

Ch1 - Vector Analysis

\* Main ideas + important equations/demonstrations copied here in low detail; see book for depth

Vector Identities:

$$\begin{aligned} \nabla(f+g) &= \nabla f + \nabla g & * \text{Note: } f, g &= \text{scalar functions} \\ \nabla \cdot (A+B) &= (\nabla \cdot A) + (\nabla \cdot B) & k &= \text{constant} \\ \nabla \times (A+B) &= (\nabla \times A) + (\nabla \times B) & A, B &= \text{vector functions} \\ \nabla(kf) &= k \nabla f \\ \nabla \cdot (kA) &= k(\nabla \cdot A) \\ \nabla \times (kA) &= k(\nabla \times A) \\ \nabla(fg) &= f(\nabla g) + g(\nabla f) \\ \nabla(A \cdot B) &= A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A \\ \nabla \cdot (fA) &= f(\nabla \cdot A) + A \cdot (\nabla f) \\ \nabla \cdot (A \times B) &= B \cdot (\nabla \times A) - A \cdot (\nabla \times B) \\ \nabla \times (fA) &= f(\nabla \times A) - A \times (\nabla f) \\ \nabla \times (A \times B) &= (B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A) \\ \nabla \cdot (\nabla \times A) &= 0 \\ \nabla \times \nabla f &= 0 \\ \nabla \times (\nabla \times A) &= \nabla(\nabla \cdot A) - \nabla^2 A \end{aligned}$$

Fundamental Theorems: Gradients  $\rightarrow \int_a^b (\nabla T) \cdot d\vec{l} = T(b) - T(a)$

Divergences  $\rightarrow \int_V (\nabla \cdot A) dV = \oint_S A \cdot d\vec{a}$

Curls  $\rightarrow \int_S (\nabla \times A) \cdot d\vec{a} = \oint_C A \cdot d\vec{l}$

## Ch 1 (cont.)

Dirac-Delta function:  $\nabla \cdot \left( \frac{\hat{r}}{r^2} \right) = 4\pi \delta^3(\vec{r})$  or  $\nabla \cdot \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = 4\pi \delta^3(\vec{r})$

$$\nabla \left( \frac{1}{r} \right) = -\frac{\hat{r}}{r^2} \Rightarrow \nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta^3(\vec{r})$$

## Ch 2 - Electrostatics

Coulombs Law:  $\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} (\hat{r}_1 - \hat{r}_2)$  (SI units)

$$\vec{F} = Q \vec{E} \Rightarrow E(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_n \frac{q_i}{r_i^2} \hat{r}_i$$

\* if we allow for a continuous charge distribution,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{r^2} \hat{r} dV$$

Gauss' Law:  $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$  (SI)  $\Leftrightarrow \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$  (SI)

$\hookrightarrow$  Derived from application of divergence theorem

\* Easiest way to calculate  $\vec{E}$ , symmetry permitting (determined by geometry)

① Spherical symmetry  $\leftrightarrow$  Gaussian sphere ( $A = 4\pi r^2$ )

② Cylindrical symmetry  $\leftrightarrow$  Coaxial cylinder ( $A = 2\pi r L$ )

③ Plane symmetry  $\leftrightarrow$  Gaussian pillbox (Area =  $2A$ )

Curl of  $\vec{E}$ : \* Simple application of Stokes theorem leads to:

$$\nabla \times \vec{E} = 0 \quad (\text{Applies only to static charge distributions})$$

Electric Potential: \* Since the curl of  $\vec{E}$  equals 0, we can use  $\nabla \times \nabla V = 0$  (and/or) the properties of line integrals (via Fundamental Theorem of gradients) to define  $\vec{E}$  as the gradient of a scalar function

$$\Rightarrow -\nabla V = \vec{E}$$

\* This allows us to reformulate our previous eqns to:

$$-\nabla^2 V = \frac{\rho}{\epsilon_0} \quad (\text{SI}) \quad (\text{Note: This is Poisson's eqn})$$

\* To calculate  $V$  from a charge distribution

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{r - r_i} \quad \text{or} \quad V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$$

Ch 2 (cont.)

Boundary Conditions: Charges create discontinuities w/in  $\vec{E}$ -field. To quantify these discontinuities:

① Apply Gauss Law  $\Rightarrow E_1^\perp - E_2^\perp = \frac{\sigma}{\epsilon_0}$  (SI)

② Apply Stokes Thm  $\Rightarrow E_1^\parallel = E_2^\parallel$

\* Remember, both  $\vec{E}$  and  $V$  go to 0 as  $r \rightarrow \infty$

\* Combined into one statement:  $E_1 - E_2 = \frac{\sigma}{\epsilon_0} \hat{n}$

\* Electric potential  $V$  is continuous across boundaries (via Stokes thm)

Work + Energy: In general,  $W = \int_a^b \vec{F} \cdot d\vec{\ell}$ , therefore it is easy to see that

$$W = Q(V(b) - V(a))$$

\* Extending this to a charge distribution, to assemble the distribution charge by charge:

$$W_1 = 0 \quad (\text{no field to work against}) \quad (\text{SI units})$$

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left( \frac{q_1}{r_{12}} \right) \quad \text{Note: } r_{ij} = |\vec{r}_i - \vec{r}_j|$$

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left( \frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right)$$

$$\hookrightarrow W = \frac{1}{4\pi\epsilon_0} \sum_i^n \sum_{j>i}^n \frac{q_i q_j}{r_{ij}}$$

$$W = \frac{1}{8\pi\epsilon_0} \sum_i^n \sum_{j \neq i}^n \frac{q_i q_j}{r_{ij}}$$

$$\boxed{W = \frac{1}{2} \sum_i q_i V(\vec{r}_i)} \quad \Leftrightarrow \quad \boxed{W = \frac{1}{2} \int \rho V dV}$$

\* Individual charges

\* Continuous distribution

\* A few substitutions to our continuous distribution yields another form of the equation:

$$W = \frac{1}{2} \int \rho V dV$$

$$= \frac{\epsilon_0}{2} \int (\nabla \cdot \vec{E}) V dV$$

$$= \frac{\epsilon_0}{2} \left[ - \int \vec{E} \cdot (\nabla V) dV + \oint V \vec{E} \cdot d\vec{a} \right]$$

$$= \frac{\epsilon_0}{2} \left[ \int E^2 dV + \oint V \vec{E} \cdot d\vec{a} \right]$$

$$= \frac{\epsilon_0}{2} \int E^2 dV \quad (\text{when integrated over all space})$$

## Ch 2 (cont.)

Conductors: \* A few quick definitions!

① Insulator - A material where electrons are tightly bound to an atom  
ex. glass, rubber

② Conductor - A material where electrons are loosely bound to individual atoms and are "free" to move around  
ex. metals

(Perfect conductors have unlimited free charge; but do not exist)

\* From the above definition, we can infer a few properties of conductors

① Inside a conductor,  $\vec{E} = 0$ . If a conductor is placed in an external  $\vec{E}$  field, the internal field will become  $-\vec{E}$  to satisfy the above rule.

② Inside a conductor,  $\rho = 0$ . This follows from Gauss' Law based on property ①

③ Any net charge on a conductor resides on its surface.

④ Conductors are equipotentials

⑤  $\vec{E}$  is  $\perp$  to the surface, just outside a conductor. This prevents flowing charge

\* Charges placed near a conductor will result in an attractive force b/w the two, as the charge polarizes the conductor

\* Charges placed w/in the cavity of a conductor will polarize conductor such that  $-q_{enc}$  charge resides on cavity surface, while  $+q$  charge exists on conductor surface. Note:  $\vec{E}$  w/in the cavity will not be 0.

\* A few important quantities we can derive for a conductor:

$$\begin{aligned}\vec{f} &= \sigma \vec{E}_{avg} \\ &= \frac{1}{2} \sigma (E_1 + E_2)\end{aligned}$$

(force per unit area)

$$P = \frac{\epsilon_0}{2} |\vec{E}|^2 \quad (\text{electrostatic pressure on conductor})$$

## Ch 2 (cont.)

Conductors: \* If we place two conductors near each other, we can define capacitance  $C$  as

$$C = \frac{Q}{\Delta V}$$

- To charge up a capacitor, the work done is:

$$W = \frac{1}{2} C V^2, \quad V = \text{final potential of capacitor}$$

## Ch 3 - Potentials

LaPlace's Eqn: \* Often, solving for  $\vec{E}$  and  $V$  according to the following are impossible

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}') dV'$$

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$$

thus, we recast the problem according to  $-\nabla^2 V = \frac{\rho}{\epsilon_0}$ , which has known solutions when boundary conditions are specified

\* The simplest form of this problem occurs when  $\rho=0$ , and is known as LaPlace's equation. From examining 1-D to 3-D solutions in Cartesian space, a few important points:

① The solution is an average of the value of the function in the local region surrounding the point

② Maxima + minima must occur at endpoints to satisfy ①

\* Since a specified set of boundary conditions is needed to solve the problem, we have a few uniqueness criteria

① The solution to LaPlace's Eqn in some volume  $V$  is uniquely determined if  $V$  is specified on the bounding surface  $S$

② In a volume  $V$  surrounded by conductors and containing a specified charge density  $\rho$ , the electric field  $\vec{E}$  is uniquely determined if the total charge on each conductor is given.

