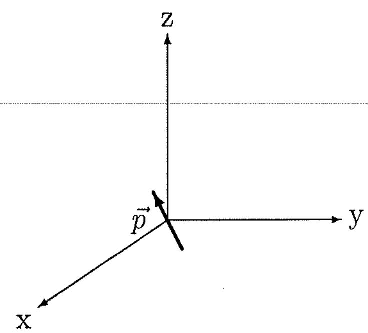
E & M Qualifier

August 18, 2011

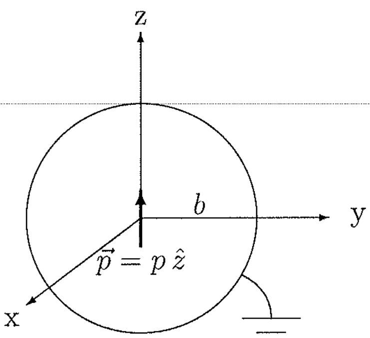
To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



part (a)



parts (b) and (c)

1. (a) {3 pts} Give the potential for a static point electric dipole, with dipole moment $\mathbf{p}$, located at the origin and pointing in an arbitrary direction (see Figure).

   (b) {3 pts} If the dipole moment points in the z-direction ($\mathbf{p} = p\,\hat{\mathbf{z}}$) and is surrounded by a thin grounded conducting sphere of radius b (see Figure), what is the electrostatic potential inside the sphere?

   (c) {4 pts}Compute the static electric charge density that exists on the inner surface of the thin conducting sphere?

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E + M #1

Gaussian

a) The general potential for an electric dipole is:

$$\Phi = \frac{\vec{p} \cdot \vec{r}}{r^3}$$

b) Since our conducting spherical shell is grounded, we proceed by the method of images to determine

2. (a) {3 pts} Use Maxwell's equations to derive the continuity equation (in differential form) relating charge ρ and current density $\mathbf{J}$.

   (b) {2 pts} Use the divergence theorem and the results of part (a) to derive the conservation of charge, $\dot{Q} = 0$, for a bounded charge distribution.

   (c) {2 pts} Show that the continuity equation can be written in 4-vector form using the 4-current J^{μ}. Define all symbols you use.

   (d) {3 pts} Use Maxwell's equations to derive Poynting's theorem, the equation analogous to the continuity of charge equation relating the Poynting vector $\mathbf{S}$ and the energy density u in the $\mathbf{E}$ and $\mathbf{B}$ fields, that represents conservation of electromagnetic energy. Assume the electric and magnetic fields are in vacuum, i.e., no charges, currents, or polarizable materials are present.

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E+M #2

Gaussian

a) We know: $\nabla \cdot D = 4\pi\rho$ $\qquad \nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$

$\nabla \cdot B = 0 \qquad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}$

Continuity Eqn: $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$

$$\frac{\partial}{\partial t} \nabla \cdot D = 4\pi \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \left(\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} \right) = \frac{4\pi}{c} \nabla \cdot J$$

$$\nabla \cdot \nabla \times H - \frac{1}{c}\frac{\partial}{\partial t} \nabla \cdot D = \frac{4\pi}{c} \nabla \cdot J$$

$$\cancelto{0}{\nabla \cdot \nabla \times H} - \frac{4\pi}{c}\frac{\partial \rho}{\partial t} = \frac{4\pi}{c} \nabla \cdot J$$

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot J$$

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot J \checkmark$$

b) In general, the Divergence theorem says: $\int_V \vec{\nabla} \cdot \vec{A}\, dV = \oint \vec{A} \cdot d\vec{a}$

