E & M Qualifier

January 11, 2012

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).

1. Dielectric Sphere

A dielectric sphere of radius R is polarized so that $\mathbf{P} = (K/r)\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ is the unit radial vector. Assume the sphere is in an empty vacuum and that the sphere's dielectric material is linear and isotropic, calculate

(a) (3 pts) the volume and the surface densities of bound charge,

(b) (2 pts) the volume density of free charge,

(c) (2 pts) the electric field inside the sphere,

(d) (3 pts) the electric field outside the sphere.

Your answers should be given in terms of $K, \chi_E, \epsilon_0, \epsilon$, and/or ϵ_r. Recall that for linear isotropic materials:

In SI units,

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \epsilon_0\chi_E\mathbf{E}$$

$$\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = 1 + \chi_E$$

In Gaussian units,

$$\mathbf{D} = \epsilon\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}$$

$$\mathbf{P} = \chi_E\mathbf{E}$$

$$\epsilon = 1 + 4\pi\chi_E = \epsilon_r$$

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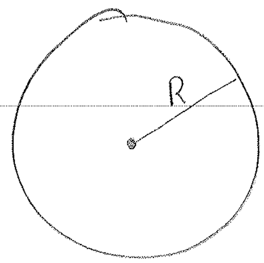
E + M #1



$$\vec{P} = \frac{K}{r}\hat{r}$$

Given a polarized, dielectric sphere (dielectric is linear + isotropic) in a vaccum;

a) Our surface charge density is:

$$\sigma_b = \vec{P}\cdot\hat{n}\,\Big|_{surface}$$
$$= \frac{K}{r}\hat{r}\cdot\hat{r}\,\Big|_{r=R}$$
$$= \frac{K}{R}$$

The volume bound charge density is:

$$\rho_b = -\nabla\cdot\vec{P}$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{K}{r}\right)$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(kr)$$
$$= -\frac{k}{r^2}$$

b) We can determine the volume density of free charge by:

$$-\nabla\cdot E = \frac{\rho_f}{\epsilon_0}$$

$\Rightarrow$ To determine the electric field inside the material, we use

$$\vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P}$$
$$(\epsilon - \epsilon_0)E = \vec{P}$$
$$\vec{E} = \frac{1}{\epsilon - \epsilon_0}\vec{P}$$
$$= \frac{K}{r(\epsilon - \epsilon_0)}\hat{r}$$

Work for part C HERE

#1 (cont.)

b) $$-\nabla \cdot E = \frac{\rho_f}{\epsilon_0}$$

$$-\nabla \cdot \frac{K}{(\epsilon-\epsilon_0)r}\hat{r} = \frac{\rho_f}{\epsilon_0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Kr}{\epsilon-\epsilon_0}\right) = \frac{\rho_f}{\epsilon_0}$$

$$\frac{K}{r^2(\epsilon-\epsilon_0)} = \frac{\rho_f}{\epsilon_0}$$

$$\hookrightarrow \rho_f = \frac{K\epsilon_0}{r^2(\epsilon-\epsilon_0)}$$

$$= \frac{K}{r^2(-1+\epsilon/\epsilon_0)}$$

$$= \frac{K}{r^2(\epsilon_r-1)}$$

d) From Gauss' Law, we know

$$\oint_S E \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

To get q_{enc}:

$$q_{enc} = \int \sigma_b \, da + \int \rho_f \, dV + \int \rho_b \, dV$$

$$= \int_0^{2\pi}\int_0^{\pi} R^2 \frac{K}{R} \sin^2\theta \, dr \, d\theta \, d\varphi + \int_0^R \int_0^{2\pi}\int_0^{\pi} \frac{K}{r^2(\epsilon_r-1)} r^2 \sin\theta \, dr \, d\theta \, d\varphi + \int_0^R \int_0^{2\pi}\int_0^{\pi} \frac{-k}{r^2} r^2 \sin\theta \, d\theta \, d\varphi \, dr$$

$$= KR \cdot 4\pi + \frac{KR}{\epsilon_r - 1} \cdot 4\pi - 4\pi KR$$

$$= \frac{4\pi KR}{\epsilon_r - 1}$$

$$\oint E \cdot d\vec{a} = \frac{4\pi KR}{\epsilon-\epsilon_0}$$

$$4\pi r^2 E = \frac{4\pi KR}{\epsilon-\epsilon_0}$$

$$E = \frac{KR}{R(\epsilon-\epsilon_0)} \cdot \frac{1}{r^2}\hat{r}$$

2. Gauge Transformation

   (a) (2 pts)
   Define the vector potential $\mathbf{A}$ and the scalar potential Φ using Maxwell's equations. (i.e. give their relationships to the $\mathbf{E}$ and $\mathbf{B}$ fields.)