\* Proofs can be found in Griffiths 3.1.5 and 3.1.6

## Ch 3 (cont.)

**Method of Images:** The method of images is a technique to solve Poisson's Eqn such that the specified boundary conditions are created by a set of "image" charges

ex. Point charge above grounded conductor

\* We want to know the potential of a point charge  $q$  a distance  $d$  above a grounded conducting plane. We must meet the following 2 conditions:

$$\textcircled{1} \quad V = 0 \quad \text{at} \quad z = 0$$

$$\textcircled{2} \quad V \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty; \quad r^2 \ll x^2 + y^2 + z^2$$

\* Since we are only interested in  $V$  above the plane, we can simply add a charge of  $-q$  at  $z = -d$  to enforce condition  $\textcircled{1}$ . Condition  $\textcircled{2}$  is automatically enforced by nature of  $V$  for point charges

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{(x^2 + y^2 + [z-d]^2)^{1/2}} - \frac{q}{(x^2 + y^2 + [z+d]^2)^{1/2}} \right]$$

\* We can also determine induced surface charge density according to:

$$\begin{aligned} \sigma &= -\epsilon_0 \frac{\partial V}{\partial n} \\ &= -\epsilon_0 \frac{\partial V}{\partial z} \Big|_{z=0} \\ &= \frac{-qd}{2\pi(x^2 + y^2 + d^2)^{3/2}} \end{aligned}$$

\* We determine the total induced charge on the surface by:

$$\begin{aligned} Q &= \int \sigma \, da \\ &= \int \frac{-qd}{2\pi(r^2 + d^2)^{3/2}} r \, dr \, d\phi \\ &= -q \end{aligned}$$

\* Other quantities we can calculate include:

$$\text{- Force: } F = \frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \hat{z}$$

$$\text{- Work: } W = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4d} \quad (\text{1/2 expected value, as } z < 0, E = 0)$$

\* More examples found in Griffiths 3.2

Ch 3 (cont.)

Separation of Variables: \*Goal is to turn a partial differential equation into a series of ordinary differential equations depending on one variable each

\*Several examples in Cartesian coordinates are found in Griffiths 3.3.1, but a general solution can be found in Jackson notes

ex. Spherical Coordinates

\*Note: This derivation assumes azimuthal symmetry. See Jackson notes for solution w/o this assumption (really Legendre polynomials become spherical harmonics)

\*Laplace's eqn in spherical coordinates is:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

⇒ with azimuthal symmetry, this reduces to

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

- Assuming a solution of the form  $V(r, \theta) = R(r) \Theta(\theta)$  and dividing by  $V$ , we see:

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

- Using the separation constant  $l(l+1)$

$$\hookrightarrow \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = l(l+1)$$

$$\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = -l(l+1)$$

we get the solutions:

$$R(r) = Ar^l + B \frac{1}{r^{l+1}}$$

$$\Theta(\theta) = P_l(\cos \theta) \text{ where } P_l = \frac{1}{2^l l!} \left( \frac{d}{dx} \right)^l (x^2 - 1)^l$$

Thus, our overall solution is:

$$V(r, \theta) = \sum_l \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

Ch 3 (cont.)

Separation of Variables:

ex. Spherical Coordinates (cont.)

\*Note: Boundary conditions will determine values of  $l, A_l, B_l$ .  
Fourier's trick (see below) is useful to help determine coefficients

$$\begin{aligned} \hookrightarrow \int_{-1}^1 P_l(x) P_{l'}(x) dx &= \int_0^\pi P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta d\theta \\ &= \begin{cases} 0 & l \neq l' \\ \frac{2}{2l+1} & l = l' \end{cases} \end{aligned}$$

Multipole Expansion:

\*This is an expansion technique to find approximate potentials for given charge configurations

\*Remember, the potential of a charge configuration is given by:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} dV'$$

- using the Law of Cosines, we see:

$$\begin{aligned} |\vec{r}-\vec{r}'| &= \sqrt{r^2 + r'^2 - 2rr'\cos(\alpha)} \\ &= r \sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha}; \quad \alpha \text{ is angle b/w } \vec{r}, \vec{r}' \\ &= r \sqrt{1 + \epsilon}; \quad \epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) \end{aligned}$$

- if our point of evaluation is far from the charge distribution,  $\epsilon \ll 1$ , thus using the binomial expansion:

$$\begin{aligned} \frac{1}{|\vec{r}-\vec{r}'|} &= \frac{1}{r\sqrt{1+\epsilon}} = \frac{1}{r} \left( 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots \right) \\ &= \frac{1}{r} \left[ 1 - \frac{1}{2} \left(\frac{r'}{r}\right) \left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8} \left(\frac{r'}{r}\right)^2 \left(\frac{r'}{r} - 2\cos\alpha\right)^2 + \dots \right] \\ &= \frac{1}{r} \left[ 1 + \left(\frac{r'}{r}\right) \cos\alpha + \left(\frac{r'}{r}\right)^2 \left(\frac{3\cos^2\alpha - 1}{2}\right) + \dots \right] \\ &= \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\alpha) \end{aligned}$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{1}{r} \int \rho(r') dV' + \frac{1}{r^2} \int r' \cos\alpha \rho(r') dV' + \dots \right]$$



## Ch 3 (cont.)

Multipole Expansion: The important terms (+ their names) in the above expansion are:

$n=0$  - Monopole term \* Normally dominant at large  $r$

$n=1$  - dipole term \* Dominant if  $Q_{tot} = 0$

$n=3$  - quadrupole term

$n=4$  - octopole term

$$V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int r' \cos \alpha \rho(\vec{r}') dV'$$

$$* \text{ but } \hat{r} \cdot \vec{r}' = r' \cos \alpha$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \int r' \rho(r') dV'$$

$$\hookrightarrow \vec{p} = \int r' \rho(r') dV' \quad \left( \text{or } \vec{p} = \sum_i q_i \vec{r}'_i \right)$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

Electric Dipole:  $V_{dip}(r, \theta) = \frac{\hat{r} \cdot \vec{p}}{4\pi\epsilon_0 r^2}$

$$= \frac{p \cos \theta}{4\pi\epsilon_0 r^2}$$

$$\vec{E} = -\nabla V$$

$$= \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta})$$

Note: This assumes spherical coordinates and a dipole orientation along + z-axis

## Ch 4 - Electric Fields in Matter

Polarization: \* Two large classes of materials: Conductors and Insulators (or dielectrics)

\* Neutral atoms placed in an external  $\vec{E}$ -field will have locations shifted by field, such that

$$\vec{p} = \alpha \vec{E}$$

where  $\alpha$  is the atomic polarizability

$$\hookrightarrow \text{for molecules, } \vec{p} = \alpha_{\perp} E_{\perp} + \alpha_{\parallel} E_{\parallel}$$

\* While neutral are only polarized by an external field, polar molecules experience a torque as well

$\vec{N} = \vec{p} \times \vec{E}$  (about center);  $N = (\vec{p} \times \vec{E})_{\perp} + (\vec{r} \times \vec{F})$  else  
and in an uneven field, a force as well

$$\vec{F} = (\vec{p} \cdot \nabla) \vec{E}$$

## Ch 4 (cont.)

(10)

Polarization: \* We define  $\vec{P}$ , the polarization as the dipole moment per unit volume

\* To examine what happens to a polarized material, we consider the potential of a single dipole such that:

$$\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \\ &= \frac{1}{4\pi\epsilon_0} \int \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV' \\ &\quad \text{* remembering } \nabla' \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \\ &= \frac{1}{4\pi\epsilon_0} \int \vec{P} \cdot (\nabla' \left[ \frac{1}{|\vec{r} - \vec{r}'|} \right]) dV' \\ &\quad \text{* integrating by parts we find} \\ &= \frac{1}{4\pi\epsilon_0} \left[ \int \nabla' \cdot \left( \frac{\vec{P}}{|\vec{r} - \vec{r}'|} \right) dV' - \int \frac{1}{|\vec{r} - \vec{r}'|} (\nabla' \cdot \vec{P}) dV' \right] \\ &\quad \text{* which after utilizing the divergence theorem becomes} \\ &= \frac{1}{4\pi\epsilon_0} \oint \frac{\vec{P}}{|\vec{r} - \vec{r}'|} \cdot d\vec{a}' - \frac{1}{4\pi\epsilon_0} \int \frac{1}{|\vec{r} - \vec{r}'|} (\nabla' \cdot \vec{P}) dV' \end{aligned}$$

\* This leads to our definitions of surface and volume bound charges

$$\sigma_b = \vec{P} \cdot \hat{n} \quad \rho_b = -\nabla \cdot \vec{P}$$

\* This implies that the potential of a polarized object is the same as that due to these bound charges. This allows us to use our previous calculation methods. For a physical interpretation of bound charge: Bound charge is the collection of charges due to long strings of induced dipoles w/in a material.

