2pm: 27-29

3pm 33-43

E & M Qualifier

$p = m\gamma v$

August 15, 2018

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta$$

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

$\epsilon = 1 + 4\pi\chi_e$

$\mu = 1 + 4\pi\chi_m$

$\nabla \cdot D = 4\pi \rho_f$ $\quad$ $\nabla \cdot B = 0$ $\quad$ $D = \epsilon E = E + 4\pi P$

$\nabla \times E - \frac{1}{c}\frac{\partial B}{\partial t} = 0$ $\quad$ $\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$ $\quad$ $B = \mu H$ $\quad$ $H = B - 4\pi M$

$\nabla \cdot J - \frac{\partial \rho}{\partial t} = 0$

$B = \nabla \times A$

$E = -\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t}$

7/10

Problem 1: Magnetostatics

√(a) Use the Biot-Savart law to calculate the magnitude and direction of the magnetic induction , **B**, on the axis of a circular loop of radius R carrying a current I. Let the z axis lie on the axis of the loop with the orgin at the center of the loop. [2 points]

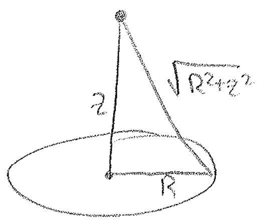
√? (b) Consider a short solenoid with N closely wound coils per meter each carrying current I. Using the result from part (a), find the magnetic induction, **B**, at the center of the solenoid. Assume the solenoid has length L, and consists of circular coils of radius R. [3 points]

√(c) Show that for any point on the axis of the solenoid, $B = \frac{1}{2}\mu_0 IN(cos(\alpha_1) + cos(\alpha_2))$ where α_1 and α_2 are the angles subtended at the point by a radius R at either end of the solenoid. [3 points]

√(d) Write expressions for $cos(\alpha_1)$ and $cos(\alpha_2)$ in terms of R, L, and x where x is the distance of the point from the center of the solenoid. [2 points]

Aug 2018 — E+M #1 — SI

a)



*We use the Biot-Savart Law to derive the magnetic field, which in SI units reads:

$$B = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}') \times (r-r')}{|r-r'|^3}\,dr'$$

$$\Rightarrow B = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}') \times (r-r')}{|r-r'|^3}\,dr'$$

*note that $J(\vec{r}')dr' = I\,dl\,\hat{\varphi}$

$(r-r') = -R\hat{r} + z\hat{z}$

$$\Rightarrow J(\vec{r}) \times (r-r')\,dr' = \begin{vmatrix} \hat{s} & \hat{\varphi} & \hat{z} \\ 0 & I\,dl & 0 \\ -R & 0 & z \end{vmatrix}$$

$$= \langle zI\,dl,\ 0,\ RI\,dl \rangle$$

*we ignore radial term, as axial symmetry will cancel this term during integration

$$B = \frac{\mu_0 I}{4\pi}\int \frac{R\,dl}{(R^2+z^2)^{3/2}}$$

$$= \frac{I R^2 \mu_0}{2\cdot(R^2+z^2)^{3/2}}\hat{z}$$

b) * To build up the field of a solenoid from current loops, we get the infinitesimal dB from a current loop of thickness dz

$$\Rightarrow dB = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\,dz'$$

We also note that $z \rightarrow z - z'$ since the source point is no longer constrained to be at $z=0$

$$dB = \frac{\mu_0 I R^2}{2(R^2+(z-z')^2)^{3/2}}\,dz'$$

and that the total current $I \rightarrow NI$

#1 (cont.)

b) $$dB = \frac{\mu_0 N I R^2}{2(R^2 + (z-z')^2)^{3/2}}\, dz'$$

* Integrating this from 0 to L in z to get the total field we find

$$B = \frac{\mu_0 N I}{2} \int_0^L \frac{R^2}{(R^2+(z-z')^2)^{3/2}}\, dz'$$

* According to Schaums: $\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}$