   (b) (3 pts) Show that when $\mathbf{A}$ and Φ undergo the gauge transformations,

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \qquad (SI) \text{ and } (Gaussian)$$

$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t}, \qquad (SI)$$

   or

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}, \qquad (Gaussian)$$

   where Λ is an arbitrary scalar, $\mathbf{B}$ and $\mathbf{E}$ are unaffected.

   (c) Two gauges used in solid-state physics for static, uniform magnetic fields $\mathbf{B}$ (i.e., constant in direction, magnitude, and time) are the Landau gauge and the circular gauge. Examples for $\mathbf{B} = B_0\hat{z}$ of each gauge respectively are:

$$\mathbf{A} = (A_x, A_y, A_z) = (0, B_0\, x, 0)$$

   and

$$\mathbf{A}' = (A'_x, A'_y, A'_z) = (-B_0 y/2, B_0 x/2, 0),$$

   with

$$\Phi = 0,$$

   for both gauges.

   i. (2 pts) Show that $\mathbf{A}$ and $\mathbf{A}'$ with $\Phi = \Phi' = 0$ describe the same $\mathbf{E}$ and $\mathbf{B}$ fields.

   ii. (3 pts) Find the scalar function Λ that produces the gauge transformation from $\mathbf{A}$ to $\mathbf{A}'$ in part (c).

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E+M #2

Gaussian

a) $\vec{\nabla} \times \vec{A} = \vec{B}$

$-\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t} = \vec{E}$

b) * If we now define $\vec{A}' = \vec{A} + \nabla \Lambda$, $\Phi' = \Phi - \frac{\partial \Lambda}{c\,\partial t}$

$$\vec{\nabla} \times \vec{A}' = \nabla \times (\vec{A} + \nabla \Lambda)$$
$$= \nabla \times A + \cancel{\nabla \times (\nabla \Lambda)}^{0}$$
$$= \vec{\nabla} \times \vec{A} \checkmark$$

$$-\nabla \Phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} = -\nabla\left(\Phi - \frac{\partial \Lambda}{c\,\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A} + \nabla \Lambda\right)$$
$$= -\nabla \Phi + \cancel{\frac{1}{c}\frac{\partial}{\partial t}\nabla \Lambda} - \frac{1}{c}\frac{\partial A}{\partial t} - \cancel{\frac{1}{c}\frac{\partial}{\partial t}\nabla \Lambda}$$
$$= -\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t} \checkmark$$

c) i) * Working with A, Φ

$$\vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t}$$
$$= 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$= \langle -\frac{\partial}{\partial z} B_0 x,\ 0,\ \frac{\partial}{\partial x} B_0 x \rangle$$
$$= B_0 \hat{z}$$

* Working with A', Φ'

$$\vec{E} = -\nabla \Phi' - \frac{1}{c}\frac{\partial A'}{\partial t}$$
$$= 0$$

$$\vec{B} = \nabla \times A'$$
$$= \langle -\frac{\partial}{\partial z}\frac{B_0}{2}x,\ \frac{\partial}{\partial z}\frac{B_0}{2}y,\ \frac{\partial}{\partial x}\frac{B_0}{2}x + \frac{\partial}{\partial y}\frac{B_0}{2}y \rangle$$
$$= \langle 0,\ 0,\ B_0 \rangle$$

