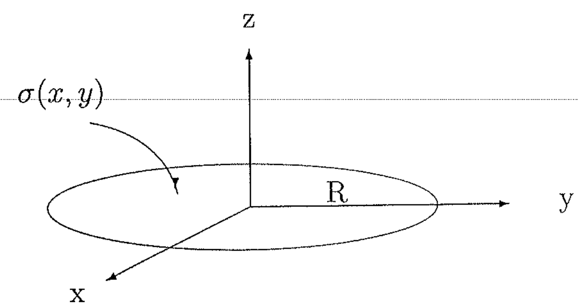
E & M Qualifier

January 14, 2010

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



Poorly written, field not symmetric about axis

1. Consider a thin nonconducting disk of radius R centered on the origin of a coordinate system, lying in the x-y plane, and carrying a surface charge density given by

$$\sigma = \sigma_o \frac{yR}{x^2 + y^2}.$$

   (a) {6 pts} Determine the electric field at a location $\vec{r} = z\hat{k}$.

   (b) {3 pts} Give an approximation to your answer to part (a) that is valid for the $z >> R$.

   (c) {1 pts} Find the force on a charge q located at a position $\vec{r} = z\hat{k}$.

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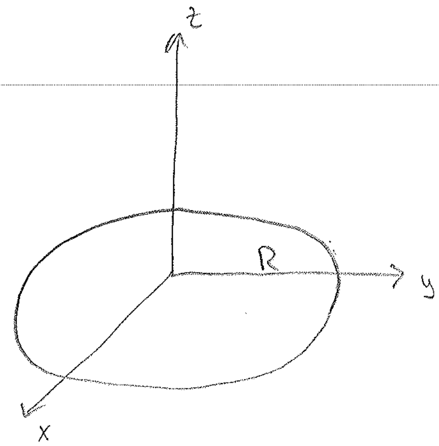
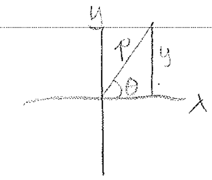
E+M #1

Gaussian

a)



$$\sigma = \sigma_0 \frac{yR}{x^2+y^2}$$



$$\sigma = \sigma_0 \frac{yR}{p^2}$$

$$= \sigma_0 \frac{p\sin\theta\, R}{p^2}$$

$$= \sigma_0 \frac{\sin\theta\, R}{p}$$

$$\int \nabla \cdot E = \int 4\pi \rho$$

$$\int \vec{E} \cdot d\vec{a} = 4\pi \int \sigma(r)\, \delta(z)\, p\,dp\,d\theta\,dz$$

$$E \cdot 2\pi p^2 = 4\pi \int \sigma_0 \sin\theta\, R\, \delta(z)\, p\,dp\,d\theta\,dz$$

$$= 4\pi\sigma_0 R \int \sin\theta\, p\,dp\,d\theta$$

$$= 2\pi\sigma_0 p \int \sin\theta\, d\theta$$

$$= -\pi\sigma_0 p \cos\theta \Big|_0^{2\pi}$$

$$= 0$$

2. Consider a linear, homogeneous, isotropic, and non-dissipative dielectric (i.e., a dielectric where $\mathbf{D} = \epsilon\mathbf{E}$ and ϵ is a constant) in the shape of a sphere of radius R with a point charge Q embedded at its center.

   (a) {2 pts} Find the electric displacement vector $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization density $\mathbf{P}$ inside the dielectric.

   (b) {2 pts} Find the bound charge volume density ρ_D inside the dielectric.

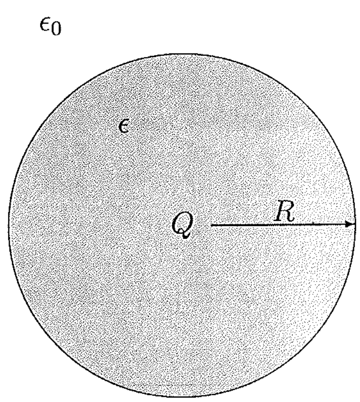
   (c) {1 pts} Find the total bound charge Q_D on the $r = R$ boundary of the dielectric.

   (d) {2 pts} Find the net charge (free plus bound) at the center of the dielectric.

   (e) {1 pts} Find the electric displacement vector $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization density $\mathbf{P}$, outside the dielectric sphere.

