E & M Qualifier

August 16, 2017

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

Problem 1: Electrostatics

A non-conducting solid sphere of radius R carries a charge density $\rho(r) = k\,r$ (where k is a constant).

(a) Find the electric field at a distance r such that $r \geq R$ [1 point]

(b) Find the electric field at a distance r such that $r \leq R$ [1 point]

(c) State the boundary conditions on the electric field components on the surface of the sphere, and show that your answers to parts a) and b) are consistent with them.
HINT: The surface charge density of the sphere is zero in this case. [1 point]

(d) Calculate the electric potential for all r using $\lim_{r\to\infty} V(r) = 0$ [2 points]

(e) Find the work required to assemble this charge [2 points]

(f) If the *non-conducting* solid sphere was replaced by a *conducting* solid sphere with the same total charge, how does that change your answer to parts a, b, and c?
Explicitly show that the new field satisfies the new boundary conditions across the surface of the sphere. [3 point]

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E+M #1

Gaussian

a) We use Gauss' Law to find the field at $r \geq R$

$$\int \vec{\nabla} \cdot \vec{E}\, dV = \int 4\pi \rho\, dV$$

$$\oint \vec{E} \cdot d\vec{a} = 4\pi \int kr \cdot r^2 \sin\theta\, dr d\theta d\varphi$$

$$\vec{E} \cdot 4\pi R^2 = 16\pi^2 k \int_0^R r^3 dr$$

$$4\pi r^2 \vec{E} = 4\pi^2 k\, r^4 \Big|_0^R$$

$$4\pi r^2 \vec{E} = 4\pi^2 k R^4$$

$$\vec{E} = \pi k \frac{R^4}{r^2} \hat{r}$$ (know $\hat{r}$ direction due to spherical symmetry of problem)

b) We proceed as above, now with $r \leq R$

$$\int \vec{\nabla} \cdot \vec{E}\, dV = \int 4\pi \rho\, dV$$

$$\oint \vec{E} \cdot d\vec{a} = 4\pi k \int r'^3 \sin\theta\, dr' d\theta d\varphi$$

$$\vec{E} \cdot 4\pi r^2 = 4\pi^2 k\, r'^4 \Big|_0^r$$

$$\vec{E} \cdot 4\pi r^2 = 4\pi^2 k r^4$$

$$\vec{E} = k\pi r^2\, \hat{r}$$

c) The boundary conditions of $\vec{E}$ are: ① $E_1^{\perp} - E_2^{\perp} = 4\pi\sigma$

② $E_1^{\parallel} = E_2^{\parallel}$

② is automatically satisfied as $\vec{E}$ only points perpendicular to surface in both cases

* Defining region 1 as inside the sphere and region 2 as outside the sphere,

$$E_1^{\perp} = \vec{E}_1 \cdot \hat{n}\big|_{r=R} \qquad\qquad E_2^{\perp} = \vec{E}_2 \cdot \hat{n}\big|_{r=R}$$

$$= E_1 \cdot \hat{r}\big|_{r=R} \qquad\qquad = \vec{E}_1 \cdot \hat{r}\big|_{r=R}$$

$$= k\pi R^2 \qquad\qquad = k\pi R^2$$

* Since we know $\sigma = 0$, condition ① is also satisfied

* See Griffiths Ch 2 for origin of formulas

#1 (cont.)

d) We calculate the potential according to:

$$V(\vec{x}) = \int \frac{p(x)}{|x-x'|} d^3x' = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

