Electrodynamics Qualifier Examination

August 21, 2008

General Instructions: **In all cases, be sure to state your system of units.** Show all your work, write only on one side of the designated paper, and if you get stuck on one part, assume a result and proceed onward. The points given for each part of each problem are indicated. Each problem carries equal weight.

1. In relativistic notation, the field strength tensor $F^{\mu\nu}$ is given by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ in terms of the 4-vector potential $A^\mu = (\phi, \mathbf{A})$. Maxwell's equations become

$$\partial_\nu F^{\mu\nu} = kJ^\nu, \quad J^\mu = (c\rho, \mathbf{J}),$$

and k is a constant depending on the system of units adopted.

a) 1 pt. Write Maxwell's equations in terms of A^μ.

b) 2pts. Show that the field strength tensor is invariant under a gauge transformation,

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \lambda,$$

where λ is any function of space and time.

c) 1 pt. How does the form of Maxwell's equations found in part a) change if we exploit the gauge freedom to impose the Lorenz condition

$$\partial_\mu A^\mu = 0?$$

d) 2pts. Show that further gauge transformations are possible provided λ' satisfies

$$\partial^2 \lambda' \equiv \partial^\mu \partial_\mu \lambda' = 0.$$

e) 2pts. In empty space, $J^\mu = 0$, impose the further condition $A^0 = 0$ and rewrite the Lorenz gauge condition to obtain the radiation or Coulomb gauge condition. Is this gauge condition Lorentz invariant?

f) 2pts. Show that the plane-wave function

$$A^\mu(x) = ae^{ip\cdot x}\epsilon^\mu(p),$$

where $p^\mu = (\omega/c, \mathbf{k})$ is the propagation or wave vector, $x^\mu = (ct, \mathbf{x})$, $x \cdot p = x_\mu p^\mu$, a is a constant, and ϵ^μ is the polarization 4-vector, satisfies the Lorenz gauge condition provided ϵ^μ satisfies a particular condition. What is this condition? If this condition is satisfied, show that the empty-space Maxwell equation is satisfied provided there is a constraint on $p^2 = p^\mu p_\mu$. What does this constraint imply about the rest mass of the photon?

2. Consider a monochromatic plane electromagnetic wave of frequency ω propagating in a non-magnetic dielectric (with index of refraction n_1), traveling in the z direction and polarized in the x direction, which impinges normally upon a second non-magnetic semi-infinite dielectric material (with index of refraction n_2), where the boundary between the two media occurs at $z = 0$, as shown in Fig. 1. The incident electric field is

$$\mathbf{E}_I(z,t) = \hat{\mathbf{x}} E_{0I} e^{i(kz-\omega t)}.$$

There are no free charges or currents in either medium.

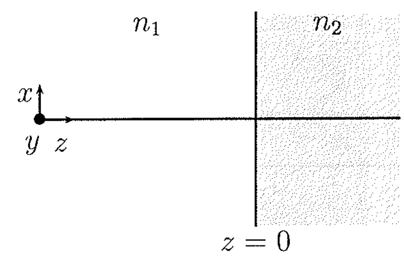

Figure 1: Plane wave normally incident on a surface separating two dielectric materials at $z = 0$. The medium in the the region $z < 0$ has index of refraction n_1 while the material in the region $z > 0$ has index of refraction n_2.

a) 1 pt. Use Maxwell's equations to determine the relation between k and ω in each region.

b) 1 pt. Use Maxwell's equations to determine the incident magnetic field, $\mathbf{B}_I(z,t)$, using the result of part b).

c) 1 pt. What are the forms of the reflected wave $\mathbf{E}_R(z,t)$, $\mathbf{B}_R(z,t)$ ($z < 0$), and of the transmitted wave $\mathbf{E}_T(z,t)$, $\mathbf{B}_T(z,t)$ ($z > 0$)?

d) 2pts. Apply the appropriate boundary conditions at the interface between the two media to obtain the equations determining the reflected amplitudes E_{0R} and B_{0R} and the transmitted amplitude E_{0T} and B_{0T} in terms of E_{0I}.