Electric Displacement: \* With the added bound charge present in dielectrics, we must slightly modify our equations before calculating the fields

$$\vec{P} = \vec{P}_{\text{free}} + \vec{P}_{\text{bound}}$$

$$\begin{aligned} \epsilon_0 (\nabla \cdot \vec{E}) &= \rho \\ &= \rho_f + \rho_b \\ &= \rho_f - \nabla \cdot \vec{P} \end{aligned}$$

Ch 4 (cont.)

Electric Displacement: \* We rewrite the above equation as:

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad (\text{Electric Displacement in SI})$$

\* This results in our new forms of Gauss' Law:

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\oint \mathbf{D} \cdot d\vec{a} = Q_{\text{free, enc}}$$

\* Note: Despite similarities to previous derivations of  $\vec{E}$ , this does not imply

~~$$D(r) = \frac{1}{4\pi} \int \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \rho_f(r') dV'$$~~ NO!

as we previously used  $\nabla \times \mathbf{E} = 0$  to derive the formula for  $\vec{E}$ , but  $\nabla \times \mathbf{D} \neq 0$ .

\* Our new boundary conditions become:

$$D_1^\perp - D_2^\perp = \sigma_f$$

$$E_1^\perp - E_2^\perp = \frac{\sigma}{\epsilon_0}$$

$$D_1^\parallel - D_2^\parallel = P_1^\parallel - P_2^\parallel$$

$$E_1^\parallel - E_2^\parallel = 0$$

Linear Dielectrics: \* Linear dielectrics where the relationship between the polarization and the electric field is of the form:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

where  $\chi_e$  is the electric susceptibility and  $\vec{E}$  is the total field, due both to an external field and the polarization of the material.

\* Thus, in linear media:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$= \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$

$$= \epsilon_0 (1 + \chi_e) \vec{E}$$

$$= \epsilon \vec{E}$$

$$> \epsilon = \epsilon_0 (1 + \chi_e)$$

where  $\epsilon$  is the permittivity of the material. We also define the relative permittivity or dielectric constant as:

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

### Ch 4 (cont.)

Linear Dielectrics: \*We can now derive a relationship between free and bound charge

$$\begin{aligned}
 P_b &= -\nabla \cdot P \\
 &= -\nabla \cdot \left( \epsilon_0 \frac{\chi_e}{\epsilon} D \right) \\
 &= \frac{-\chi_e}{1+\chi_e} P_f
 \end{aligned}$$

which allows us to formulate our boundary conditions in terms of free charges

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f$$

$$V_1 = V_2$$

\*Other formulas we can re-evaluate include:

- Work to charge capacitor

$$W = \frac{1}{2} CV^2 \rightarrow W = \frac{\epsilon_r}{2} CV^2$$

- General Work

$$W = \frac{\epsilon_0}{2} \int E^2 dV \rightarrow W = \frac{1}{2} \int D \cdot E dV$$

Proof:  $\Delta W = \int (\Delta P_f) V dV'$

$$= \int (\nabla \cdot \Delta D) V dV'$$

$$= \int \nabla \cdot (\Delta D) V dV' + \int \Delta D \cdot \nabla V dV'$$

$$= 0 + \int \Delta D \cdot E dV'$$

$$W = \frac{1}{2} \int D \cdot E dV$$

- Force:

\* See Griffiths 4.4.4 for full derivation

$$F = +\frac{1}{2} V^2 \frac{dC}{dx} \quad \left( \text{from } dW = -F dx + V dQ \right)$$

## Ch 5 - Magnetostatics

Lorentz Force Law: \* In the presence of magnetic fields only,  $\vec{F} = q(\vec{v} \times \vec{B})$ , and explains why current carrying wires attract/repel each other when their currents are parallel/anti-parallel. The full law is  $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

\* Lorentz force law explains the cyclotron motion of charged particles in a magnetic field. Note: Any velocity parallel to field is unaffected. Solving for the equations of motion via Newton's 2<sup>nd</sup> Law will yield  $\omega = \frac{qB}{m}$  for the cyclotron frequency

\* Solving for the amount of work done by a magnetic field via  $dW_{\text{mag}} = \vec{F}_{\text{mag}} \cdot d\vec{\ell}$

$$\begin{aligned} \hookrightarrow dW_{\text{mag}} &= \vec{F}_{\text{mag}} \cdot d\vec{\ell} \\ &= q(\vec{v} \times \vec{B}) \cdot \vec{v} dt \\ &= 0 \end{aligned}$$

\* We define the current in a wire as the charge per unit time passing a specified point. We define it as  $\vec{I} = \lambda \vec{v}$  where  $\lambda$  is the charge density and  $\vec{v}$  is the flow velocity. This allows us to define  $\vec{F}$  using  $\vec{I}$  as:

$$\begin{aligned} \vec{F} &= \int (\vec{v} \times \vec{B}) dq \\ &= \int (\vec{v} \times \vec{B}) \lambda d\ell \\ &= \int (\vec{I} \times \vec{B}) d\ell \\ &= \int I (d\vec{\ell} \times \vec{B}) \quad (\text{since } \vec{I}, d\vec{\ell} \text{ are parallel}) \end{aligned}$$

\* We define the surface ( $\vec{K}$ ) and volume ( $\vec{J}$ ) currents as:

$$\vec{K} = \sigma \vec{v} \qquad \vec{J} = \rho \vec{v}$$

\* To satisfy conservation of charge, we need the continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

## Ch 5 (cont.)

Biot-Savart Law: \*Steady state currents (ie  $\frac{\partial}{\partial t} = 0$ ) defines magnetostatics

\*The Biot-Savart defines the magnetic field due to a line current

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dl' \quad (\text{SI})$$

- A few useful fields due to specific geometries:

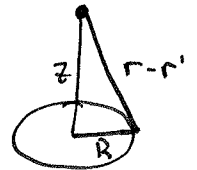
① Straight wire segment -  $\vec{B} = \frac{\mu_0 I}{4\pi s} (\sin \theta_2 - \sin \theta_1) \quad (\text{SI})$



② Current loop, evaluated on axis -  $\vec{B} = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \quad (\text{SI})$

(Derivation b/c I always fuck this up; Gaussian units)

$$\begin{aligned} \vec{B} &= \frac{4\pi}{c} \int \frac{\vec{J} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dr' \\ &= \frac{4\pi}{c} I_0 \int \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \end{aligned}$$



- evaluating the cross product (in cylindrical coordinates)

$$d\vec{l} \times (\vec{r} - \vec{r}') = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ 0 & dl & 0 \\ -R & 0 & z \end{vmatrix}$$

$$= \langle z dl, 0, R dl \rangle$$

However, b/c we know by symmetry our field will point in  $\pm \hat{z}$  direction, we only consider z-component

$$= \frac{4\pi}{c} I_0 \int \frac{R dl}{(R^2 + z^2)^{3/2}}$$

$$= \frac{4\pi}{c} I_0 \frac{2\pi R^2}{(R^2 + z^2)^{3/2}}$$

- In more dimensions, the Biot-Savart Law becomes:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} da' \quad \text{or} \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'$$

Ch5 (cont.)

Divergence + Curl of  $\vec{B}$ : \* Looking at a long straight wire + its magnetic field reveals a non-zero curl

$$\hookrightarrow \oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

But we can also derive  $I_{enc}$  vica:  $I_{enc} = \int \vec{J} \cdot d\vec{a}$

$$\hookrightarrow \int (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a} \Rightarrow \boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$$

\* The above boxed equation is known as Ampere's Law (formal derivation in Sec. 5.3.2) and works in a similar manner to Gauss' Law in electrostatics

ex. Field of long, straight wire

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

$$\begin{aligned} \oint \vec{B} \cdot d\vec{\ell} &= B \int d\ell \quad (\text{"equipotential" around Amperian loop in cylindrical}) \\ &= B \cdot 2\pi r \end{aligned}$$

$$\hookrightarrow B \cdot 2\pi r = \mu_0 I_{enc}$$

$$B = \frac{\mu_0 I_{enc}}{2\pi r} \quad (\text{as expected})$$

\* Note: Only in specific geometries is Ampere's Law useful, including:

- ① Infinite straight lines (ex. 5.7)
- ② Infinite planes (ex. 5.8)
- ③ Infinite solenoids (ex. 5.9)
- ④ Toroids (ex. 5.10)

\* Since the divergence of  $\vec{B} = 0$ , our magnetostatics equations are:

$$\nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$

## Ch 5 (cont.)

Magnetic Vector Potential: \* Similar to electrostatics + vector identities allow a ~~vector~~ <sup>potential</sup> formulation of electrostatics, we can also derive a potential formulation for magnetostatics using  $\nabla \cdot \mathbf{B} = 0$  to define  $\vec{B}$  in terms of  $\vec{A}$  as:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

