E & M Qualifier

January 10, 2018

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

G!

$\nabla \cdot E = 4\pi \rho$ $\qquad$ $\nabla \cdot B = 0$

$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$ $\qquad$ $\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c} J$

$\nabla \cdot D = 4\pi \rho_f$ $\qquad$ $\nabla \cdot B = 0$

$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t}$ $\qquad$ $\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$

$D = E + 4\pi P$

$H = B + 4\pi M$

Boundary's

$E_{\parallel} = 0 \qquad D_{\perp} = \sigma_b$

$B_{\perp} = 0 \qquad H_{\parallel} = \frac{4\pi}{c} K \times \hat{n}$

$S = E \times B$

$= E \times H$

Problem 1: Electrostatics

A wire of radius R_1 is insulated with a dielectric of outer radius R_2 that is itself enclosed in a grounded conducting sheath. Let the charge per unit length on the wire be λ.

1. ✓ Find an expression for the electric field, $\vec{E}$, on the wire at a radius ρ from the center of the wire. [3 points]
2. ✓ Find the voltage, V, between the inner and outer conductors. [2 points]
3. Calculate the force per unit volume on the insulating material in the coaxial cable. [3 points]
4. Estimate the size of the force for $R_1 = 1$ mm, $R_2 = 5$ mm, $\epsilon_r = 2.5$, and $V = 25,000$ volts. Is this force larger than the force of gravity if the dielectric has the same density as water (10^3 *kilograms/meter*^{3})? [2 points]

[Hint: The force per unit volume on a dielectric is given by $\frac{1}{2}(\epsilon - \epsilon_0)\nabla E^2$; also, $\epsilon_0 = 8.85 \times 10^{-12}\text{C}^2/(\text{Nm}^2)$.]

$h_1 = 1 \quad h_2 = s \quad h_3 = 1$

$$\nabla \Phi = \left\langle \frac{1}{h_1}\frac{\partial \Phi}{\partial s}, \frac{1}{h_2}\frac{\partial \Phi}{\partial \varphi}, \frac{1}{h_3}\frac{\partial \Phi}{\partial z} \right\rangle$$

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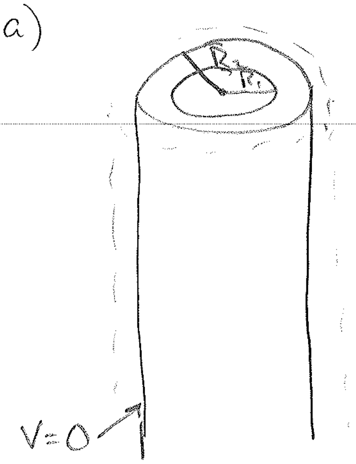
E + M #1

Gaussian

a)



$$\nabla \cdot D = 4\pi \rho_f$$

* if $s < R_1$:

$$\int \nabla \cdot E \, dV = \oint E \cdot d\vec{a} = \int 4\pi \rho_f$$

$$E \cdot 2\pi s L = 4\pi \lambda L \frac{s^2}{R_1^2}$$

$$\vec{E} = 2\lambda \frac{s}{R_1^2} \hat{\varphi}$$

* if $R_1 < s < R_2$:

$$\oint \vec{D} \cdot d\vec{a} = \int 4\pi \rho_f$$

$$D \cdot 2\pi s L = 4\pi \lambda L$$

$$\vec{D} = \frac{2\lambda}{s} \hat{\varphi}$$

* Assuming dielectric is linear, $D = \epsilon E$

$$\hookrightarrow E = \frac{2\lambda}{\epsilon s} \hat{\varphi}$$

* if $s > R_2$:

$\vec{E} = 0$ (identically true for conductors)

#1 (cont.)

b) $\Delta \underline{V} = -\int_a^b \vec{E} \cdot dl$

$= -\int_{R_2}^{R_1} \vec{E} \cdot dl$

$= -\int_{R_2}^{R_1} \frac{2\lambda}{\epsilon s}\, ds$

$= -\frac{2\lambda}{\epsilon} \ln(s) \Big|_{R_2}^{R_1}$

$= \frac{2\lambda}{c} \ln(R_1/R_2)$

c) $F = qE \quad \Rightarrow \quad \frac{F}{V} = \frac{qE}{V}$

$\Rightarrow \quad \frac{F}{V} = \frac{\lambda E}{\pi(R_2^2 - R_1^2)}$

$\frac{F}{V} = \frac{2\lambda^2}{\pi \epsilon s (R_2^2 - R_1^2)}$

Problem 2: Magnetostatics

An infinitely long circular cylinder of radius R (with its axis along the z-direction) carries a magnetization $\vec{M} = ks^2\hat{\phi}$, where k is a constant, s is the distance from the axis, and $\hat{\phi}$ is the azimuthal unit vector.

