E & M Qualifier

January 14, 2016

To insure that the your work is graded correctly you MUST:

1. use only the reference material supplied (Schaum's Guides),
2. use only the blank answer paper provided,
3. write only on one side of the page,
4. put your alias **(NOT YOUR REAL NAME)** on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
   (d) try to answer every problem, but if you don't please include a single numbered page stating that you have skipped that problem.
7. **DO NOT** staple your exam when done. Paper clips will be provided.

1. Consider a Lorentz frame K containing no polarizable materials in which there is a magnetic induction $\mathbf{B} = B^x\,\hat{\mathbf{x}} + B^y\,\hat{\mathbf{y}} + B^z\,\hat{\mathbf{z}}$ but no electric field.

   (a) [1 pt] For the above magnetic induction, write down the 4-dimensional electromagnetic field tensor $F^{\alpha\beta}$ in frame K as a matrix.

   (b) [1 pt] Write down a homogeneous Lorentz boost $\Lambda^{\alpha}{}_{\beta}$ in the y-direction from frame K to another frame K′ which is moving with velocity $\mathbf{v} = v_0\,\hat{\mathbf{y}}$ as seen by observers that are at rest in frame K.

   (c) [2 pt] Apply the boost $\Lambda^{\alpha}{}_{\beta}$ to $F^{\alpha\beta}$ to find $F'^{\alpha\beta}$, the field strength tensor as seen in the moving frame K′ .

   (d) [2 pt] What are the electric field components E'^x, E'^y, and E'^z and the magnetic induction components B'^x, B'^y, and B'^z in frame K′?

   (e) [4 pt] Consider explicitly a **B** field in the K frame caused by an **uncharged** infinitely long and thin wire centered on the y-axis $(x, z) = (0, 0)$ which carries a current I in the $+y$ direction. Assume that no polarizable materials are present, i.e., assume $\epsilon_r = 1$ and $\mu_r = 1$. What are $\mathbf{B}'(x', y', z')$ and $\mathbf{E}'(x', y', z')$ in the K′ frame, written as functions of the K′-coordinates? Where does $\mathbf{E}'$ point?

Jan 2016

E+M #1

Gaussian

a) Given $E=0$, $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad \text{(in general)}$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B_z & B_y \\ 0 & B_z & 0 & -B_x \\ 0 & -B_y & B_x & 0 \end{bmatrix}$$

b) For a Lorentz boost in y-direction

$$\Lambda^{\alpha}{}_{\beta} = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) In matrix form: $F' = \Lambda F \Lambda^T$

$$F' = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B_z & B_y \\ 0 & +B_z & 0 & -B_x \\ 0 & -B_y & +B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\beta\gamma B_z & 0 & \beta\gamma B_x \\ 0 & 0 & -B_z & B_y \\ 0 & \gamma B_z & 0 & -\gamma B_x \\ 0 & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

#1 (cont.)

c) $$F'^{\alpha\beta} = \begin{bmatrix} 0 & -\beta\gamma B_z & 0 & \beta\gamma B_x \\ \beta\gamma B_z & 0 & -\gamma B_z & B_y \\ 0 & \gamma B_z & 0 & -\gamma B_x \\ -\beta\gamma B_x & -B_y & \gamma B_x & 0 \end{bmatrix}$$

d) $\vec{E}' = \beta\gamma(B_z\hat{x} - B_x\hat{z}) \Rightarrow E'_x = \beta\gamma B_z \quad E'_y = 0 \quad E'_z = -\beta\gamma B_x$

$\vec{B}' = \gamma B_x\hat{x} + B_y\hat{y} + \gamma B_z\hat{z} \Rightarrow B'_x = \gamma B_x \quad B'_y = B_y \quad B'_z = \gamma B_z$

e) We find $\vec{B}$ according to Ampere's Law

$$\oint B \cdot dl = \frac{4\pi}{c} I_{enc}$$

$$B \cdot 2\pi r = \frac{4\pi}{c} I$$

$$B = \frac{2I}{cr} \langle \cos\varphi\,\hat{x} + \sin\varphi\,\hat{z} \rangle$$