- Plugging the above into Ampere's Law gives:

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\ &= \mu_0 \mathbf{J} \end{aligned}$$

- From this, we require  $\nabla \cdot \mathbf{A} = 0$ , which is always possible

- This then allows us to determine  $\vec{A}$  via Poisson's eqn ( $-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$ ) which has the known solution:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

- While not as useful as  $V$ ,  $\vec{A}$  can still be easier to use than Biot-Savart

\* See ex. 5.11 and 5.12 for deriving  $\vec{A}$  from specific geometries

Boundary Conditions: \* We still must be aware of our boundary conditions when solving magnetostatics problems. From  $\nabla \cdot \mathbf{B} = 0$  and the divergence theorem, we determine

$$B_1^\perp = B_2^\perp$$

Similarly from  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$  and Stokes Thm, we see

$$B_1^\parallel - B_2^\parallel = \mu_0 \vec{K} \cdot \hat{n}$$

Just like in electrostatics, our potential is continuous

$$A_1 = A_2 \quad (\text{at boundary})$$

Multipole Expansion: \* We can again get an approximate potential via multipole expansion by:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{(r^2 + r'^2 - 2rr' \cos \alpha)^{1/2}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos \alpha)$$



### Ch 5 (cont.)

Multipole Expansion: - Therefore, our equation for  $\vec{A}$  becomes:

$$\begin{aligned} \vec{A} &= \frac{\mu_0 I}{4\pi} \oint \frac{dl}{|r-r'|} \\ &= \frac{\mu_0 I}{4\pi} \sum_l \frac{1}{r^{l+1}} \oint (r')^l P_l(\cos\alpha) dl' \\ &= \frac{\mu_0 I}{4\pi} \left[ \frac{1}{r} \oint dl' + \frac{1}{r^2} \oint r' \cos\alpha dl' + \frac{1}{r^3} \oint (r')^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) dl' + \dots \right] \end{aligned}$$

where the following terms are named

$$\frac{1}{r} - \text{monopole}, \quad \frac{1}{r^2} - \text{dipole}, \quad \frac{1}{r^3} - \text{quadrupole}$$

\* Here, it is important to note the monopole term is always 0. Thus, we care most about the dipole term:

$$\begin{aligned} \vec{A}_{\text{dip}} &= \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha dl' \\ &= \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot r') dl' \\ &\quad * \text{ using } \oint (\hat{r} \cdot r') dl' = -\hat{r} \times \int da' \\ &= \frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{r}, \quad \vec{m} = I \int da' \quad (\text{magnetic dipole moment}) \end{aligned}$$

### Ch 6 - Magnetic Fields in Matter

Magnetization: \* Similar to polarization, magnetization occurs when a material is placed in an external  $\vec{B}$ -field. These different materials are categorized as:

- ① Paramagnets - Magnetization parallel to  $\vec{B}$
- ② Diamagnets - Magnetization opposite to  $\vec{B}$
- ③ Ferromagnets - Maintain magnetization after removal from  $\vec{B}$ -field

\* Magnetic dipoles experience a torque when placed in an external field,

$$\vec{N} = \vec{m} \times \vec{B}$$

Since the torque aligns the dipole with the field, this is how paramagnetism is generated. Due to Pauli exclusion principle, paramagnetism often only occurs in materials with odd numbers of valence electrons.

Ch 6 (cont.)

Magnetization: \* Remember that in a uniform  $\vec{B}$ -field, the net force on a current loop is 0. However, if the field is non-uniform, then the force becomes

$$\vec{F} = \nabla (m \cdot \vec{B})$$

\* Magnetization (noted  $\vec{M}$ ) is the magnetic dipole moment per unit volume.

Fields of Magnetic Objects: \* Given the above definition of  $\vec{M}$ , we can now define  $\vec{A}$  as:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{M} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'$$

$$\text{- Again exploiting that } \nabla' \frac{1}{|\vec{r} - \vec{r}'|} = \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$= \frac{\mu_0}{4\pi} \int \vec{M} \times (\nabla' \frac{1}{|\vec{r} - \vec{r}'|}) dV'$$

$$= \frac{\mu_0}{4\pi} \left[ \int \frac{1}{|\vec{r} - \vec{r}'|} (\nabla' \times \vec{M}) dV' - \int \nabla' \times \left( \frac{\vec{M}}{|\vec{r} - \vec{r}'|} \right) dV' \right]$$

$$= \frac{\mu_0}{4\pi} \left[ \int \frac{1}{|\vec{r} - \vec{r}'|} (\nabla' \times \vec{M}) dV' - \oint \frac{1}{|\vec{r} - \vec{r}'|} \vec{M} \cdot d\vec{a}' \right]$$

$$= \frac{\mu_0}{4\pi} \left[ \int \frac{1}{|\vec{r} - \vec{r}'|} \vec{J}_b dV' - \oint \frac{1}{|\vec{r} - \vec{r}'|} \vec{K}_b da \right]$$

where  $\vec{J}_b = \nabla \times \vec{M}$  is the bound volume current and

$\vec{K}_b = \vec{M} \times \hat{n}$  is the bound surface current.

ex. Magnetic Field of Uniformly Magnetized Sphere

\* Note: We choose to orient  $\vec{M}$  along z-axis such that  $\vec{M} = M \hat{z}$

$$\hookrightarrow \vec{J}_b = \nabla \times \vec{M} = 0$$

$$\vec{K}_b = \vec{M} \times \hat{n} = M \sin \theta \hat{\phi}$$

\* Previously, we solved a rotating spherical shell of charge where (ex 5.11)

$$\vec{K} = \sigma \vec{v} = \sigma \omega R \sin \theta \hat{\phi}$$

Thus our answers should agree and

$$\vec{B} = \begin{cases} \frac{2}{3} \mu_0 \vec{M} & \text{inside} \\ \frac{1}{3} \mu_0 \vec{M} & \text{outside} \end{cases}$$

## Ch 6 (cont.)

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Fields: \* For a physical interpretation of bound current, imagine many infinitely small current loops within a material. For loops completely internal, there is always a corresponding side from another loop to cancel it out. Only on the edge is the current not cancelled. When magnetization is non-uniform, these internal currents don't cancel, resulting in a volume bound current as well.

Auxiliary Field  $\vec{H}$ : \* Again, similar to our approach in electrostatics, we fully define the current as:

$$\vec{J} = \vec{J}_{\text{free}} + \vec{J}_{\text{bound}}$$

Plugging this back into Ampere's Law yields:

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{J}_f + \vec{J}_b$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{J}_f + \nabla \times \vec{M}$$

$$\nabla \times \left( \frac{1}{\mu_0} \vec{B} - \vec{M} \right) = \vec{J}_f$$

$$\nabla \times \vec{H} = \vec{J}_f ; \quad \vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

(Integral form:  $\oint \vec{H} \cdot d\vec{l} = I_{\text{free, enc}}$ )

ex. Uniform Rod

\* Assuming a rod w/ uniform free current  $I$

$$\oint \vec{H} \cdot d\vec{l} = I_{\text{free}}$$

$$H \cdot 2\pi s = I \cdot \pi s^2 / \pi R^2$$

$$H = \begin{cases} \frac{I s}{2\pi R^2} & R < s \\ \frac{I}{2\pi s} & R > s \end{cases} \quad \left( B = \frac{\mu_0 I}{2\pi s} \right)$$

\* In media, our boundary conditions now become:

$$B_1^\perp - B_2^\perp = 0$$

$$H_1^\parallel - H_2^\parallel = \vec{K}_f \times \hat{n} \quad (B_1^\parallel - B_2^\parallel = \mu_0 \vec{K}_f \times \hat{n})$$

## Ch 6 (cont.)

(Non)/Linear Media: \*For linear media, we define  $\vec{M}$  in terms of  $\vec{H}$  via:

$$\vec{M} = \chi_m \vec{H}$$

where  $\chi_m$  is the magnetic susceptibility. Continuing forward, we see:

$$\begin{aligned}\vec{B} &= \mu_0 (\vec{H} + \vec{M}) \\ &= \mu_0 (1 + \chi_m) \vec{H} \\ &= \mu \vec{H}\end{aligned}$$

where  $\mu$  is the permeability of the material.

\*In linear media only  $J_b \propto J_f$

$$J_b = \nabla \times M = \nabla \times \chi_m H = \chi_m J_f$$

## Ch 7 - Electrodynamics

Electromotive Force: \*In order to make charges move, you must apply a force. Thus  $J \propto f$ , the force per unit charge. Using the Lorentz force law, we see:

$$\vec{J} = \sigma (\vec{E} + \vec{v} \times \vec{B})$$

However,  $\vec{B}$  and  $\vec{v}$  are typically much weaker than  $\vec{E}$ , thus Ohm's Law becomes  $\vec{J} = \sigma \vec{E}$ . We are typically more familiar w/  $V = IR$  for Ohm's Law.