* Both sets match

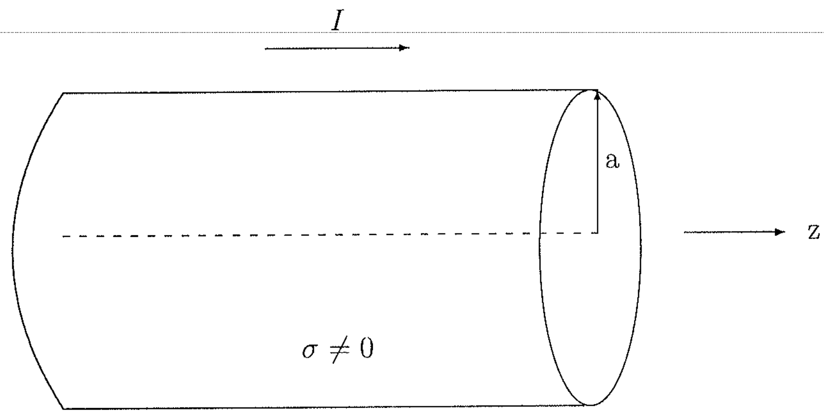
ii) $A' = A + \nabla \Lambda \implies \nabla \Lambda = A' - A$

* Note: $\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

$\hookrightarrow \nabla \Lambda = \langle -\frac{B_0}{2}y,\ -\frac{B_0}{2}x,\ 0 \rangle$

$$\Lambda = -\frac{B_0}{2}xy + \text{Constant}$$

3. Poynting Vector



A straight metal wire of conductivity σ and cross-sectional area $A = \pi a^2$ carries a uniform, steady current I.

(a) (2 pts) Calculate $\mathbf{E}$ at the surface of the wire.

(b) (2 pts) Calculate $\mathbf{B}$ at the surface of the wire.

(c) (1 pts) Calculate the direction and magnitude of the Poynting vector at the surface of the wire.

(d) (3 pts) Integrate the normal component of the Poynting vector over the surface of the wire for a segment of length L.

(e) (2 pts) compare your result for (d) with the Joule heat produced in this segment.

The Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

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E+M #3

a) From Ohms Law, we know $\vec{J} = \sigma \vec{E}$, and that the total current I is related to $\vec{J}$ via $I = \oint \vec{J} \cdot d\vec{a}$

$$\Rightarrow \quad I = \oint \sigma E \cdot d\vec{a}$$

$$I = \sigma \oint E \cdot d\vec{a}$$

$$I = \sigma E \cdot \pi a^2$$

$$\frac{I}{\sigma \pi a^2} \hat{z} = \vec{E}$$

b) We can calculate $\vec{B}$ using Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_{enc}$$

$$B \cdot 2\pi a = \frac{4\pi}{c} I$$

$$\vec{B} = \frac{2I}{ca} \hat{\varphi} \quad \text{(from RHR)}$$

c) In Gaussian units, $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H}$, $B = H + 4\pi M$

$$\vec{S} = \frac{c}{4\pi} \begin{vmatrix} \hat{\rho} & \hat{\varphi} & \hat{z} \\ 0 & 0 & \frac{I}{\pi \sigma a^2} \\ 0 & \frac{2I}{ca} & 0 \end{vmatrix}$$

$$= \left\langle -\frac{2I^2}{\pi c \sigma a^2}, 0, 0 \right\rangle$$

$$= \frac{-2I^2}{\pi c \sigma a^2} \cdot \frac{c}{4\pi}$$

$$= \frac{-I^2}{2\pi^2 \sigma a^3} \hat{\rho}$$

#3 (cont.)

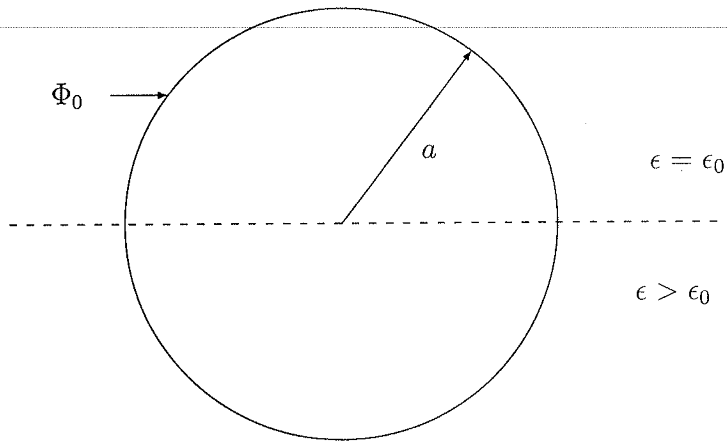
d) We want to calculate $\oint S \cdot d\vec{a}$

$$\oint \vec{S} \cdot d\vec{a} = S \oint d\vec{a}$$

$$= \frac{-I^2}{2\pi^2 \sigma a^3} (2\pi a L)$$

$$= \frac{-I^2 L}{\sigma a^2}$$

4. Half Submerged Conducting Sphere



An originally uncharged thin spherical conducting shell of radius a is brought to a potential Φ_0. The shell floats half submerged in a dielectric liquid of dielectric constant $k = \epsilon_r \equiv \epsilon/\epsilon_0$.

