E & M Qualifier

January 7, 2015

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

1. (a) [2 pts] Write down all four of Maxwell's equations in differential form for the four fields $\mathbf{E}, \mathbf{H}, \mathbf{D}$, and $\mathbf{B}$. Which two are homogeneous equations and which two are non-homogeneous equations? Which one is called Faraday's law of induction, which one is called Ampère's law, and which one is equivalent to Gauss's law?

   (b) [1 pts] Maxwell's equations simplify for fields defined in static, nonconducting, homogeneous, isotropic, and linear materials. Eliminate $\mathbf{H}$ and $\mathbf{D}$ from two of the four equations and simplify these two equations by assuming $\mathbf{B} = \mu\mathbf{H}$, and $\mathbf{D} = \epsilon\mathbf{E}$, where the permittivity ϵ and permeability μ are both real positive constants. (Do not assume the free charge density or the free current density vanishes.)

   (c) [2 pts] Using your Maxwell equations from part (a) explain exactly why $\mathbf{E}$ and $\mathbf{B}$ can be replaced by potentials ϕ and $\mathbf{A}$. What freedom (non-uniqueness) exists in ϕ and $\mathbf{A}$ for a given pair of fields $\mathbf{E}$ and $\mathbf{B}$?

   (d) [2 pts] Replace $\mathbf{E}$ and $\mathbf{B}$ by ϕ and $\mathbf{A}$ in your four Maxwell equations of part (b) and explain how it is possible to solve them for ϕ and $\mathbf{A}$ if these quantities are not unique?

   (e) [3 pts]Given charge and current densities $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ bounded in space (i.e., contained entirely in $r < R$) and static before some early time $t = t_0$, simplify your Maxwell equations from part (d) by using the Coulomb ($\nabla \cdot \mathbf{A} = 0$) gauge constraint. Give the retarded solution for ϕ and $\mathbf{A}$ to your Maxwell equations as 3-d spatial integrals.

2. Consider a capacitor composed of two thin concentric spherical metal shells, the inner one with radius a and the outer one with radius b. The region between the spherical metal shells is filled with a linear dielectric with permittivity $\epsilon = k/r^2$. A charge $+Q$ exists on the inner metallic shell and $-Q$ on the outer metallic shell.

(a) [2 pts] Find the electric displacement $\mathbf{D}$ everywhere in space.

(b) [3 pts] Find the capacitance of the configuration.

(c) [5 pts] Calculate the bound charge densities within the dielectric and on its surfaces, and verify that the total net bound charge is zero.

3. In this problem you will construct the 4-dimensional (4-d) electromagnetic stress-energy-momentum tensor from the 4-d electromagnetic field tensor $F^{\mu\nu}$ in **Gaussian** units. Recall that $F^{\mu\nu}$ is antisymmetric ($F^{\mu\nu} = -F^{\nu\mu}$) and is constructed from components of the electric and magnetic induction fields $\mathbf{E}$ and $\mathbf{B}$ by choosing

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon^{ijk} B^k.$$

Here we use the Einstein convention of summing over repeated indices, where Greek letters run from 0 to 3, while Latin letters run from 1 to 3. The symbol ϵ^{ijk} is the totally anti-symmetric 3-dimensional Levi-Civita symbol and satisfies $\epsilon^{123} = +1$. The time coordinate is given by $x^0 = ct$, where c is the speed of light and the 4-d metric used to raise and lower Greek indices is $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

(a) [3 pts] Define the 4-current J^μ and show that in a region containing no polarizable materials ($\epsilon = \mu = 1$) Maxwell equations are written in 4-d form as

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.$$

(b) [1 pts] From Maxwells equations prove that charge is conserved, i.e., show that

$$\partial_\mu J^\mu = 0.$$

(c) [3 pts] The 4-d stress-energy-momentum tensor is a traceless symmetric second-rank tensor, quadratic in the field strengths defined by

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\beta\alpha}\right].$$

Show that the 4 parts of $T^{\mu\nu}$ can be identified with the electromagnetic energy density u by $T^{00} = u$, the momentum density $\mathbf{g}$ and the Poynting vector $\mathbf{S}$ by $T^{0i} = T^{i0} = cg^i = S^i/c$, and the 3-d Maxwell stress tensor $\overleftrightarrow{\mathbf{T}}_M$ by $T^{ij} = -T_M^{ij}$. Be sure to give u, $\mathbf{g} = \mathbf{S}/c^2$, and $\overleftrightarrow{\mathbf{T}}_M$ as functions of $\mathbf{E}$ and $\mathbf{B}$.