e) 2pts. Solve these equations for the reflection and transmission coefficients, $r = E_{0R}/E_{0I}$, $t = E_{0T}/E_{0I}$ in terms of the indices of refraction of the two media.

f) 2pts. Show that the averaged energy flux in a plane wave of amplitude E_0 moving in a medium with index of refraction n is given by (Gaussian units)

$$S = \frac{c}{8\pi} n |E_0|^2.$$

Show that the relative reflected and transmitted energy fluxes are

$$R = \frac{S_R}{S_I} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2, \quad T = \frac{S_T}{S_I} = \frac{4n_1 n_2}{(n_1 + n_2)^2}.$$

g) 1 pt. Show that $R + T = 1$. Why is this as expected?

3. A *relativistic* particle of rest mass m and charge e is moving in a uniform (constant and static) magnetic field $\mathbf{B}$. The equations of motion for the particle momentum $\mathbf{p}$ and its energy E are (Gaussian units)

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c}\mathbf{v} \times \mathbf{B}, \quad \frac{dE}{dt} = 0.$$

a) 1 pt. Why is the particle energy conserved?

b) 1 pt. Express $\mathbf{p}$ in terms of m and the particle velocity $\mathbf{v}$, and E in terms of m and $\mathbf{v}$.

c) 3pts. Show that these equations of motion can be written as

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\omega} \times \mathbf{v},$$

and express $\boldsymbol{\omega}$ in terms of e, E, and $\mathbf{B}$. This says that the velocity vector precesses with angular velocity $\boldsymbol{\omega}$.

d) 3pts. Now suppose the motion is confined to the plane perpendicular to $\mathbf{B}$, that is, $\mathbf{B} \perp \mathbf{v}$. Then show that the particle moves with angular speed ω in a circle of radius R. Give an equation for R in terms of v, E, e, and B.

e) 2pts. Now give an equation relating the magnitude of the particle momentum p to the radius R found in part d). Thus show that a measurement of the radius of the orbit determines the particle momentum. If the velocity of the particle is independently known, we can then determine the mass m of the particle, according to the relation given in part b).

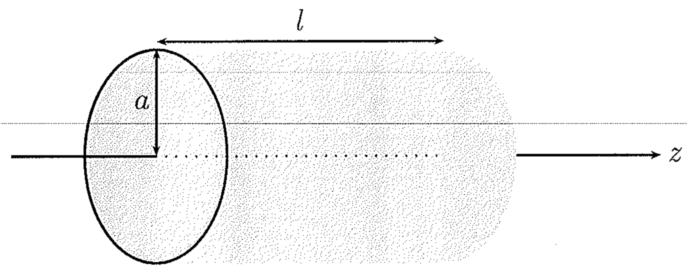

Figure 2: Hollow cylinder (radius a and length l) containing uniform gas flowing along the axis, the z direction, with velocity v. Protons are injected into the cylinder with velocity V parallel to the axis. As a result of magnetic forces, they are brought to a focus at a point on the z axis a distance p far from the cylinder, $p \gg l$.

4. Consider a hollow cylinder of radius a and length l filled with a completely ionized gas of uniform charge density ρ which is moving parallel to the axis of the cylinder with velocity v.

a) 3pts. Find the magnetic field (magnitude and direction) at a distance r from the axis of the cylinder, for $r < a$; assume that we are well inside the cylinder and that $l \gg r$ so that we can neglect edge effects. Assume that the gas is nonmagnetic.

b) 3pts. Suppose a beam of nonrelativistic protons of mass m and velocity V are sent into this cylinder with their initial velocities parallel to the z axis. Neglect electrostatic, edge effects, and collisions between protons and the gas. Show that while in the gas-filled cylinder, the protons experience a force pushing them toward the axis of the cylinder. Calculate the radial velocity V_r acquired by the protons when they exit the cylinder. Assume that the distance moved toward the axis while in the cylinder is negligible.

c) 2pts. After the protons leave the cylinder, they continue to move toward the z axis with constant radial velocity V_r. Calculate the time T required for the protons to reach the axis.

d) 2pts. As a result, the protons will travel through the cylinder and be focused at a point p on the z axis beyond the cylinder where $p \gg l$. Find p and show that it is independent of the initial distance of the protons from the axis when they enter the cylinder.

