Classical Mech Main Points

Qualifier

Winter 2012

1 Newtonian Mechanics

- Set $F_N = 0$ to find the point when two objects separate (ex. ball rolls off hemisphere)
- Momentum ($p = mv$, $L = I\omega$) is conserved for all collisions; energy is conserved for elastic collisions
- Force $= -\nabla U$
- For periodic motion, if the equation of motion is $\ddot{x} + \xi x = 0$, the frequency is $\omega = \sqrt{\xi}$. If the equation has a term linear in $\dot{x}$, that is a damping term.
- Power: $P = \frac{dE}{dt} = \frac{\Delta W}{\delta t} = \vec{F} \cdot \vec{v} = \vec{\tau} \cdot \vec{\omega}$

1.1 Angular Motion

- Use $v = \omega r$, $x = \theta r$, $a = \alpha r$ for basic angular motion
- Circular motion: $ma = \frac{mv^2}{r} = m\omega^2 r$
- **Torque**: $\frac{dL}{dt} = \tau = \vec{r} \times \vec{F} = I\alpha = Fd\sin\theta$
- Period $T = \frac{2\pi}{\omega}$
- Remember: it's often easier to find $d\sin\theta$ than to find d and θ separately
- To derive moment of inertia: $I = \int r^2 dm$; solve for dm in terms of dr
- Can still also use $\Sigma F = ma$ if it helps. Consider all forces acting at same point (point particle)
- **Orbits**: $\frac{\partial^2 V_{eff}}{\partial r^2} > 0$ for **stable orbits**. Use $\frac{\partial V}{\partial r} = 0$ for circular orbits
- **Parallel Axis Theorem**: $I_{new} = I_{original} + MR^2$

Helpful moments of inertia:

- **sphere**: $I = \frac{2}{5}MR^2$
- **disc**: $I = \frac{1}{2}MR^2$

Rocket Ships: Use m = mass of ship, dm'=ejected mass, v=velocity of ship, $-u$=ejected mass velocity relative to ship. Then we have:

$$p_i = p_f \rightarrow 0 = (m - dm')(v + dv) + dm'(v - u) \tag{1}$$

Set $v = 0$ for simplicity, and $dm = -dm'$. After that it's mostly algebra/calculus.

2 Virtual Work

The principle of virtual work presents an alternative to Newtonian solutions for force problems. This method uses the equations:

$$\delta W = \sum_i \vec{F}_i^a \cdot \delta \vec{r}_i = 0 \qquad \delta W = \sum_i Q_i^a \delta q_i = 0 \tag{2}$$

In these equations, $\vec{F}_i^a$ represent the net applied forces, and Q_i^a represent the differentiated constraint equations. Transform the Q_i^a equation into the generalized (simplest) coordinates, and solve the resulting equations.

For example, if the constraint equation is for two blocks connected by a massless rod: $x^2 + y^2 - l^2 = 0$, with $x = l\cos\theta$ and $y = l\sin\theta$:

$$\delta W = \sum_i Q_i^a \delta q_i = 0 \rightarrow 2x\delta x + 2y\delta y = 0 \rightarrow \delta x \cos\theta + \delta y \sin\theta = 0 \tag{3}$$

2.1 D'Alembert's Principle

The virtual work method given previously works for systems in static equilibrium. To generalize this method to dynamic systems, D'Alembert introduced a new "force of inertia" that modifies the virtual work equation that governs forces:

$$\delta W = \sum_i \left[\vec{F}_i^a - m_i \ddot{\vec{r}}_i\right] \cdot \delta \vec{r}_i = 0 \tag{4}$$

3 Lagrangian & Hamiltonian

3.1 Lagrangian

- $L = T - U$
- Euler Lagrange Equation:
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{5}$$
- We can always add a total time derivative of a function to the Lagrangian for free (without changing equations of motion):
$$L' = L + \frac{dF(q, \dot{q}, t}{dt} \tag{6}$$
This kind of trick can give a simplified Hamiltonian, even making it a constant of the motion.
- A variable q_i is **cyclic** if it does not appear in the Lagrangian. In that case, the associated momentum p_i is conserved/constant, and subtracting the associated $p_i \dot{q}_i$ transforms the Lagrangian into the Routhian:
$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \alpha_i \quad \rightarrow \quad R = L - \alpha_i \dot{q}_i \tag{7}$$

3.2 Hamiltonian

- Legendre Transformation: $H = p\dot{q} - L$
- $p_q = \frac{\partial L}{\partial \dot{q}}$
- Hamilton's equations of motion: $\dot{p}_q = -\frac{\partial H}{\partial q}$ and $\dot{q} = \frac{\partial H}{\partial p_q}$
- Solve for q(t) using the E-L equation or Hamilton's equations of motion (take $\frac{d\dot{q}}{dt}$ and plug in for $\dot{p}_q$)
- We can see that H is conserved (thus representing the total energy) if $\frac{\partial H}{\partial t} = 0$ and if it includes no terms that depend *linearly* on a momentum variable (only quadratically).

- We can go farther, and write a momentum-space "Lagrangian", similar to how we did the first Legendre transform: $K(p, \dot{p}, t) = q_i \dot{p}_i + H(q, p, t)$
- $KE_{cylindrical} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right)$
- $KE_{spherical} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)$

3.3 Undetermined Multipliers

If we can't include some constraints when writing the Lagrangian, we have to take these constraints into account in the Euler-Lagrange equation as undetermined multipliers:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i^a + \sum_{j=1}^{m} \lambda_j a_{ji} \tag{8}$$

Each λ_j corresponds to each constraint equation f_j, and each a_{ji} corresponds to $\frac{\partial f_j}{\partial q_i}$.
Each Q_i^a corresponds to applied forces that cannot be written as part of the potential energy:

$$Q_i = \frac{\partial \vec{r}_j}{\partial q_i} \cdot \vec{F}_j \tag{9}$$

A constraint is *holonomic* if:

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \tag{10}$$

3.4 Canonical Transformations

"Guess" the Q & P to transform into in order to make $\frac{\partial H}{\partial t} = 0$. Show canonical by $[Q, P]_{q,p} = 1$

Use existing p and q definitions to find generating functions:

$$p = \frac{\partial F_1(q, Q)}{\partial q} \qquad P = -\frac{\partial F_1(q, Q)}{\partial P} \tag{11}$$

$$p = \frac{\partial F_2(q, P)}{\partial q} \qquad Q = \frac{\partial F_2(q, P)}{\partial P} \tag{12}$$

$$q = -\frac{\partial F_3(Q, p)}{\partial p} \qquad P = -\frac{\partial F_3(Q, p)}{\partial Q} \tag{13}$$

$$q = -\frac{\partial F_4(p, P)}{\partial q} \qquad Q = \frac{\partial F_4(p, P)}{\partial P} \tag{14}$$

The generating function(s) result in a new Hamiltonian:

$$K(Q, P, t) = H(q, p, t) + \frac{\partial F_2}{\partial t} \tag{15}$$

The new Hamiltonian results in corresponding new equations of motion:

$$\dot{P} = -\frac{\partial K}{\partial Q} \qquad \dot{Q} = \frac{\partial K}{\partial P} \tag{16}$$

$H = T + U$ if $\frac{\partial H}{\partial t} = 0$, no explicit time dependence, AND no terms linear in momentum/velocity

3.5 Small Oscillations with Effective Potentials

To find frequency of small oscillations:

1. Write the Hamiltonian and find the effective potential, V_{eff} (all terms that depend on q)
2. Find $\frac{\partial^2 V_{eff}}{\partial q^2}|_{q=qmin}$ where q represents the variable with small oscillations
3. Write the V matrix as:
$$V = \frac{1}{2}\tilde{V}q^2 = \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial q^2}|_{qmin}q^2 \tag{17}$$
4. Write the T matrix as:
$$T = \frac{1}{2}\tilde{T}\dot{q}^2 \tag{18}$$
5. Solve for the frequency using $\tilde{V}$ and $\tilde{T}$:
$$\tilde{V} - \omega^2\tilde{T} = 0 \tag{19}$$

Quick way to get frequency: Make the Lagrangian look like: $L = \frac{1}{2}m'\dot{\eta}^2 - \frac{1}{2}k'\eta^2$. Then $\omega = \sqrt{\frac{k'}{m'}}$

3.6 Variational Calculus

The Euler-Lagrange equation can also solve other physics of path minimization, such as the brachistone problem of minimizing time for a particle in a force field to travel between two points. To use the E-L for this type of problem:

1. Write an equation that describes the motion and the element to minimize, such as $dt = \frac{ds}{v}$. The element to minimize should be alone on the LHS.
2. Add an integration symbol on both sides: $t = \int \frac{ds}{v}$
3. Write the RHS differential in terms of path variables, such as dx and dy, in order to evaluate the integral, such as: $t = \int \frac{\sqrt{1+x'^2}}{\sqrt{2gy}}dy$
4. Use the E-L equation on the integrand, using the appropriate variables, such as: $\frac{\partial F}{\partial x} - \frac{d}{dy}\frac{\partial F}{\partial x'} = 0$
5. Solve the resulting equation by separation of variables, such as $x(y) = \int \sqrt{\frac{y}{(c^2/2g)-y}}dy$

4 Vector Potentials

Remember that the vector potential due to a particle in a magnetic field is:

$$\vec{A} = -\frac{1}{2}B_0(y\hat{x} - x\hat{y}) \tag{20}$$

And to find the potential, use:

$$U = q\phi - q\vec{A}\cdot\vec{v} \tag{21}$$

where ϕ represents the electric potential.

5 Small Oscillations

Standard coordinates define how the blocks are displaced *relative to each other*, while small coordinates (usually η) define how the blocks are displaced *relative to their original equilibrium position.* Start by writing the Lagrangian in standard coordinates, then transform to small coordinates. Then use these notations:

$$L = \frac{1}{2}\mathbf{T}\dot{\eta}_i\dot{\eta}_j - \frac{1}{2}\mathbf{V}\eta_i\eta_j \tag{22}$$

Use $\frac{\partial V}{\partial q_i}|_{q0i} = 0$ to find the minimum point q_{0i}, and $\mathbf{V} = \frac{\partial^2 V_{eff}}{\partial q_i^2}|_{q0i} = \frac{\partial^2 V_{eff}}{\partial \eta_i \eta_j}|_0$ to find $\mathbf{V}$.