Determine the following:

(a) (2 pts) The electric potential Φ everywhere **outside** the shell,

(b) (2 pts) The electric field $\mathbf{E}$ everywhere **outside** the shell,

(c) (2 pts) The free surface charge density σ on the shell,

(d) (4 pts) The net electrostatic force $\mathbf{F}$ acting on the shell.

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E+M #4

5. Capacitor Plates

Consider a very large parallel plate capacitor with the positive plate at $z = d/2$, the negative plate at $z = -d/2$ and no dielectric material in between. If the respective surface charge densities are $\pm\sigma$ compute the *force/area on the positive plate* in the following two ways:

(a) (4 pts) Calculate it directly from σ and the electric field $\mathbf{E}$. Give a logical explanation of why your answer is correct.

(b) (6 pts) Calculate it using the Maxwell stress tensor

$$T_M^{ij} = \epsilon_0 \left[E^i E^j - \frac{1}{2}\delta^{ij}\vec{E}\cdot\vec{E} \right], \qquad (SI)$$

$$T_M^{ij} = \frac{1}{4\pi} \left[E^i E^j - \frac{1}{2}\delta^{ij}\vec{E}\cdot\vec{E} \right]. \qquad (Gaussian)$$

6. E&M Waves

A monochromatic, plane polarized, plane electromagnetic wave traveling in the z-direction in the lab (in a vacuum) can be written in the following 3+1 dimensional form:

$$\mathbf{E} = E_0 \; \hat{\mathbf{x}} \; e^{i(k\,z - \omega\,t)},$$

$$\mathbf{B} = B_0 \; \hat{\mathbf{y}} \; e^{i(k\,z - \omega\,t)}.$$

(a) (3 pts) Combine this $\mathbf{E}$ and $\mathbf{B}$ into a single electromagnetic field tensor $F^{\alpha\beta}$ and use Maxell's equations in the 4-dimensional form

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0,$$

$$\partial_\alpha F^{\alpha\beta} = 0$$

to find all constraints on the 4 constants $E_0, B_0, k,$ and ω (i.e., the above wave won't satisfy Maxwell's equations for arbitrary values of all four of these parameters). Depending on your choice of conventions: $x^\alpha = (x^0, x^1, x^2, x^3)$ with $x^0 = ct$ or $x^\alpha = (x^1, x^2, x^3, x^4)$ with $x^4 = ct$ and $x^1 = x, x^2 = y, x^3 = z$.

(b) (1 pts) What are the values of the invariants $F^{\alpha\beta}F_{\alpha\beta}$ and $\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$ for this wave?

(c) (3 pts) Use a Lorentz boost to find $F'^{\alpha\beta}$ in a frame moving in the $+z$ direction with a speed v. Don't forget to express your answer in terms of the moving coordinates ct' and x', y', z'.

(d) (2 pts) What is the frequency and the wavelength of this wave in the moving frame?

(e) (1 pts) How have the electric and magnetic fields changed in direction and/or magnitude?

E+M #6

Gaussian

a) $\vec{E} = \langle E_0, 0, 0 \rangle \exp[i(kz - \omega t)]$

$\vec{B} = \langle 0, B_0, 0 \rangle \exp[i(kz - \omega t)]$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} \exp[i(kz - \omega t)]$$

