1. (a) [4 pts] Use δ-functions to give volume charge densities ρ_f for each of the following:

   i. Give $\rho_f(\rho, \phi, z)$ in cylindrical-polar coordinates for a cylindrical shell of charge of a radius $\rho = b$, centered on the z-axis, which has a surface charge density $\sigma_f(\phi, z)$.

   ii. Give $\rho_f(\rho, \phi, z)$ in cylindrical-polar coordinates for a line of charge located at $\rho = b, \phi = \beta$ which has a charge/length $= \lambda_f(z)$.

   iii. Give $\rho_f(r, \theta, \phi)$ in spherical-polar coordinates for a spherical shell of charge of radius $r = a$, centered on the origin, which has a surface density $\sigma_f(\theta, \phi)$.

   (b) [2 pts] Use Gauss's law to compute the electric field caused by the cylindrically symmetric charge density

$$\rho_f(\rho) = \frac{\lambda_0}{\pi b^2} e^{-\rho^2/b^2}.$$

   (c) [2 pts] What charge density produces an electrostatic potential

$$\Phi(z) = V_0 e^{-z^2/a^2}.$$

   (d) [2 pts] What charge density produces an electrostatic potential

$$\Phi(z) = -E_0 \, |z|.$$

Aug 2015 # E+M #1 Gaussian

a) i) $\rho_f(\rho, \varphi, z) = \sigma_f(\varphi, z)\,\delta(\rho - b)$

ii) $\rho_f(\rho, \varphi, z) = \lambda_f(z)\,\delta(\rho - b)\,\delta(\varphi - \beta) \cdot \frac{1}{b}$

iii) $\rho_f(r, \theta, \varphi) = \sigma_f(\varphi, z)\,\delta(r - a)$

b) The integral form of Gauss' Law states

$$\int \vec{\nabla} \cdot \vec{E} = \int 4\pi \rho$$

$$\int \vec{E} \cdot d\vec{a} = 4\pi \int \frac{\lambda_0}{\pi b^2} \exp\left[-\frac{\rho^2}{b^2}\right] dV$$

$$E \cdot 2\pi \rho L = 4\pi \int \frac{\lambda_0}{\pi b^2} \exp\left[-\frac{\rho^2}{b^2}\right] \rho \, d\rho \, d\varphi \, dz$$

$$E \cdot 2\pi \rho L = 4\lambda_0 L \int \frac{\rho}{b^2} \exp\left[-\frac{\rho^2}{b^2}\right] d\rho \, d\varphi$$

$$E \cdot 2\pi \rho L = 8\pi \lambda_0 L \int \frac{\rho}{b^2} \exp\left[-\frac{\rho^2}{b^2}\right] d\rho$$

$$E \cdot 2\pi \rho L = 8\pi \lambda_0 L \left(1 - \exp\left[-\frac{\rho^2}{b^2}\right]\right)$$

$$E = \frac{4\lambda_0}{\rho}\left(1 - \exp\left[-\frac{\rho^2}{b^2}\right]\right) \hat{\rho}$$

c) $-\nabla^2 \Phi = 4\pi \rho$

* In Cartesian Coordinates $\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$

$$-\nabla^2\left(V_0 \exp\left[-z^2/a^2\right]\right) = 4\pi \rho$$

$$-\frac{d^2}{dz^2}\left(V_0 \exp\left[-z^2/a^2\right]\right) = 4\pi \rho$$

$$-\frac{d}{dz}\left(V_0\left[\frac{-2z}{a^2} \exp\left[z^2/a^2\right]\right)\right) = 4\pi \rho$$

$$-V_0\left(-\frac{2}{a^2} \exp\left[\frac{z^2}{a^2}\right] + \frac{4z^2}{a^4} \exp\left[z^2/a^2\right]\right) = 4\pi \rho$$

$$\hookrightarrow \rho = \frac{-V_0}{4\pi} \exp\left[z^2/a^2\right]\left(\frac{z^2 4}{a^4} - \frac{1}{a^2}\right)$$

#1 (cont.)

d) * Similiarly to part c

$$-\nabla^2 \Phi = 4\pi \rho$$

$$-\frac{d^2}{dz^2}(E_0|z|) = 4\pi\rho$$

$\hookrightarrow$ if $z \neq 0$, $\Phi = 0$

* Due to the discontinuity at $z=0$, we proceed via Gauss Law

$$-\nabla \Phi = E$$

$$\hookrightarrow \quad E = \begin{cases} E_0 \hat{z} & z>0 \\ -E_0 \hat{z} & z<0 \end{cases}$$

$$\int E \cdot d\vec{a} = 4\pi \rho$$

$$2E_0 A = 4\pi \rho$$

$$= 4\pi \sigma A$$

$$\hookrightarrow \quad \sigma = \frac{E_0}{2\pi}$$

$$\hookrightarrow \rho = \frac{E_0}{2\pi}\delta(z)$$