5. Consider an infinitely long, solid, nonmagnetic conducting rod of radius a centered on the z axis. An infinitely long, hollow, conducting cylinder with inner radius $b > a$ and outer radius d is coaxial with the rod. Let r be the radial distance perpendicular to the axis of the rod and the cylinder. The region between the conducting rod and the conducting cylinder (that is, $a < r < b$) is filled with a nonconducting, linear, isotropic magnetic material with a constant relative permeability $K = \mu/\mu_0$, where μ is the permeability of the material, and μ_0 is the permeability of free space ($\mu_0 = 1$ in Gaussian units).

   The rod carries a current I in the $+z$ direction while the concentric cylinder carries a current I in the $-z$ direction. We assume that the current density $\mathbf{j}$ is uniform and of the same magnitude in both the rod and the cylinder,

$$j = \frac{I}{\pi a^2} = \frac{I}{\pi(d^2 - b^2)}.$$

a) 3 pts. Calculate the magnetic field $H(r)$ for the four regions

$$\text{I: } r \leq a, \quad \text{II: } a \leq r \leq b, \quad \text{III: } b \leq r \leq d, \quad \text{IV: } d \leq r.$$

b) 3 pts. Calculate the magnetic flux (per unit length in the z direction) crossing a half-plane extending from the axis of the coaxial system and extending to infinity, that is, the surface defined by $x > 0$, $y = 0$, $-\infty < z < \infty$. Use this result to find the self-inductance L per unit length of the coaxial conductor.

c) 2 pts. Compute the magnetic energy U per unit length along the z axis stored in the region filled with the linear magnetic material, that is for region II, $a < r < b$.

d) 2 pts. Using the result from part c), show that the contribution to L coming from the region $a \leq r \leq b$, $L_{\rm II}$, is consistent with the contribution from the same region that you calculated in part b) above. That is, compute $\frac{1}{2}L_{\rm II}I^2$ and compare with the result of part c).

E+M #5

Aug 2008 Gaussian

a) We can compute the magnetic field via Ampere's Law (from $\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$)

$$\int \nabla \times B \, da = \int \frac{4\pi}{c} \vec{J} \cdot d\vec{a}$$

$$\int B \cdot d\vec{l} = \frac{4\pi}{c} \int \vec{J} \cdot da$$

$H = \vec{B} - 4\pi\vec{M}$

$B = \mu H$, $\mu = 1 + 4\pi \chi_m$

$M = \chi_m H$ *for linear, isotropic material

- We have 4 regions to consider:

Region I: $r \le a$

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c} \int \left(\frac{I}{\pi a^2}\right) \cdot d\vec{a}$$

$$B \cdot \cancel{2\pi} r = \frac{{}^2\cancel{4\pi}}{c} \frac{I}{\cancel{\pi} a^2} \cdot \cancel{\pi} r^2$$

$$\vec{B} = \frac{2Ir}{ca^2} \hat{\varphi}$$ (direction determined via symmetry)

$$\vec{H} = \vec{B} - 4\pi\vec{M}$$

$$\vec{H} = \frac{2Ir}{ca^2}\hat{\varphi} - 0$$ (non-magnetic material)

$$\vec{H} = \frac{2Ir}{ca^2}\hat{\varphi}$$

Region II: $a \le r \le b$

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c} \int \frac{I}{\pi a^2} \cdot d\vec{a}$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \frac{I}{\pi a^2} \cdot \pi a^2$$

$$B = \frac{2I}{cr}\hat{\varphi}$$

* Because our magnetization is non-zero, $M = \chi_m H$, $H = \frac{1}{\mu} B$, $\mu = 1 + 4\pi\chi_m$

$$H = B - 4\pi M$$

$$H = B - 4\pi \chi_m H$$

$$H(1 + 4\pi\chi_m) = B$$

$$H\mu = B$$

$$H = \frac{1}{\mu} B$$

$$\Rightarrow H = \frac{2I}{\mu c r}\hat{\varphi}$$

#5 (cont.)