↳ If we want to show conservation of charge, we define a surface that encloses all of our charge with $J_{bound} = 0$

$$\Rightarrow Q_{enc} = \frac{1}{4\pi} \int \rho \cdot dV$$

$$\dot{Q} = 0 = \frac{1}{4\pi} \frac{\partial}{\partial t} \int \rho \cdot dV$$

$$= \frac{-1}{4\pi} \int \nabla \cdot J \, dV$$

$$= \frac{-1}{4\pi} \int \cancelto{0}{J \cdot d\vec{a}}$$

$$= 0 \checkmark$$

c) $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}$

* if $J^0 = c\rho$, $J^i = j_i$, $i \in \{x, y, z\}$

↳ $\frac{\partial}{\partial x^\mu} J^\mu = 0 \checkmark$

#2 (cont.)

d) In Gaussian units: Poynting vector: $\vec{S} = \frac{c}{4\pi} E \times H$

Energy density: $u = \frac{1}{4\pi}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$

$$\vec{E}\cdot\left(\nabla\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\right) = \vec{E}\cdot\frac{4\pi}{c}\vec{J}$$

$$\vec{E}\cdot\nabla\times\vec{H} - \frac{1}{c}\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \cancel{\frac{4\pi}{c}\vec{E}\cdot\vec{J}}$$ 0 b/c no current

$$-\nabla\cdot(E\times H) + H\cdot(\nabla\times E) - \frac{1}{c}E\cdot\frac{\partial D}{\partial t} = 0$$

$$-\nabla\cdot(E\times H) + H\cdot\left(-\frac{1}{c}\frac{\partial B}{\partial t}\right) - \frac{1}{c}E\cdot\frac{\partial D}{\partial t} = 0$$

$$-\nabla\cdot(E\times H) - \frac{1}{c}\left[H\cdot\frac{\partial B}{\partial t} + E\cdot\frac{\partial D}{\partial t}\right] = 0$$

$$-\nabla\cdot(E\times H) - \frac{1}{c}\frac{\partial}{\partial t}\left(E\cdot D + H\cdot B\right) = 0 \leftarrow$$ b/c E, H have no time dependence

$E = \epsilon D$, $H = \frac{1}{\mu}B$

* multiplying by $\frac{-c}{4\pi}$

$$\nabla\cdot\vec{S} + \frac{\partial u}{\partial t} = 0 \checkmark$$

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E + M #2

Gaussian

a) The generalized Maxwell Eqns are:

$$\nabla \cdot D = 4\pi \rho_f \qquad \nabla \cdot B = 0$$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$$

$$\nabla \cdot \left(\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t}\right) = \nabla \cdot \frac{4\pi}{c}\vec{J}_f$$

$$\cancel{\nabla \cdot (\nabla \times H)}^{0} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot D) = \frac{4\pi}{c}\nabla \cdot \vec{J}_f$$

$$-\cancel{\frac{4\pi}{c}}\frac{\partial}{\partial t}\rho_f = \cancel{\frac{4\pi}{c}}\nabla \cdot \vec{J}_f$$

$$-\frac{\partial}{\partial t}\rho_f = \nabla \cdot \vec{J}_f \checkmark$$

b) The divergence theorem states: $\int \nabla \cdot A \, dV = \oint A \cdot d\vec{a}$

$$\int \nabla \cdot \vec{J}_f \, dV = -\frac{\partial}{\partial t}\int \rho_f \, dV$$

$$\oint \vec{J}_f \cdot d\vec{a} = -\frac{\partial}{\partial t} Q$$

* if our arbitrary surface totally encloses region of $\vec{J}_f$, the flux is 0

$$0 = \dot{Q} \checkmark$$

c) We define J^μ as follows

$$J^\mu = \begin{bmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{bmatrix} \Rightarrow \quad 0 = \frac{1}{c}\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}J_i$$

$$= \frac{1}{\partial x^\mu} J^\mu \checkmark$$

#2 (cont.)

d) * Remember that $\vec{S} = \frac{c}{4\pi} E \times H$

$$U = \frac{1}{4\pi}(E \cdot D + B \cdot H)$$

$$\Rightarrow \quad E \cdot \left(\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} \right) = E \cdot 0 \quad \text{(b/c in vaccum)}$$

$$E \cdot \nabla \times H - \frac{1}{c}\frac{\partial}{\partial t} E \cdot D = 0$$