* for $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$, the only non-zero terms are: F_{01}, F_{13}

$$\partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}(-B_0 \exp[i(kz-\omega t)]) + \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial z}(E_0 \exp[i(kz-\omega t)]) = 0$$

$$-\frac{i\omega}{c} B_0 \exp[i(kz-\omega t)] + ik E_0 \exp[i(kz-\omega t)] = 0$$

$$k E_0 = \frac{\omega}{c} B_0 \quad ①$$

* for $\partial_\alpha F^{\alpha\beta} = 0$, $\beta = 1$ to get both E_0, B_0

$$\partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}(E_0 \exp[i(kz-\omega t)]) + 0 + 0 + \frac{\partial}{\partial z}(B_0 \exp[i(kz-\omega t)]) = 0$$

$$-\frac{i\omega}{c} E_0 \exp[i(kz-\omega t)] + ik B_0 \exp[i(kz-\omega t)] = 0$$

$$\hookrightarrow \frac{\omega}{c} E_0 = k B_0 \quad ②$$

* Combining ① and ② yields

$$\frac{\omega}{c} E_0 = k\left(\frac{ck}{\omega} E_0\right)$$

$$\frac{\omega^2}{c^2} = k^2 \Rightarrow \boxed{k = \pm \frac{\omega}{c}}$$

$$\frac{\omega}{c} E_0 = \pm \frac{\omega}{c} B_0 \Rightarrow \boxed{E_0 = \pm B_0}$$

#6 (cont.)

b) $$F^{\alpha\beta}F_{\alpha\beta} = \begin{bmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & E_0 & 0 & 0 \\ -E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} E_0^2 & 0 & 0 & -E_0B_0 \\ 0 & E_0^2-B_0^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_0B_0 & 0 & 0 & -B_0^2 \end{bmatrix}$$

* but we want the trace of this matrix

$$= 2(E_0^2 - B_0^2)$$

* Should be $2(B_0^2 - E_0^2)$

$$\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} = 2\mathcal{F}^{\gamma\delta}F_{\gamma\delta}$$

$$= \begin{bmatrix} 0 & 0 & -B_0 & 0 \\ 0 & 0 & 0 & 0 \\ B_0 & 0 & 0 & E_0 \\ 0 & 0 & -E_0 & 0 \end{bmatrix}\begin{bmatrix} 0 & E_0 & 0 & 0 \\ -E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} 2$$

$$= 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2B_0E_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$= 0$, makes sense b/c $\hat{x}\cdot\hat{y} = 0$

* Should be $-8(B\cdot E)$

#6 (cont.)

c) $L = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$

$$F' = L^T F L$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & B_0 \\ 0 & 0 & 0 & 0 \\ 0 & -B_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\gamma+\beta\gamma & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \beta\gamma-\gamma & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} E_0 \exp[i(kz-\omega t)]$$

$$= \begin{bmatrix} 0 & -\gamma+\beta\gamma & 0 & 0 \\ \gamma-\beta\gamma & 0 & 0 & \gamma-\beta\gamma \\ 0 & 0 & 0 & 0 \\ 0 & -\gamma+\beta\gamma & 0 & 0 \end{bmatrix} E_0 \exp[i(kz-\omega t)]$$

* but we must now transform the coordinates, which we can rewrite according to

$$kz - \omega t = -k_\sigma x^\sigma, \qquad k_\sigma = \frac{\omega}{c}\langle ct, 0, 0, -z\rangle$$
$$= k\langle 1, 0, 0, -1\rangle$$

$$k'_\lambda x'^\lambda = k_\sigma B^{-1\sigma}{}_\lambda$$

$$= [1\ 0\ 0\ {-1}] \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix}$$

$$= [\gamma(1-\beta) \quad 0 \quad 0 \quad -\gamma(1-\beta)]$$

$$\Rightarrow k' = k\gamma(1-\beta) \quad \Rightarrow \quad \frac{\omega'}{c} = \frac{\omega}{c}\gamma(1-\beta)$$

#6 (cont.)

c) Since wave will have same form in both frames

$$E' = E_0 \gamma(1-\beta) \exp[i(k'z' - \omega' t')]$$

$$B' = B_0 \gamma(1-\beta) \exp[i(k'z' - \omega' t')]$$

d) where $k' = k\gamma(1-\beta)$, $\omega' = \omega\gamma(1-\beta) = \omega\sqrt{\frac{1-\beta}{1+\beta}} \Rightarrow \lambda' = \lambda\sqrt{\frac{1+\beta}{1-\beta}}$

e) Direction is same but magnitude has changed

$$E_0' = E_0 \gamma(1-\beta)$$

$$B_0' = B_0 \gamma(1-\beta)$$

$$= E_0 \gamma(1-\beta)$$