(d) [3 pts] Use Maxwell's equations to compute $\partial_\mu T^{\mu\nu}$. Show that $\partial_\mu T^{\mu\nu} = 0$ in a region where $J^\mu = 0$ and that this one 4-d vector equation is equivalent to the local conservation of electromagnetic energy and momentum in 3-d, i.e., that

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = 0,$$

and

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \overleftrightarrow{\mathbf{T}}_M.$$

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E+M #3

a) We define the 4-current J^{μ} as:

$$J^{\mu} = \langle c\rho, \vec{j} \rangle$$

* To prove $\partial_{\nu} F^{\nu\mu} = \frac{4\pi}{c} J^{\mu}$, we consider two cases:

$\mu = 0$:

$$\partial_{\nu} F^{\nu 0} = \frac{4\pi}{c} J^{0}$$

$$\partial_0 F^{00} + \partial_i F^{i0} = \frac{4\pi}{c} c\rho$$

$$0 + \nabla \cdot E = 4\pi\rho \checkmark$$

$\mu = i$:

$$\partial_0 F^{0i} + \partial_i F^{ij} = \frac{4\pi}{c} \vec{j}^{i}$$

$$-\frac{1}{c} \frac{\partial}{\partial t}(\vec{E}) - \partial_j \epsilon_{ijk} B_k = \frac{4\pi}{c} j^i$$

$$-\frac{1}{c} \frac{\partial E_i}{\partial t} + \epsilon_{jik} \frac{\partial}{\partial x^i} B_k = \frac{4\pi}{c} j^i$$

$$-\frac{1}{c} \frac{\partial}{\partial t} \vec{E} + \nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} \checkmark$$

* To prove $\partial_{\lambda} F_{\mu\nu} + \partial_{\mu} F_{\nu\lambda} + \partial_{\nu} F_{\lambda\mu} = 0$, we again consider two cases

$\lambda, \mu, \nu \in \{1,2,3\}$:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

$$\frac{\partial}{\partial x}(-B_x) + \frac{\partial}{\partial y}(-B_y) + \frac{\partial}{\partial z}(-B_z) = 0$$

$$\nabla \cdot B = 0 \checkmark$$

$\lambda, \mu, \nu \in \{0, i, j\}$:
$i \neq j$

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0$$

$$\partial_0(-\epsilon_{ijk} B_k) - \partial_i(E_j) + \partial_j(E_i)$$

$$\frac{1}{c} \frac{\partial}{\partial t} \vec{B} + \nabla \times E = 0 \checkmark$$

#3 (cont.)

b) The equation of charge conservation says $\partial_\mu J^\mu = 0$

$$\partial_\mu J^\mu = \partial_\mu(\partial_\nu F^{\nu\mu})$$

$$= \partial_\nu \partial_\mu F^{\nu\mu} \quad \text{(order of partial differentiation doesn't matter)}$$

$$= -\partial_\nu \partial_\mu F^{\mu\nu} \quad \text{(anti-symmetric exchange} \Rightarrow F^{\nu\mu} = -F^{\mu\nu})$$

$$\Rightarrow \partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu} \quad \rightarrow \text{only } 0 \text{ is equal to its own negative}$$

$$\hookrightarrow \partial_\mu J^\mu = 0 \checkmark$$

c) Given $T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\beta\alpha}\right]$