Then use $\mathbf{T}$ and $\mathbf{V}$ to solve for the frequency(s):

$$|\mathbf{V} - \lambda\mathbf{T}| = 0 \tag{23}$$

where $\lambda = \omega^2$, to solve for the frequencies ω_i. To find the eigenvectors:

$$(\mathbf{V} - \lambda_i\mathbf{T})\,\vec{c}_i = 0 \tag{24}$$

these $\vec{c}_i$ also make up the amplitude ratios for λ_i, $\frac{A_1}{A_2}$:

$$(\mathbf{V} - \lambda_i\mathbf{T})\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \tag{25}$$

To normalize the eigenvectors:

$$\vec{C}_i = N_i\vec{c}_i \rightarrow \vec{C}_i^T\mathbf{T}\vec{C}_i = 1 \tag{26}$$

Solve for N_i. Finally, to write the displacement of the system as a function of time:

$$A_i = \vec{C}_i^T\mathbf{T}\eta(0) \tag{27}$$

$$\omega_i^2 > 0 \rightarrow \omega_i B_i = \vec{C}_i^T\mathbf{T}\dot{\eta}(0) \tag{28}$$

$$\omega_i = 0 \rightarrow B_i = \vec{C}_i^T\mathbf{T}\dot{\eta}(0) \tag{29}$$

The general solution can now be written as:

$$\vec{\eta}(t) = \sum_{\omega_i^2>0} \vec{C}_i(A_i cos\omega_i t + B_i \sin\omega_i t) + \sum_{\omega_i^2=0} \vec{C}_i(A_i + B_i t) \tag{30}$$

Smaller ω's correspond to more symmetry in the oscillation mode.

6 Central Forces & the Hamilton Jacobi Equation

Whenever we have two masses exerting a force on each other, we can move into the center of mass reference frame and consider the reduced mass combination acted on by a central force, since the center of mass of the system does not move.

Orbits & Stability

- A circular orbit is stable if $\frac{\partial^2 V_{eff}}{\partial r^2} > 0$
- To find the radius for circular orbit, set $\frac{\partial V}{\partial r} = 0$ and solve for r (can also use Hamilton's equations)
- To find the condition on the radius for circular orbit, find $\frac{\partial^2 V_{eff}}{\partial r^2} > 0$ and substitute in the radius for circular orbit

Steps for Solving Motion with the Hamilton-Jacobi

1. *Background*: We can transform H without loss of generality to $K = H + \frac{\partial S}{\partial t} = 0$. Assuming then that S, Hamilton's principle/generating function is separable ($S(q,t) = S_1(t) + S_2(q)$) and $p = \frac{\partial S}{\partial q}$, we can rearrange K to be:
$$\frac{1}{2m}\left(\frac{\partial S_2}{\partial q}\right)^2 + V(q) = -\frac{\partial S_1}{\partial t} \tag{31}$$
Now the variables are separated, and we can set both sides equal to a constant, E. This makes solving for S_1 and S_2 a matter of maths.
2. Write Hamilton's equation, and substitute $\frac{\partial S_2}{\partial q}$ for each p_q term. (S_2 is sometimes referred to as W)
3. Separate variables - this usually entails writing everything not dependent on r inside a bracket, and setting that bracket equal to α_3. (This is usually the total angular momentum, which we can see is a constant of the motion by finding $[L, H] = 0$). Or solve so that r is on one side of the equation, and θ and ϕ are on the other side, then set both sides equal to α_3.
4. Assuming W is separable (example $W(r,\theta,\phi) = W_r + W_\theta + W_\phi$), find integrals defining each component of W.
5. Use $p_q = \frac{\partial W}{\partial q}$ to find the meaning of α_2 and α_3.
6. Use the form $\frac{\partial W}{\partial E} = t + \beta$ to solve for the motion of r depending on E and α's.
7. *Additional*: It may be useful to also remember that $Q = \frac{\partial S_2}{\partial P} = \frac{\partial S_2}{\partial E}$ and $\dot{Q} = \frac{\partial H}{\partial P} = \frac{\partial H}{\partial E}$.

The "action", J is equivalent to $S_2(q)$ as long as $S(q,t)$ is separable:
$$J = \int p dq = \int P dQ \tag{32}$$
Given this J, the frequency of motion is:
$$\nu_i = \frac{\partial E}{\partial J_i} \tag{33}$$
where E came from integrating the action J and solving for $E(J)$.

7 The Poisson Bracket

The poisson bracket is a good method of determining which elements associated with a Hamiltonian are constants of motion:
$$\frac{du}{dt} = [u, H]_{qi,pi} + \frac{\partial u}{\partial t} \tag{34}$$
$$[u, H]_{qi,pi} = \sum_i^n \left(\frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{35}$$
For example, given angular momentum $J = q_1 p_2 - q_2 p_1$, the poisson bracket of J with H quickly shows that the angular momentum is conserved:
$$\frac{dJ}{dt} = [J, H]_{qi,pi} = 0 \tag{36}$$
In general, to find whether an element is a constant of motion:

1. Write the element A in terms of q_i and p_i
2. Write the Hamiltonian according to the physical description
3. Find $\frac{dA}{dt} = [A, H]_{qi,pi} + \frac{\partial A}{\partial t}$

For canonical variables:
$$[q_i, q_j] = 0 \quad [q_i, p_j] = \delta_{ij} \quad [p_i, p_j] = 0 \tag{37}$$
The poisson bracket also helps verify that transformations are properly canonical:
$$[Q, P]_{q,p} = 1 \tag{38}$$

8 Extra

8.1 Conservative Forces

A force is conservative if $\vec{\nabla} \times \vec{F} = 0$. In Cartesian coordinates, can find this as: $\frac{\partial F_i}{\partial j} = \frac{\partial F_j}{\partial i}$

8.2 Nonhomogeneous Equations

Solving a non-homogeneous equation requires the combination of a *particular* and a *complementary* solution:

$$\dot{y} + ay = b \quad \rightarrow \quad y(t) = y_p(t) + y_c(t) \tag{39}$$

1. The particular solution should be of the form $y_p(t) = At^2 + Bt + C$, keeping only the terms so that $y_p(t)$ is a polynomial of the same order as the right hand side of the original equation. So in this example, $y_p(t) = C$.
2. The complementary solution solves $y(t)$ for the right hand side equalling zero: $\dot{y} + ay = 0$. Solve this the usual way, including the constant of integration.
3. Write $y(t) = y_p(t) + y_c(t)$, and substitute these results back into the original equation. Use the original equation and initial conditions to solve for the constants of integration.

Remember that a second derivative equation of motion can be handled as a first derivative equation by writing it in terms of velocity instead of position: $\ddot{y} + a\dot{y} = b \rightarrow \dot{v}_y + av_y = b$

9 Coordinate Systems

9.1 Cartesian

Convert to spherical: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Convert to cylindrical: $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

9.2 Spherical

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{40}$$

$$\hat{\theta} = \frac{\partial\hat{r}}{\partial\theta} \;\&\; \hat{\phi} = \frac{\partial\hat{r}}{\partial\phi} \tag{41}$$

Derivation of a small chunk of circular area (such as in Kepler's law for orbits):

$$S = r\theta \rightarrow dS = rd\theta \rightarrow dA = R^2 d\theta \tag{42}$$

9.3 Cylindrical

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \tag{43}$$

$$\hat{\theta} = \frac{\partial\hat{r}}{\partial\theta} \tag{44}$$

Stat Mech Main Points

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1 Basic Thermo

Definitions:

- Adiabatic: No heat is exchanged in or out of system. $dQ = 0$
- Quasi-static: Uniform pressure throughout system
- Isoentropic: Adiabatic, quasi-static, and constant entropy
- Reversible: constant entropy

Equations:

- Most all basic thermo relations can be derived from these two equations/derivations:

$$F = E - TS \rightarrow dF = dE - TdS - SdT \tag{1}$$

$$dU = TdS - PdV + \Sigma\mu dN \tag{2}$$

- Internal energy:

$$\Delta U = Q - W \tag{3}$$

- Work done **by** the system is calculated as:

$$W = \int PdV \tag{4}$$

- Relationship between pressure, force and area:

$$P = \frac{F}{A} \tag{5}$$

- **Ideal Gases**:
  - Ideal gas law equation: $PV = NkT = nRT$. $N = \#$ of particles, $n = \#$ of moles of particles
  - Energy of an ideal gas $= \frac{N}{2}kT$, where N represents degrees of freedom.
    Energy of a **monatomic** ideal gas $= \frac{3}{2}NkT$
    Energy of a **diatomic** ideal gas $= \frac{5}{2}NkT$

- Clausius-Clapeyron Equation:

$$\frac{dP}{dT} = \frac{L}{T(V_G - V_L)} \tag{6}$$

- We can also see from the free energy equation that we can relate the pressure derivative to the entropy at equilibrium:

$$\frac{dP}{dT} = \frac{S_G - S_L}{V_G - V_L} \tag{7}$$

- Enthalpy:

$$H = E - pV \tag{8}$$

1.1 Extensivity

A thermo property such as entropy is properly **extensive** if: $S(\lambda N, \lambda E, \lambda V) = \lambda S$. Equations of state can be given as **intensive** instead. That means they are given as a per-particle measurement - such as:

$$e = \frac{E}{N} \quad v = \frac{V}{N} \quad s = \frac{S}{N} \tag{9}$$

All the same derivative relations still hold for these intensive properties.

1.2 Basic Temperature & Entropy Relationships

- Heat to raise/lower temp: $Q = C_p m \int dT$
- Heat to melt ice: $Q = mL$
- $\Delta S = \frac{Q}{T}$; plug in Q and then integrate (the integral in definition of Q)
- C_v and C_p for a liquid are basically the same
- At high temperatures, we expect the macrostate of a system to be in its most random state. That means that every microstate should have equal probability of occurring.