   (f) {2 pts} Are $\mathbf{D}$ and $\mathbf{E}$ continuous at $r = R$? If not explain why.

   (If you use Gaussian units you can put $\epsilon_0 = 1$.)



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E+M #2

Gaussian

a) * As a reminder, our electrostatic Maxwell's law in media are:

$$\nabla \cdot D = 4\pi\rho \qquad D = \vec{E} + 4\pi\vec{P}\,, \qquad D = \epsilon\vec{E}$$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad P = \chi_e E \qquad \hookrightarrow \epsilon = 1 + 4\pi\chi_e$$

* We solve for D using Gauss Law

$$\int \nabla \cdot D = \int 4\pi\rho$$

$$D \cdot 4\pi r^2 = 4\pi Q$$

$$D = \frac{Q}{r^2}\hat{r}$$

$$\Rightarrow \vec{E} = \frac{D}{\epsilon} = \frac{Q}{\epsilon r^2}\hat{r}$$

$$\Rightarrow \vec{P} = \frac{\vec{D} - \vec{E}}{4\pi} = \frac{1}{4\pi}\left(\frac{Q}{r^2} - \frac{Q}{\epsilon r^2}\right)\hat{r} = \frac{Q}{4\pi r^2}\left(1 - \frac{1}{\epsilon}\right)\hat{r}$$

b) The bound charge density is defined as:

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{Q}{4\pi r^2}\left(1 - \frac{1}{\epsilon}\right)\right) = Q\,\delta(\vec{r})\left(1 - \frac{1}{\epsilon}\right)$$

c) To find the total surface bound charge,

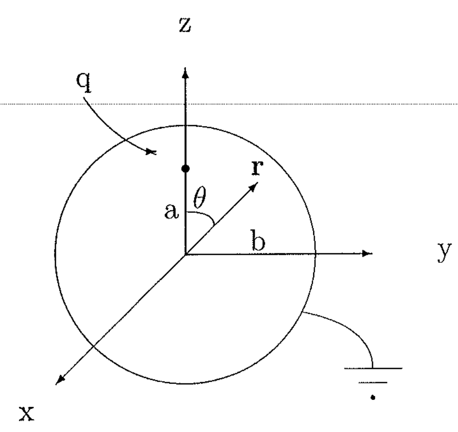
$$\sigma_b = P \cdot \hat{n} = P \cdot \hat{r} = \frac{Q}{4\pi r^2}\left(1 - \frac{1}{\epsilon}\right)\Big|_{r=R} = \frac{Q}{4\pi R^2}\left(1 - \frac{1}{\epsilon}\right)$$

#2 (cont.)

c) We must integrate over the surface to find:

$$Q_D = Q\left(1 - \frac{1}{6}\right)$$

d)



3. A thin grounded hollow conducting sphere of radius ‘b’ is centered at the origin. A point charge q is located on the z-axis at $z = a < b$ INSIDE the sphere.

   (a) {5 pts} Write the total potential for this system as a sum,

   $$\Phi = \Phi_{sphere} + \Phi_q,$$

   where Φ_q is the potential due to the point charge and Φ_{sphere} (in spherical polar coordinates) is the appropriate linear combination of Legendre polynomials $P_\ell(\cos(\theta))$. Evaluate the coefficients of the $P_\ell(\cos(\theta))$ in the Φ_{sphere} expansion. Recall that the Legendre polynomials are independent orthogonal functions satisfying

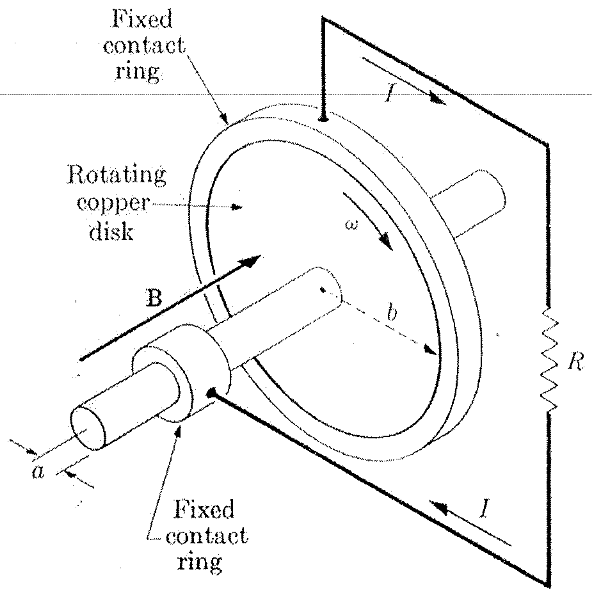
   $$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