* This is best done in cylindrical coordinates, but professors are stupid question writers. φ measured ccw from $+x$ axis in x-z plane

$$\Rightarrow \vec{E}' = \langle \beta\gamma\sin\varphi \frac{2I}{cr}, \ 0, \ -\beta\gamma\cos\varphi\frac{2I}{cr} \rangle$$

$$\vec{B}' = \langle \gamma\sin\varphi\frac{2I}{cr}, \ 0, \ \gamma\cos\varphi\frac{2I}{cr} \rangle$$

2. Consider a very long hollow cylinder made of iron that is placed with its axis perpendicular to a uniform external magnetic induction $\mathbf{B}_0 = B_0\hat{\mathbf{x}}$. Assume the inner radius of the hollow cylinder is a and the outer radius is b. Also assume the permeability μ of the iron is a constant. The goal of this problem is to calculate the magnetic induction $\mathbf{B}$ inside the hollow region $(0 \le \rho \equiv \sqrt{x^2 + y^2} < a)$.

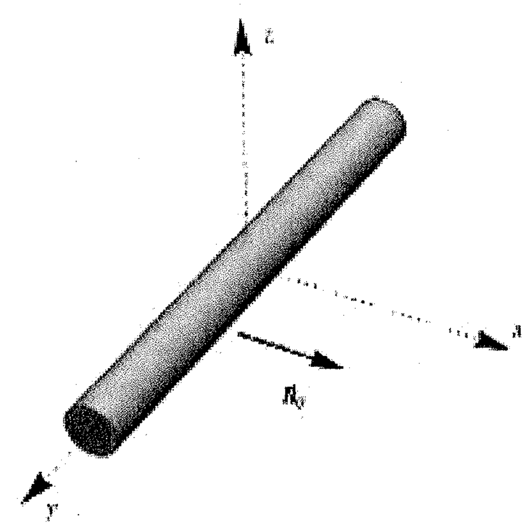
   (a) [3 pt] Starting with Maxwell's equations for static $\mathbf{B}$ and $\mathbf{H}$ fields and assuming that there is no free current density, $\mathbf{J}_f = 0$, prove that the field $\mathbf{H}$ can be written as the negative gradient of a magnetic scalar potential Φ_M that satisfies the Poisson equation with an appropriate source term. For this particular problem the Poisson equation reduces to the Laplace equation except at the cylinder's boundaries.

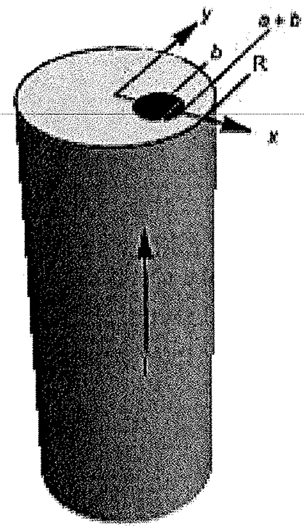
   (b) [3 pt] Derive the appropriate boundary conditions to be satisfied by the scalar potential Φ_M and the magnetic field $\mathbf{H}$ at $\rho = a$ and $\rho = b$.

   (c) [4 pt] Solve for the $\mathbf{H}$ field in the interior region $\rho < a$. Hint: solve the Laplace equation for Φ_M in the three regions $0 \le r < a$, $a < r < b$, and $b < r < \infty$, and appropriately match these solutions at the cylinder's boundaries. Show that for large μ, (i.e., when $\mu \to \infty$) the iron provides complete shielding from the magnetic field, i.e., $\mathbf{H} \to 0$ for $\rho < a$.

