3-5

Problem 5: Special relativity/Compton effect

A photon with four-momentum k and energy $E = \hbar\omega$ is incident in the z direction upon an electron of mass m and four-momentum p which is at rest. The photon recoils with four-momentum k' at an angle θ and frequency ω' after scattering whilst the recoil electron has four-momentum p'. You may adopt units with $\hbar = c = 1$ if you wish.

✓(a) Write out a diagram depicting the initial and final states of the particles. [1 point]

✓(b) What are their 4-momentum vectors? (You may orient the y-axis such that all scattering takes place in the $x - z$ plane) [1 point]

(c) Derive an expression relating the final state photon wavelength λ' to λ and its angle of scatter. [4 points]

✓(d) Is the final state photon red-shifted or blue-shifted? Why? [1 point]

(e) Plot the final state photon wavelength as a function of scattering angle. [1 point]

(f) How is the final state photon energy ω' related to ω and its angle of scatter? [2 points]

$$E = \hbar\omega$$
$$= \frac{\hbar c}{\lambda}$$

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E+M #5

a) Initial   Final

b) *If oriented so all scattering occurs in x-z plane, the 4-momentum vectors become:

$$P_{i,\gamma} = \langle \hbar\omega/c,\ \hbar\omega v_x/c,\ \hbar\omega v_y/c,\ \hbar\omega v_z/c \rangle$$

$$P_{f,\gamma} = \langle \hbar\omega'/c,\ \hbar\omega' v_x/c^2,\ \hbar\omega' v_y/c^2,\ \hbar\omega' v_z/c^2 \rangle$$

> Note: $v_x^2 + v_y^2 + v_z^2 = c^2$

*Note: 4-momentum of photon is null vector, therefore $|p|^2 = 0$

$$\left(\frac{\hbar\omega}{c}\right)^2 - \left[\left(\frac{\hbar\omega v_x}{c^2}\right)^2 + \left(\frac{\hbar\omega v_y}{c^2}\right)^2 + \left(\frac{\hbar\omega v_z}{c^2}\right)^2\right]c^2 = 0 \checkmark$$

$$P_{i,e^-} = \langle m_e c\ 0,\ 0,\ 0 \rangle$$

$$P_{f,e^-} = \langle m_e c,\ v_x,\ v_y,\ v_z \rangle$$

c) *We can derive the scattering formula by the applications of energy + momentum conservation

① Energy Conservation

$$E_{i,\gamma} + E_{i,e^-} = E_{f,\gamma} + E_{f,e^-}$$

$$\hbar\omega + m_e c^2 = \hbar\omega' + (m_e^2 c^4 + P_{f,e^-}^2 c^2)^{1/2}$$

② Momentum Conservation

$$\vec{P}_{i,\gamma} + \vec{P}_{i,e^-} = \vec{P}_{f,\gamma} + \vec{P}_{f,e^-}$$

$$\hookrightarrow \vec{P}_{f,e^-} = \vec{P}_{i,\gamma} - \vec{P}_{f,\gamma}$$

*Making use of $P_\alpha P^\alpha$ being invariant

$$P_{f,e^-}^2 = P_{i,\gamma}^2 + P_{f,\gamma}^2 - 2 P_{i,\gamma} P_{f,\gamma} \cos\theta_1$$

#5 (cont.)

c) $$p_{f,e^-}^2 c^2 = p_{\gamma,i}^2 c^2 + p_{f,\gamma}^2 c^2 - 2c^2 p_{i,\gamma} p_{f,\gamma} \cos\theta_1$$

$$= (\hbar\omega)^2 + (\hbar\omega')^2 - 2(\hbar\omega)(\hbar\omega')\cos\theta_1$$

*if we manipulate our energy conservation equation, we see:

$$p_{f,e^-}^2 c^2 = [\hbar\omega - \hbar\omega' + m_e c^2]^2 - m_e^2 c^4$$

$$[\hbar\omega - \hbar\omega' + m_e c^2]^2 - m_e^2 c^4 = (\hbar\omega)^2 + (\hbar\omega')^2 - 2(\hbar\omega)(\hbar\omega')\cos\theta_1$$

$$(\cancel{\hbar\omega})^2 + (\cancel{\hbar\omega'})^2 + \cancel{m_e^2 c^4} + 2\hbar\omega m_e c^2 - 2\hbar\omega' m_e c^2 - 2(\hbar\omega)(\hbar\omega') - \cancel{m_e^2 c^4} = (\cancel{\hbar\omega})^2 + (\cancel{\hbar\omega'})^2 - 2(\hbar\omega)(\hbar\omega')\cos\theta_1$$

$$2(\hbar\omega)(\hbar\omega')[1-\cos\theta_1] = -2\hbar\omega' m_e c^2 + 2\hbar\omega m_e c^2$$

*dividing by $2\hbar\omega\omega' m_e c$ yields

$$\frac{\hbar}{m_e c}[1-\cos\theta_1] = \frac{c}{\omega'} - \frac{c}{\omega}$$

$$\frac{h}{m_e c}[1-\cos\theta_1] = \lambda' - \lambda$$

d) Red shifted, as photon loses energy (or wavelength gets longer by formula in part c)

f) $\frac{\hbar}{m_e c}[1-\cos\theta_1] = \frac{c}{\omega'} - \frac{c}{\omega}$

e)