The constituent parts of the equation are:

$$F^{\alpha\beta} F_{\beta\alpha} = 2(|E|^2 - |B|^2)$$

$$F^{0\lambda} F_\lambda{}^0 = |E|^2$$

$$F^{0\lambda} F_\lambda{}^i = (-E^j)[-\epsilon_{ijk} B_k)$$

$$= (E \times B)^i$$

$$F^{i\lambda} F_\lambda{}^j = F^{i0} F_0{}^j + F^{ik} F_k{}^j$$

$$= (E_i)(-E_j) + (-\epsilon^{ikm} B^m)(\epsilon^{kjp} B^p)$$

$$= -E_i E_j - (\delta^j_m \delta^i_p - \delta^j_i \delta^p_m) B^m B^p$$

$$= -E_i E_j - B^i B^j + \delta^{ij} |B|^2$$

#3 (cont)

c) $T^{00} = u$

$$= \frac{1}{4\pi}\left[F^{00}F_0^{\ 0} - \frac{1}{4}g^{00}F^{\alpha\beta}F_{\beta\alpha}\right]$$

$$= \frac{1}{4\pi}\left[|E|^2 - \frac{1}{4}(2[|E|^2 - |B|^2])\right]$$

$$= \frac{1}{4\pi}\left[|E|^2 - \frac{1}{2}|E|^2 - \frac{1}{2}|B|^2\right]$$

$$= \frac{1}{8\pi}(|E|^2 + |B|^2) \qquad \text{(Energy Density)}$$

$$T^{0i} = T^{i0} = cg^i = \frac{1}{c}S^i$$

$$= \frac{1}{4\pi}\left[F^{0i}F_i^{\ 0} - \frac{1}{4}g^{01}F^{\alpha\beta}F_{\beta\alpha}\right]$$

$$= \frac{1}{4\pi}(E \times B)^i$$

$$\Rightarrow \quad \frac{S^i}{c} = \frac{1}{4\pi}(E \times B)^i$$

$$S^i = \frac{c}{4\pi}(E \times B)^i \qquad \text{(Poynting vector)}$$

$$T^{ij} = -T^{ji}$$

$$= \frac{1}{4\pi}\left[F^{ij}F_j^{\ i} - \frac{1}{4}g^{ij}F^{\alpha\beta}F_{\beta\alpha}\right]$$

$$= \frac{1}{4\pi}\left[-E_iE_j - B_iB_j + \delta_{ij}|B|^2 - \frac{1}{4}\delta^{ij}(2|E|^2 - 2|B|^2)\right]$$

$$= -\frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}(|E|^2 - |B|^2)\right] \qquad \text{(Maxwell Stress Tensor}$$

#3 (cont.)

d) $$\partial_\mu T^{\mu\nu} = \partial_\mu \left[\frac{1}{4\pi} \left(F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\beta\alpha} \right) \right]$$

$$= \frac{1}{4\pi} \left[(\partial_\mu F^{\mu\lambda}) F_\lambda{}^\nu + F^{\mu\lambda} (\partial_\mu F_\lambda{}^\mu) - \frac{1}{4} g^{\mu\nu} \left[(\partial_\mu F^{\alpha\beta}) F_{\beta\alpha} + F^{\alpha\beta} (\partial_\mu F_{\beta\alpha}) \right] \right) \right]$$

$$= \frac{1}{4\pi} \left[-J^\lambda F_\lambda{}^\nu + F^{\alpha\beta} (\partial_\alpha F_\beta{}^\nu) - \frac{1}{4} g^{\mu\nu} \left[(\partial_\mu F^{\alpha\beta}) F_{\beta\alpha} + F^{\alpha\beta} (\partial_\mu F_{\beta\alpha}) \right] \right]$$

4. A solution of dextrose, which is optically active, is characterized by a polarization vector $\mathbf{P} = \gamma \nabla \times \mathbf{E}$ where γ is a real constant that depends on the concentration of dextrose. The solution is non-conducting ($\mathbf{J}_{\text{free}} = 0$) and non-magnetic ($\mathbf{M} = 0$). Consider a plane electromagnetic wave of angular frequency ω propagating along the $+z$-axis in such a solution.