Hint:

$$\nabla^2 \Phi_M = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi_M}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi_M}{\partial \phi^2} + \frac{\partial^2 \Phi_M}{\partial z^2}.$$



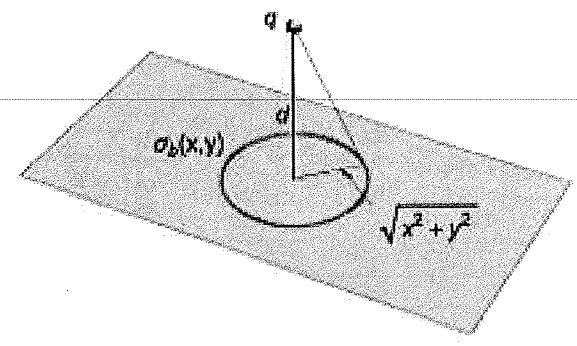


3. A very long straight conductor has a circular cross section of radius R and carries a current I. Inside the conductor, there is a cylindrical hole of radius a whose axis is parallel to the axis of the conductor and a distance b from it ($a + b < R$). The goal of this problem is to show that the magnetic induction $\mathbf{B}(x, y)$ inside the hole is uniform and to calculate its value. Assume the wire of radius R is centered on the z axis, i.e., at $(x, y) = (0, 0)$ and the cylindrical hole of radius a is centered at $(x, y) = (b, 0)$. Assume the current I is uniformly distributed in the conducting material.

   (a) [3 pts]
   Ignoring the hole, use Amperés Law to find the magnetic induction, $\mathbf{B}_R(x, y)$, inside a homogeneous cylindrical wire of radius R that carries a uniform current density $J_R = I_R/\pi R^2$ in the $+z$ direction.

   (b) [4 pts] Ignoring the current in the wire of radius R assume an imaginary wire of radius a located at $(x, y) = (b, 0)$ carries a current density $J_a = I_a/\pi a^2$ in the $-z$ direction. Use Amperes Law to find the magnetic induction, $\mathbf{B}_a(x, y)$, inside the imaginary wire of radius a caused by J_a.

   (c) [3 pts]
   By adjusting the two current densities to have the same magnitude, and superimposing the two magnetic inductions, find the resultant $\mathbf{B}(x, y)$ field inside the hole in the original conductor that carries a current I described at the beginning of this problem.



4. Consider a large flat interface at $z = 0$ between a dielectric and free space. The region where $z < 0$ is filled with a uniform linear dielectric material with a relative permittivity ϵ_r (equivalently a dielectric constant ϵ_r). If the only free charge present is a point charge $q > 0$ situated a distance d from the origin at $\mathbf{r}_q = (0, 0, d)$, where $d > 0$, answer the following 5 questions.

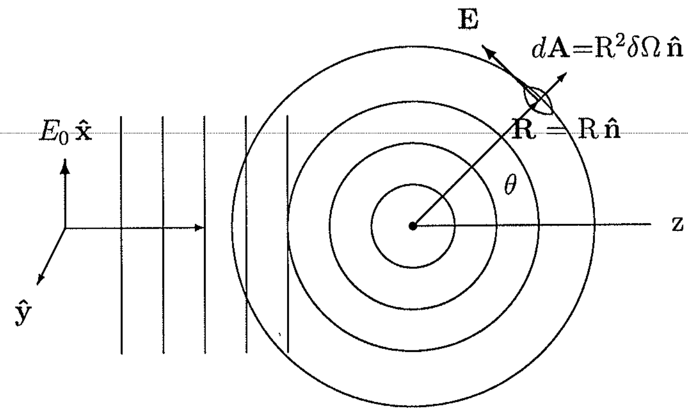
   To answer them you should look at the electric field as a sum of two fields, a coulomb part $\mathbf{E}_q$ caused by the point charge q and a second part $\mathbf{E}_b$ caused by the bound surface charge $\sigma_b(x, y)$ located on the $z = 0$ interface.

   (a) [2 pts] Write two expressions for the z component of the total electric field $E^z = E^z_q + E^z_b$, one just above the dielectric's surface and one just below the dielectric's surface. The E^z_b part is directly related to σ_b by Gauss's law.

   (b) [3 pts] Use the two electric fields from part (a) and the continuity of the normal part of the displacement vector ϵE^z to solve for $\sigma_b(x, y)$ as a function of the known coulomb field $E^z_q(x, y, 0)$.