   and

   $$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{\ell=0}^{\ell=\infty} \frac{(r_<)^\ell}{(r_>)^{\ell+1}} P_\ell(\cos(\gamma))$$

   where γ is the angle between the two directions $\mathbf{r}$ and $\mathbf{r}'$.

   (b) {5 pts} Show that your expression for Φ_{sphere} is equivalent to the potential of a point charge. Where is the point charge located and what is it’s charge?



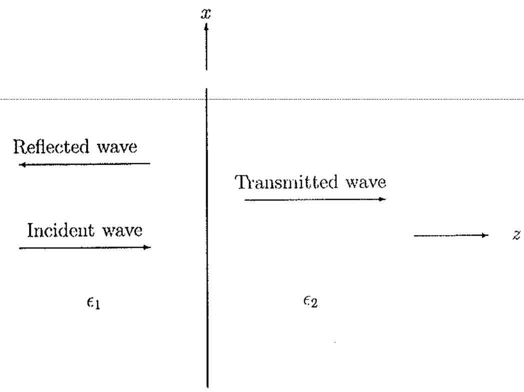
4. The Homopolar Generator consists of a flat copper disk of radius b and thickness t, mounted on an axle of radius a, which mechanically rotates the disk with angular speed ω in the presence of an orthogonal magnetic induction $\mathbf{B}$. A stationary contact ring with inner radius b and negligible resistance surrounds the rotating disk making good electrical and frictionless contact with it. As shown in the figure, the closed electrical circuit consists of the disk and a load resistor R connected by wires between the axle and the stationary contact ring. (Assume the load resistor R is much greater than the resistance of the disk, the contact ring, and the wires.) A constant magnetic induction $\mathbf{B}$ perpendicular to the disk (parallel to the rotation axis) exists between the radii a and b and is zero elsewhere in the circuit.

   (a) {4 pts} Find the current I that flows in the circuit as a function of B, a, b, ω, and R.

   (b) {2 pts}What is the magnitude of the current density $J(r)$ in the rotating disc.

   (c) {2 pts} What torque would you have to apply to the rotating wheel to keep ω from slowing down.

   (d) {2 pts} If σ is the conductivity of copper and t is the thickness of the disk, find the electrical resistance R_d of the disk between the radii a and b. Recall that the resistance of a small length $\Delta\ell$ of conducting material with cross sectional area A is $\Delta R = \Delta\ell/(\sigma A)$.



5. A plane-polarized harmonic ($e^{-i\omega t}$) plane electromagnetic wave traveling to the right in a homogeneous dielectric medium described by an dielectric constant ϵ_1, strikes a second homogeneous dielectric material described by dielectric constant $\epsilon_2 > \epsilon_1$ (see the figure). Assume that both materials have the same magnetic permeability μ_0 and that the incidence angle is 0^o (i.e., the wave is traveling perpendicular to the junction). Assume the incoming wave is polarized in the $\hat{x}$ direction and that its electric field amplitude is E_0, i.e., assume the incoming electric field is the real part of

$$\mathbf{E} = E_0\, e^{i(kz-\omega t)}\, \hat{x}.$$

   (a) {3 pts} Give the magnetic induction $\mathbf{B}$ associated with the above incoming wave. Make sure your wave satisfies Maxwell's equations, e.g., give k as a function of ω, the direction of $\mathbf{B}$, and the amplitude of $\mathbf{B}$ as a function of E_0.

   (b) {1 pts} Give similar expressions for the $\mathbf{E}$ and $\mathbf{B}$ components of the reflected and transmitted waves. Use E_0'' and E_0' for the respective amplitudes of reflected and transmitted waves.

   (c) {2 pts} In general, what conditions must be satisfied at the junction between two materials by the electromagnetic fields $\mathbf{E}, \mathbf{B}, \mathbf{D}$, and $\mathbf{H}$, if Maxwell's equations are to be satisfied?