1.3 Maxwell Relations

Knowing the dU and dF equations listed previously, we can find additional physics by taking mixed partial derivatives. Since partial derivatives can switch order without changing the result, doing so can lead to additional physics relationships. For example:

$$\frac{\partial}{\partial E}\frac{\partial S}{\partial V} = \frac{\partial}{\partial V}\frac{\partial S}{\partial E} \rightarrow \frac{\partial}{\partial E}\frac{P}{T} = \frac{\partial}{\partial V}\frac{1}{T} \rightarrow P = \frac{\partial E}{\partial V} \tag{10}$$

1.4 Engines

- Work done by an engine is positive if the **area under the curve of a P-V diagram** is **positive**.
- A **Carnot cycle** consists of **two isotherms** and **two adiabats**.
- An adiabat line goes from a lower isotherm to a higher isotherm line (so an adiabat line is **steeper** than an isotherm line).
- Adiabat: $dQ = 0$; and since $dS = \frac{dQ}{t}$, we find that $dS = 0$.
  Also for an adiabat:
  $$PV^{\gamma} = constant \quad \& \quad TV^{\gamma - 1} = constant \tag{11}$$
  Where $\gamma = \frac{f+2}{f} = \frac{C_P}{C_V}$ and f represents the degrees of freedom (3 for monatomic ideal gas).
- Isotherm: $dT = 0$
- Isochore: $dV = 0$, and $Q = C_v m \Delta T$

Efficiency:

$$\eta = \frac{benefit}{cost} = \frac{W}{Q_h} = 1 - \frac{Q_c}{Q_h} \tag{12}$$

2 Classical Statistical Mechanics

Classical stat mech systems are usually **distinguishable** and energy levels can be **either discrete or continuous**. Usually the question must give information about the particles' energy levels and distinguishability.

Given an average measurement per time $\bar{n}_s$, we find the **variance** as:

$$\overline{(\Delta n_s)^2} = < (n_s - < n_s >)^2 > = < n_s^2 > - \bar{n}_s^2 \tag{13}$$

2.1 Ensembles

2.1.1 Microcanonical Ensemble (E, N, V fixed)

Entropy:

$$S = k \ln \Omega \tag{14}$$

Where Ω represents multiplicity, which is the number of possible states present.

Partition function is $\frac{1}{\Omega}$

Binomial distribution:

$$\binom{N}{n} = \frac{N!}{n!(N-n)!} \tag{15}$$

If asked to calculate the multiplicity Ω or other derivations in the MCE for a more complicated system, such as a classical harmonic oscillator ($H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2$), remember that Ω represents the total number of states present - so integrate with a step function in order to include all energies up to the given energy:

$$\Omega = \left[\int \frac{d^3x d^3p}{h^3} \Theta \left(E - \sum_i \left(\frac{p_i^2}{2m} + \frac{m}{2}\omega^2 x^2 \right) \right) \right]^N \tag{16}$$

We can write this as a 6-dimensional sphere of radius $\sqrt{E}$ in phase space. (Probably won't encounter it.)

2.1.2 Canonical Ensemble (N, V, T fixed)

The canonical ensemble represents a system in equilibrium with a reservoir.

Use this partition function for a single particle:

$$z = \sum_{n=0}^{\infty} e^{-\beta n} \tag{17}$$

and for N particles (add an $\frac{1}{N!}$ if indistinguishable):

$$Z = z^N \tag{18}$$

Along with the relations derived from the dF and dU equations, we also have:

$$E = -\frac{\partial}{\partial \beta} \ln Z \qquad F = -\frac{1}{\beta} \ln Z \qquad C_V = \frac{\partial E}{\partial T} \tag{19}$$

More CE details:

- If given energy in term of p or k (in CE or GCE), as long as it is not bosons or fermions, integrate over all phase space to find the partition function. This means the partition function in 3D looks like:

  $$z = \frac{1}{h^3} \int e^{-\beta E(x,p)} d^3x d^3p \tag{20}$$

  Remember that if integrating over momentum space, p, divide by a factor of h for each dimension in the integration. If integrating over k space instead, only divide by a factor of 2π for each dimension.

- If a system has multiple sources of energy, the partition function is the multiplicative combination of each energy source. For example, for a system of indistinguishable particles with internal energy as well as translational energy (if they are point particles):

  $$Z = Z_{trans} Z_{internal} = \frac{1}{N!} (z_{trans} z_{internal})^N \tag{21}$$

  These z's can be calculated independently. For example, if given point particles with two internal energy states, 0 and Δ:

  $$Z = \frac{1}{N!} \left[e^0 + e^{-\beta \Delta} \right]^N \left[\int \frac{d^3x d^3p}{h^3} e^{-\beta p^2/2m} \right]^N \tag{22}$$

- Similarly, if we have a gas made up of two or more particles, the combined partition function is the multiplicative combination of each kind of particle. For example, the partition function for a gas made up of two particles, A and B, looks like:

$$Z = Z_A Z_B \tag{23}$$

- If two or more gases/liquids are in equilibrium in the same volume, their chemical potentials must be equal.
- If asked for less traditional calculations (such as the average height of atoms in a gravitational field), look at the partition function and see what derivative to take to bring that quantity down from the exponent (and divide by any extraneous terms that would come down). For example, for the average height of atoms in a gravitational field:

$$Z = \frac{1}{N!}\left(\int \frac{d^3x d^3p}{h^3} e^{-\beta(p^2/2m - mgz)}\right)^N \tag{24}$$

$$\langle z \rangle = -\frac{1}{N\beta m}\frac{\partial}{\partial g}\ln Z \tag{25}$$

2.1.3 Grand Canonical Ensemble (V, T, μ fixed)

Along with the dU and dF equations, use the grand potential $\mathcal{G}$ to derive relations for the GCE:

$$\mathcal{G} = F - \mu N = E - TS - \mu N = -pV = -kT\ln Q = -\frac{1}{\beta}\ln Q \tag{26}$$

Use the grand partition function with this ensemble:

$$Q = \Sigma e^{\beta\mu N}\Sigma e^{-\beta E_n N} \tag{27}$$

Another entity used in GCE is the **grand canonical potential**, ψ (is actually derivable from $d\mathcal{G}$ as $\psi = \frac{PV}{kT}$):

$$\psi = \ln Q \tag{28}$$

As with CE, if given a composite substance (such as electrons and positrons), it's easiest to write the partition function as a product of the two partition functions:

$$Q_{tot} = Q_+ Q_- \tag{29}$$

This also works if given an energy that varies based on spin, such as $H = \frac{\hbar^2 k^2}{2m} + m_z B$, where $m_z = \pm 1$:

$$Q_{tot} = Q(m_z = 1) \times Q(m_z = -1) \tag{30}$$

3 Quantum Statistical Mechanics

Quantum statistical mechanics particles usually have **discrete energy levels** and are usually **indistinguishable**, but in the case of *bosons* or *fermions*, we take care of indistinguishability when we calculate the partition function according to **distribution functions** rather than just using the $\frac{1}{N!}$ that we used for classical particles.

Usually we use the Grand Canonical Ensemble to analyze quantum systems, but we can no longer sum over N in the partition function (Z), since we need to sum over *states* rather than *particles.*

For example, for a Fermi gas with a particular defined energy:

$$Q = \sum_{N=0}^{\infty}\sum_{\{\sigma\}} e^{\beta\mu N} e^{-\beta E_\sigma} \tag{31}$$

We must convert our sum over N to a sum over states, so $N = \sum_{i=0}^{\infty} n_i$:

$$Q = \sum_{n_i=0}^{\infty} e^{\sum_i n_i\beta(\mu - E_i)} = \prod_i \left(\sum_{n_i=0}^{\infty} e^{-n_i\beta(E_i-\mu)} \right) \tag{32}$$

It's easiest to calculate this for spin zero particles, in which case $n_i = 0, 1$:

$$Q = \prod_i \left(1 + e^{-\beta(E_i-\mu)}\right) \tag{33}$$

For nonzero spin, the n_i should be multiplied by a factor of $2S+1$. So for example, for a particle with spin $\frac{5}{2}$, $2S+1 = 6$ and the grand partition function becomes:

$$Q = \prod_i \left(1 + e^{-\beta(E_i-\mu)}\right)^6 \tag{34}$$

As a general rule, quantum effects should dominate at low temperatures; classical effects should dominate at high temperatures.

3.1 Photons

Remember that **N is not conserved for photons**, and likewise, $\mu = 0$.

3.2 Condensates

How to tell if we have a condensate:
Set $\mu = 0$ and $T = 0$. If $N \to \infty$, no condensate. If $N \to 0$ or $N \to$ a number, condensate.

Or, solve for μ, set $T = 0$ and see if we have any restrictions on N to make μ equal the ground state energy. No restrictions on N = no condensate. Essentially if you can even find μ by itself, there's no condensate.

How to find critical temperature:
Set $\mu = 0$ and solve for T as a function of N. This is the critical temperature.

3.3 Distribution Functions

We can use the probability distribution $f(\epsilon_i)$ for bosons and fermions to calculate various useful things. (We can derive these distribution functions from the canonical ensemble, where Z has an added $\frac{1}{N!}$ for bosons.) In these formulas:

- The distribution function $f(\epsilon_i)$ represents the **number of particles with energy** ϵ_i. When called a "probability function", it does not include g_s in the numerator; when used to calculate "population", it does. Be careful with wording and note which case you are assuming.

- Spin degeneracy g_s represents the **spin degeneracy of state i** being calculated. This is $2S+1$ except in the case of massless particles, which have spin degeneracy $2S$. If the energy ϵ_i depends on the spin, this is just 1.

- ϵ_i represents an **energy** that the state can have.

- $g(k)$ and $g(\epsilon)$ represent the **density of states**. These must be calculated depending on the energy and dimensions of the question. The easiest way to do that is:

$$g(\epsilon) = \int \left(\frac{L}{2\pi}\right)^n d\vec{k}\, \delta(\epsilon - \epsilon_k) \tag{35}$$

  In this definition for density of states, $d\vec{k}$ should be written according to the number of dimensions, and ϵ_k is the reference energy relating ϵ and k (for example, $\epsilon_k = pc = \hbar kc$). Write $d\vec{k}$ in terms of energy and the integral becomes trivial.

Remember this definition does not include **spin degeneracy** g_s, which some people do include in their definition of density of states.

The following calculations assume that g_s is **not** included in the distribution function $f(\epsilon)$ or the density of states $g(\epsilon)$.