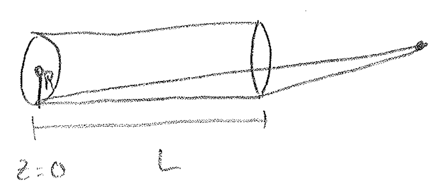
↳ if $a^2 = R^2$

$x^2 = (z-z')^2 \qquad dx = dz'$

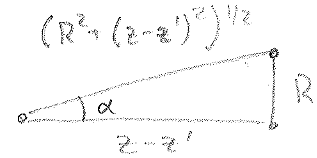
$$B = \frac{\mu_0 N I R^2}{2} \left(\frac{z-z'}{R^2(R^2+(z-z')^2)^{1/2}} \right) \Bigg|_0^L$$

$$= \frac{\mu_0 N I}{2} \left(\frac{z-L}{(R^2+(z-L)^2)^{1/2}} - \frac{z}{(R^2+z^2)^{1/2}} \right) \hat{z}$$

c)



⇒



$\cos\alpha = \frac{z-z'}{(R^2+(z-z')^2)}$

* For the point at the base of the solenoid, $z-z' = z$, and for the point at the top of the solenoid, $z-z' = z-L$

$$\Rightarrow B = \frac{\mu_0 I N}{2} \left(\cos(\alpha_2) - \cos(\alpha_1) \right)$$

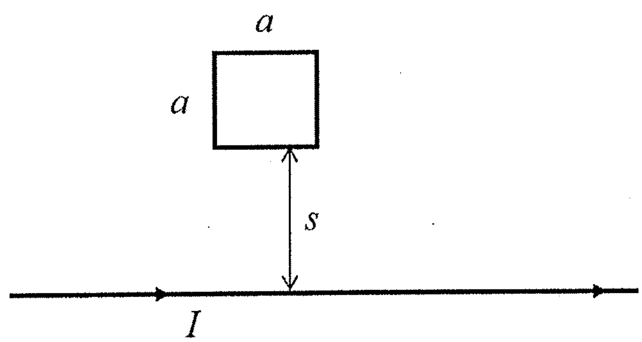
d) From our work above: $\cos(\alpha_2) = \frac{z-L}{(R^2+(z+L)^2)^{1/2}}$

$$\cos(\alpha_1) = \frac{z}{(R^2+z^2)^{1/2}}$$

2-3

Problem 2: Maxwell equations

Consider a square loop of side a and with resistance R. As shown below, the loop is a distance s from an infinite straight wire that carries a current I.



(a) Find the magnetic field as a function of distance from the wire. [1 point]

(b) Find the magnetic flux through the square loop. [1 point]

Griffiths 7.18

(c) Suppose the infinite wire is cut, so its current drops to zero. Explain why this will cause a current to flow through the loop. In what direction will the current flow through the loop? [1 points]

(d) For the situation described in part c), what total charge passes a given point in the loop during the time the current flows? [2 points]

An alternating current $I = I_o \cos \omega t$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a.

Griffiths 7.16

(e) Does the induced electric field point in the radial, circumferential, or longitudinal direction? Assuming that the field goes to zero as s goes to infinity, find $E(s,t)$. [2 points]

(f) Find the displacement current density. Integrate your answer to get the total displacement current. [3 points]

Aug 2018

E+M #2

Gaussian

a) We can determine the magnetic field from Ampere's Law

$$\int (\nabla \times B) \cdot d\vec{a} = \frac{4\pi}{c} \int J(\vec{x})\, da$$

$$\oint B \cdot dl = \frac{4\pi}{c} I_{enc}$$

$$B \cdot 2\pi s = \frac{4\pi}{c} I$$

$$\vec{B} = \frac{2I}{cs} \hat{\varphi}$$ (direction determined by RHR)

b) Φ_B, the magnetic flux thru the loop is:

$$\Phi_B = \int \vec{B} \cdot d\vec{a}$$

$$= \int \frac{2I}{cs}\, ds\, dz$$

$$= \frac{2Ia}{c} \int \frac{1}{s} ds$$

$$= \frac{2Ia}{c} \ln\left(\frac{s+a}{s}\right)$$

c) Nature abhors a change in flux. As such, the induced current will flow in the direction necessary to counteract the change. Since the RHR says our initial field points out of the page, so will the induced field. Thus the current flows counter clockwise within our loop.