\*Also of importance is Joule's Heating Law:  $P = VI = I^2 R$

\*If we want to investigate what forces current to flow in a circuit, we divide the total force into the source force and the electrostatic force such that

$$\vec{F} = \vec{F}_s + \vec{E}$$

If we integrate the force around the loop, we get the EMF  $\mathcal{E}$

$$\mathcal{E} = \oint \vec{F} \cdot d\vec{\ell} = \oint \vec{F}_s \cdot d\vec{\ell} \quad (\oint \vec{E} \cdot d\vec{\ell} = 0)$$

In an ideal EMF, the net force is 0, ( $\vec{E} = -\vec{F}_s$ ), therefore

$$V = - \int_a^b \vec{E} \cdot d\vec{\ell} = \int_a^b \vec{F}_s \cdot d\vec{\ell} = \mathcal{E}$$

Ch 7 (cont.)

EMF: \* One example of EMF's are generators, or wires moving through magnetic fields.

**Physics 221A**  
**Fall 2017**  
**Appendix A**  
**Gaussian, SI and Other Systems of Units**  
**in Electromagnetic Theory**

## 1. Introduction

Most students are taught SI units in their undergraduate courses in electromagnetism, but in this course we will use Gaussian units or other systems of units derived from them (atomic units, natural units, etc). This Appendix is intended to help you become familiar with Gaussian units, assuming you have a background in SI units. We will also mention briefly other systems of units, which you may encounter in other courses. We will make only limited use of SI units in this course, mainly for expressing the answers to homework problems.

This Appendix is intended to be as practical as possible for the purposes of this course. We will address the most common problems you will encounter first, and then discuss some general considerations that lie behind the choices of units. We will concentrate on the problems you will likely encounter in this course on quantum mechanics, and neglect other issues. For reference you may also wish to look at Jackson's discussion of units (an appendix in *Classical Electrodynamics*).

In this appendix vectors  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{D}$  and  $\mathbf{H}$  have their usual meanings, while  $\mathbf{P}$  and  $\mathbf{M}$  are the electric and magnetic dipole moments per unit volume, respectively, and  $\mathbf{p}$  and  $\mathbf{m}$  are the electric and magnetic dipole moments of individual dipoles, respectively. The scalar and vector potentials are  $\Phi$  and  $\mathbf{A}$ , respectively.

## 2. A Comparison of SI and Gaussian Units

The main advantage of SI units is popularity. The entire engineering world runs on volts, ohms, farads, etc (all SI units), so almost any discussion of electrical equipment or experimental apparatus will be in terms of SI units. The main advantage of Gaussian units is that they make fundamental physical issues and theoretical relations involving electromagnetic phenomena more clear. For example, special relativity and quantum electrodynamics are simpler, more transparent and more elegant in Gaussian units than in SI units, and generally the various formulas of electromagnetism are simpler and easier to remember in Gaussian units than in SI units. Gaussian units are really the simpler system of units, and they would be better for pedagogical purposes were it not for the fact that students must deal with SI units some day anyway.

SI units have gradually been winning the popularity contest against Gaussian and other systems of units. Almost all engineers, most chemists and many physicists now use SI units almost exclusively,

and undergraduate courses in electricity and magnetism are now normally taught in SI units. Book publishers have contributed to this trend; they want books written in SI units so engineers will buy them. Nevertheless, it is unlikely that Gaussian units will ever be completely abandoned, because they are so superior for fundamental physical questions.

We will now list several aspects in which SI and Gaussian (and other systems of units) differ.

A fairly trivial difference is the system of units used for mass, distance and time (leaving aside charge for the moment). In SI units, the MKS (meter-kilogram-second) system is used, while the Gaussian units use the cgs (centimeter-gram-second) system. Thus, converting mechanical quantities (energy, force, etc) from SI to Gaussian units only involves various powers of 10. See Table 1.

	MKS unit	cgs unit	Conversion
mass	kilogram	gram	1kg = 10 <sup>3</sup> gm
distance	meter	centimeter	1m = 10 <sup>2</sup> cm
energy	Joule	erg	1J = 10 <sup>7</sup> erg
force	Newton	dyne	1N = 10 <sup>5</sup> dyne

Table 1. Converting mass, distance, and mechanical units only involves powers of 10.

Another fairly trivial aspect in which SI and Gaussian units differ is the placement of the  $4\pi$ 's in the formulas. In any system of units for electromagnetism factors of  $4\pi$  will appear somewhere; the usual choices are in Maxwell's equations, or else in the force laws (Coulomb and Biot-Savart). The difference is brought about by either absorbing factors of  $\sqrt{4\pi}$  into the definition of the unit of charge or splitting them off. Units in which the  $4\pi$ 's have been eliminated from Maxwell's equations are called *rationalized*; SI units are an example of rationalized units, in which the  $4\pi$ 's appear in Coulomb's law and the Biot-Savart law (in the factors  $1/4\pi\epsilon_0$  and  $\mu_0/4\pi$ ), but not in Maxwell's equations. Gaussian units are not rationalized, so the  $4\pi$ 's appear in Maxwell's equations. See Eqs. (14)–(17).

Another difference between SI and Gaussian units, this one not so trivial, is the definition of the unit of charge. In Gaussian units, the unit of charge is defined to make Coulomb's law look simple, that is, with a force constant equal to 1 (instead of the  $1/4\pi\epsilon_0$  that appears everywhere in SI units). This leads to a simple rule for translating formulas of electrostatics (without  $\mathbf{D}$ ) from SI to Gaussian units: just replace  $1/4\pi\epsilon_0$  by 1. Thus, there are no  $\epsilon_0$ 's in Gaussian units (see Sec. 4). There are no  $\mu_0$ 's either, since these can be expressed in terms of the speed of light by the relation

$$\epsilon_0\mu_0 = \frac{1}{c^2}. \quad (1)$$

Instead of  $\epsilon_0$ 's and  $\mu_0$ 's, one sees only factors of  $c$  in Gaussian units. In SI units one could use Eq. (1) to eliminate one of the three constants  $\epsilon_0$ ,  $\mu_0$  and  $c$ , but not in a symmetrical manner, so in

practice all three constants are retained. The result is that formulas in SI units can be written in a nonunique manner.

In SI units one sees expressions such as the permittivity, permeability or impedance of “free space.” These are not fundamental physical properties of free space, but rather artifacts of the SI system of units, which disappear (along with the  $\epsilon_0$ 's and  $\mu_0$ 's) in Gaussian units.

In Gaussian units the unit of charge (the statcoulomb) is defined as the charge such that two charges of one statcoulomb each at one centimeter distance will feel an electrostatic force of one dyne. Thus, in Gaussian units, the electrostatic force constant in Coulomb's law is 1 (see Eq. (4)), and it is possible to solve Coulomb's law for the charge, dimensionally speaking, to express the unit of charge in terms of other units. This gives

$$Q = \left( \frac{ML^3}{T^2} \right)^{1/2}, \quad (2)$$

where  $M$ ,  $L$ ,  $T$  and  $Q$  stand for mass, distance, time and charge, respectively. That is, one statcoulomb is the same as one  $\text{gm}^{1/2}\text{cm}^{3/2}/\text{sec}$ . It is not practical to do this with SI units (since the force constant is not 1), so the SI unit of charge (the Coulomb) is usually regarded as an independent unit in the SI system. You may not like the fractional powers that appear in Eq. (2), but you are always free to treat the statcoulomb as an independent unit also in Gaussian units. Many formulas appear with the square of charge, so there are no fractional powers. In any case, dimensional relations are much simpler in Gaussian units than in SI units.

Another difference between SI and Gaussian units is the fact that the magnetic field  $\mathbf{B}$  is defined with an extra factor of  $c$  in Gaussian units compared to SI units. This same factor of  $c$  percolates down to all derived magnetic quantities, including  $\mathbf{A}$ ,  $\mathbf{M}$ ,  $\mathbf{H}$  and  $\mathbf{m}$ , and it means that all these quantities have different dimensions in SI units than in Gaussian units, even after the relation (2) is taken into account. This is the main difference that makes it difficult to remember how to convert magnetic formulas from SI units to Gaussian units.

It has the benefit, however, that in Gaussian units all the fields  $\mathbf{E}$ ,  $\mathbf{P}$ ,  $\mathbf{D}$ ,  $\mathbf{B}$ ,  $\mathbf{M}$  and  $\mathbf{H}$  have the same dimensions, while in SI units the dimensions are all different. In addition, the scalar and vector potentials  $\Phi$  and  $\mathbf{A}$  have the same dimensions in Gaussian units, but not in SI units. The uniform dimensions for fields in Gaussian units make it easy to remember formulas in that system, and it makes fundamental physical relations more transparent. For example, dielectrics convert  $\mathbf{E}$  into  $\mathbf{D}$ , and relativity converts  $\mathbf{E}$  into  $\mathbf{B}$ , with coefficients that are dimensionless in Gaussian units. In SI units, it is necessary to insert factors of  $\epsilon_0$ ,  $\mu_0$ , etc. into the conversion formulas, which makes them harder to remember.