Bosons:

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \tag{36}$$

Fermions:

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \tag{37}$$

Boltzmann Distribution:

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}} \tag{38}$$

The following are particularly helpful for quantum systems:

Average internal energy:

$$U = \int \epsilon \, g_s \, g(\epsilon) f(\epsilon) d\epsilon \tag{39}$$

Number of particles:

$$N = \int g_s \, g(\epsilon) f(\epsilon) d\epsilon \tag{40}$$

Specific heat capacity:

$$C = \frac{\partial}{\partial T} \int \epsilon \, g_s \, g(\epsilon) f(\epsilon) d\epsilon \tag{41}$$

The average of any random quantity n, if we know that quantity in terms of ϵ, $n(\epsilon)$:

$$\langle n \rangle = \int n(\epsilon) \, f(\epsilon) \, g(\epsilon) \, d\epsilon \tag{42}$$

We can also find the average of a quantity by integrating over phase space:

$$\langle \epsilon \rangle = \int \epsilon_k \, f(k) \, g(k) \, d^3k d^3x \tag{43}$$

At $T = 0K$, we can find a few more properties:

$$N = \int_0^{\epsilon_f} g(\epsilon) d\epsilon \tag{44}$$

$$\mu = \epsilon_f \;\; at \;\; T = 0K \tag{45}$$

4 Phase Transitions & Mean Field Theory

Magnetization:

$$M = -\frac{\partial F}{\partial B} = \mu_0 (N_+ - N_-) \tag{46}$$

Magnetic Susceptibility :

$$\chi = \frac{\partial M}{\partial B} \tag{47}$$

Evidence of **Phase Transition**:

- Divergence of E or M (or divergence of their derivatives)
- Multiple states for a fixed temperature (example, high and low density states simultaneously present)

To find the solution(s) of a transcendental equation, set the *slope* of each side equal to each other.

For example, when looking for spontaneous magnetization, remember that the slope of $\tanh(\alpha x)$ at $x = 0$ is α (or just take the derivative of the function to get the slope! evaluate result at M=0 for most linear case). So, to find solutions for $\beta x = \tanh(\alpha x)$, set the slopes equal: $\beta = \alpha$ in this case.

5 Helpful Maths

Taylor expansion:

$$F(x_0 + dx) = F(x_0) + dx\frac{dF}{dx}|_{x=x_0} + dx^2\frac{d^2F}{dx^2}|_{x=x_0} + ... \tag{48}$$

Don't forget the chain rule for derivatives:

$$\frac{dT}{dz} = \frac{\partial T}{\partial P}\frac{\partial P}{\partial z} \tag{49}$$

Stirling's approximation:

$$\ln N! = N \ln N - N \tag{50}$$

To take the derivative of a function that has multiple variables, such as $S(T, N, V)$:

$$dS(T, N, V) = \frac{\partial S}{\partial T}dT + \frac{\partial S}{\partial N}dN + \frac{\partial S}{\partial V}dV \tag{51}$$

We can think of this from taking the whole time derivative and "canceling out" the dt from each term:

$$\frac{d}{dt}S(T, N, V) = \frac{\partial S}{\partial T}\frac{dT}{dt} + \frac{\partial S}{\partial N}\frac{dN}{dt} + \frac{\partial S}{\partial V}\frac{dV}{dt} \tag{52}$$

Expansions for small x (useful for temperature limits!):

$$(1 + x)^n \simeq 1 + nx \tag{53}$$

$$e^x \simeq 1 + x + ... \tag{54}$$

Helpful summation tricks

$$\sum_{N=0}^{\infty} a^N = \frac{1}{1-a} \qquad a \ll 1 \tag{55}$$

$$\sum_{N=0}^{\infty} \frac{1}{N!}x^N = e^x \tag{56}$$

Sometimes a partition function (grand partition functions especially) cannot be easily calculated. Then it helps to look at the high and low temperature limits. For example:

$$Z \sim \sum e^{-\beta E n^2} \tag{57}$$

For $T \to \infty$, $\beta \to 0$, so exponent gets small. **Integrate.**

For $T \to 0$, $\beta \to \infty$, so $e^{-\beta E}$ gets small. **Sum**, keeping only the first couple terms.

Remember this integral for help solving N in grand canonical ensemble problems:

$$\int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1} = \Gamma(n) g_n(z) \tag{58}$$

Standard deviation:

$$\Delta n^2 = \langle n^2 \rangle - \langle n \rangle^2 \tag{59}$$

Gamma function:

$$\Gamma(n+1) = n\Gamma(n) \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{60}$$

$\eta = \frac{W_{out}}{Q_H} = 1 - \frac{Q_C}{Q_H}$

* Isentropic = adiabatic + reversible

Table 19.1 summarizes the results for specific gas processes. This table shows W_s, the work done *by* the system, so the signs are opposite those in Chapter 17.

TABLE 19.1 Summary of ideal-gas processes

Process	Gas law	Work W_s	Heat Q	Thermal energy
Isochoric	$p_i/T_i = p_f/T_f$	0	$nC_V\Delta T$	$\Delta E_{th} = Q$
Isobaric	$V_i/T_i = V_f/T_f$	$p\Delta V$	$nC_P\Delta T$	$\Delta E_{th} = Q - W_s$
Isothermal	$p_iV_i = p_fV_f$	$nRT\ln(V_f/V_i)$ $pV\ln(V_f/V_i)$	$Q = W_s$	$\Delta E_{th} = 0$
Adiabatic	$p_iV_i^\gamma = p_fV_f^\gamma$ $T_iV_i^{\gamma-1} = T_fV_f^{\gamma-1}$	$(p_fV_f - p_iV_i)/(1-\gamma)$ $-nC_V\Delta T$	0	$\Delta E_{th} = -W_s$
Any	$p_iV_i/T_i = p_fV_f/T_f$	area under curve		$\Delta E_{th} = nC_V\Delta T$

There is one entry in this table that you haven't seen before. The expression

$$W_s = \frac{p_fV_f - p_iV_i}{1-\gamma} \quad \text{(work in an adiabatic process)} \tag{19.12}$$

for the work done in an adiabatic process follows from writing $W_s = -\Delta E_{th} = -nC_V\Delta T$, which you learned in Chapter 17, then using $\Delta T = \Delta(pV)/nR$ and the definition of γ. The proof will be left for a homework problem.

You learned in Chapter 18 that the thermal energy of an ideal gas depends only on its temperature. Table 19.2 lists the thermal energy, molar specific heats, and specific heat ratio $\gamma = C_P/C_V$ for monatomic and diatomic gases.

TABLE 19.2 Properties of monatomic and diatomic gases

	Monatomic	Diatomic
E_{th}	$\frac{3}{2}nRT$	$\frac{5}{2}nRT$
C_V	$\frac{3}{2}R$	$\frac{5}{2}R$
C_P	$\frac{5}{2}R$	$\frac{7}{2}R$
γ	$\frac{5}{3} = 1.67$	$\frac{7}{5} = 1.40$

A Strategy for Heat-Engine Problems

8.12, 8.13 ActivPhysics ONLINE

The engine of Example 19.1 was not a realistic heat engine, but it did illustrate the kinds of reasoning and computations involved in the analysis of a heat engine. A basic strategy for analyzing a heat engine follows.

PROBLEM-SOLVING STRATEGY 19.1 Heat-engine problems (MP)

MODEL Identify each process in the cycle.

VISUALIZE Draw the pV diagram of the cycle.

SOLVE There are several steps in the mathematical analysis.

- Use the ideal-gas law to complete your knowledge of n, p, V, and T at one point in the cycle.
- Use the ideal-gas law and equations for specific gas processes to determine p, V, and T at the beginning and end of each process.
- Calculate Q, W_s, and ΔE_{th} for each process.
- Find W_{out} by adding W_s for each process in the cycle. If the geometry is simple, you can confirm this value by finding the area enclosed within the pV curve.
- Add just the *positive* values of Q to find Q_H.
- Verify that $(\Delta E_{th})_{net} = 0$. This is a self-consistency check to verify that you haven't made any mistakes.
- Calculate the thermal efficiency η and any other quantities you need to complete the solution.

ASSESS Is $(\Delta E_{th})_{net} = 0$? Do all the signs of W_s and Q make sense? Does η have a reasonable value? Have you answered the question?

Thermodynamics Review

Ch 16 - Macroscopic Description of Matter

- Thermal equilibrium is defined by non-changing state variables (ex. P, V, T)
- At phase equilibrium, any amount of each state possible at those points can exist
  - critical point is limit where gas + liquid can be distinguished

* Ideal gas systems present good systems to develop thermodynamic basics on

$PV = nRT$ (Ideal Gas Law) alternatively, $PV = Nk_BT$; n = # of moles, N = # of Particles

- Manipulations of the Ideal Gas Law yield:

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$ * # of particles constant

- Processes that occur in thermal equilibrium are known as quasi-static. They are also reversible
- Constant volume processes are isochoric
- Constant pressure processes are isobaric
- Constant temperature processes are isothermic

Typically represented on a P-V diagram

Ch 17 - Work, Heat, + the First Law of Thermodynamics

- Redefining the work-energy theorem

$W_{ext} = \Delta K + \Delta U + \Delta E_{th}$ where $E_{th} = K_{micro} + U_{micro}$

* However, not all changes in energy go towards work, some is lost to heat

$\Rightarrow \Delta E = W + Q$

- Note: Work is not a state variable;

$$W = \int \vec{F} \cdot d\vec{s}$$

$$= -\int P\,dV \quad \text{(for gases)}$$ * P is typically a function of V *

* Work is not path-independent quantity *

- Heat is energy transferred b/w a system + the environment due to a difference in temperature.
  * no motion of the system is required

$\Delta E_{th} = W + Q$ (First Law of Thermodynamics)

* simply energy conservation

- A process is adiabatic if $Q = 0$
  $\Rightarrow$ This does not imply that $\Delta T = 0$
- The specific heat of a substance is the amount of energy necessary to raise 1kg of it by 1K
  * Assuming $W = 0$,

$$Q = Mc\Delta T = nC\Delta T$$

molar specific heat

- The heat of transformation is the amount of energy necessary to transform 1kg of a substance from one phase to another

  $Q = \pm ML$, where L is heat of transformation
- Gases have two types of specific heat; C_p and C_v
  $\Rightarrow C_p = C_v + R$
  * for adiabatic processes, $PV^{\gamma} = \text{constant}$; $\gamma = C_p/C_v$