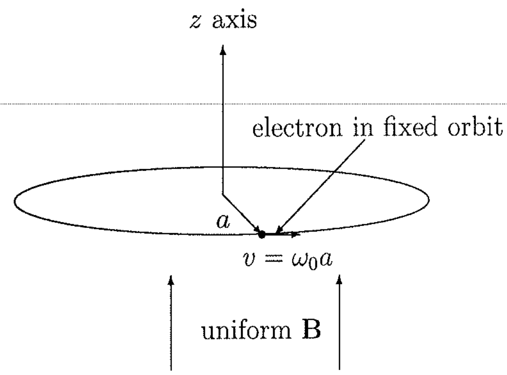
$$-\nabla \cdot (E \times H) + H \cdot (\nabla \times E) - \frac{1}{c}\frac{\partial}{\partial t} E \cdot D = 0$$

$$-\nabla \cdot (E \times H) + H \cdot \left(-\frac{1}{c}\frac{\partial B}{\partial t} \right) - \frac{1}{c}\frac{\partial}{\partial t} E \cdot D = 0$$

$$-\nabla \cdot (E \times H) - \frac{1}{c}\frac{\partial}{\partial t}(E \cdot D + H \cdot B) = 0$$

$$\frac{c}{4\pi}\left(\cdot \nabla \cdot (E \times H) + \frac{1}{4\pi}\frac{\partial}{\partial t}(E \cdot D + H \cdot B) \right) = 0$$

$$\nabla \cdot S + \frac{\partial}{\partial t} U = 0 \checkmark$$

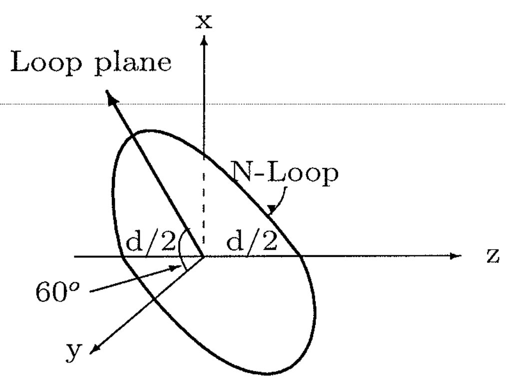


3. An electron is <u>constrained</u> to revolve (without friction) in a circular orbit of radius a and does so with an initial angular velocity ω_0 (assume $\omega_0 a << c$).

   (a) {3 pts} What is the magnetic dipole moment due to the electron's motion?

   (b) {5 pts}
   A uniform magnetic field **B**, parallel to the angular momentum of the electron is slowly turned on. Derive an expression for $\delta\omega(B)$, the change in angular speed of the electron as a function of the magnetic induction B (Hint: Use Faraday's Law of induction and Newtonian mechanics).

   (c) {2 pts} Does $\delta\omega$ increase or decrease the magnetic dipole moment?



4. In vacuum, a plane electromagnetic wave of angular frequency $\omega = c\,k = 2\pi\, c/\lambda$ travels parallel to the z-axis. The wave has an electric field given by

$$\mathbf{E} = \hat{\mathbf{j}}\ E_0 e^{i(kz-\omega t)},$$

A small N-loop (i.e., an N-turn circular coil) of diameter d very much smaller than the wavelength λ ($d << \lambda$) acting as an antenna is located with its center at the origin. It is oriented so that a diameter of the coil lies along the z-axis and the plane of the coil makes an angle $\theta = 60^o$ with the y-axis.

(a) {3 pts} Use Maxwell's equations to obtain the **B** field associated with the above wave?

(b) {3 pts} Compute the Magnetic flux through the N turn coil as a function of time.

(c) {4 pts} What is the peak EMF induced in the antenna?

5. A charged particle with charge q and mass m starts from rest on the inner plate (radius a) of a cylindrical capacitor and is accelerated towards the outer plate (radius b). Orient the coordinates so that q starts at $(x, y) = (a, 0)$ and is accelerated along the +x direction until it reaches $(x, y) = (b, 0)$.

