi) &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}, \\
Y_1^{0}(\theta, \phi) &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\
Y_1^{1}(\theta, \phi) &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}.
\end{aligned}
$$

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E+M #5

a) The general solution to La Place's eqn, expanded in spherical coordinates is:

$$\Phi(r,\theta,\varphi) = \sum_{l} \sum_{m=-l}^{l} \left(A_{lm} r^{l} + \frac{B_{lm}}{r^{l+1}}\right) Y_l^m(\theta,\varphi)$$

Since we are in empty space inside the sphere, $B_{lm}=0$ since $\Phi(0,\theta,\varphi)=0$

$$\hookrightarrow \Phi(r,\theta,\varphi) = \sum_{l} \sum_{m=-l}^{l} A_{lm} r^{l} Y_l^m(\theta,\varphi)$$

We can then determine the values of A_{lm} by the orthogonality of the spherical harmonics:

$$V(\theta,\varphi) = \sum_{l} \sum_{m=-l}^{l} A_{lm} a^{l} Y_l^M(\theta,\varphi)$$

$$\int_0^{2\pi}\int_0^{\pi} V(\theta,\varphi) Y_l^{*m}(\theta,\varphi) = A_{lm} a^{l}$$

$$\hookrightarrow A_{lm} = \iint V(\theta,\varphi) Y_l^{*m}(\theta,\varphi)\, d\Omega \cdot \frac{1}{a^{l}}$$

b) We proceed similarly to part a, except here $A_{lm}=0$ b/c $\Phi \to 0$ at $r\to\infty$

$$\hookrightarrow \Phi(r,\theta,\varphi) = \sum_{l} \sum_{m=-l}^{l} B_{lm} \frac{1}{r^{l+1}} Y_l^M(\theta,\varphi)$$

Again, similar to above, we use orthogonality of spherical harmonics to determine B_{lm}

$$\hookrightarrow B_{lm} = a^{l+1} \iint V(\theta,\varphi) Y_l^{*m}(\theta,\varphi)\, d\Omega$$

c) If we now specify that $\Phi(a,\theta,\varphi) = V_0 \sin\theta \sin\varphi$

* Rewriting Φ in terms of spherical harmonics:

$$\sin\theta \sin\varphi = \sin\theta \left(\frac{1}{2i} e^{i\varphi} - e^{-i\varphi}\right)$$

$$= \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1 + Y_1^{-1}\right)$$

#5 (cont.)

c) * For $r < a$

$$A_{\ell m} = \iint V(\theta,\varphi)\, Y_\ell^{*m}\, d\Omega \cdot \frac{1}{a^\ell}$$

$$= \frac{1}{a} V(r) \iint \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1 + Y_1^{-1}\right) Y_\ell^{*m}\, d\Omega$$

$$= \frac{1}{a} V(r) \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1 + Y_1^{-1}\right)$$

$$= \frac{1}{a} V(r) \sin\theta \sin\varphi$$

$$\Rightarrow \Phi(r,\theta,\varphi) = \frac{r}{a} V(r) \sin\theta \sin\varphi$$

* For $r \geq a$

$$B_{\ell m} = \iint a^{\ell+1}\, V(\theta,\varphi)\, Y_\ell^{*m}\, d\Omega$$

$$= a^2 V(r) \iint \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1 + Y_1^{-1}\right) Y_\ell^{*m}\, d\Omega$$

$$= a^2 V(r) \sin\theta \sin\varphi$$

$$\Rightarrow \Phi(r,\theta,\varphi) = \left(\frac{a}{r}\right)^2 V(r) \sin\theta \sin\varphi$$