A final difference between Gaussian and SI units are the different definitions of the fields  $\mathbf{D}$  and  $\mathbf{H}$  (in terms of  $\mathbf{E}$  and  $\mathbf{P}$ , and  $\mathbf{B}$  and  $\mathbf{M}$ ), and likewise different definitions of electric and magnetic susceptibilities. We will not make much use of  $\mathbf{D}$  and  $\mathbf{H}$  in this course (they are important for the properties of bulk matter), but for reference the differences are described in Sec. 6.



Other systems of units involve different choices. For example, the Heaviside-Lorentz system is similar to the Gaussian system, but it is rationalized. There are no  $\epsilon_0$ 's or  $\mu_0$ 's in the Heaviside-Lorentz system. Heaviside-Lorentz units are favored by field theorists, who prefer not to see factors of  $4\pi$  in the field Lagrangian.

There are many other choices. For example, it would be easy to construct a Gaussian system of units for electromagnetism that uses MKS units for mass, distance and time (and a correspondingly modified definition of the statcoulomb, so the electrostatic force constant would still be unity). Such a system would be nonstandard, but it would have all the advantages of the Gaussian cgs system and none of the disadvantages of the SI system.

Constant	Symbol	SI	Gaussian
Speed of light	$c$	$2.998 \times 10^8 \text{m/sec}$	$2.998 \times 10^{10} \text{cm/sec}$
Planck's constant	$\hbar$	$1.055 \times 10^{-34} \text{J-sec}$	$1.055 \times 10^{-27} \text{erg-sec}$
Boltzmann constant	$k$	$1.381 \times 10^{-23} \text{J/K}$	$1.381 \times 10^{-16} \text{erg/K}$
Avogadro number	$N_A$	$6.022 \times 10^{23}$	$6.022 \times 10^{23}$
Fine structure constant	$\alpha$	1/137.0	1/137.0
Proton charge	$e$	$1.602 \times 10^{-19} \text{C}$	$4.803 \times 10^{-10} \text{statcoul}$
Electron mass	$m_e$	$9.109 \times 10^{-31} \text{kg}$	$9.109 \times 10^{-28} \text{gm}$
Proton mass	$m_p$	$1.673 \times 10^{-27} \text{kg}$	$1.673 \times 10^{-24} \text{gm}$
Neutron mass	$m_n$	$1.675 \times 10^{-27} \text{kg}$	$1.675 \times 10^{-24} \text{gm}$
Permittivity of free space	$\epsilon_0$	$8.854 \times 10^{-12} \text{F/m}$	—
Permeability of free space	$\mu_0$	$1.257 \times 10^{-6} \text{N/A}^2$	—

**Table 2.** Numerical values of physical constants in SI and Gaussian units, to four significant figures. Source: Phys. Rev. D **50**, 1233(1994).

### 3. Converting Numerical Values

The values of various fundamental physical constants in the two systems of units are summarized in Table 2. Naturally if you carry out a calculation consistently in one system of units, the answer emerges in that system of units. For example, an electric field calculated in Gaussian units comes out in statvolts/centimeter. Usually calculations in the Gaussian system are easier, because there are fewer constants (no  $\epsilon_0$ 's or  $\mu_0$ 's).

Once a numerical answer is calculated in one system of units, you may need to convert it to another. Table 1 can be used to convert mechanical quantities, and Table 3 will help in converting electromagnetic quantities.

Quantity	SI unit	Gaussian unit	Conversion
Charge $q$	Coulomb	statcoulomb	1 C = $(3) \times 10^9$ statcoulombs
Potential $\Phi$	Volt	statvolt	1 statvolt = $(3)00$ Volts
Electric field $\mathbf{E}$	Volt/m	statvolt/cm	1 statvolt/cm = $(3) \times 10^4$ V/m
Magnetic field $\mathbf{B}$	Tesla	Gauss	1 Tesla = $10^4$ Gauss

**Table 3.** Conversion of electromagnetic quantities between SI and Gaussian units. The notation  $(3)$  represents  $2.9979\dots$ , the same number (apart from a power of 10) that occurs in the speed of light. Note that one statvolt/cm is the same as one Gauss.

#### 4. Useful Things to Remember

The following is a list of useful things to remember when converting numbers, equations, or concepts between SI and Gaussian units.

- It is useful to remember that 1 statvolt is (approximately) 300 Volts, and that 1 Tesla is exactly  $10^4$  Gauss.
- It is worthwhile memorizing Maxwell's equations in both systems of units, both the vacuum equations (14)–(17), and the macroscopic equations (40)–(41).
- Equations of electrostatics without  $\mathbf{D}$  can be converted from SI to Gaussian units by replacing  $1/4\pi\epsilon_0$  by 1. Gaussian electrostatic equations (without  $\mathbf{D}$ ) containing  $e^2$  can be converted to SI units by replacing  $e^2$  by  $e^2/4\pi\epsilon_0$ . Probably the easiest way to convert other equations in electrostatics and all magnetic equations is to look them up in a table (see Sec. 5.)
- In Gaussian units, all the fields  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{P}$ ,  $\mathbf{M}$ ,  $\mathbf{D}$  and  $\mathbf{H}$  have the same dimensions. In particular, this means that in Gaussian units one statvolt/cm (the unit of  $\mathbf{E}$ ) is identical to 1 Gauss (the unit of  $\mathbf{B}$ ). Also, in Gaussian units the scalar and vector potentials  $\Phi$  and  $\mathbf{A}$  have the same dimensions. This makes it easy to remember where to put the factors of  $c$  in Maxwell's equations and other places.
- The electric polarization vector  $\mathbf{P}$  is defined as the electric dipole moment per unit volume, and the magnetization vector  $\mathbf{M}$  is defined as the magnetic dipole moment per unit volume, in *both* Gaussian and SI units.
- The energy of a electric dipole  $\mathbf{p}$  or a magnetic dipole  $\mathbf{m}$  in an external electric or magnetic field is given by  $-\mathbf{p} \cdot \mathbf{E}$  or  $-\mathbf{m} \cdot \mathbf{B}$  in *both* SI and Gaussian units. However, the dimensions of  $\mathbf{m}$  and  $\mathbf{B}$  are different in the two systems (because of the  $1/c$  factor discussed above).

#### 5. Dictionary of Electromagnetic Equations in SI and Gaussian Units

It is possible to give general rules for converting an equation from Gaussian to SI units or vice versa, but in practice it is easier to look things up in a dictionary. In this section we list the principal equations of electromagnetism in both SI and Gaussian units, apart from equations involving  $\mathbf{D}$  and

$\mathbf{H}$ , which are given in Sec. 6. The SI equations are on the left and the Gaussian equations on the right. If an equation is the same in both systems, it is listed only once.

The continuity equation:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (\text{SI, G}). \quad (3)$$

Coulomb's law:

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \quad (\text{SI}), \quad F = \frac{q_1 q_2}{r^2} \quad (\text{G}), \quad (4)$$

where  $F$  is the force between the two particles, directed along the line joining the particles and considered positive if repulsive, negative if attractive. Potential produced by charge distribution  $\rho(\mathbf{r})$  in electrostatics:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \quad (\text{SI}), \quad \Phi(\mathbf{r}) = \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \quad (\text{G}). \quad (5)$$

Electric field in terms of potential in electrostatics:

$$\mathbf{E} = -\nabla\Phi \quad (\text{SI, G}). \quad (6)$$

Potential in terms of electric field in electrostatics:

$$\Phi(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}' \quad (\text{SI, G}), \quad (7)$$

where the path connecting  $\mathbf{r}_0$  and  $\mathbf{r}$  is arbitrary. Poisson equation in electrostatics:

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \quad (\text{SI}), \quad \nabla^2\Phi = -4\pi\rho \quad (\text{G}). \quad (8)$$

In magnetostatics the current is steady and satisfies  $\nabla \cdot \mathbf{J} = 0$ . Vector potential in terms of current in magnetostatics:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \quad (\text{SI}), \quad \mathbf{A}(\mathbf{r}) = \frac{1}{c} \int d^3\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \quad (\text{G}), \quad (9)$$

where  $\mathbf{A}$  is in Coulomb gauge ( $\nabla \cdot \mathbf{A} = 0$ ).

Electric and magnetic fields in terms of potentials in the general (time-dependent) case:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} \quad (\text{SI}), \quad \mathbf{E} = -\nabla\Phi - \frac{1}{c} \frac{\partial\mathbf{A}}{\partial t} \quad (\text{G}), \quad (10)$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (\text{SI, G}). \quad (11)$$

These are valid in any gauge.

Gauge transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla g \quad (\text{SI, G}), \quad (12)$$

$$\Phi' = \Phi - \frac{\partial g}{\partial t} \quad (\text{SI}), \quad \Phi' = \Phi - \frac{1}{c} \frac{\partial g}{\partial t} \quad (\text{G}), \quad (13)$$

where  $g$  is the *gauge scalar* converting one choice of potentials  $(\Phi, \mathbf{A})$  into another choice  $(\Phi', \mathbf{A}')$ .