d) The total charge in the loop is determined by:

$$Q = \frac{1}{R} \nabla \Phi$$

$$= \frac{1}{R} \cdot \frac{2Ia}{c} \ln\left(\frac{s+a}{s}\right)$$

#2 (cont.)

e) * We can use our samework from part a, only $I \to I_0 \cos(\omega t)$ where $I_{enc} \neq 0$. Thus the magnetic field for this configuration is:

$$\vec{B} = \begin{cases} \dfrac{2 I_0 \cos(\omega t)}{cs} \hat{\varphi} & a < s < b \\ 0 & \text{elsewhere} \end{cases}$$

* From Lenz's law, we know $\vec{E}$ points in the $\hat{z}$ direction. The $\hat{\varphi}$ component must be 0 since $\oint E \cdot dl = 0$ for an amperian loop, and $E(\hat{r}) = 0$ b/c current reversal is equivalent to flipping the ends of the wire in our problem, which will have no impact on the radial symmetry.

* To find $\vec{E}$:

$$\int \nabla \times E\, da = -\int \frac{1}{c} \frac{\partial B}{\partial t}\, da$$

$$\oint E \cdot dl = \frac{-1}{c} \int \frac{\partial B}{\partial t}\, da$$

$$E \cdot l = \frac{-1}{c} \int \frac{-2 I_0 \omega \sin(\omega t)}{cs}\, da$$

$$E \cdot l = \frac{2 I_0 \omega \sin(\omega t)}{c^2} \int \frac{1}{s}\, ds\, dz$$

$$E = \frac{2 I_0 \omega \sin(\omega t)}{c^2} \ln\left(\frac{a}{s}\right)$$

$$\Rightarrow E = \begin{cases} \dfrac{2 I_0 \omega \sin(\omega t)}{c^2} \ln\left(\dfrac{a}{s}\right) & 0 < s < a \\ 0 & s > a \end{cases}$$

#2 (cont.)

f) * The displacement current is defined as:

$$I_d = \frac{1}{4\pi}\frac{\partial E}{\partial t}$$

$$= \frac{1}{4\pi}\frac{2I_0 \omega^2 \cos(\omega t)}{c^2}\ln\left(\frac{a}{s}\right)$$

$$= \frac{I_0 \omega^2 \cos(\omega t)}{2\pi c^2}\ln\left(\frac{a}{s}\right)$$

* Integrating over a volume of length L, we get

$$\vec{J}_d = \int \frac{I_0 \omega^2 \cos(\omega t)}{2\pi c^2}\ln\left(\frac{a}{s}\right) s\,ds\,d\varphi\,dz$$

$$= \frac{I_0 \omega^2 \cos(\omega t) L}{c^2}\int \ln\left(\frac{a}{s}\right) s\,ds$$

$$= \frac{I_0 \omega^2 \cos(\omega t) L}{c^2}\left[\frac{1}{2}x^2 \ln\left(\frac{a}{x}\right) + \frac{x^2}{4}\right]\Big|_s^a$$

$$= \frac{I_0 \omega^2 \cos(\omega t) L}{c^2}\left[\frac{a^2}{4} - \frac{1}{2}s^2 \ln\left(\frac{a}{s}\right) - \frac{s^2}{4}\right]$$

6-8

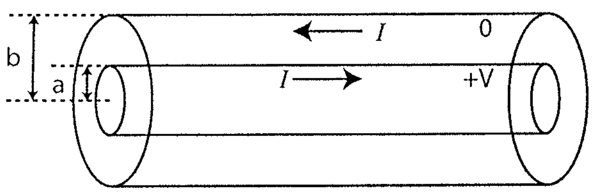
Problem 3: E/p in EM field

? (a) The work required to assemble n point charges q_i is $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$. What is the physical meaning of the factor $\frac{1}{2}$ that appears in this expression? [0.5 pts]

√(b) Generalizing to a volume charge density ρ, the work becomes $W = \frac{1}{2}\int \rho V d^3r$. Use Gauss' law and integration by parts to show from this that the energy stored in the electric field is $W = \frac{\epsilon_0}{2}\int E^2 d^3r$. [1 point]