1. Find the bound currend densities ($\vec{K}_b$ and $\vec{J}_b$). [2 points]
2. Verify that the total bound current in the cylinder is zero. [2 points]
3. Find the magnetic field $\vec{B}$, due to $\vec{M}$, inside and outside the cylinder. [3 points]
4. Verify the boundary conditions for $\vec{B}$ at the interface ($s = R$). [3 points]

Jan 2018 # E+M #2 Gaussian

a) $\vec{K}_b = \hat{M} \times \hat{n}$ $\vec{M} = \langle 0, ks^2, 0 \rangle$

$\vec{J}_b = -\nabla \times M$

$$J_b = \langle \tfrac{1}{s} \partial_\varphi M_z - \partial_z M_\varphi,\ \partial_z M_s - \partial_s M_z,\ \tfrac{1}{s} \partial_s (s M_\varphi) - \partial_\varphi M_s \rangle$$

$$= \langle \tfrac{1}{s} \partial_\varphi (0) - \partial_z (ks^2),\ \partial_z (0) - \partial_s (0),\ \tfrac{1}{s} \partial_s (s ks^2) - \partial_\varphi (0) \rangle$$

$$= \langle 0, 0, 3ks \rangle$$

$$\vec{K}_B = \begin{vmatrix} \hat{s} & \hat{\varphi} & \hat{z} \\ 0 & ks^2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, 0, -ks^2 \rangle$$

b) $$0 = \oint K_B \cdot d\vec{a} + \int J_B \, dV$$

$$= \int -ks^2 \cdot d\vec{a} + \int_0^s \int_0^{2\pi} \int_0^L 3ks \; s\,ds\,d\varphi\,dz$$

$$= -ks^2 \cdot 2\pi s L + ks^3 \Big|_0^s 2\pi L$$

$$= -2\pi L k s^3 + 2\pi L k s^3$$

$$= 0 \checkmark$$

c) $$\nabla \times B = \frac{4\pi}{c} J_f + J_b$$

$$\int B \cdot dl = \frac{4\pi}{c} \int J_b \cdot d\vec{a}$$

$*$ if $s < R'$

$$B \cdot 2\pi s = \frac{4\pi}{c} \int 3ks \; s\,ds\,d\varphi$$

$$2\pi s B = \frac{8\pi^2}{c} 3k \cdot \frac{1}{3} s^3 \Big|_0^s$$

$$2\pi s B = \frac{8\pi^2 k s^3}{c} \Rightarrow B = \frac{4\pi k s^2}{c} \hat{\varphi}$$

#2 (cont)

c) * if $s > R$,

$\vec{B} = 0$ b/c total bound current is 0, as is free current

d) Our boundary conditions are:

$$B_1^{\parallel} - B_2^{\parallel} = \frac{4\pi}{c}\vec{K}$$

$$B_1^{\perp} - B_2^{\perp} = 0$$

* Defining region 1 as inside the cylinder and region 2 as outside

$$B_1^{\perp} - B_2^{\perp} = 0$$

$$0 - 0 = 0 \checkmark$$

$$B_1^{\parallel} - B_2^{\parallel} = \frac{4\pi}{c}\vec{K}$$

$$\frac{4\pi}{c} k s^2 - 0 = \frac{4\pi}{c} k s^2 \checkmark$$

5-10

Problem 3: Waves

Consider an electromagnetic wave propagating in a vacuum where there are no charges or electric currents, with electric field of $\vec{E}(z,t) = E_0 e^{i(kz-\omega t)}\hat{x}$.

1. Show that three of Maxwell's equations can be combined to give $\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$ for a region with no charges or electric currents. Show that three of Maxwell's equations can also lead to an analogous equation for the magnetic field $\vec{B}$. *Hint:* Start by calculating $\nabla \times (\nabla \times \vec{E})$. [2 points]

2. In the equation for $\vec{E}(z,t)$ how are ω and k related to μ_0 and ϵ_0? What is the equation 2.5? for $\vec{B}(z,t)$ of the electromagnetic wave? Be explicit about how the amplitude, direction and phase of $\vec{B}$ are related to those of $\vec{E}$. [3 points]

3. Now suppose the wave propagates from vacuum into a dielectric material with permittivity of $\epsilon = \kappa\epsilon_0$, where κ is a positive constant. Assuming a normal angle of incidence at the vacuum/dielectric interface, calculate the amplitude of the electric field in the dielectric material. Express your answer in terms of E_0 and κ. [3 points]