Ch 18: The Macro/Micro Connection

- Kinetic theory states that the macroscopic properties of a system are related to the avg. behavior of the molecules/atoms that compose the system
- The mean free path of a molecule/atom is:

  $\lambda = \frac{1}{4\sqrt{2}\pi(N/V)r^2}$, where $\frac{N}{V}$ is # density of particles

- The pressure of a gas is:

  $P = \frac{F}{A} = \frac{1}{3}nmv_{rms}^2$, $v_{rms} = \sqrt{\langle v^2 \rangle}$, A = Area, n = # density

$\epsilon_{avg} = \frac{3}{2}k_BT \Rightarrow T = \frac{2}{3k_B}\epsilon_{avg}$

  $\rightarrow$ Temperature measures avg translational kinetic energy
- For ideal, monatomic gases, $C_v = \frac{3}{2}R$; ideal diatomic gases $C_v = \frac{5}{2}R$
- The equipartition theorem states, that the thermal energy of a system is equally divided

  ex. rotational energy = kinetic energy

  $v_x = v_y$
- Heat will be exchanged by two systems in contact until they reach thermal equilibrium, i.e the avg energy of a particle in each system is the same

  $\Rightarrow \epsilon_{avg,1} = \epsilon_{avg,2}$, Not $E_1 = E_2$ (only if $N_1 = N_2$)

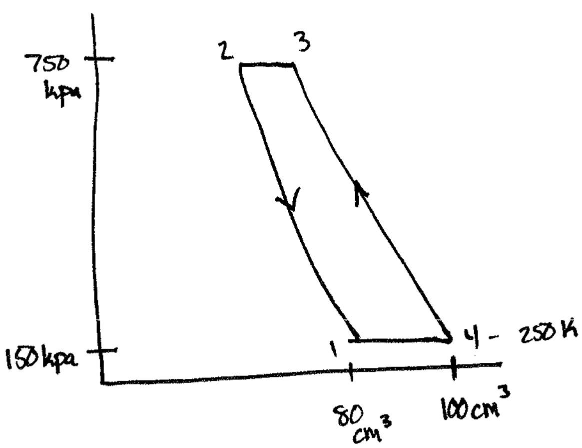
- Equilibrium is the most probable state of a system

- The Second Law of Thermodynamics states that entropy (the measure of disorder in a system) never decreases
  - ⇒ Implies that heat is always transferred from hot to cold
  - ⇒ Irreversible processes imply that entropy only increases in one direction

Ch 19 - Heat Engines + Refrigerators

- Heat engine refers to a cyclical process that transforms heat into work
- Refrigerators is a device that uses work to move heat from a cold object to a hot object
- Devices that turn heat into work (ie heat engines) must return to their initial state at the end of the process + be able to repeat the process

ex. Brayton-cycle



* He gas

$P_1 = 150$ kPa	$V_1 = 80\ cm^3$	$T_1 = 200K$
$P_2 = 750$ kPa	$V_2 =$	$T_2 = 381$ K
$P_3 = 750$ kPa	$V_3 =$	$T_3 = 476$ K
$P_4 = 150$ kPa	$V_4 = 100\ cm^3$	$T_4 = 250$ K

$$\frac{V_1}{T_1} = \frac{V_4}{T_4}$$

$$\Rightarrow T_1 = T_4 V_1 / V_4$$

$$= 200K$$

$$P_4 V_4^{\gamma} = P_3 V_3^{\gamma}$$

* but $V = \frac{nRT}{P}$, $\gamma = \frac{C_P}{C_V}$

$$\Rightarrow P_4^{\gamma-1} T_4^{\gamma} = P_3^{\gamma-1} T_3^{\gamma}$$

$$\Rightarrow T^3 = T^4 \left(\frac{P_4}{P_3}\right)^{\frac{\gamma-1}{\gamma}}$$

$$= 476\ K$$

$$P_2^{\gamma-1} T_2^{\gamma} = P_1^{\gamma-1} T_1^{\gamma}$$

$$P_2^{\frac{\gamma-1}{\gamma}} T_2 = P_1^{\frac{\gamma-1}{\gamma}} T_1$$

$$T_2 = \left(\frac{P_1}{P_2}\right)^{\frac{\gamma-1}{\gamma}} T_1$$

$$= 381\ K$$

$$Q_{3\to2} = nC_P \Delta T$$

$$= 14.3\ J$$

$$Q_{1\to4} = nC_P \Delta T$$

$$= 7.5\ J$$

$$\Rightarrow W = 14.3 - 7.5 = 6.8\ J$$

$$K = \frac{Q_c}{W_{in}} = \frac{7.5}{6.8} = 1.1$$

Classical / Stat Mech Qualifier Topics

Exam: Spring 2008

Question #1 – Forces/Newtons Laws, Friction (all types), Kinematics, Rotational motion
Question #2 – Lagrangian Mechanics, Oscillations, Constraint forces, moments of motion
Question #3 – Hamiltonian Mechanics, Poisson Brackets, Orbital motion
Question #4 – Heat Engine (Work, efficiency, direction of cycle)
Question #5 – Partition Function + related quantities
Question #6 – Grand Partition Function, Quantum stat mech (occupation #'s)

Exam: Fall 2008

Question #1 – Pendulum, Collisions, Kinematics
Question #2 – Harmonic Oscillator, Lagrangian mechanics,
Question #3 – Hamilton–Jacobi Theory, Canonical Variables, Characteristic Functions
Question #4 – Helmholtz Free Energy, Pistons, Equilibrium, Heat Reservoir
Question #5 – One-dimensional gas, quantum limit, classical limit
Question #6 – Fermionic Gas, massless gas, thermodynamic quantities

Exam: Spring 2009

Question #1: Newtons Laws, Lagrangian Mechanics, Oscillations, Equations of motion
Question #2: Central Force, Kepler's Laws, Orbits, Oscillations
Question #3: Power, Maximization, Equations of motion
Question #4: Heat Engine (Work, Efficiency, direction of cycle)
Question #5: Free Energy, Ideal Gas, thermodynamic quantities,
Question #6: Boson Gas, Grand Canonical Free Energy

Exam: Fall 2009

Question #1: Rotational Motion, Friction
Question #2: Lagrangian Mechanics, Equations of motion
Question #3: Lagrangian Mechanics, Equations of motion, Hamiltonian Mechanics, Canonical variables
Question #4: Helmholtz Free Energy, Carnot Cycle, Heat Engine
Question #5: Free Energy, Thermodynamic quantities
Question #6: –

Exam: Spring 2010

Question #1: Pendulum, Oscillations, Rotational Motion, Forces/Newton's Laws
Question #2: Rotational Motion, Kinematics, Newton's Laws / Forces
Question #3: Lagrangian Mechanics, Hamiltonian Mechanics, Forces/Newtons Laws
Question #4: Hydrostatic Equilibrium, Adiabatic Processes
Question #5: Boson gas, Fermionic Gas, Boltzmann Gas, Thermodynamic Quantities
Question #6: Lattice Gas, Grand Canonical Ensemble, Thermodynamic Quantities, Phase Transition

Exam: Fall 2010

Question #1: Oscillations, Eigenvalue/Eigenvectors, Forces/Newtons Laws, Equations of Motion
Question #2: Rotational Motion
Question #3: Lagrangian Mechanics, Hamiltonian Mechanics, Canonical Transform, Equations of Motion
Question #4: Cooling, Entropy
Question #5: Electron gas, Density states, Fermi energy, Thermodynamic quantities
Question #6: Statistics (Avg, variance), Clump/Anti-clump, Instrumentation

Exam: Fall 2011

Question #1: Rotational Motion, Kinematics, Forces/Newton's Laws, Friction
Question #2: Central Force, Angular momentum, Orbits
Question #3: Lagrangian Mechanics, Hamiltonian Mechanics, Equations of motion
Question #4: Heat Engine (Work, efficiency, direction of cycle)
Question #5: Partition Function, Thermodynamic quantities, Equipartition Thm, quantum limit
Question #6: Dispersion relationship, Grand Canonical Ensemble, Fermi energy, Entropy

Exam: Spring 2012

Question #1: Rotational Motion, Kinematics, Forces/Newton's Laws, Friction
Question #2: Pendulum, Lagrangian Mechanics, Equations of Motion
Question #3: Lagrangian Mechanics, Canonical variables, Action principles, Calculus of variations
Question #4: Maxwell relations, Thermodynamic variables (P, V, T, S, μ, N, etc)
Question #5: D-dimensional boson gas, Condensation
Question #6: Lattice gas, partition function, Helmholtz Free Energy, Thermodynamic quantities

Exam: Fall 2012

Question #1: Rotational Motion, Kinematics, Forces/Newton's Laws, Friction, Equations of motion
Question #2: Rotational Motion, Lagrangian Mechanics, Equations of motion, Oscillations
Question #3: Hamiltonian Mechanics, Characteristic Function, Residue Thm, Action Principle, Orbits
Question #4: Helmholtz Free Energy, thermodynamic variables, thermodynamic quantities, Thermo Laws
Question #5: Lattice gas, Grand Potential, canonical ensemble, microcanonical ensemble
Question #6: D-dimensional Fermi gas, Fermi energy, Grand Potential, Ideal Gas Law

Exam: Spring 2013

Question #1: Rotational Motion, Kinematics, Forces/Newtons Law, Work-Energy Thm
Question #2: Collisions, Rotational Motion, Kinematics, Torque?
Question #3: Lagrangian Mechanics, Lagrange multipliers, Equations of motion
Question #4: Thermodynamic Laws, Maxwell Relations, Free Energy
Question #5: Partition Function, microcanonical ensemble, canonical ensemble, free energy, thermo quant
Question #6: Thermodynamic quantities, equation of state, partition function, Helmholtz Free Energy

Exam: Fall 2013

Question #1: Rotational Motion, Forces/Newton's Laws, Kinematics
Question #2: Lagrangian Mechanics, Rotational Motion, Equations of motion, Oscillations, Forces/Newton
Question #3: Pendulum, Lagrangian Mechanics, Hamiltonian Mechanics, Harmonic motion, Equations of Motion
Question #4: Heat Engine (Work, efficiency, direction of cycle)
Question #5: Partition function, high/low temp limits,
Question #6: Thermodynamic Equilibrium, temp limits, Entropy, thermodynamic quantities