√(c) Using the expression for W from part 2, find the energy of a uniformly charged spherical shell of total charge Q and radius R. [1 point]

√(d) The work required to set up a current density $\mathbf{J}$ is $W = \frac{1}{2}\int \mathbf{A}\cdot\mathbf{J}\, d^3r$, where $\mathbf{A}$ is the vector potential due to the current, and the integral is over all space. Show from this expression, via integration by parts, that the energy stored in the magnetic field is $W = \frac{1}{2\mu_0}\int B^2\, d^3r$. [1 point]



√(e) For this and the following parts of this problem, consider a long coaxial cable that carries current I along the surface of the inner cylinder, radius a, flowing down in one direction, and the same current I along the outer cylinder surface, radius b, flowing back up in the opposite direction, as shown in the above figure. Find the energy stored in the *magnetic* field of a section of length l of this cable, as a function of I, l, b and a. [2 points]

\5? √(f) What is the energy flux density (magnitude of Poynting vector $\mathbf{S}$) in the cable if the inner conductor is held at a positive potential V with respect to the outer conductor, and current I flows down one conductor and back up the other conductor as shown. Express $|\mathbf{S}|$ as a function of linear charge density λ, radial distance s, I, a and b. [2 points]

$S = \frac{c}{4\pi} E \times H = \frac{1}{\mu_0} E \times B$

√(g) Which direction does the Poynting vector point? [0.5 points]

? (h) What is the power (energy per unit time) transported down the cable, as a function of linear charge density λ, b, and a? [1 point]

? √(i) Express the power transported in the cable in terms of the current I and potential difference V. [1 point]

$P = IV^2$

Aug 2018

E+M #3

SI

a) The physical meaning of the ½ in the formula $W = \frac{1}{2}\sum_i^{n}{}' q_i V(\vec{r}_i)$ is that it avoids double counting when $V(\vec{r}_i) = \sum_j{}' \frac{q_j}{4\pi\epsilon_0 r_{ij}}$ as $W_{12} = W_{21}$

b)
$$W = \frac{1}{2}\int \rho V d^3r$$
$$= \frac{1}{2}\int \epsilon_0 \frac{\rho}{\epsilon_0} V d^3r$$
$$= \frac{\epsilon_0}{2}\int (\nabla\cdot E)\, V d^3r$$
$$= \frac{\epsilon_0}{2}\left[\int E\cdot\nabla V d^3r - \oint \underbrace{\cancel{EV} d^3r}_{0}\right]$$

0 if integrated over all space ($V=0$ at ∞)

$$= \frac{\epsilon_0}{2}\int E\cdot E\, d^3r$$
$$= \frac{\epsilon_0}{2}\int |E|^2 d^3r$$

c) From Gauss Law, we know the field of a uniformly charged spherical shell is:

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} & r \geq R \\ 0 & r < R \end{cases}$$

$$\Rightarrow W = \frac{\epsilon_0}{2}\int E^2 d^3r$$
$$= \frac{\epsilon_0}{2}\int \frac{Q^2}{16\pi^2\epsilon_0^2 r^4} r^2 dr\, d\Omega$$
$$= \frac{Q^2}{8\pi\epsilon_0}\left(\frac{-1}{r}\right)\Big|_R^{\infty}$$
$$= \frac{Q^2}{8\pi\epsilon_0 r}$$

#3 (cont.)

d) $W = \frac{1}{2}\int A \cdot J \, d^3r$

$= \frac{1}{2\mu_0}\int A \cdot (\nabla \times B)\, d^3r$

$= \frac{1}{2\mu_0}\left[-\cancelto{0}{\oint A B \, d^3r} + \int (\nabla \times A) \cdot B \, d^3r \right]$ $\quad (A = 0 \text{ at } \infty)$

$= \frac{1}{2\mu_0}\int |B|^2 \, d^3r$

e) We determine the magnetic field using Ampere's Law

$$\int \vec{\nabla} \times \vec{H} \, da = I_{enc}$$

$$\oint \vec{H} \, dl = I_{enc}$$

$$\vec{H} \, 2\pi s = I$$

$$\vec{H} = \frac{I}{2\pi s} \hat{\varphi}$$ (direction determined by symmetry / RHR)

*Applying our general formula above to our specific situation where $B = \mu H$.

