1. Dielectric Sphere

   A dielectric sphere of radius R is polarized so that $\mathbf{P} = (K/r)\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ is the unit radial vector. Assume the sphere is in an empty vacuum and that the sphere's dielectric material is linear and isotropic, calculate

   (a) (3 pts) the volume and the surface densities of bound charge,

   (b) (2 pts) the volume density of free charge,

   (c) (2 pts) the electric field inside the sphere,

   (d) (3 pts) the electric field outside the sphere.

   Your answers should be given in terms of $K, \chi_E, \epsilon_0, \epsilon$, and/or ϵ_r. Recall that for linear isotropic materials:

   In SI units,

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \epsilon_0\chi_E\mathbf{E}$$

$$\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = 1 + \chi_E$$

   In Gaussian units,

$$\mathbf{D} = \epsilon\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}$$

$$\mathbf{P} = \chi_E\mathbf{E}$$

$$\epsilon = 1 + 4\pi\chi_E = \epsilon_r$$

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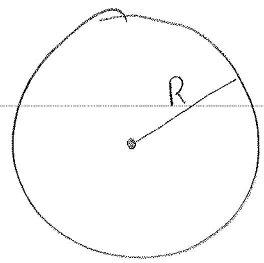
E + M #1



$\vec{P} = \frac{K}{r}\hat{r}$

Given a polarized, dielectric sphere (dielectric is linear + isotropic) in a vacuum:

a) Our surface charge density is:

$$\sigma_b = \vec{P}\cdot\hat{n}\,\Big|_{surface}$$
$$= \frac{K}{r}\hat{r}\cdot\hat{r}\,\Big|_{r=R}$$
$$= \frac{K}{R}$$

The volume bound charge density is:

$$\rho_b = -\nabla\cdot\vec{P}$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{K}{r}\right)$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(kr)$$
$$= -\frac{k}{r^2}$$

b) We can determine the volume density of free charge by:

$$-\nabla\cdot E = \frac{\rho_f}{\epsilon_0}$$

⇒ To determine the electric field inside the material, we use

$$\vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P}$$
$$(\epsilon - \epsilon_0)E = \vec{P}$$
$$\vec{E} = \frac{1}{\epsilon-\epsilon_0}\vec{P}$$
$$= \frac{K}{r(\epsilon-\epsilon_0)}\hat{r}$$

$\boxed{\text{Work for part C HERE}}$

#1 (cont.)

b) $$-\nabla \cdot E = \frac{\rho_f}{\epsilon_0}$$

$$-\nabla \cdot \frac{K}{(\epsilon-\epsilon_0)r}\hat{r} = \frac{\rho_f}{\epsilon_0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Kr}{\epsilon-\epsilon_0}\right) = \frac{\rho_f}{\epsilon_0}$$

$$\frac{K}{r^2(\epsilon-\epsilon_0)} = \frac{\rho_f}{\epsilon_0}$$

$$\hookrightarrow \rho_f = \frac{K\epsilon_0}{r^2(\epsilon-\epsilon_0)}$$

$$= \frac{K}{r^2(-1+\epsilon/\epsilon_0)}$$

$$= \frac{K}{r^2(\epsilon_r-1)}$$

d) From Gauss' Law, we know

$$\oint_S E \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

To get q_{enc}:

$$q_{enc} = \int \sigma_b\, da + \int \rho_f\, dV + \int \rho_b\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi} R^2 \frac{K}{R} \sin^2\theta\, dr\, d\theta\, d\varphi + \int_0^R\int_0^{2\pi}\int_0^{\pi} \frac{K}{r^2(\epsilon_r-1)} r^2 \sin\theta\, dr\, d\theta\, d\varphi + \int_0^R\int_0^{2\pi}\int_0^{\pi} \frac{-K}{r^2} r^2 \sin\theta\, d\theta\, d\varphi\, dr$$

$$= KR\cdot 4\pi + \frac{KR}{\epsilon_r-1}\cdot 4\pi - 4\pi KR$$

$$= \frac{4\pi KR}{\epsilon_r-1}$$

$$\oint E\cdot d\vec{a} = \frac{4\pi KR}{\epsilon-\epsilon_0}$$

$$4\pi r^2 E = \frac{4\pi KR}{\epsilon-\epsilon_0}$$

$$E = \frac{KR}{R(\epsilon-\epsilon_0)}\cdot\frac{1}{r^2}\hat{r}$$