* due to spherical symmetry, it is easier to use the second form and integrate along a line radially pointing inwards from infinity to r

$$V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

* since the field changes at $r = R$, we must proceed with 2 cases

* if $r > R$

$$V(r) = -\int_{\infty}^{r} \frac{\pi k R^4}{r'^2} dr'$$

$$= \left. \frac{\pi k R^4}{r'} \right|_{\infty}^{r}$$

$$= \frac{\pi k R^4}{r}$$

* if $r < R$

$$V(r) = -\left(\int_{\infty}^{R} \frac{\pi k R^4}{r'} dr' + \int_{R}^{r} k\pi r'^2 dr' \right)$$

$$= \left. \frac{\pi k R^4}{r'} \right|_{\infty}^{R} - \left. \frac{\pi k r'^3}{3} \right|_{R}^{r}$$

$$= \pi k R^3 - \left(\tfrac{1}{3} \pi k r^3 - \pi k R^3 \right)$$

$$= \tfrac{4}{3} \pi k R^3 - \tfrac{1}{3} \pi k r^3$$

e) We can find the work necessary to assemble this charge distribution according to

$$W = \frac{1}{8\pi} \int E^2 dV$$

$$= \frac{1}{2} \int_{0}^{\infty} E^2 r'^2 dr'$$

* again we must consider our two regions

#1 (cont.)

e) $W = \frac{1}{2}\left[\int_0^R E_1^2 r'^2 dr' + \int_R^\infty E_2^2 r'^2 dr'\right]$

$$= \frac{1}{2}\left[\int_0^R k^2\pi^2 r'^6 dr' + \int_R^\infty k^2\pi^2 \frac{R^8}{r'^4} r'^2 dr'\right]$$

$$= \frac{1}{2}\left[k^2\pi^2 \frac{1}{7} r'^7 \Big|_0^R + k^2\pi^2 R^8 \left(\frac{-1}{r'}\Big|_R^\infty\right)\right]$$

$$= \frac{k^2\pi^2}{2}\left(\frac{1}{7}R^7 + R^7\right)$$

$$= \frac{4k^2\pi^2 R^7}{7}$$

f) If we replace the non-conducting sphere with a conducting sphere of equal charge

a) Field outside conductor remains unchanged

b) Field goes to 0 inside conductor

c) Boundary conditions are same, except all charge accumulates on surface

$$\sigma = \frac{1}{4\pi R^2}\int \rho \, dV$$

$$= \frac{1}{4\pi R^2} 4\pi k \cdot \frac{1}{4} R^4$$

$$= \frac{1}{4} k R^2$$

$$E_1^\perp - E_2^\perp = 4\pi\sigma$$

$$0 - \frac{\pi k R^4}{r^2} = 4\pi\left(\frac{1}{4}kR^2\right)$$

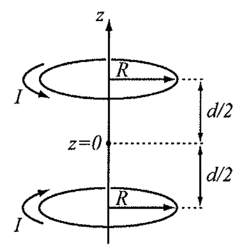
$$-\pi k R^2 = \pi k R^2 \checkmark$$

Problem 2: Magnetostatics

Consider a circular loop of radius R which carries a steady current I.

(a) Calculate the magnetic field a distance z above the center of the loop. [3 points]

(b) Now consider a configuration composed of two circular loops a distance d apart and with currents flowing in opposite directions, as shown in the figure. This configuration is know as anti-Helmholtz coils. Calculate the magnetic field along the z-axis as a function of z. [2 points]



(c) For what value of z will the the magnetic field due to the anti-Helmholtz coils be equal to zero? Give a physical explanation for your result. [2 points]

(d) Calculate the magnetic dipole moment of the configuration. [3 points]

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E+M #2

a) We calculate the magnetic field according to the Biot-Savart Law

$$\vec{B} = \frac{4\pi}{c} \int \frac{\vec{J} \times (\vec{r} - \vec{r}')}{|r - r'|^3} dr'$$

$$= \frac{4\pi}{c} I_0 \int \frac{dl \times (\vec{r} - \vec{r}')}{|r - r'|^3}$$

$$= \frac{4\pi}{c} I_0 \int \frac{R\,dl}{\sqrt{(R^2 + z^2)^3}} \hat{z}$$ (from evaluation of cross product + symmetry of problem, leaving only z-component non-zero)

$$= \frac{4\pi}{c} I_0 \frac{2\pi R^2}{(R^2 + z^2)^{3/2}} \hat{z}$$

$$= \frac{8\pi^2 I_0 R^2}{c(R^2 + z^2)^{3/2}} \hat{z}$$

b) In our current configuration, we call the loop with the CCW current loop 1, and the loop with CW current loop 2

$$\Rightarrow B_t = B_1 + B_2$$

$$= \frac{8\pi^2 I_0 R^2}{c(R^2 + [z - d/2]^2)^{3/2}} \hat{z} + \frac{8\pi^2 (-I_0) R^2}{c(R^2 + [z + d/2]^2)^{3/2}} \hat{z}$$

