2. Gauge Transformation

   (a) (2 pts)
   Define the vector potential $\mathbf{A}$ and the scalar potential Φ using Maxwell's equations. (i.e. give their relationships to the $\mathbf{E}$ and $\mathbf{B}$ fields.)

   (b) (3 pts) Show that when $\mathbf{A}$ and Φ undergo the gauge transformations,

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \qquad (SI) \text{ and } (Gaussian)$$

$$\Phi' = \Phi - \frac{\partial\Lambda}{\partial t}, \qquad (SI)$$

   or

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}, \qquad (Gaussian)$$

   where Λ is an arbitrary scalar, $\mathbf{B}$ and $\mathbf{E}$ are unaffected.

   (c) Two gauges used in solid-state physics for static, uniform magnetic fields $\mathbf{B}$ (i.e., constant in direction, magnitude, and time) are the Landau gauge and the circular gauge. Examples for $\mathbf{B} = B_0\hat{z}$ of each gauge respectively are:

$$\mathbf{A} = (A_x, A_y, A_z) = (0, B_0\, x, 0)$$

   and

$$\mathbf{A}' = (A'_x, A'_y, A'_z) = (-B_0 y/2, B_0 x/2, 0),$$

   with

$$\Phi = 0,$$

   for both gauges.

      i. (2 pts) Show that $\mathbf{A}$ and $\mathbf{A}'$ with $\Phi = \Phi' = 0$ describe the same $\mathbf{E}$ and $\mathbf{B}$ fields.

      ii. (3 pts) Find the scalar function Λ that produces the gauge transformation from $\mathbf{A}$ to $\mathbf{A}'$ in part (c).

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E+M #2

Gaussian

a) $\vec{\nabla} \times \vec{A} = \vec{B}$

$-\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t} = \vec{E}$

b) * If we now define $\vec{A}' = \vec{A} + \nabla \Lambda$, $\Phi' = \Phi - \frac{\partial \Lambda}{c\,\partial t}$

$$\vec{\nabla} \times \vec{A}' = \nabla \times (\vec{A} + \nabla \Lambda)$$
$$= \nabla \times A + \cancelto{0}{\nabla \times (\nabla \Lambda)}$$
$$= \vec{\nabla} \times \vec{A} \checkmark$$

$$-\nabla \Phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} = -\nabla\left(\Phi - \frac{\partial \Lambda}{c\,\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A} + \nabla \Lambda\right)$$
$$= -\nabla \Phi + \cancel{\frac{1}{c}\frac{\partial}{\partial t}\nabla \Lambda} - \frac{1}{c}\frac{\partial A}{\partial t} - \cancel{\frac{1}{c}\frac{\partial}{\partial t}\nabla \Lambda}$$
$$= -\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t} \checkmark$$

c) i) * Working with A, Φ

$$\vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t}$$
$$= 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$= \left\langle -\frac{\partial}{\partial z} B_0 x,\ 0,\ \frac{\partial}{\partial x} B_0 x \right\rangle$$
$$= B_0 \hat{z}$$

* Working with A', Φ'

$$\vec{E} = -\nabla \Phi' - \frac{1}{c}\frac{\partial A'}{\partial t}$$
$$= 0$$

$$\vec{B} = \nabla \times A'$$
$$= \left\langle -\frac{\partial}{\partial z}\frac{B_0}{2}x,\ \frac{\partial}{\partial z}\frac{B_0}{2}y,\ \frac{\partial}{\partial x}\frac{B_0}{2}x + \frac{\partial}{\partial y}\frac{B_0}{2}y \right\rangle$$
$$= \langle 0,\ 0,\ B_0 \rangle$$

* Both sets match

ii) $A' = A + \nabla \Lambda \implies \nabla \Lambda = A' - A$

* Note: $\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

$\hookrightarrow \nabla \Lambda = \left\langle -\frac{B_0}{2} y,\ -\frac{B_0}{2} x,\ 0 \right\rangle$

$$\Lambda = -\frac{B_0}{2} xy + \text{Constant}$$