- If $s < a$, $\vec{B} = 0$ b/c $I_{enc} = 0$
- If $a \leq s \leq b$, $\vec{B} = \frac{\mu I}{2\pi s} \hat{\varphi}$
- If $s \geq b$ $\vec{B} = 0$ b/c $I_{enc} = 0$

$$\Rightarrow \vec{B} = \begin{cases} \frac{\mu I}{2\pi s} \hat{\varphi} & a \leq s \leq b \\ 0 & \text{elsewhere} \end{cases}$$

$W = \frac{1}{2\mu_0}\int B^2 \, d^3r$

$= \frac{1}{2\mu_0}\int \frac{\mu^2 I^2}{4\pi^2 s^2} \, s\,ds\,d\varphi\,dz$

$= \frac{\mu^2 I^2 L}{4\pi \mu} \ln\left(b/a\right)$

#3 (cont.)

f) * Assuming a steady current, the amount of charge Q within our region of interest is constant.

* Again because we know nothing about our medium, we assume $\vec{D} = \epsilon \vec{E}$

Using Gauss Law, we find

$$\int \nabla \cdot D \, dV = \int \rho_f \, dV$$

$$\oint D \cdot d\vec{a} = Q$$

$$D \cdot 2\pi r L = Q$$

$$\vec{D} = \frac{Q}{2\pi r L} \hat{r} \quad \text{(Direction from RHR)}$$

$$\Rightarrow \vec{E} = \begin{cases} \frac{Q}{2\pi s L \epsilon} \hat{r} & a < s < b \\ 0 & \text{elsewhere} \end{cases}$$

$$\vec{S} = \frac{1}{\mu_0} E \times B$$

$$= \frac{1}{\mu_0} |E||B| \hat{z}$$

$$= \frac{1}{\mu_0} \frac{Q}{2\pi s L \epsilon} \cdot \frac{\mu I}{2\pi s} \hat{z}$$

$$= \frac{\mu Q I}{\mu_0 4\pi^2 s^2 L \epsilon} \hat{z}$$

$$= \frac{\mu Q \lambda}{4\pi^2 \mu_0 \epsilon s^2} \hat{z}$$

g) $\hat{z}$ direction

h) $P = \int S \cdot d\vec{a}$

$$= \frac{\mu Q \lambda}{4\pi^2 \mu_0 \epsilon} \int \frac{1}{s^2} s \, ds \, d\varphi$$

$$= \frac{\mu Q \lambda}{2\pi \mu_0 \epsilon} \ln\left(\frac{b}{a}\right)$$

j) $P = \frac{\mu I}{2\pi L} \cdot \frac{V}{b-a}$

10

Problem 4: Interaction forces and energies

Consider a spherical shell of radius a and charge q uniformly distributed over its surface.

(a) Find the electric field everywhere in space. [1 point]

(b) Calculate the energy of the configuration. [3 points]

(c) Consider now the case in which we add a second spherical shell of radius b ($b > a$) and total charge $-q$ uniformly distributed over its surface. Calculate the energy of the configuration if the two spherical shells are concentric. [3 points]

(d) Does the superposition principle apply to the energy? That is, is the energy of the concentric spherical shells equal to the sum of the energy of two spherical shells taken individually? Justify your answer. [3 points]