   (c) [3 pts] Calculate the electric field at the position of the charge q caused by the bound surface charge σ_b. You simply have to integrate a superposition of coulomb fields. From symmetry the resultant field points in the $\pm z$ direction.

   (d) [2 pts] Show that this resultant bound charge field at $(0, 0, d)$ can be interpreted as the field of a single image charge q' located at point $\mathbf{r}_{q'} = (0, 0, -d)$. What is the value of q'?



5. In this question a monochromatic linearly polarized plane wave is scattered by a free electron. If the initial speed of the particle is non-relativistic (i.e., $\beta \ll 1$) and the frequency of the plane wave satisfies $h\nu \ll m_e c^2$, then the electron is accelerated by the plane wave's electric field in accord with Newton's 2^{nd} law, but its speed remains non-relativistic. Due to its acceleration, the electron emits radiation in all directions thus scattering the original plane wave. See the figure.

   (a) [2 pts] Assume the plane wave travels in the z-direction and is polarized in the x-direction as shown in the figure. Compute the acceleration, $\dot{\boldsymbol{\beta}}(t) = \dot{\mathbf{v}}(t)/c$, of the electron caused by the plane wave's electric field.

   (b) [3 pts] Compute the electric field $\mathbf{E}$, the magnetic induction $\mathbf{B}$, and the Poynting vector $\mathbf{S}$ of the radiated wave. { Hint: In Gaussian units $\mathbf{E}_G = q[\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\boldsymbol{\beta}})]/(cR)|_{ret}$, $\mathbf{B}_G = \hat{\mathbf{n}} \times \mathbf{E}$, and $\mathbf{S}_G = (c/4\pi)\mathbf{E} \times \mathbf{H}$. In SI units $\mathbf{E}_{SI} = (1/4\pi\epsilon_0)\mathbf{E}_G$, $\mathbf{B}_{SI} = (1/c)\mathbf{B}_G$, and $\mathbf{S}_{SI} = \mathbf{E} \times \mathbf{H}$. }

   (c) [3 pts] Use your results to compute the differential scattering cross section

$$\frac{d\sigma(\theta,\phi)}{d\Omega} = \frac{< \mathbf{S} \cdot d\mathbf{A} >}{| < \mathbf{S}_0 > | \delta\Omega}.$$

   In the above $<>$ stands for a time average and $| < \mathbf{S}_0 > |$ is the magnitude of the time averaged Poynting vector of the incoming plane wave. The detector area element $d\mathbf{A}$ subtends a solid angle $\delta\Omega$ at the radiating electron and is typically of the form

$$d\mathbf{A} = R^2 \delta\Omega\, \hat{\mathbf{n}}.$$

   (d) [2 pts] Integrate your differential cross section over all (θ, ϕ) directions to obtain the total Thompson cross section σ_T.

6. (a) [2 pts] In a homogeneous, linear and isotropic conducting material whose electromagnetic properties (at low frequencies) are described by constant (and real) values of the permittivity, permeability, and conductivity respectively ϵ, μ, and σ, show that Maxwell's equations require that the electric field satisfy the telegraph equation

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0, \qquad \text{(SI)}$$

$$\nabla^2 \mathbf{E} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \mathbf{E}}{\partial t} = 0. \qquad \text{(Gaussian)}$$

(b) [3 pts] Given a linearly polarized plane wave of angular frequency ω whose electric field is of the form

$$\mathbf{E}(z,t) = \text{Real}\left\{E_0\, e^{i(kz-\omega t)}\right\} \hat{\mathbf{x}},$$

evaluate k^2 as a function of ϵ, μ, σ, and ω.

(c) [2 pts] Find the real and imaginary parts of k assuming $\sigma >> \omega\epsilon$.

(d) [3 pts] Using your results from (c) find the skin depth δ of the conductor. The skin depth is defined by the depth at which the wave's amplitude decreases by e^{-1}, i.e.,

$$\frac{|\mathbf{E}(z+\delta,t)|}{|\mathbf{E}(z,t)|} = \frac{1}{e}$$