$-I_0$ b/c CW current
$\pm d/2$ added to reflect shift from origin

$$= \frac{8\pi^2 I_0 R^2}{c} \left[\frac{1}{(R^2 + [z - d/2]^2)^{3/2}} - \frac{1}{(R^2 + [z + d/2]^2)^{3/2}} \right] \hat{z}$$

c) $\vec{B} = 0$ at $z = 0$. This is due to the fields generated by each coil pointing in the opposite direction

d) The formula for the magnetic dipole moment is:

$$\vec{m} = I \int d\vec{a}$$

$$\hookrightarrow \vec{m}_{tot} = \vec{m}_1 + \vec{m}_2$$

$$= I_0 \pi R^2 \hat{z} - I_0 \pi R^2 \hat{z}$$

$$= 0 \ ? \quad \text{or} \quad 2 I_0 \pi R^2$$ (depending on if we have positive or negative normal on $\vec{m}_2$ integral)

Problem 3: Waves

An electromagnetic wave with an angular frequency of ω passes from medium 1, through a slab of medium 2 (with thickness d), and into medium 3. All three media are linear and homogeneous, and have a permeability of μ_o. The index of refraction is n_1, n_2, and n_3 for medium 1, 2, and 3, respectively.

(a) In medium 1, there is an incident plane wave (electric field of amplitude E_I) and a reflected plane wave (electric field of amplitude E_R). In medium 3, there is a transmitted plane wave (electric field of amplitude E_T). In medium 2, there is a plane wave going toward medium 3 (electric field of amplitude E_r) and a plane wave going toward medium 1 (electric field of amplitude E_l). Write down expressions that describe the electric and magnetic fields in medium 1, medium 2, and medium 3. [4 points]

(b) Apply the boundary conditions for the electric and magnetic fields at the interface between medium 1 and medium 2. Express E_I in terms of E_r, E_l, n_1, and n_2. [1.5 points]

(c) Apply the boundary conditions for the electric and magnetic fields at the interface between medium 2 and medium 3. Express E_r in terms of E_T, n_2, n_3 and d. Also express E_l in terms of E_T, n_2, n_3 and d. [1.5 points]

(d) Combine your answers for part b and c to get an expression for E_T/E_I in terms of n_1, n_2, n_3, and d. What percentage of the incident wave is transmitted into medium 3? Express your answer in terms of n_1, n_2, n_3 and d. [1 points]

(e) Suppose medium 1 is water ($n_1 = 4/3$), medium 2 is glass ($n_2 = 3/2$), and medium 3 is air ($n_3 = 1$). Will the thickness of the glass make much difference in how well you (in medium 3) can see the fish (in medium 1)? How would your answer change if instead you were in the water and the fish was in the air? [2 points]

Problem 4: ED in media

In this problem we consider the propagation of an electromagnetic wave through an optically active media; a media that causes the direction of polarization to rotate about the direction of propagation. Such a media can be describe using the susceptibility tensor

$$\boldsymbol{\chi} = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix}.$$

Furthermore, recall that the polarization of the media and the electric field are related by $\vec{P} = \epsilon_0 \boldsymbol{\chi} \vec{E}$ in SI units. *You can assume a non-magnetic linear dielectric material with no free charges or currents.*

(a) Starting from the Maxwell equations, the expressions given in the statement of the problem, and assuming a transverse electromagnetic wave propagating in the z direction, derive the magnitude of the wave vector k for the two possible polarizations in terms of the components of $\boldsymbol{\chi}$. [4 points]