   (a) {2 pts} If the charge/length on the capacitor λ_0 is constant find the electric field causing the acceleration by using Gauss's law. Assume the capacitor is very long compared to the radii a and b and assume the electric field can is cylindrically symmetric.

   (b) {4 pts} Write down the 4-D [or (3+1) D] special relativistic version of Newton's equations for the motion of a point charge experiencing the Lorentz force (the force due to an arbitrary external **E** and **B** field).

$$\frac{dp^\mu}{d\tau} = ?$$

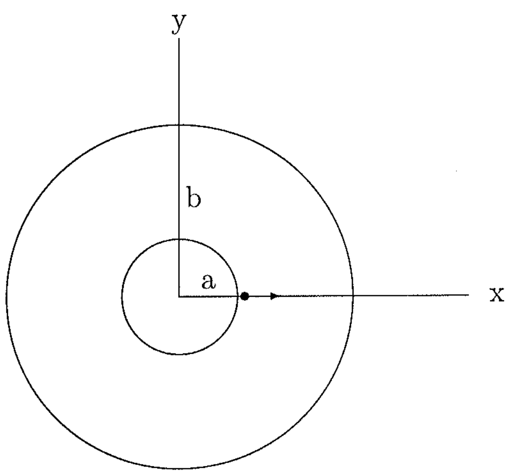
   or

$$\frac{dm\gamma c^2}{dt} = ? \qquad \text{and} \qquad \frac{d\vec{p}}{dt} = ?$$

   Be sure to define p^μ and $\vec{p}$ as well as the Lorentz force terms that appear on the right hand sides of the above equations.

   (c) {4 pts} Apply your answer to part (b) to the field you found in part (a) and integrate your dynamical equations to obtain the particle's energy when it reaches the outer plate. You can easily compute $\gamma(x)$ from Newton's equations even though the acceleration is not constant.

   You are **not** asked to compute $x(t)$ or $x(\tau)$ nor how long it took to reach $x = b$.



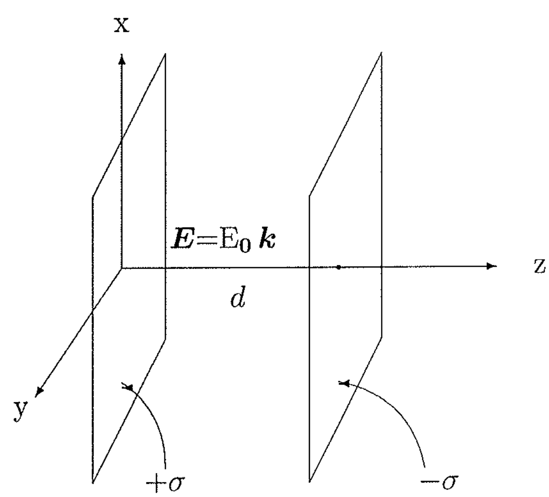
6. A uniform static electric field $\mathbf{E} = E_0\mathbf{k}$ exists between two large thin conducting metal plates. The positive plate is at $z = 0$ and the negative plate is at $z = a$. You can assume the plates are infinitely large in the x-y directions.

(a) {1 pts} Use Maxwell's equations to relate the value of E_0 to the surface charge density $\pm\,\sigma$ on the plates.

(b) {4 pts} Lorentz transform $F^{\alpha\beta}$ to obtain the $\mathbf{E}$ and $\mathbf{B}$ fields seen by an observer moving between the plates with velocity c/2 $\hat{\imath}$?

(c) {2 pts} What are the 4-current densities J^σ of the plates in the rest frame and in a frame moving with the observer?

(d) {3 pts} Show that Maxwell's inhomogeneous equations are satisfied by your fields and charge-currents in the moving frame. (Hint: Using $\mathbf{E}$ and $\mathbf{B}$ rather than $F^{\alpha\beta}$ is probably easier.)