E+M #4

a) For a spherical shell of radius a and total charge q:

$$E = \begin{cases} 0 & r<a \\ \frac{q}{4\pi\epsilon_0 r^2}\hat{r} & r\geq a \end{cases} \quad \text{(from Gauss Law)}$$

$$\int E\cdot da = \frac{q_{enc}}{\epsilon_0}$$

$$E\cdot 4\pi r^2 = \frac{q_{enc}}{\epsilon_0}$$

$$E = \frac{q_{enc}}{4\pi\epsilon_0 r^2}\hat{r}$$

b) The energy stored in the electric field is:

$$W = \frac{\epsilon_0}{2}\int |E|^2 d^3r$$

$$= \frac{\epsilon_0}{2}\int \frac{q^2}{16\pi^2\epsilon_0^2 r^4} r^2 dr d\Omega$$

$$= \frac{q^2}{32\pi^2\epsilon_0}\int \frac{1}{r^2} dr d\Omega$$

$$= \frac{q^2}{8\pi\epsilon_0}\int_a^\infty \frac{1}{r^2} dr$$

$$= \frac{q^2}{8\pi\epsilon_0}\left(\frac{-1}{r}\right)\Big|_a^\infty$$

$$= \frac{q^2}{8\pi\epsilon_0 a}$$

c) If we now add a second shell of charge $-q$ at radius b, $(b>a)$, our total electric field becomes (again by Gauss' Law):

$$\vec{E} = \begin{cases} 0 & r<a & (q_{enc}=0) \\ \frac{q}{4\pi\epsilon_0 r^2} & a\leq r\leq b & \\ 0 & r\geq b & (q_{enc}=0) \end{cases}$$

#4 (cont.)

c) We calculate the energy the same way as in part b:

$$W = \frac{\epsilon_0}{2}\int |E|^2 d^3r$$

$$= \frac{\epsilon_0}{2}\int \frac{q^2}{16\pi^2\epsilon_0^2 r^4} r^2 dr\, d\Omega$$

$$= \frac{q^2}{32\pi^2\epsilon_0}\int \frac{1}{r^2} dr\, d\Omega$$

$$= \frac{q^2}{8\pi\epsilon_0}\int_a^b \frac{1}{r^2} dr$$

$$= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

d) In general, the superposition principle does not apply to energy. Since we know the superposition principle applies to fields ($\vec{E}_T = \vec{E}_1 + \vec{E}_2$), we can see:

$$W = \frac{\epsilon_0}{2}\int |E|^2 d^3r$$

$$= \frac{\epsilon_0}{2}\int (E_1 + E_2)^2 d^3r \stackrel{?}{=} \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2)\, d^3r$$

$$= \frac{\epsilon}{2}\int (E_1^2 + 2E_1E_2 + E_2^2) d^3r \neq \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2)\, d^3r$$

3-5

Problem 5: Special relativity/Compton effect

A photon with four-momentum k and energy $E = \hbar\omega$ is incident in the z direction upon an electron of mass m and four-momentum p which is at rest. The photon recoils with four-momentum k' at an angle θ and frequency ω' after scattering whilst the recoil electron has four-momentum p'. You may adopt units with $\hbar = c = 1$ if you wish.

✓(a) Write out a diagram depicting the initial and final states of the particles. [1 point]

✓(b) What are their 4-momentum vectors? (You may orient the y-axis such that all scattering takes place in the $x - z$ plane) [1 point]

(c) Derive an expression relating the final state photon wavelength λ' to λ and its angle of scatter. [4 points]

✓(d) Is the final state photon red-shifted or blue-shifted? Why? [1 point]

(e) Plot the final state photon wavelength as a function of scattering angle. [1 point]

(f) How is the final state photon energy ω' related to ω and its angle of scatter? [2 points]

$$E = \hbar\omega$$
$$= \frac{\hbar c}{\lambda}$$

Aug 2018

E+M #5

a) Initial Final

b) *If oriented so all scattering occurs in x-z plane, the 4-momentum vectors become:

$$P_{i,\gamma} = \langle \hbar\omega/c,\ \hbar\omega v_x/c,\ \hbar\omega v_y/c,\ \hbar\omega v_z/c \rangle$$

$$P_{f,\gamma} = \langle \hbar\omega'/c,\ \hbar\omega' v_x/c^2,\ \hbar\omega' v_y/c^2,\ \hbar\omega' v_z/c^2 \rangle$$