   (d) {2 pts}Apply these junction conditions to the combined incoming, reflected, and transmitted wave to compute E_0'' and E_0' as functions of E_0 and the two dielectric constants ϵ_1 and ϵ_2.

   (e) {2 pts} Evaluate the time averages of the Poynting vectors of the incident, reflected, and transmitted waves. Recall that

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} \equiv \frac{1}{4\pi} \mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

   The sum of the magnitudes of the reflected and transmitted time averaged Poynting vectors should equal the magnitude of the incident wave's time averaged Poynting vector.

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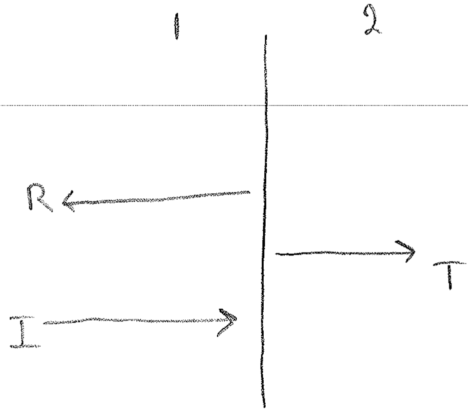
E+M #5

Gaussian

a)



$\vec{E} = E_0 \exp[i(k_1 z - \omega t)]\,\hat{x}$

$$\vec{B} = n\hat{k}_1 \times \vec{E}$$
$$= \sqrt{\epsilon_1}\, E_0 \exp[i(k_1 z - \omega t)]\,\hat{y}$$

* The relevant Maxwell equations in Gaussian units are:

① $\nabla \cdot D = 4\pi \rho_f$ ③ $\nabla \cdot B = 0$

② $\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$ ④ $\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$

* Starting w/ equation ③:

$$\nabla \cdot B = \frac{\partial}{\partial x} B_x + \frac{\partial}{\partial y} B_y + \frac{\partial}{\partial z} B_z$$
$$= \frac{\partial}{\partial y}\left(-\sqrt{\epsilon_1}\, E_0 \exp[i(k_1 z - \omega t)]\right)$$
$$= 0 \checkmark$$

* Now equation ②:

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$

$$\hookrightarrow \nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = \langle 0, +\partial/\partial z\, E_x, -\frac{\partial}{\partial y} E_x \rangle$$

$$= \langle 0, ik_1 \; E_0 \exp[i(k_1 z - \omega t)], 0 \rangle$$

#5 (cont.)

a) $\frac{1}{c}\frac{\partial}{\partial t}B = \frac{1}{c}\frac{\partial}{\partial t}\left(-\sqrt{\epsilon_1}\, E_0 \exp[i(k_1 z - \omega t)]\right)\hat{y}$

$= \frac{-1}{c}\sqrt{\epsilon_1}\, i\omega\, E_0 \exp[i(k_1 z - \omega t)]\,\hat{y}$

$\Rightarrow \nabla \times \vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} \stackrel{?}{=} 0$

$= -ik \quad E_0 \exp[i(kz - \omega t)] + -i\frac{\omega}{c}\sqrt{\epsilon_1}\, E_0 \exp[i(kz-\omega t)]$

$= -i E_0 \exp[i(k_1 z - \omega t)]\left(-k + \frac{\omega}{c}\sqrt{\epsilon_1}\right)$

$\hookrightarrow$ Only true if $k = \frac{\omega}{c}\sqrt{\epsilon_1}$

* Finally equation (4)

$\nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi}{c}\vec{J}_f$

$\vec{J}_f = 0$

$\nabla \times H = \nabla \times \frac{1}{\mu_0}\vec{B}$

$= \frac{1}{\mu_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & B_y & 0 \end{vmatrix} = \langle -\partial_z B_y,\ 0,\ \partial_x B_y \rangle$

$= \frac{1}{\mu_0}\frac{\partial}{\partial z}\left(\sqrt{\epsilon_1}\, E_0 \exp[i(k_1 z - \omega t)]\right)\hat{x}$

$= \frac{-\sqrt{\epsilon_1}}{\mu_0} E_0 (ik_1) \exp[i(k_1 z - \omega t)]\,\hat{x}$