   (a) [5 pts] Using Maxwell's equations show that left and right circularly polarized waves travel at 2 distinct speeds ($v_\pm$) in this medium. Calculate the indices of refraction $n_\pm = (ck_\pm)/\omega = c/v_\pm$ as a function of ω and γ for left and right circularly polarized waves. Recall that left ($+$) and right ($-$) circularly polarized waves are of the form

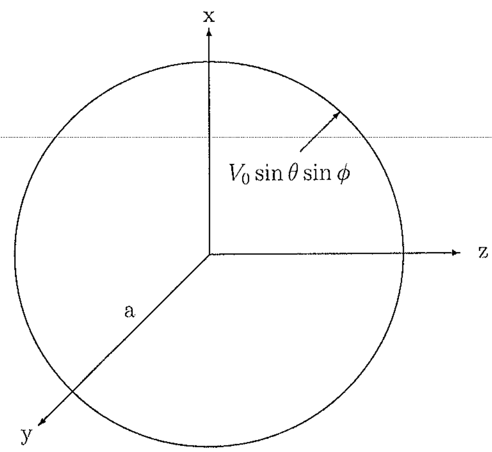
$$\mathbf{E} = E_0 \left(\hat{\mathbf{x}} \pm i \, \hat{\mathbf{y}} \right) e^{i(kz - \omega t)}$$

   (b) [5 pts] Suppose linearly polarized light is incident on the dextrose solution. After traveling a distance L through the solution, the light is still linearly polarized but its direction of polarization rotated by an angle $\Delta\phi$. Calculate $\Delta\phi$ in terms of L, γ, and ω. Hint: Write $k_\pm = \overline{k} \pm \Delta k$ where

$$\overline{k} \equiv \frac{k_+ + k_-}{2} \quad \text{and} \quad \Delta k \equiv \frac{k_+ - k_-}{2}.$$

   Also recall that the amplitude of a wave linearly polarized at an angle ϕ relative to the x-direction can be written as a combination of circularly polarized amplitudes as

$$\left(\cos\phi \; \hat{\mathbf{x}} + \sin\phi \; \hat{\mathbf{y}} \right) = e^{i\phi} \left(\frac{\hat{\mathbf{x}} - i \, \hat{\mathbf{y}}}{2} \right) + e^{-i\phi} \left(\frac{\hat{\mathbf{x}} + i \, \hat{\mathbf{y}}}{2} \right).$$



5. Assume that in spherical polar coordinates (r, θ, ϕ), the potential on the surface of a sphere of radius a, centered on the origin, is known to be $V(\theta, \phi)$.

   (a) [2 pts] If the space inside the sphere is empty give an expression for the potential $\Phi(r, \theta, \phi)$ everywhere inside as an expansion in spherical harmonics with arbitrary constants. If you knew the potential $V(\theta, \phi)$ on the surface how would you evaluate the constants in your expansion?

   (b) [2 pts] If the space outside the sphere is empty give an expression for the potential $\Phi(r, \theta, \phi)$ everywhere outside as an expansion in spherical harmonics with arbitrary constants. If you knew the potential $V(\theta, \phi)$ on the surface how would you evaluate the constants in your expansion?

   (c) [6 pts] If $V(\theta, \phi) = V_0 \sin\theta \sin\phi$ give exact expressions for $\Phi(r, \theta, \phi)$ inside and outside the sphere.

   The spherical harmonics are ortho-normal on the sphere and for $\ell = 1$

$$
\begin{aligned}
Y_1^{-1}(\theta, \phi) &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}, \\
Y_1^{0}(\theta, \phi) &= \sqrt{\frac{3}{4\pi}} \cos\theta, \\
Y_1^{1}(\theta, \phi) &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}.
\end{aligned}
$$

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E+M #5

a) The general solution to LaPlace's eqn, expanded in spherical coordinates is:

$$\Phi(r,\theta,\varphi) = \sum_{l} \sum_{m=-l}^{l} \left(A_{lm} r^{l} + \frac{B_{lm}}{r^{l+1}}\right) Y_l^m(\theta,\varphi)$$

Since we are in empty space inside the sphere, $B_{lm}=0$ since $\Phi(0,\theta,\varphi)=0$

$$\hookrightarrow \Phi(r,\theta,\varphi) = \sum_{l} \sum_{m=-l}^{l} A_{lm} r^{l} Y_l^m(\theta,\varphi)$$