Vacuum Maxwell equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (\text{SI}), \quad \nabla \cdot \mathbf{E} = 4\pi\rho \quad (\text{G}), \quad (14)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad (\text{SI}), \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \quad (\text{G}), \quad (15)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{SI}), \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad (\text{G}), \quad (16)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{SI, G}). \quad (17)$$

Force on a charged particle of charge  $q$  in an electric and magnetic field,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (\text{SI}), \quad \mathbf{F} = q\left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}\right) \quad (\text{G}), \quad (18)$$

where  $\mathbf{v}$  is the velocity of the particle.

Electric dipole moment of a static charge distribution  $\rho(\mathbf{r})$ :

$$\mathbf{p} = \int d^3\mathbf{r} \mathbf{r} \rho(\mathbf{r}) \quad (\text{SI, G}). \quad (19)$$

The electric polarization vector  $\mathbf{P}$  is the electric dipole moment per unit volume in both systems of units. It gives rise to a bound charge density,

$$\rho_b = -\nabla \cdot \mathbf{P}. \quad (\text{SI, G}) \quad (20)$$

Energy of an electric dipole  $\mathbf{p}$  in an external electric field:

$$W = -\mathbf{p} \cdot \mathbf{E} \quad (\text{SI, G}). \quad (21)$$

See Sec. 6 for the electric susceptibility, the permittivity and dielectric constant, and the displacement vector  $\mathbf{D}$ .

Magnetic dipole moment of static current distribution  $\mathbf{J}(\mathbf{r})$ :

$$\mathbf{m} = \frac{1}{2} \int d^3\mathbf{r} \mathbf{r} \times \mathbf{J}(\mathbf{r}) \quad (\text{SI}), \quad \mathbf{m} = \frac{1}{2c} \int d^3\mathbf{r} \mathbf{r} \times \mathbf{J}(\mathbf{r}) \quad (\text{G}). \quad (22)$$

The magnetization vector  $\mathbf{M}$  is the magnetic dipole moment per unit volume in both systems of units. The bound current density  $\mathbf{J}_b$  depends on both  $\mathbf{M}$  and  $\mathbf{P}$  in time-dependent systems,

$$\mathbf{J}_b = \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \quad (\text{SI}), \quad \mathbf{J}_b = c \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \quad (\text{G}). \quad (23)$$

Energy of a magnetic dipole  $\mathbf{m}$  in an external magnetic field:

$$W = -\mathbf{m} \cdot \mathbf{B} \quad (\text{SI, G}). \quad (24)$$

See Sec. 6 for the magnetic susceptibility, the permeability, and the magnetic field vector  $\mathbf{H}$ .

The vacuum energy density of the electromagnetic field:

$$u = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \quad (\text{SI}), \quad u = \frac{1}{8\pi}(E^2 + B^2) \quad (\text{G}). \quad (25)$$

The vacuum energy flux (Poynting vector):

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad (\text{SI}), \quad \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \quad (\text{G}). \quad (26)$$

The vacuum momentum density  $\mathbf{g}$  is  $(1/c^2)$  times the Poynting vector in both systems of units,

$$\mathbf{g} = \epsilon_0 \mathbf{E} \times \mathbf{B} \quad (\text{SI}), \quad \mathbf{g} = \frac{1}{4\pi c} \mathbf{E} \times \mathbf{B} \quad (\text{G}). \quad (27)$$

Elementary treatments of  $u$ ,  $\mathbf{S}$  and  $\mathbf{g}$  in the presence of matter are often flawed by the neglect of thermodynamic considerations. Many of the equations found in textbooks involving these quantities are only correct at zero temperature, or under conditions of local thermal equilibrium and adiabaticity.

## 6. The Macroscopic Fields $\mathbf{D}$ and $\mathbf{H}$

The vectors  $\mathbf{P}$ ,  $\mathbf{D}$ ,  $\mathbf{M}$  and  $\mathbf{H}$  apply in the presence of bulk matter and are defined by some kind of macroscopic and/or statistical averaging process. The fields  $\mathbf{E}$  and  $\mathbf{B}$  can be either microscopic or macroscopic, depending on context.

The definitions of  $\mathbf{D}$  in terms of  $\mathbf{E}$  and  $\mathbf{P}$  and of  $\mathbf{H}$  in terms of  $\mathbf{B}$  and  $\mathbf{M}$  in Gaussian and SI units cannot be reconciled by taking the relation (2) into account, or (in the case of  $\mathbf{H}$ ,  $\mathbf{B}$  and  $\mathbf{M}$ ) the extra factor of  $c$  in the definitions of magnetic quantities. Instead, one must take into account an extra factor of  $4\pi$  that has been inserted into the definitions in one of the two systems of units. This is an extra layer of confusion in the comparison between the two systems of units. In practice it is probably easiest to look formulas up.

The displacement vector  $\mathbf{D}$  in terms of the electric field  $\mathbf{E}$  and the polarization  $\mathbf{P}$ :

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (\text{SI}), \quad \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} \quad (\text{G}). \quad (28)$$

In a linear medium, the polarization  $\mathbf{P}$  is a linear function of the electric field  $\mathbf{E}$ ,

$$P_i = \epsilon_0 \sum_j (\chi_e)_{ij} E_j \quad (\text{SI}), \quad P_i = \sum_j (\chi_e)_{ij} E_j \quad (\text{G}), \quad (29)$$

where the dimensionless electric susceptibility tensor  $(\chi_e)_{ij}$  is characteristic of the medium. In an isotropic medium,  $\chi_e$  becomes a scalar, and we have

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad (\text{SI}), \quad \mathbf{P} = \chi_e \mathbf{E} \quad (\text{G}). \quad (30)$$

Although  $\chi_e$  is dimensionless, it does not have the same value in the two systems of units:

$$\chi_e^{\text{SI}} = 4\pi \chi_e^{\text{G}}. \quad (31)$$

In a linear medium (for simplicity taken to be isotropic),  $\mathbf{D}$  is proportional to  $\mathbf{E}$ ,

$$\mathbf{D} = \epsilon \mathbf{E} \quad (\text{SI, G}), \quad (32)$$

where  $\epsilon$ , the permittivity or dielectric constant of the medium, is given in terms of the susceptibility by

$$\epsilon = \epsilon_0(1 + \chi_e) \quad (\text{SI}), \quad \epsilon = 1 + 4\pi\chi_e \quad (\text{G}). \quad (33)$$

The permittivity  $\epsilon$  is dimensionless in Gaussian units. The same quantity is sometimes called the *relative* permittivity in SI units, and denoted  $\epsilon_r = \epsilon/\epsilon_0$ .

The magnetic field vector  $\mathbf{H}$  in terms of  $\mathbf{B}$  and magnetization  $\mathbf{M}$ :

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad (\text{SI}), \quad \mathbf{H} = \mathbf{B} - 4\pi \mathbf{M} \quad (\text{G}). \quad (34)$$

In a linear medium, the magnetization  $\mathbf{M}$  is a linear function of the magnetic field  $\mathbf{H}$ ,

$$M_i = \sum_j (\chi_m)_{ij} H_j \quad (\text{SI, G}), \quad (35)$$

where the dimensionless magnetic susceptibility tensor  $(\chi_m)_{ij}$  is characteristic of the medium. In an isotropic linear medium,  $\chi_m$  becomes a scalar, and we have

$$\mathbf{M} = \chi_m \mathbf{H} \quad (\text{SI, G}). \quad (36)$$

Although  $\chi_m$  is dimensionless, it does not have the same value in the two systems of units:

$$\chi_m^{\text{SI}} = 4\pi \chi_m^{\text{G}}. \quad (37)$$

In a linear medium (for simplicity taken to be isotropic),  $\mathbf{B}$  is proportional to  $\mathbf{H}$ ,

$$\mathbf{B} = \mu \mathbf{H} \quad (\text{SI, G}), \quad (38)$$

where  $\mu$ , the permeability of the medium, is given in terms of the magnetic susceptibility by

$$\mu = \mu_0(1 + \chi_m) \quad (\text{SI}), \quad \mu = 1 + 4\pi\chi_m \quad (\text{G}). \quad (39)$$

The permeability  $\mu$  is dimensionless in Gaussian units. The same quantity is sometimes called the *relative* permeability in SI units, and denoted  $\mu_r = \mu/\mu_0$ .

The inhomogeneous, macroscopic Maxwell equations (in the presence of matter) are

$$\nabla \cdot \mathbf{D} = \rho_f \quad (\text{SI}), \quad \nabla \cdot \mathbf{D} = 4\pi \rho_f \quad (\text{G}), \quad (40)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \quad (\text{SI}), \quad \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_f + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \quad (\text{G}), \quad (41)$$

where  $\rho_f$  and  $\mathbf{J}_f$  are the free charge and current densities, respectively. The homogeneous Maxwell equations are the same as in vacuum, Eqs. (16)–(17).