$-\frac{1}{c}\frac{\partial D}{\partial t} = \frac{-1}{c}\frac{\partial}{\partial t}\epsilon_1 \vec{E}$

$= -\frac{\epsilon_1}{c}(-i\omega) E_0 \exp[i(k_1 z - \omega t)]\,\hat{x}$

$\hookrightarrow \nabla \times H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{i\omega\epsilon_1}{c} E_0 \exp[i(k_1 z - \omega t)]\hat{x} + -i\frac{\sqrt{\epsilon_1}k}{\mu_0} E_0 \exp[i(k_1 z - \omega t)]$

$= 0$ if $\frac{\omega\epsilon_1}{c} - \frac{\sqrt{\epsilon_1}k}{\mu_0} = 0 \Rightarrow k = \frac{\omega\sqrt{\epsilon_1}}{c}$ as before ✓

#5 (cont.)

b) * For the transmitted wave:

$$\vec{E}' = E_0' \exp[i(k_2 z - \omega t)]\,\hat{x}$$

$$\vec{B}' = \sqrt{\epsilon_2}\, E_0' \exp[i(k_2 z - \omega t)]\,\hat{y}$$

$$k_2 = \frac{\omega\sqrt{\epsilon_2}}{c}$$

* For the reflected waves

$$\vec{E}'' = E_0'' \exp[i(k_1(-z) - \omega t)]\,\hat{x}$$

$$\vec{B}' = +\sqrt{\epsilon_1}\, E_0'' \exp[i(-k_1 z - \omega t)]\,\hat{y}$$

$$k_1 = \frac{\omega\sqrt{\epsilon_1}}{c}$$

c) The general boundary conditions are:

$$D_1^{\perp} - D_2^{\perp} = 4\pi\sigma_f \qquad B_1^{\perp} - B_2^{\perp} = 0$$

$$E_1^{\parallel} = E_2^{\parallel} \qquad H_1^{\parallel} - H_2^{\parallel} = 4\pi \vec{K}_f$$

d) Applying these boundary conditions to our situation

$$E_1^{\parallel} - E_2^{\parallel} = 0$$

$$(\vec{E}_I^{\parallel} + \vec{E}_R^{\parallel}) - \vec{E}_T^{\parallel} = 0$$

$$[(E_I + E_R) - E_T] \times \hat{z} = 0 \quad \Rightarrow \text{picks out } \hat{y} \text{ components}$$

$$-E_0 \exp[i(k_1 z - \omega t)] + E''\exp[i(-k_1 z - \omega t)] - E'\exp[i(k_2 z - \omega t)] = 0$$

* if we set our boundary to be $z = 0$

$$-E_0 + E_0'' - E_0' = 0$$

#5 (cont.)

d) $H_1^{\parallel} - H_2^{\parallel} = \cancel{4\pi \vec{K}_f} \to 0$

$\frac{1}{\mu_0}(B_1^{\parallel} - B_2^{\parallel}) = 0 \quad \Rightarrow \quad B_1^{\parallel} - B_2^{\parallel} = 0$

$\hookrightarrow \quad B_1^{\parallel} - B_2^{\parallel} = 0$

$B_I^{\parallel} + B_R^{\parallel} - B_T^{\parallel} = 0$

$[(B_I + B_R) - B_T] \times \hat{z} = 0 \quad \Rightarrow$ picks off x-component

$-E_0\sqrt{\epsilon_1}\exp[i(k_1 z - \omega t)] + E_0''\sqrt{\epsilon_1}\exp[i(-k_1 z - \omega t)] - E_0'\sqrt{\epsilon_2}\exp[i(k_2 z - \omega t)]$

$(-E_0 + E_0'')\sqrt{\epsilon_1} - \sqrt{\epsilon_2}E_0' = 0$

$\hookrightarrow \quad E_0' = \sqrt{\frac{\epsilon_1}{\epsilon_2}}(-E_0 + E_0'')$

* Using our equations to solve for E_0', E_0'' in terms of E_0:

$E_0' = \sqrt{\frac{\epsilon_1}{\epsilon_2}}(-E_0 + E_0'')$

$= \sqrt{\frac{\epsilon_1}{\epsilon_2}}(-E_0 + (E_0' + E_0))$

6. Maxwell's equations in 4 dimensions

(a) {2 pts} Write the Maxwell equations in the absence of polarizable materials using 4-vector notation, making use of the field strength tensor $F_{\mu\nu}$.