> Note: $v_x^2 + v_y^2 + v_z^2 = c^2$

*Note: 4-momentum of photon is null vector, therefore $|p|^2 = 0$

$$\left(\frac{\hbar\omega}{c}\right)^2 - \left[\left(\frac{\hbar\omega v_x}{c^2}\right)^2 + \left(\frac{\hbar\omega v_y}{c^2}\right)^2 + \left(\frac{\hbar\omega v_z}{c^2}\right)^2\right]c^2 = 0 \checkmark$$

$$P_{i,e^-} = \langle m_e c\ 0,\ 0,\ 0 \rangle$$

$$P_{f,e^-} = \langle m_e c,\ v_x,\ v_y,\ v_z \rangle$$

c) *We can derive the scattering formula by the applications of energy + momentum conservation

① Energy Conservation

$$E_{i,\gamma} + E_{i,e^-} = E_{f,\gamma} + E_{f,e^-}$$

$$\hbar\omega + m_e c^2 = \hbar\omega' + (m_e^2 c^4 + P_{f,e^-}^2 c^2)^{1/2}$$

② Momentum Conservation

$$\vec{P}_{i,\gamma} + \vec{P}_{i,e^-} = \vec{P}_{f,\gamma} + \vec{P}_{f,e^-}$$

$$\hookrightarrow \vec{P}_{f,e^-} = \vec{P}_{i,\gamma} - \vec{P}_{f,\gamma}$$

*Making use of $P_\alpha P^\alpha$ being invariant

$$P_{f,e^-}^2 = P_{i,\gamma}^2 + P_{f,\gamma}^2 - 2 P_{i,\gamma} P_{f,\gamma} \cos\theta_1$$

#5 (cont.)

c) $$P_{f,e^-}^{2} c^2 = P_{\gamma,i}^2 c^2 + P_{f,\gamma}^2 c^2 - 2c^2 P_{i,\gamma} P_{f,\gamma} \cos\theta_1$$

$$= (\hbar\omega)^2 + (\hbar\omega')^2 - 2(\hbar\omega)(\hbar\omega')\cos\theta_1$$

* if we manipulate our energy conservation equation, we see:

$$P_{f,e^-}^2 c^2 = [\hbar\omega - \hbar\omega' + m_e c^2]^2 - m_e^2 c^4$$

$$[\hbar\omega - \hbar\omega' + m_e c^2]^2 - m_e^2 c^4 = (\hbar\omega)^2 + (\hbar\omega')^2 - 2(\hbar\omega)(\hbar\omega')\cos\theta_1$$

$$\cancel{(\hbar\omega)^2} + \cancel{(\hbar\omega')^2} + \cancel{m_e^2 c^4} + 2\hbar\omega m_e c^2 - 2\hbar\omega' m_e c^2 - 2(\hbar\omega)(\hbar\omega') - \cancel{m_e^2 c^4} = \cancel{(\hbar\omega)^2} + \cancel{(\hbar\omega')^2} - 2(\hbar\omega)(\hbar\omega')\cos\theta_1$$

$$2(\hbar\omega)(\hbar\omega')[1-\cos\theta_1] = -2\hbar\omega' m_e c^2 + 2\hbar\omega m_e c^2$$

* dividing by $2\hbar\omega\omega' m_e c$ yields

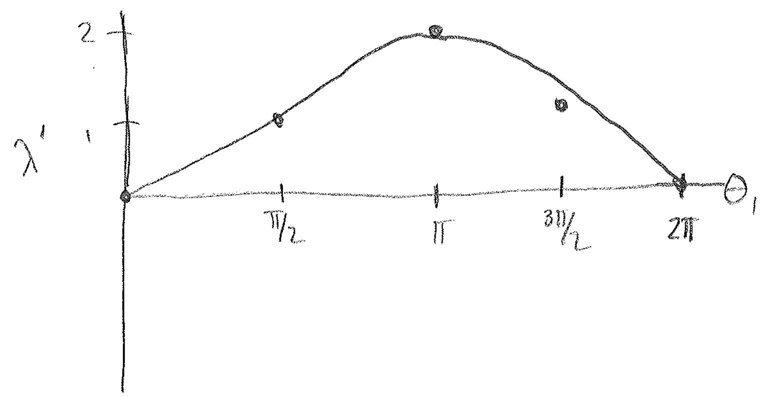
$$\frac{\hbar}{m_e c}[1-\cos\theta_1] = \frac{c}{\omega'} - \frac{c}{\omega}$$

$$\frac{h}{m_e c}[1-\cos\theta_1] = \lambda' - \lambda$$

d) Red shifted, as photon loses energy (or wavelength gets longer by formula in part c)

f) $\frac{\hbar}{m_e c}[1-\cos\theta_1] = \frac{c}{\omega'} - \frac{c}{\omega}$

e)