We can then determine the values of A_{lm} by the orthogonality of the spherical harmonics:

$$V(\theta,\varphi) = \sum_{l} \sum_{m=-l}^{l} A_{lm} a^{l} Y_l^m(\theta,\varphi)$$

$$\int_0^{2\pi}\int_0^{\pi} V(\theta,\varphi) Y_l^{*m}(\theta,\varphi) = A_{lm} a^{l}$$

$$\hookrightarrow A_{lm} = \iint V(\theta,\varphi) Y_l^{*m}(\theta,\varphi)\, d\Omega \cdot \frac{1}{a^l}$$

b) We proceed similarly to part a, except here $A_{lm}=0$ b/c $\Phi \to 0$ at $r\to\infty$

$$\hookrightarrow \Phi(r,\theta,\varphi) = \sum_{l} \sum_{m=-l}^{l} B_{lm} \frac{1}{r^{l+1}} Y_l^m(\theta,\varphi)$$

Again, similar to above, we use orthogonality of spherical harmonics to determine B_{lm}

$$\hookrightarrow B_{lm} = a^{l+1} \iint V(\theta,\varphi) Y_l^{*m}(\theta,\varphi)\, d\Omega$$

c) If we now specify that $\Phi(a,\theta,\varphi) = V_0 \sin\theta \sin\varphi$

* Rewriting Φ in terms of spherical harmonics:

$$\sin\theta\sin\varphi = \sin\theta\left(\frac{1}{2i} e^{i\varphi} - e^{-i\varphi}\right)$$

$$= \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1 + Y_1^{-1}\right)$$

#5 (cont.)

c) * For $r < a$

$$A_{lm} = \iint V(\theta,\varphi)\, Y_l^{*m}\, d\Omega \cdot \frac{1}{a^l}$$

$$= \frac{1}{a} V(r) \iint \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1 + Y_1^{-1} \right) Y_l^{*m}\, d\Omega$$

$$= \frac{1}{a} V(r) \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1 + Y_1^{-1} \right)$$

$$= \frac{1}{a} V(r) \sin\theta \sin\varphi$$

$$\Rightarrow \Phi(r,\theta,\varphi) = \frac{r}{a} V(r) \sin\theta \sin\varphi$$

* For $r \geq a$

$$B_{lm} = \iint a^{l+1}\, V(\theta,\varphi)\, Y_l^{*m}\, d\Omega$$

$$= a^2 V(r) \iint \sqrt{\frac{2\pi}{3}}\, i \left(Y_1^1 + Y_1^{-1} \right) Y_l^{*m}\, d\Omega$$

$$= a^2 V(r) \sin\theta \sin\varphi$$

$$\Rightarrow \Phi(r,\theta,\varphi) = \left(\frac{a}{r}\right)^2 V(r) \sin\theta \sin\varphi$$

6. In this problem you are to describe properties of waves penetrating into conductors. If the conductor is static, homogeneous, isotropic, linear, and ohmic, you can replace $\mathbf{D}, \mathbf{H}$, and $\mathbf{J}$ in Maxwell's equations using

$$\mathbf{D} = \epsilon \mathbf{E}, \ \mathbf{B} = \mu \mathbf{H}, \ \mathbf{J} = \sigma \mathbf{E},$$

where ϵ, μ, and σ are real positive constants. For simplicity you can also assume the wave is a harmonic plane wave propagating in the z-direction, e.g., its electric field and magnetic induction are of the form

$$\mathbf{E} = E_0 e^{i(kz-\omega t)} \hat{\mathbf{x}},$$

$$\mathbf{B} = B_0 e^{i(kz-\omega t)} \hat{\mathbf{y}}.$$

(a) [4 pts] Use Maxwell's equations to relate k to ω. Explain what the imaginary part of $k = k_{Re} + ik_{Im}$ does to the amplitude of the wave.

(b) [3 pts] Use Maxwell's equations to relate B_0 to E_0. Explain what the phase of $k = |k|e^{i\phi}$ does to the phase of $\mathbf{B}$ as compared to the phase of $\mathbf{E}$.

(c) [3 pts] If the conductor is a "good" conductor, i.e., if for low frequencies, $\epsilon \ll \sigma/\omega$, what does k simplify to, what is the attenuation distance (skin depth) of the wave, and what is the phase delay of the magnetic induction $\mathbf{B}$ relative to the electric field $\mathbf{E}$?