(b) {4 pts} Show that the equations of part (a) reduce to the usual form of Maxwell's equations in 3-vector notation.

(c) {2 pts} The Lagrangian density of the EM field is given by

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}, \qquad (SI)$$

or

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}. \qquad (Gaussian)$$

Recall that all repeated Greek indices are summed over 4-dimensions (1 time and 3 space). Show that the Lagrangian density is invariant under a gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha(x)$, where α is an arbitrary function of spacetime $x \equiv (ct, \vec{x})$.

(d) {2 pts} If we add an interaction term $\mathcal{L} \to \mathcal{L} + \Delta\mathcal{L}$ where

$$\Delta\mathcal{L} = j^\mu A_\mu, \qquad (SI)$$

or

$$\Delta\mathcal{L} = \frac{1}{c} j^\mu A_\mu, \qquad (Gaussian)$$

to the Lagrangian– where j^μ is some spatially bounded and conserved 4-current density– how does the action $I \equiv \int \mathcal{L} d^4 r$ change under a gauge transformation and do the resulting equations of motion change?

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E+M #6

Gaussian

a) In free space, 4-D Maxwell equations in tensor form are:

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \qquad (\text{Dual Tensor } (B \to E,\ E \to B))$$

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

b) To show these reduce to the 3 vector form:

$$\nabla \cdot E = 4\pi\rho \qquad \nabla \cdot B = 0$$

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad \nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}$$

It is easiest to show with the inhomogeneous equations first

$$J^\beta = \begin{bmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{bmatrix} \qquad F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

* If $\beta = 0$: $\quad \partial_\alpha F^{\alpha 0} = \frac{4\pi}{c} J^0$

$$\partial_0 F^{00} + \partial_i F^{i0} = \frac{4\pi}{c}(c\rho)$$

$$0 + \nabla \cdot E = 4\pi\rho \ \checkmark$$

If $\beta = i$: $\quad \partial_\alpha F^{\alpha i} = \frac{4\pi}{c} J^i$

$(i \in \{1,2,3\})$

$$\partial_0 F^{0i} + \partial_j F^{ji} = \frac{4\pi}{c} J^i$$

$$\frac{1}{c}\frac{\partial E}{\partial t} + \nabla \times B = \frac{4\pi}{c}\vec{J} \ \checkmark$$

* Proceeding to the homogeneous eqns:

$$\mathcal{F}^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_x & E_y \\ B_y & E_x & 0 & E_z \\ B_z & E_y & E_z & 0 \end{bmatrix}$$

#6 (cont.)

b) *If $\beta = 0$: $\partial_\alpha F^{\alpha 0} = 0$

$$\partial_0 F^{00} + \partial_i F^{i0} = 0$$

$$0 + \nabla \cdot B = 0 \checkmark$$

*if $\beta = i$, $i \in \{1,2,3\}$: $\partial_\alpha F^{\alpha i} = 0$

$$\partial_0 F^{0i} + \partial_j F^{ji} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{B} - \nabla \times E = 0$$

$$-\frac{1}{c}\frac{\partial}{\partial t}B + \nabla \times E = 0 \checkmark$$

c) * Remember that $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $F^{\alpha\beta} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$

$$\Rightarrow \mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

*if $A_\mu \to A_\mu + \partial_\mu \alpha(x)$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} \left(\partial_\mu [A_\nu - \partial_\nu \alpha(x)] - \partial_\nu [A_\mu - \partial_\mu \alpha(x)]\right)\left(\partial_\mu [A_\nu - \partial_\nu \alpha(x)] - \partial_\nu [A_\mu - \partial_\mu \alpha(x)]\right)$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} \left(\partial_\mu A_\nu - \cancel{\partial_\mu \partial_\nu \alpha(x)} - \partial_\nu A_\mu + \cancel{\partial_\nu \partial_\mu \alpha(x)}\right)\left(\partial_\mu A_\nu + \cancel{\partial_\mu \partial_\nu \alpha(x} - \partial_\nu A_\mu + \cancel{\partial_\nu \partial_\mu \alpha(x)}\right)$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\mu\nu} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \checkmark$$