Exam: Spring 2014

Question #1: Rotational Motion, Forces/Newtons Laws, Kinematics, Energy
Question #2: Pendulum, Lagrangian Mechanics, Equations of motion, Oscillations
Question #3: Poisson Brackets, Angular Momentum, generalized coordinates
Question #4: Pistons, Thermodynamic Equilibrium, Thermodynamic quantities
Question #5: Entropy, canonical ensemble, partition function, adiabatic processes, Thermo equilibrium
Question #6: Density of states, Fermi energy, temp limits, thermodynamic quantities

Exam: Fall 2014

Question #1: Rotational Motion, Forces/Newton's Laws, Oscillations
Question #2: Coriolis force, Rotational Motion, Forces/Newton's Law, Moving coordinate systems
Question #3: Hamiltonian Mechanics, Hamilton-Jacobi eqn, Hamilton principle function, eqn of motion
Question #4: Phase transition, Entropy, Gibbs Energy
Question #5: Lattice gas, Grand Canonical Partition Function, Mean field, Thermodynamic quantities
Question #6: Photon gas, Thermodynamic Functions, Thermodynamic quantities

Exam: Spring 2015

Question #1: Collisions, Forces/Newtons Laws, Kinematics
Question #2: Rotational Motion, Eqn of motion, Lagrangian Mechanics, Pendulum
Question #3: Poisson Bracket, Orbits, Hamiltonian Mechanics, Runge-Lenz vector
Question #4: Ideal gas, Thermodynamic eqns, Thermodynamic quantities
Question #5: Lattice, Thermodynamic eqns, Entropy
Question #6: Boson gas, dispersion relations, Density of states, Thermodynamic eqns, Condensation

Exam: Fall 2015

Question #1: Rotational Motion, Kinematics, Forces/Newton's Laws
Question #2: Forces/Newtons Laws, Kinematics, Eqn of motion
Question #3: Hamiltonian Mechanics, Poisson Brackets, Canonical coordinates
Question #4: Thermodynamic Equilibrium, Entropy
Question #5: Partition function, Thermodynamic quantities, Thermodynamic eqns
Question #6: Fermion gas, Density of States, Fermi energy, Thermodynamic eqns

Exam: Spring 2016

Question #1: Rotational Motion, Kinematics, Forces/Newton's Laws, Friction
Question #2: Lagrangian Mechanics, Eqn of motion, Oscillations, Rotational Motion
Question #3: Double pendulum, Lagrangian Mechanics, eigenvalue/eigenvector
Question #4: Heat engine (efficiency, work)
Question #5: Partition function, temp limits, thermodynamic quantities
Question #6: Boson gas, density of states, thermodynamic eqns, thermodynamic quantities

Classical Mechanics and Statistical/Thermodynamics

January 2008

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \equiv \zeta(p)$$

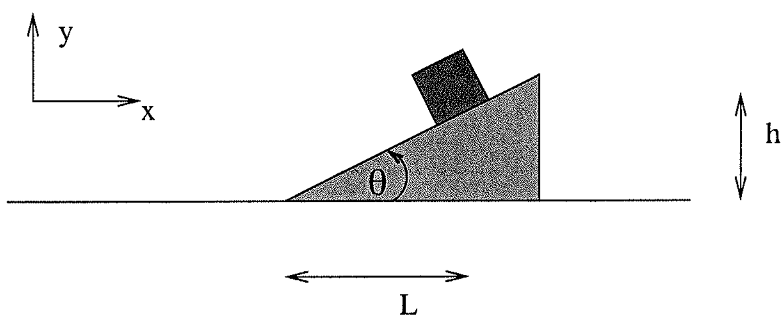
$$\sum_{n=1}^{\infty} \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^{\infty} (-1)^p\,\frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = 0.0833333$$
$$\zeta(2) = 1.64493 \qquad \zeta(-2) = 0$$
$$\zeta(3) = 1.20206 \qquad \zeta(-3) = 0.0083333$$
$$\zeta(4) = 1.08232 \qquad \zeta(-4) = 0$$

Classical Mechanics

1. A block of mass m_1 sits atop a triangular wedge of mass m_2, which is itself on a frictionless plane, as shown. The two are initially at rest, and the block is a height h above the surface of the plane, a horizontal distance L from the bottom edge of the wedge. The wedge has an opening angle θ, as shown.



   (a) Assume that there is no friction between the block and the wedge. The block slides down the wedge. What are the velocities (measured with respect to the fixed inertial reference frame denoted by the x and y axes shown) of the block and wedge just as the block reaches the lower edge of the wedge? (3 points).

   (b) Now replace the block by a ball of radius R (and mass m_1). The ball rolls down the wedge without slipping. What are the velocities of the ball and wedge just as the ball reaches the lower edge of the wedge? (3 points).

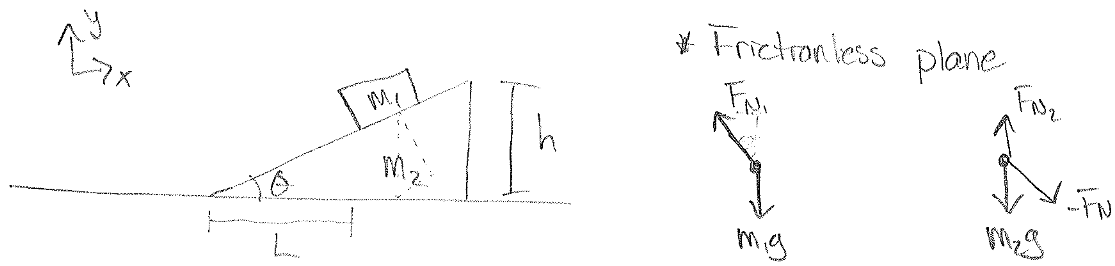
   (c) Return to the block problem, but now assume that the coefficients of static and kinetic friction between the block and the wedge are μ (they have the same value). What is μ_{min}, the minimum value of μ for which the system is stable? (1 point).

   (d) If $\mu < \mu_{\text{min}}$, calculate the minimum **horizontal** force that can be applied to the wedge such that the block will not accelerate down the wedge. (3 points).

Note: you can neglect the finite size of the block in your calculation, and you are asked for the velocities before the block or ball make contact with the frictionless plane.

Jan 2008

Classical #1



a) * Assume no friction b/w block + wedge

* Find velocities when block reaches bottom of wedge

- Forces

  * Block $= \langle -F_{N_1}\sin\theta, -m_1 g + F_{N_1}\cos\theta \rangle$

  * Wedge $= \langle F_N \sin\theta, -m_2 g - F_{N_1}\cos\theta + F_{N_2} \rangle$

- Energy

  $\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = mgh$

- Kinematics

  $r_{1,f} = r_{1,i} + \frac{1}{2}a_i t^2$

  $v_{1,f}^2 = 2a\Delta r \qquad v_1 = a_1 t$

$r_{1,f} = \langle x_{f,1}, 0 \rangle$ * Note: $-m_2 g - F_{N_1}\cos\theta + F_{N_2} = 0$

$\Rightarrow \cancel{y_{1,f}}^{0} = y_{1,i} + \cancel{v_{1,i} t}^{0} + \frac{1}{2}a_{1y} t^2$

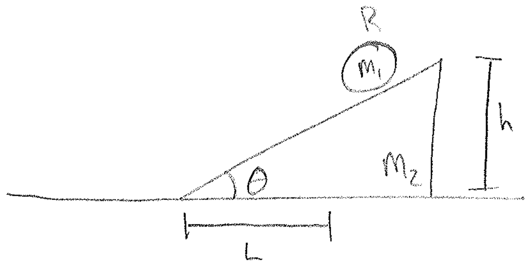
$0 = h + \frac{1}{2}a_{1,y} t^2$

$\left(\frac{-2h\, m_1}{F_{N_1}\cos\theta - m_1 g}\right)^{1/2} = t$

$\Rightarrow v_1 = at$

$v_1 = \left(\frac{-2h}{F_{N_1}\cos\theta - m_1 g}\right)\langle -F_{N_1}\sin\theta, F_{N_1}\cos\theta - m_1 g \rangle, \quad v_2 = \left(\frac{-2hm_1}{m_2(F_{N_1}\cos\theta - m_1 g)}\right)\langle F_{N_1}\sin\theta, 0 \rangle$

b) * Block is now ball rolling w/o slipping (ie. $V_{tangent} = V_{cm}$)



* Frictionless Plane

- Forces

* Ball $= \langle -F_{N_1}\sin\theta,\ -m_1 g + F_{N_1}\cos\theta \rangle$

* Wedge $= \langle F_{N_1}\sin\theta,\ -m_2 g - F_{N_1}\cos\theta + F_{N_2} \rangle$

- Energy

$$m_1 g h = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} I_1 \omega_1^2 + \tfrac{1}{2} m_2 v_2^2 \quad , \quad I = \tfrac{2}{5} m_1 R^2$$

- Kinematics

$$\vec{r}_f = \vec{r}_i + \tfrac{1}{2}\vec{a}t^2$$

$$\vec{\theta}_f = \vec{\theta}_i + \tfrac{1}{2}\vec{\alpha}t^2$$

$v = \omega R$

$a = \alpha R$

$$\vec{v}_f^{\,2} = 2\vec{a}\,\Delta\vec{r}$$

$$\vec{v}_f = \vec{a}t$$

2. Consider a point particle of mass m constrained to move on a parabola in the x-z plane, i.e.,

$$z = \frac{\alpha}{2}x^2.$$

Assume the constraint force is frictionless and gravity acts vertically ($F_z = -mg$).