4. What fraction of the incident energy is transmitted across the boundary? [2 points]

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E+M #3

a) In Gaussian units, Maxwell's eqns are:

① $\nabla \cdot E = 4\pi \rho$ ③ $\nabla \cdot B = 0$

② $\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$ ④ $\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$

*If we take $\nabla \times$ ②

$$\nabla \times \nabla \times E + \nabla \times \left(\frac{1}{c}\frac{\partial B}{\partial t}\right) = \nabla \times 0$$

$$\nabla(\nabla \cdot E) - \nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times B) = 0$$

$$\nabla(4\pi\rho) - \nabla^2 E + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t}\right) = 0$$

*Since we are in a vacuum, $\rho = 0$, $\vec{J} = 0$

$$-\nabla^2 E + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \checkmark$$ *Note: $c^2 = \frac{1}{\mu_0\epsilon_0}$

b) *From Jackson, $\frac{\omega}{k} = \frac{c}{n} = \frac{1}{\sqrt{\mu\epsilon}}$

↳ $\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ in a vacuum

*To calculate $\vec{B}$, we simply plug E into Maxwell's eqns

↳ Remember, in SI units (forced by problem), Maxwell's eqns are:

$\nabla \cdot E = \frac{\rho}{\epsilon}$ $\nabla \cdot B = 0$

$\nabla \times E + \frac{\partial B}{\partial t} = 0$ $\nabla \times B - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0 \vec{J}$

$$\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = \langle 0, \frac{\partial}{\partial z}E_x, 0\rangle = \langle 0, ikE_0 \exp[i(kz-\omega t)], 0\rangle$$

$$\Rightarrow ikE_0 \exp[i(kz-\omega t)]\hat{y} = -\frac{\partial B}{\partial t}$$

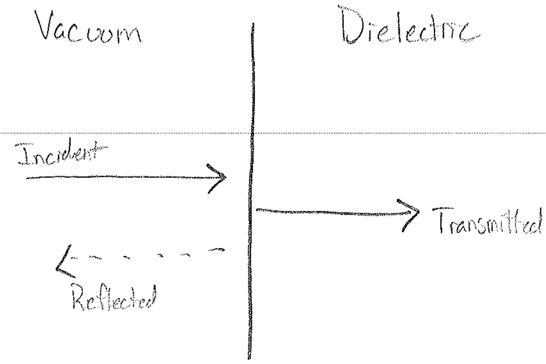
$$\int ikE_0 \exp[i(kz-\omega t)]\,\hat{y}\,dt = -\vec{B}$$

$$-\frac{k}{\omega}E_0 \exp[i(kz-\omega t)]y = -\vec{B}$$

$$\therefore \quad \vec{B} = \sqrt{\mu_0\epsilon_0}\,E_0 \exp[i(kz-\omega t)]\,\hat{y}$$

#3 (cont.)

c)



*Across all boundaries,

$$\epsilon_1 E_{1,\perp} = \epsilon_2 E_{2\perp}$$

$$E_{1,\parallel} = E_{2,\parallel}$$

$$B_{1,\perp} = B_{2,\perp}$$

$$\frac{1}{\mu_1} B_{1,\parallel} = \frac{1}{\mu_2} B_{2,\parallel}$$

2 ↔ 7

Problem 4: Maxwell Eq'n in 4-d

For this problem, use the following metric: $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$, and $g_{ij} = 0$ for $i \neq j$ and consider Maxwell's equations in four dimensions (4D)

$$\sum_\nu \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$$

$$\sum_\nu \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

✓ 1. (0.5 pt) Write down the field strength tensor $F^{\mu\nu}$, its dual tensor $G^{\mu\nu}$, and the 4-vector charge-current density J^μ in vector form in terms of $\mathbf{E}, \mathbf{B}, \rho, \mathbf{j}$, and c.

✓ 2. (0.5 pt) Explicitly derive the equation $\sum_\mu \partial J^\mu / \partial x^\mu = 0$ from Maxwell's equations in 4D. What is the physical meaning of this equation?

✓ 3. (1.0 pt) Show that the field tensor can be written as $F^{\mu\nu} = \partial A^\nu / \partial x_\mu - \partial A^\mu / \partial x_\nu$ by introducing a 4-vector potential A^μ.

? 4. (1.0 pt) Show that with the introduction of the 4-vector potential A^μ the 4D Maxwell equation involving $G^{\mu\nu}$ is automatically satisfied.