Region III: $b \le r \le d$

*Must separate $\vec{J}$ - integral to account for multiple currents

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c} \int \vec{J} \cdot da$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \left[\int \vec{J}_{rod} \cdot da_{rod} + \int \vec{J}_{shell} \cdot da_{shell} \right]$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \left[\frac{I}{\pi a^2} \cdot \pi a^2 + \frac{-I}{\pi (d^2 - b^2)} \cdot \pi (r^2 - b^2) \right]$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \left[I - \frac{I (r^2 - b^2)}{d^2 - b^2} \right]$$

$$\vec{B} = \frac{2I}{cr} \left[1 - \frac{r^2 - b^2}{d^2 - b^2} \right] \hat{\varphi}$$

* We again must consider magnetization in region II; following above work,

$$H = \frac{1}{\mu} B$$

$$= \frac{2I}{c\mu r} \left[1 - \frac{r^2 - b^2}{d^2 - b^2} \right] \hat{\varphi}$$

Region IV: $r \ge d$

$$\oint B \cdot d\vec{l} = \frac{4\pi}{c} \int \vec{j} \cdot d\vec{a}$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \left[\int \vec{J}_{rod} \cdot da_{rod} + \int \vec{J}_{shell} \cdot da_{shell} \right]$$

$$B \cdot 2\pi r = \frac{4\pi}{c} \left[\frac{I}{\pi a^2} (\pi a^2) + \frac{-I}{\pi (d^2 - b^2)} (d^2 - b^2) \pi \right]$$

$$\vec{B} = \frac{4\pi}{c} \left[I - I \right] \cdot \frac{1}{2\pi r}$$

$$\vec{B} = 0$$

$$\vec{H} = \frac{1}{\mu} \vec{B}$$

$$\vec{H} = 0$$

#5 (cont.)

b) *We know that the self-inductance L is related to the flux via: $\Phi = LI$

To get total flux of system, we sum the fluxes from the individual regions

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^{\ell} dz \int dx \; B$$

Region I: $r \le a$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^{\ell} dz \int_0^a \frac{2I\sqrt{x^2+y^2}}{ca^2}\, dx$$

$$= \int_0^r \frac{2I}{ca^2}\, x\, dx$$

$$= \frac{I}{ca^2} x^2 \Big|_0^a$$

$$= \frac{I}{c}$$

Region II: $a \le r \le b$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^{\ell} dz \int_a^b \frac{2I}{c\sqrt{x^2+y^2}}\, dx$$

$$= \int_a^b \frac{2I}{cx}\, dx$$

$$= \frac{2I}{c} \ln(x) \Big|_a^b$$

$$= \frac{2I}{c}\left(\ln(b/a)\right)$$

Region III: $b \le r \le d$

$$\frac{\Phi}{\ell} = \frac{1}{\ell}\int_0^{\ell} dz \int_b^d \frac{2I}{cr}\left[1 - \frac{r^2-b^2}{a^2-b^2}\right] dx$$

$$= \int_b^d \frac{2I}{cx}\left[1 - \frac{x^2-b^2}{d^2-b^2}\right] dx$$

$$= \frac{2I}{c}\int\left[\frac{1}{x} - \frac{x}{d^2-b^2} + \frac{b^2}{x(d^2-b^2)}\right] dx$$

$$= \frac{2I}{c}\left[\ln(x) - \frac{1}{2}\frac{x^2}{d^2-b^2} + \frac{b^2}{d^2-b^2}\ln(x)\right]\Big|_b^d$$

#5 (cont.)

b) $\frac{\Phi}{l} = \frac{2I}{c}\left[\ln\left(\frac{d}{b}\right) - \frac{1}{2} + \frac{b^2}{d^2-b^2}\ln\left(\frac{d}{b}\right)\right]$

$= \frac{2I}{c}\left[\left(1+\frac{b^2}{d^2-b^2}\right)\ln\left(\frac{d}{b}\right) - \frac{1}{2}\right]$