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E+M #6

Gaussian

a) Using Gauss Law, we see

$$\int \vec{E} \cdot d\vec{a} = 4\pi q_{enc}$$

$$E \cdot 2A = 4\pi \sigma A$$

$$E = 2\pi\sigma \quad \text{(for one plate)}$$

$$\hookrightarrow \vec{E}_0 = 4\pi\sigma$$

b) $F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

* In matrix form $F' = \Lambda F \Lambda^T$

$$F' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & +\beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\gamma E_z \\ 0 & 0 & 0 & \beta\gamma E_z \\ 0 & 0 & 0 & 0 \\ \gamma E_z & -\beta\gamma E_z & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{E}' = \gamma \vec{E}$$

$$\vec{B}' = -\beta\gamma E_z \hat{y}$$

* If $\beta = \frac{v}{c} = \frac{1}{2}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{1}{4}}}$

$$= \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2}{\sqrt{3}}$$

#6 (cont.)

c) The 4-current vector J^α is defined as $J^\alpha = \langle c\rho, \vec{J} \rangle$ and transforms according to: $J'^\beta = \Lambda^\beta_\alpha J^\alpha$

$$J^\alpha = \langle c[\sigma\delta(z) - \sigma\delta(z-d)], 0 \rangle$$

$$J' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\rho \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J' = \begin{bmatrix} \gamma c\rho \\ -\beta\gamma c\rho \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c\rho' \\ J'_x \\ J'_y \\ J'_z \end{bmatrix} \Rightarrow \begin{aligned} \rho' &= \gamma\sigma(\delta(z) - \delta(z-d)) \\ \vec{J} &= \langle -\beta\gamma c\sigma(\delta(z) - \delta(z-d)), 0, 0 \rangle \end{aligned}$$

d) The inhomogeneous Maxwell equations in Gaussian units are:

$$\nabla \cdot E' = 4\pi\rho'$$

$$\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$$

– To test Gauss Law

$$\begin{aligned} \nabla \cdot E' &= \frac{\partial}{\partial z} \frac{2}{\sqrt{3}}(4\pi\sigma) \\ &= \frac{8\pi}{\sqrt{3}} \frac{\partial}{\partial z}\left(\sigma[\Theta(z) - \Theta(z-d)]\right) \leftarrow \text{Assumption works, not sure why since } \sigma \text{ only exists on plates} \\ &= \frac{8\pi}{\sqrt{3}}\sigma(\delta(z) - \delta(z-d)) \\ &= 4\pi\left(\frac{2}{\sqrt{3}}\sigma[\delta(z) - \delta(z-d)]\right) \\ &= 4\pi\rho' \checkmark \end{aligned}$$

6 (cont)

d) - Evaluating Ampere's Law

$$\nabla \times B' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & -\beta\gamma E_z & 0 \end{vmatrix} = \langle -\partial_z(\beta\gamma E_z),\ 0,\ \partial_x(-\beta\gamma E_z) \rangle$$

$$= \langle -\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{3}} 4\pi\sigma[\Theta(z) - \Theta(z-d)]\right),\ 0,\ \frac{\partial}{\partial x}\left(\frac{-1}{\sqrt{3}} 4\pi\sigma[\Theta(z) - \Theta(z-d)]\right) \rangle$$

$$= \langle -\frac{4\pi\sigma}{\sqrt{3}}(\delta(z) - \delta(z-d)),\ 0,\ 0 \rangle$$

$$= \langle -\frac{4\pi}{c}\left(\frac{c\sigma}{\sqrt{3}}[\delta(z) - \delta(z-d)]\right),\ 0,\ 0 \rangle$$

$$= \langle -\frac{4\pi}{c}(\beta\gamma c E_z),\ 0,\ 0 \rangle$$

$$= -\frac{4\pi}{c} \langle J'_x,\ 0,\ 0 \rangle \checkmark$$

(* Note: $\frac{1}{c}\frac{\partial E}{\partial t} = 0$ b/c no time dependence in formula)