5. (1.0 pt) Impose the Lorentz gauge on A^μ and show that, with this gauge, the 4D Maxwell's equations reduce to the inhomogeneous 4D wave equation for the 4-vector potential.

? 6. (1.0 pt) Consider the Minkowski force acting on a charge q, $K^\mu = q\eta_\nu F^{\mu\nu}$, where η_ν is the proper velocity. Find the $\mu = 1, 2, 3$ components of K^μ in terms of $\mathbf{E}, \mathbf{B}$, q and c.

? 7. (0.5 pt) What is the physical meaning of the Minkowski force expression for $\mu = 1, 2, 3$?

? 8. (1.0 pt) Find the $\mu = 0$ component of the Minkoswki force in terms of $\mathbf{E}$ and $\mathbf{B}$, q and c.

? 9. (0.5 pt) What is the physical meaning of the Minkowski force expression for $\mu = 0$?

10. (3.0 pt) Consider a particle starting from rest at the origin under the influence of a constant Minkowski force in the x-direction. Find an implicit relativistic expression for the particle velocity v. Leave your answer in implicit form (t as a function of v).

Problem 5: Radiation

A current source $\vec{J}(\vec{x}, t)$ is localized within a sphere of radius a near the origin of a coordinate system and oscillates with harmonic time dependence $e^{-i\omega t}$. Would an oscillating charge density all by itself ($\vec{J} = 0$) contribute to the power radiated into the radiation zone? Why or why not? [1 point] ?

The vector potential (in SI units) is given by

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(x', t')}{|\vec{x} - \vec{x}'|} \delta(t' - (t - \frac{|\vec{x} - \vec{x}'|}{c})) d^3x' dt' \tag{1}$$

1. Integrate out the time dependence to find $\vec{A}(\vec{x})$. [1 point]

2. In the radiation zone, $|\vec{x} - \vec{x}'| \simeq r - \hat{n} \cdot \vec{x}'$. What is the vector potential in the radiation zone? [2 points]

3. What approximation must be made to gain the electric dipole contribution to $\vec{A}$? [2 points]

4. The electric dipole moment is given by $\vec{p} = \int \vec{x}' \rho(\vec{x}') d^3x'$. What is the vector potential in the radiation zone in terms of the EDP moment $\vec{p}$? [2 points]
   (*Hint:* $\partial_i(x_k J_i) = \delta_{ik} J_i + x_k \partial_i J_i$ and the equation of continuity for harmonic time dependence is given by $\nabla \cdot \vec{J} = i\omega\rho$.)

5. For a dipole $\vec{p}$ oriented along the z-axis, what angular distribution do you expect in the power radiated from EDP radiation? (You need not actually do the calculation.) [2 points]

Problem 6: Stress tensor

The manifestly covariant form of the electromagnetic field Lagrangian is given by $\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}J_\mu A^\mu$ in Gaussian units where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

? (a) For $\mathcal{L} = \mathcal{L}(A^\nu, \partial^\mu A^\nu)$, write down the Euler-Lagrange equations. [1 point]

(b) Apply these to derive the covariant form of the inhomogeneous Maxwell equations. [1 point]

$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$

√ (c) From the Maxwell equations, show that the equation of continuity $\partial_\mu J^\mu = 0$ is satisfied. [2 points]

(d) List at least three important steps in deriving the symmetrized electromagnetic stress-energy tensor $\Theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}f_{\mu\nu}F^{\mu\nu}\right)$. [2 points]

(e) Express $\Theta^{\alpha\beta}$ in matrix form in terms of the EM energy density u, momentum density $c\vec{g}$ and the Maxwell stress tensor $T^M_{ij} = \frac{1}{4\pi}\left(E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}(\vec{E}^2 + \vec{B}^2)\right)$. [2 points]

(f) Express the zeroth component of the conservation equation $\partial_\mu\Theta^{\mu\nu} = 0$ in terms of u and the Poynting vector $\vec{S} = c^2\vec{g}$. What is the significance of this equation? [2 points]

$c^2\, E \times B$

$\partial_\mu \sum_\nu \frac{\partial F^{\mu\nu}}{\partial x_\nu} = \partial_\mu \mu_0 J^\mu$

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E+M #6

a) General Euler-Lagrang eqn: $\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$

$$\Rightarrow \quad \frac{\partial \mathcal{L}(A^\nu, \partial^\mu A^\nu)}{\partial A^\nu} = \frac{d}{dt}\frac{\partial \mathcal{L}(A^\nu, \partial^\mu A^\nu)}{\partial(\partial^\mu A^\nu)}$$

b) The covariant form of the inhomogeneous Maxwell Eqns is: $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$

Given: $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$