(b) Prove that the allowed wave modes correspond to circularly polarized waves. [2 points]

(c) Under the assumption that $\chi_{12} \ll \chi_{11}$, derive an expression for the amount that the polarization rotates over a distance ℓ. The approximation $\sqrt{1-\epsilon} \approx 1 + \epsilon/2$ might be useful. [4 points]

Problem 5: Radiation

Two oscillating dipole moments $\vec{d_1}$ and $\vec{d_2}$ are oriented parallel to each other in the direction of the y-axis and are separated by a distance L. They oscillate in phase at the same frequency ω. For an observer at a distance r with $r \gg L$ located at an angle θ with respect to the y-axis in the plane of the two dipoles.

(a) Show that [9 points]

$$\frac{dP}{d\Omega} = \frac{\omega^4 \sin\theta^2}{8\pi c^3}(d_1^2 + 2d_1 d_2 \cos\delta + d_2^2), \tag{1}$$

where

$$\delta = \frac{\omega L \sin\theta}{c} \tag{2}$$

and where P is the time-averaged power. The units are Gaussian.

(b) Show that when $L \ll \lambda$, the radiation can be approximated as from a single oscillating dipole of amplitude $d_1 + d_2$. [1 point]

Problem 6: Relativity

(a) Write down both the homogeneous and inhomogeneous Maxwell's equations in manifestly Lorentz covariant form using the 2nd rank field strength tensor $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. (State which system of units you are using) [2 points]

(b) Write down the components (in matrix form) of $F_{\mu\nu}$. [2 points]

(c) Lorentz transformations on a four-vector are given by $x'_\mu = \Lambda^\nu_\mu x_\nu$. Write down the form of Λ^μ_ν for a boost to velocity $\beta_x = v_x/c$ along the x-direction. [2 points]

(d) Using Λ^μ_ν, calculate the Lorentz transformation relations for $\vec{E}$ and $\vec{B}$ for a boost along the x-direction. [2 points]

(e) Depict the lines of electric field $\vec{E}$ from a point charge a) at rest and b) moving with some large velocity β_x. [2 points]

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E+M #6

a) The manifestly covariant form of Maxwell's eqns are (in SI units)

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

$$\partial_\alpha \left(\tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}\right) = 0$$

* Rest of problem in Gaussian

b) $$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

c) $$\Lambda^\mu_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d) $F'^{\alpha\beta} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F_{\mu\nu} \iff F' = \Lambda F \Lambda^T$ in matrix form

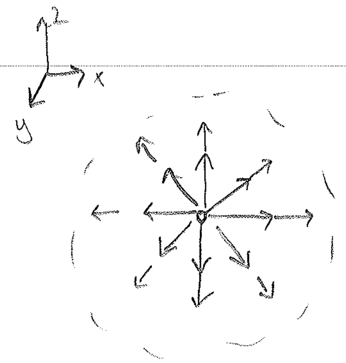
$$F' = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \beta\gamma E_x & \gamma E_x & \gamma E_y + \beta\gamma B_z & \gamma E_z - \beta\gamma B_y \\ -\gamma E_x & -\beta\gamma E_x & -\beta\gamma E_y - \gamma B_z & -\beta\gamma E_z + \gamma B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (\underbrace{-\beta^2\gamma^2 + \gamma^2}_{1})E_x & \gamma E_y + \beta\gamma B_z & \gamma E_z - \beta\gamma B_y \\ (\underbrace{-\gamma^2 + \beta^2\gamma^2}_{-1})E_x & 0 & -\beta\gamma E_y - \gamma B_z & -\beta\gamma E_z + \gamma B_y \\ -\gamma E_y - \beta\gamma B_z & \beta\gamma E_y + \gamma B_z & 0 & -B_x \\ -\gamma E_z + \beta\gamma B_y & \beta\gamma E_z - \gamma B_y & B_x & 0 \end{bmatrix}$$

#6 (cont.)

e) For a point charge at rest



For a rapidly moving point charge