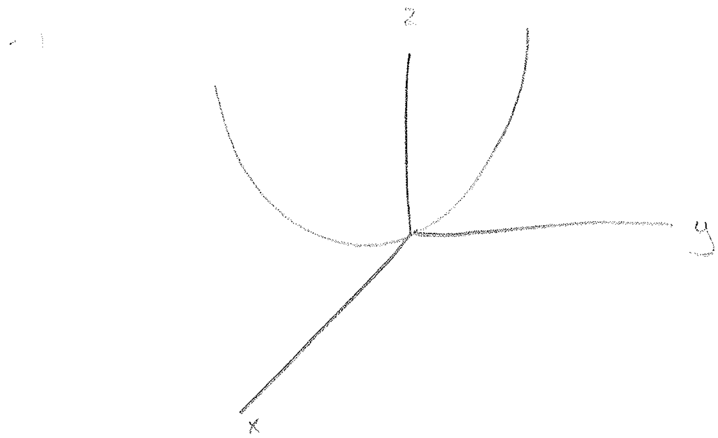
(a) Use Lagrangian mechanics to write a second order differential equation for $x(t)$. (2 points)

(b) Find a first integral of this equation (any way you can) and evaluate the constant of integration using the maximum value x_{max} reached by x. (4 points)

(c) Assume that the particle is pulled a short distance from the origin and allowed to oscillate. Calculate the period in the limit of small oscillations, $\epsilon \equiv \alpha x_{\text{max}} << 1$. (4 points)

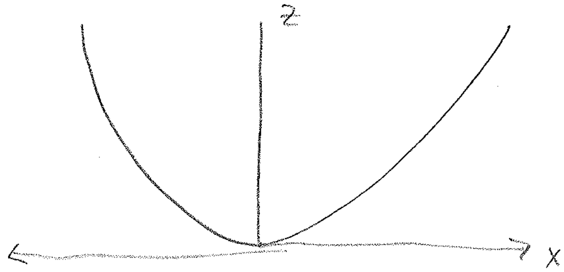
Jan 2008

Classical # 2



* Particle of mass m constrained to move on parabola in x-z plane
  $\Rightarrow z = \frac{\alpha}{2} x^2$
* Constraint force is frictionless

$F_z = -mg$



a) $\mathcal{L} = T - U$

$$T = \tfrac{1}{2} m v^2 \qquad U = mgz$$
$$= \tfrac{1}{2} m (\dot{x}^2 + \dot{z}^2) \qquad = mg \tfrac{\alpha}{2} x^2$$
$$= \tfrac{1}{2} m (\dot{x}^2 + (\alpha \dot{x})^2)$$
$$= \tfrac{1}{2} m \dot{x}^2 (1 + \alpha)$$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}(m\dot{x}^2(1+\alpha) - mg\alpha x^2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

$$mg\alpha x = \frac{\partial}{\partial t}\left(m\dot{x}(1+\alpha)\right)$$

$$mg\alpha x = m(1+\alpha)\ddot{x}$$

$$g\alpha x = (1+\alpha)\ddot{x}$$

$$x = \frac{1+\alpha}{g\alpha}\ddot{x}$$

b)

3. **Angular momentum and the Rungé-Lenz vector:** Given a point particle of mass m, trajectory $\vec{r}(t)$, and momentum $\vec{p}(t)$, we can define the angular momentum

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

and the Rungé-Lenz vector

$$\vec{\mathcal{A}} = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

We consider the explicit case of a $1/r$ potential, so that

$$H = \frac{p^2}{2m} - \frac{1}{r}$$

(a) Prove that the Poisson bracket of H and $\vec{L}$ is zero, that is:

$$\left\{H, \vec{L}\right\} = 0.$$

(3 points).

(b) Prove that the Poisson bracket of H and $\vec{\mathcal{A}}$ is zero, that is:

$$\left\{H, \vec{\mathcal{A}}\right\} = 0.$$

(3 points)

(c) What do your results in parts (a) and (b) imply about the behavior of $\vec{\mathcal{A}}$ and $\vec{L}$? (1 point)

(d) Evaluate $\vec{r} \cdot \vec{\mathcal{A}} = r\mathcal{A}\cos\theta$, using the explicit form for $\vec{\mathcal{A}}$ above. Use this to calculate the orbital motion of the particle (that is, a relationship between r and θ as the particle moves about its orbit). (3 points)

Jan 2008

Classical #3

$\vec{L} = \vec{r} \times \vec{p}$

$\vec{A} = \frac{1}{m} \vec{p} \times \vec{L} - \hat{r}$

$H = \frac{p^2}{2m} - \frac{1}{r}$

a) Prove Poisson bracket, $\{H, L\} = 0$

$$H = \frac{p^2}{2m} - \frac{1}{r}$$
$$= \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$L = \vec{r} \times p$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \langle y p_z - z p_y, -(x p_z - z p_x), x p_y - y p_x \rangle$$

$$\{H, L\} = \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\partial L}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial L}{\partial q_i} \right)$$
$$= \frac{\partial H}{\partial x}\frac{\partial L}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial L}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial L}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial L}{\partial y} + \frac{\partial H}{\partial z}\frac{\partial L}{\partial p_z} - \frac{\partial H}{\partial p_z}\frac{\partial L}{\partial z}$$

$\frac{\partial H}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_x} = \frac{p_x}{m}$

$\frac{\partial L}{\partial p_x} = \langle 0, z, -y \rangle$ $\qquad$ $\frac{\partial L}{\partial x} = \langle 0, -p_z, p_y \rangle$

$\frac{\partial H}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_y} = \frac{p_y}{m}$

$\frac{\partial L}{\partial p_y} = \langle -z, 0, x \rangle$ $\qquad$ $\frac{\partial L}{\partial y} = \langle p_z, 0, -p_x \rangle$

$\frac{\partial H}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_z} = \frac{p_z}{m}$

$\frac{\partial L}{\partial p_z} = \langle y, -x, 0 \rangle$ $\qquad$ $\frac{\partial L}{\partial z} = \langle -p_y, p_x, 0 \rangle$

$$= \frac{x}{(\ )^{3/2}} \langle 0, z, -y \rangle - \frac{p_x}{m} \langle 0, -p_z, p_y \rangle +$$
$$\frac{y}{(\ )^{3/2}} \langle -z, 0, x \rangle - \frac{p_y}{m} \langle p_z, 0, -p_x \rangle +$$
$$\frac{z}{(\ )^{3/2}} \langle y, -x, 0 \rangle - \frac{p_z}{m} \langle -p_y, p_x, 0 \rangle$$

$\langle 0, 0, 0 \rangle$ ✓

b) Show $\{H, A\} = 0$

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$A = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

$$= \frac{1}{m}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ p_x & p_y & p_z \\ yp_z - zp_y & zp_x - xp_z & xp_y - yp_x \end{vmatrix} - \langle x, y, z \rangle \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{1}{m}\langle p_y(xp_y - yp_x) - p_z(zp_x - xp_z),\ p_z(yp_z - zp_y) - p_x(xp_y - yp_x),\ p_x(zp_x - xp_z) - p_y(yp_z - zp_y) \rangle - \langle x, y, z \rangle$$

$$\{H, \hat{A}\} = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial A}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial A}{\partial q_i}$$

$$= \frac{\partial H}{\partial x}\frac{\partial A}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial A}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial A}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial A}{\partial y} + \frac{\partial H}{\partial z}\frac{\partial A}{\partial p_z} - \frac{\partial H}{\partial p_z}\frac{\partial A}{\partial z}$$

$$\frac{\partial H}{\partial x} = -\left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x\right) = \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

$$\frac{\partial A}{\partial x} = \frac{1}{m}\langle p_y^2 + p_z^2, -p_xp_y, -p_xp_z \rangle - \langle 1, 0, 0 \rangle$$

$$\frac{\partial A}{\partial p_x} = \frac{1}{m}\langle -yp_y - zp_z, -xp_y + 2yp_x, 2zp_x - xp_z \rangle$$

$$\frac{\partial H}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\frac{\partial A}{\partial y} = \frac{1}{m}\langle -p_xp_y, p_z^2 + p_x^2, -p_yp_z \rangle - \langle 0, 1, 0 \rangle$$

$$\frac{\partial A}{\partial p_y} = \frac{1}{m}\langle 2xp_y - yp_x, -zp_z - xp_x, -yp_z + 2zp_y \rangle$$

$$\frac{\partial H}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

$$\frac{\partial A}{\partial z} = \frac{1}{m}\langle -p_xp_z, -p_yp_z, p_x^2 + p_y^2 \rangle - \langle 0, 0, 1 \rangle$$

$$\frac{\partial A}{\partial p_z} = \frac{1}{m}\langle -zp_x + 2xp_z, 2yp_z - zp_y, -xp_x - yp_y \rangle$$

$$\Rightarrow \{H, A\}_3 = \frac{x}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle -yp_y - zp_z,\ -xp_y + 2yp_x,\ 2zp_x - xp_z \rangle$$

$$- \frac{p_x}{m}\left(\frac{1}{m}\langle p_y^2 + p_z^2,\ -p_xp_y,\ -p_xp_z \rangle - \langle 1, 0, 0 \rangle\right)$$

$$+ \frac{y}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle 2xp_y - yp_x,\ -zp_z - xp_x,\ -yp_z + 2zp_y \rangle$$

$$- \frac{p_y}{m}\left(\frac{1}{m}\langle -p_xp_y,\ p_z^2 + p_x^2,\ -p_yp_z \rangle - \langle 0, 1, 0 \rangle\right)$$

$$+ \frac{z}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle 2xp_z - zp_x,\ 2yp_z - zp_y,\ -xp_x - yp_y)$$

$$- \frac{p_z}{m}\left(\frac{1}{m}\langle -p_xp_z,\ -p_yp_z,\ p_x^2 + p_y^2 \rangle - \langle 0, 0, 1 \rangle\right)$$

$$= \frac{1}{mr^3} \langle x(-yp_y - zp_z) + y(2xp_y - yp_x) + z(2xp_z - zp_x),\ x(-xp_y + 2yp_x) + y(-zp_z - xp_x) + z(2yp_z - zp_y)$$
$$,\ x(2zp_x - xp_z) + y(-yp_z + 2zp_y) + z(-xp_x - yp_y) \rangle$$

$$+ \frac{1}{m}\langle p_x, p_y, p_z \rangle - \frac{1}{m^2}\langle p_x(p_y^2 + p_z^2) + p_y(-p_xp_y) + p_z(-p_xp_z),\ p_x(-p_xp_y) + p_y(p_x^2 + p_z^2) + p_z(-p_yp_z)$$
$$,\ p_x(-p_xp_z) + p_y(-p_yp_z) + p_z(p_x^2 + p_y^2) \rangle$$

$$= \frac{1}{mr^3} \langle -xyp_y - xzp_z + 2xyp_y - y^2p_x + 2xzp_z - z^2p_x,\ -x^2p_y + \cancel{2}yxp_x - \cancel{yzp_z} - \cancel{xyp_x} + \cancel{2}yzp_z - z^2p_y,$$
$$\cancel{2}xzp_x - x^2p_z - y^2p_z - \cancel{2yzp_y} - \cancel{xzp_x} - \cancel{yzp_y} \rangle$$

$$+ \frac{1}{m}\langle p_x, p_y, p_z \rangle - \frac{1}{m^2}\langle \cancel{p_xp_y^2} + \cancel{p_xp_z^2} - \cancel{p_xp_y^2} - \cancel{p_xp_z^2},\ -\cancel{p_x^2p_y} + \cancel{p_x^2p_y} + \cancel{p_yp_z^2} - \cancel{p_yp_z^2},$$
$$,\ -\cancel{p_x^2p_z} - \cancel{p_y^2p_z} + \cancel{p_x^2p_z} + \cancel{p_y^2p_z} \rangle$$

$$= \frac{1}{mr^3} \langle xyp_y + xzp_z - y^2p_x - z^2p_x,\ xyp_x + yzp_z - x^2p_y - z^2p_y,\ xzp_x + yzp_y - x^2p_z - y^2p_z \rangle$$

$$+ \frac{1}{m}\langle p_x, p_y, p_z \rangle$$

c) If $\{H, L\}$ and $\{H, A\}$ are both 0, then both $\vec{L}$ and $\vec{A}$ are constants of the motion.