8.5 - 10

Problem 6: Gauges and 4-potentials

(a) Write down the Maxwell equations in terms of 3-vectors in a vacuum including sources. These provide eight coupled PDEs for $\vec{E}$ and $\vec{B}$ given the source functions $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$. [1 point]

(b) Introduce the vector potential $\vec{A}(\vec{x},t)$. How is this related to $\vec{B}$ and how does it help solve the Maxwell equations? [2 points]

1-1.5? (c) Plugging $\vec{A}(\vec{x},t)$ into Faraday's equation allows introduction of the scalar potential $\Phi(\vec{x},t)$. How is Φ related to $\vec{A}$ and $\vec{E}$ and how does it help to solve the Maxwell equations? [2 points]

(d) Write the inhomogeneous Maxwell equations in terms of the potentials Φ and $\vec{A}$. What is the Lorenz gauge condition and how does it help to solve the Maxwell equations? [2 points]

(e) If the electro- and magneto-static solutions to Maxwell equations are $\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$ and $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$ what are the corresponding electro*dynamic* (*i.e.* time-dependent) solutions in the Lorenz gauge? What do they have to do with causality? [3 points]

Aug 2018

E+M #6

Gaussian

a) In Gaussian units, the 3-vector form of Maxwell's equations are:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

b) Introducing the vector potential $\vec{A}$, which is related to $\vec{B}$ by:

$$\vec{\nabla}\times\vec{A} = \vec{B}$$

We can use the vector potential to reformulate Maxwell's equations for magneto statics in terms of Poisson's eqn $\left(-\nabla^2\vec{A} = \frac{4\pi}{c}\vec{J}\right)$, which is often times easier to solve than the Biot-Savart Law or Ampere's Law, which only applies to certain geometries

c) Substituting the above into Faraday's equation $\left(\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0\right)$ yields

$$\nabla\times E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times A) = 0$$

$$\nabla\times\left(E + \frac{1}{c}\frac{\partial}{\partial t}A\right) = 0$$

Since the curl of a gradient of a scalar function is idetically 0, we can define Φ as:

$$-\nabla\Phi = \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \iff \vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

Again, this helps us to solve Maxwell's equations by allowing us in electrostatics to reformulate our equation in terms of Poisson's equation $(-\nabla^2\Phi = 4\pi\rho)$, which is easier to solve than Coloumbs Law, and works in more geometries than Gauss' Law.

d) Maxwell's inhomogeneous equations reformulated in terms of the scalar/vector potenials are:

$$\left.\begin{array}{l} -\nabla^2\Phi = 4\pi\rho \\ -\nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} \end{array}\right\} -\Box A^\mu = \frac{4\pi}{c}\vec{J}^\mu$$

#6 (cont.)

d) The Lorenz guage condition specifies that $\partial_\mu A^\mu = 0$. This condition helps to solve Maxwell's equations by allowing use of the Poisson equation in all media and to pair down its solutions to a family of solutions related only by a guage transformation. It is also consistent with time-dependent potentials.

e) In the case of electrodynamics, our potentials become the Lienard-Wiechert potentials, described by:

$$\Phi(t,\vec{r}) = 4\pi \int \frac{\rho\left(t - \frac{|r-r'|}{c}, \vec{r}'\right)}{|r-r'|} d^3r'$$

$$\vec{A}(t,\vec{r}) = \frac{1}{c} \int \frac{J\left(t - \frac{|r-r'|}{c}, \vec{r}'\right)}{|r-r'|} d^3r'$$

These solutions account for the fact that any effects from changes in the fields require time to propogate before being felt by the test particles, since no information can travel faster than the speed of light.