## 7. The Physics Behind the Choices

Fundamental physical concepts are independent of any choice of units, but can be lost in the mass of conventions that are employed. Therefore it is worthwhile looking at some of the basic physics of electromagnetism and the rationales that led to choices listed above.

Coulomb's law states that two charges exert a force on one another that is proportional to the product of the charges and inversely proportional to the distance between the charges,

$$F = k_e \frac{Q_1 Q_2}{d^2}, \quad (42)$$

where  $Q_1$  and  $Q_2$  are the charges and  $d$  is the distance. The force  $F$  is a scalar in this equation but the force vector is directed along a line between the particles and is repulsive (attractive) if  $F > 0$  ( $F < 0$ ).

The constant  $k_e$  obviously depends on the units chosen for charge. One possibility is to use an arbitrary definition, for example, the amount of charge deposited on a standard glass rod by three rubs with a standard cat fur, or the amount of charge needed to deposit so many grams of silver in an electrochemical cell. Arbitrary systems of units like this were once common, for example, the original definition of the second was 1/86,400 of a day (the rotational period of the earth), and the meter was originally defined (by Napoleon's commission) as 1/10,000,000 of the distance from the equator to the north pole of the earth. Later these definitions were changed several times to enhance the reproducibility of the standards. Now the second is defined in terms of the period of the radiation produced in a hyperfine transition of the cesium 133 atom, and the meter is defined so that the speed of light in vacuum is 299,792,458 m/sec exactly.

If an arbitrary unit of charge is chosen, then the constant  $k_e$  must be measured experimentally. Another possibility is to define the unit of charge so that the constant  $k_e$  takes on a simple value. The value  $k_e = 1$  is especially simple, and this is the choice made in the Gaussian system of units (with the cgs system used for units of distance and force). That is, the statcoulomb is the charge such that two charges of one statcoulomb each at a distance of one centimeter exert a force of one dyne on one another. In SI units, the unit of charge (the Coulomb) causes the electrostatic force constant  $k_e$  not to have a particularly simple value, so that insofar as electrostatics is concerned, the definition of the Coulomb appears to be arbitrary. Actually, the Coulomb is defined so that the magnetic force law looks (somewhat) simple, a point we will return to below. In SI units, the constant  $k_e$  is written as  $1/4\pi\epsilon_0$ . Given that the value of  $k_e$  is not particularly simple, there is the question as to why this constant is written in such a complicated way. The purpose of the  $4\pi$  is to rationalize Maxwell's equations, but the manner of writing  $\epsilon_0$  in the constant and the 0-subscript are unfortunate, since they make the simple physics of Coulomb's law look complicated.

Next we consider the definition of the electric field. We begin with an imaginary experiment, in which a set of charges are present in some region of space. We treat these charges as if they were the only charges in the universe (any others are assumed to be far enough away that they have no effect on our experiment). The charges may be in an arbitrary state of motion. We pick out one

charge  $Q$ , call it the “test” charge, and measure the force on it under different circumstances. When  $Q$  is stationary, we call the force  $\mathbf{F}_e$  on  $Q$  the “electric” force, and we find experimentally that it is proportional to  $Q$ . It also depends on the position  $\mathbf{x}$  of  $Q$  and the time  $t$  at which the measurement is made. Then we define the electric field  $\mathbf{E}(\mathbf{x}, t)$  as the force per unit charge on  $Q$ , so that

$$\mathbf{F}_e = Q\mathbf{E}(\mathbf{x}, t). \quad (43)$$

This definition of electric field (electric force per unit charge) is the same in both SI and Gaussian units.

Suppose now that the test charge  $Q$  in our imaginary experiment is in motion with velocity  $\mathbf{v}$ . Then we define the “magnetic force” on  $Q$  as the total force  $\mathbf{F}$  minus the electric force  $\mathbf{F}_e$  (the force that  $Q$  would experience if it were stationary),

$$\mathbf{F}_m = \mathbf{F} - \mathbf{F}_e. \quad (44)$$

Then we can take the following as experimental facts. First,  $\mathbf{F}_m$  (like  $\mathbf{F}_e$ ) is proportional to  $Q$ . Second, it is linear in the velocity  $\mathbf{v}$ , that is, if we do several measurements with the same  $Q$  but with different velocities, we find that if velocity  $\mathbf{v}_1$  gives magnetic force  $\mathbf{F}_{m1}$  and velocity  $\mathbf{v}_2$  gives magnetic force  $\mathbf{F}_{m2}$ , then velocity  $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$  gives magnetic force  $a_1\mathbf{F}_{m1} + a_2\mathbf{F}_{m2}$ . Third, the magnetic force is orthogonal to the velocity,  $\mathbf{v} \cdot \mathbf{F}_m = 0$ . The first and second properties mean that there is a matrix  $\mathbf{M}$ , a function of the position  $\mathbf{x}$  of  $Q$  and the time  $t$  of the measurement, such that

$$\mathbf{F}_m = Q\mathbf{M}\mathbf{v}, \quad (45)$$

while the third property means that  $\mathbf{M}$  is antisymmetric,  $M_{ij} = -M_{ji}$ . An antisymmetric matrix has 3 independent components, which are conveniently expressed in terms of a vector  $\mathbf{X} = (X_x, X_y, X_z)$  by

$$\mathbf{M} = \begin{pmatrix} 0 & X_z & -X_y \\ -X_z & 0 & X_x \\ X_y & -X_x & 0 \end{pmatrix}, \quad (46)$$

so that Eq. (45) has the form

$$\mathbf{F}_m = Q\mathbf{v} \times \mathbf{X}. \quad (47)$$

Now we define the magnetic field. Unlike the definition of the electric field, the magnetic field is defined differently in the SI and Gaussian systems. In the SI system, the magnetic field is defined by  $\mathbf{B} = \mathbf{X}$ , whereas in the Gaussian system, it is defined by  $\mathbf{B} = c\mathbf{X}$ . This explains the extra factor of  $c$  in the Gaussian force law (18). The Gaussian definition causes  $\mathbf{E}$  and  $\mathbf{B}$  to have the same dimensions (the statvolt/cm and the Gauss are actually identical units).

In these imaginary experiments we have not said how the force on the test charge  $Q$  is specified by the rest of the charges (the “source” charges). Two cases are simple. If the source charges are stationary (and have been so for a long time), then the electric force on the test charge is given by

$$\mathbf{F}_e = k_e Q \int d^3\mathbf{x}' \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}, \quad (48)$$



where  $\rho$  is the source charge density,  $k_e$  is the electric constant introduced in Eq. (42) and  $\mathbf{x}$  is the position of the test charge. If the source charges are in motion but the source current is steady (and has been so for a long time), then the magnetic force on the test charge is given by

$$\mathbf{F}_m = k_m Q \mathbf{v} \times \int d\mathbf{x}' \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}, \quad (49)$$

where  $\mathbf{J}$  is the source current density and  $k_m$  is a new constant (the magnetic force constant). Equations (48) and (49) have been written purely in terms of forces, without reference to the electric or magnetic fields (the latter of which is defined differently in the two systems of units), because we wanted to have expressions for the electric and magnetic forces that were as independent of conventions as possible. No assumptions have been made about the unit of charge or the definitions of the electric and magnetic fields in these equations.

Taking the ratio, dimensionally speaking, of Eqs. (48) and (49) shows that the ratio  $k_e/k_m$  has dimensions of velocity<sup>2</sup>. Therefore the value of this ratio is independent of the choice made for the unit of charge. In fact, both  $k_e$  and  $k_m$  can be measured in laboratory experiments (using some arbitrary unit of charge, if necessary), and their ratio computed. This was done in the early history of electromagnetism, and it was found that

$$\frac{k_e}{k_m} = c^2, \quad (50)$$

to experimental accuracy. Now we believe that the relation (50) is exactly true (at least, it is a fundamental tenet of electromagnetic theory). Thus, the electric and magnetic force constants are not independent. In Gaussian units, Eq. (50) implies  $k_m = 1/c^2$ , and in SI units, where  $k_m$  is usually written  $\mu_0/4\pi$ , it implies  $\epsilon_0\mu_0 = 1/c^2$ . The situation is summarized in Table 4.

Units	$k_e$	$k_m$
Gaussian	1	$\frac{1}{c^2}$
SI	$\frac{1}{4\pi\epsilon_0}$	$\frac{\mu_0}{4\pi}$

**Table 4.** Electric and magnetic force constants in the different systems of units.

In SI units, the unit of charge (the Coulomb) is chosen so as to make the magnetic force constant  $k_m = \mu_0/4\pi$  take on the value  $10^{-7}$  (exactly). One could say this is a way of making the magnetic force law look simple (instead of the electric force law, the choice made in Gaussian units). Of course,  $10^{-7}$  is not as simple as 1, but the result is that the Ampere (one Coulomb/sec) is a reasonable unit in simple laboratory experiments and common electrical devices.