Region IV: $r \geq d$

$B = 0 \Rightarrow \frac{\Phi}{l} = 0$

* Thus our overall flux per unit length is:

$$\frac{\Phi}{l} = \frac{I}{c} + \frac{2I}{c}(\ln(b/a)) + \frac{2I}{c}\left[\left(1+\frac{b^2}{d^2-b^2}\left(\ln\left(\frac{d}{b}\right)\right) - \frac{1}{2}\right]$$

$$= \cancel{\frac{I}{c}} + \frac{2I}{c}\ln(b/a) + \frac{2I}{c}\ln\left(\frac{d}{b}\right) + \frac{2I}{c}\frac{b^2}{d^2-b^2}\ln\left(\frac{d}{b}\right) - \cancel{\frac{I}{c}}$$

$$= \frac{2I}{c}\left[\ln(b/a) + \left(1+\frac{b^2}{d^2+b^2}\right)\ln\left(\frac{d}{b}\right)\right]$$

$$\frac{\Phi}{l} = \frac{LI}{l} \Rightarrow \frac{L}{l} = \frac{\Phi}{lI}$$

$$\Rightarrow \frac{L}{l} = \frac{2}{c}\left[\ln(b/a) + \left(1+\frac{b^2}{d^2+b^2}\right)\ln\left(\frac{d}{b}\right)\right]$$

c) $U = \frac{1}{8\pi}B^2$

$U = \frac{1}{8\pi}\left(\frac{2I}{cr}\hat{\varphi}\right)^2$

$= \frac{1}{8\pi}\frac{4I^2}{c^2r^2}$

$U = \int \frac{I^2}{4\pi c^2 r^2}\, r\, dr\, d\varphi\, dz$

$= \frac{lI^2}{2c^2}\int_a^b \frac{1}{r}\,dr$

$= \frac{lI^2}{2c^2}\ln(b/a)$

$\frac{U}{l} = \frac{I^2}{2c^2}\ln(b/a)$

#5 (cont.)

d) $\frac{1}{2} L_{II} I^2 = \frac{1}{2}\left(\frac{2}{c}\ln(b/a)\right) I^2$

$= \frac{I^2}{c}\ln(b/a)$ * off by factor $\frac{1}{2c}$, likely due to units being Gaussian v SI

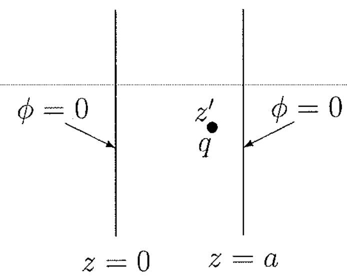

Figure 3: Geometry of point charge placed between grounded, parallel, conducting plates. The plates extend infinitely in the x and y directions.

6. Consider a point charge q placed between two parallel conducting plates, as shown in Fig. 3. The electrostatic potential ϕ vanishes at the two plates, located on the planes $z = 0$ and $z = a$.

a) 2pts. Because the physics has translational symmetry in the x-y plane, show that the potential at a point $\mathbf{r}$ between the plates due to a point charge at $\mathbf{r}'$ can be written in the form (Gaussian units)

$$\phi(\mathbf{r}) = 4\pi q \int \frac{(d^2\mathbf{r}_\perp)}{(2\pi)^2} e^{i\mathbf{r}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp} g(z, z'; k_\perp),$$

with $\mathbf{r}_\perp = (x, y)$, $\mathbf{r}'_\perp = (x', y')$, where the function g satisfies

$$\left(-\frac{\partial^2}{\partial z^2} + k_\perp^2\right) g(z, z'; k_\perp) = \delta(z - z').$$

What are the boundary conditions on $g(z, z'; k_\perp)$ at $z = 0$ and $z = a$?

b) 3pts. Solve this differential equation explicitly in closed form by solving it in two regions, I: $0 < z < z'$ and II: $z' < z < a$, and matching the solutions appropriately to reproduce the δ-function in the differential equation.

c) 2pts. What is the relationship between the electric field at the surface of a conductor and the surface charge density on the surface?

d) 3pts. Determine the normal component of the electric field just to the right of the plate at $z = 0$ (that is, at $z = 0 + \epsilon$) and just to the left of the plate at $z = a$ (that is, at $z = a - \epsilon$). By integrating this field over the surfaces, and using the result of part c), determine the total charge on each of the conducting surfaces. Is the sum of the charges on the two plates as expected?