d) $\vec{r} \cdot \vec{A} = rA\cos\theta$

$$\vec{r} \cdot \left(\frac{1}{m}\vec{p} \times \vec{L} - \hat{r}\right) = \frac{1}{m}\left(\vec{r} \cdot (\vec{p} \times \vec{L})\right) - \vec{r} \cdot \hat{r}$$

$$= \frac{1}{m}\vec{L} \cdot (\vec{r} \times \vec{p}) - r$$

$$= \frac{1}{m}L^2 - r$$

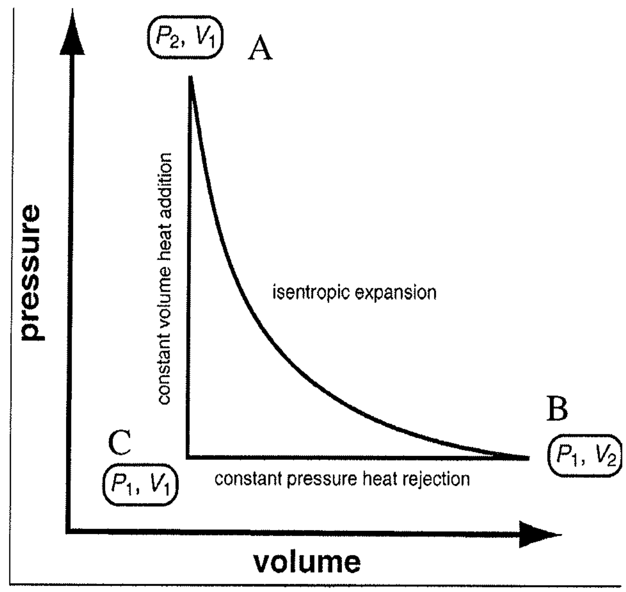
$$\frac{L^2}{m} - r = rA\cos\theta$$

$$\frac{L^2}{m} = r(A\cos\theta + 1)$$

$$\frac{L^2}{m} \cdot \frac{1}{A\cos\theta + 1} = \vec{r}$$

Statistical Mechanics

4. **Heat Engines:** A pulse jet operates under a Lenoir cycle. This consists of an adiabat, an isobar, and an isochore, as shown.



Assuming that the working fluid is an ideal 3D monoatomic gas of N particles:

(a) Find the work done in one complete cycle. (3 points)

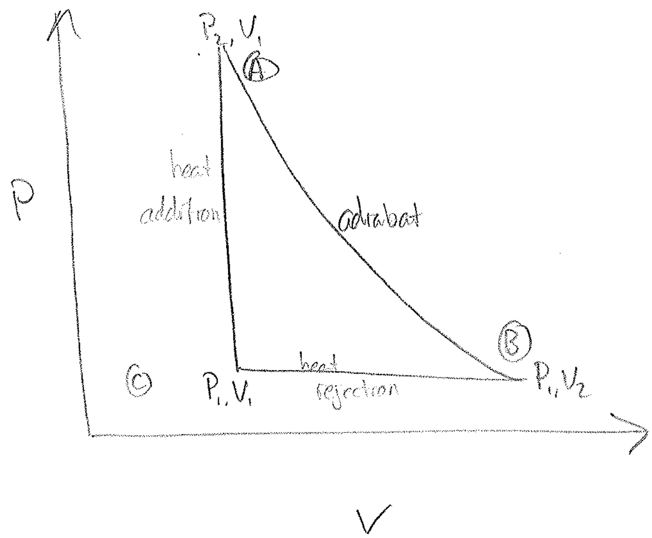
(b) Find the heat exchanged in each step in the cycle. (3 points)

(c) Find the efficiency of the engine. Express your answer in terms of pressures and volutmes. (3 points)

(d) To produce work, should the engine cycle operate clockwise ($A \to B \to C \to A$) or counterclockwise ($A \to C \to B \to A$) ? (1 point)

Jan 2008

Stat Mech #1



* Assume 3-D monatomic gas of N particles

$\Rightarrow PV = Nk_BT$

a) Find work done in one complete cycle

$P_A = P_2$ $\quad P_B = P_1$ $\quad P_c = P_1$

$V_A = V_1$ $\quad V_B = V_2$ $\quad V_c = V_1$

$T_A = \frac{Nk_B}{P_2V_1}$ $\quad T_B = \frac{Nk_B}{P_1V_2} = \frac{Nk_B}{P_2V_1}$ $\quad T_c = \frac{Nk_B}{V_2P_2}$

$$\frac{V_B}{T_B} = \frac{V_c}{T_c}$$

$$\Rightarrow T_c = \frac{V_cT_B}{V_B}$$

$$T_c = \frac{V_1}{V_2}\frac{Nk_B}{P_2V_1} = \frac{Nk_B}{V_2P_2}$$

$W = \int P dV$

$W_{A\to C} = 0, \quad dV = 0$

$W_{C\to B} = P_1(V_2 - V_1)$

$$W_{B\to A} = \frac{P_AV_A - P_BV_B}{1-\gamma} = \frac{P_2V_1 - P_1V_2}{1 - 5/3} = -\frac{3}{2}(P_2V_1 - P_1V_2)$$

$$W_{TOT} = 0 + P_1V_2 - P_1V_1 - \frac{3}{2}(P_2V_1 - P_1V_2)$$
$$= \frac{5}{2}P_1V_2 - P_1V_1 - \frac{3}{2}P_2V_1$$
$$= \frac{5}{2}P_1V_2 - V_1\left(P_1 + \frac{3}{2}P_2\right)$$

b) Find the heat exchanged in each step of the cycle.

$$Q_{A\to C} = nC_v\Delta T$$

$$= \frac{N}{6.02\cdot 10^{23}}\cdot \frac{3}{2}R\cdot\left(\frac{nR}{V_2P_2}-\frac{nR}{P_2V_1}\right)$$

$$= \frac{3nR}{2}\cdot\frac{nR}{P_2}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)$$

$$= \frac{3n^2R^2}{2P_2}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)$$

$$Q_{C\to B} = nC_p\Delta T$$

$$= \frac{5nR}{2}\left(\frac{nR}{P_2V_1}-\frac{nR}{V_2P_2}\right)$$

$$= \frac{5n^2R^2}{2P_2}\left(\frac{1}{V_1}-\frac{1}{V_2}\right)$$

$Q_{B\to A} = 0$ b/c adiabatic

$$Q_{TOT} = \frac{3n^2R^2}{2P_2}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)+\frac{5n^2R^2}{2P_2}\left(\frac{1}{V_1}-\frac{1}{V_2}\right)$$

$$= \frac{n^2R^2}{P_2}\left(\frac{1}{V_1}-\frac{1}{V_2}\right)$$

c) Find the efficiency of the engine

$$\eta = \frac{W_{out}}{Q_{in}}$$

* Note: Q_{in} occurs in $A\to C$

$$\eta = \frac{\frac{5}{2}P_1V_2 - V_1\left(P_1+\frac{3}{2}P_2\right)}{\frac{3n^2R^2}{2P_2}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)}$$

$$= \frac{5P_1V_2-2V_1P_1-3V_1P_2}{\frac{3n^2R^2}{P_2}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)}$$

$$= \frac{5P_1P_2V_2-2P_1P_2V_1-3P_2^2V_1}{3n^2R^2\left(\frac{1}{V_2}-\frac{1}{V_1}\right)}$$

d) To produce work, does engine operate clockwise or counter-clockwise?

CW

5. Consider a classical ideal gas in 3D that feels a linear gravitational potential,

$$V(z) = mgz$$

where m is the mass of a single gas atom and $0 < z < \infty$. This is not an interaction between gas atoms, it is simply their gravitational potential energy near the surface of the Earth.

The gas is in a box of dimensions L_x, L_y, and L_z, so that:

$$\begin{aligned} 0 < & \; z \; < L_z \\ 0 < & \; x \; < L_x \\ 0 < & \; y \; < L_y \end{aligned}$$

(a) Calculate the partition function in the canonical ensemble. (3 points)

(b) Determine the internal energy of the gas. (3 points)

(c) Calculate the specific heat c_v. (3 points)

(d) Explain the behavior of the specific heat when $\beta mgL_z >> 1$ and when $\beta mgL_z << 1$. (The approximation for the gravitational potential may or may not be valid for large L_z. Don't worry about that.) (1 point)

Stat Mech #2

*Consider a classical ideal gas in 3-D w/ linear gravitational potential $V(z) = mgz$. Note: m is mass of single atom, $0 < z < \infty$. Dimensions of box are: $0 < z < L_z$, $0 < x < L_x$, $0 < y < L_y$

a) Calculate the partition function in the classical ensemble

$$Z = \left[\frac{1}{h^3}\int dp^3 dq^3 e^{-\beta E}\right]^N$$

$$* \text{ let } E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + mgz$$

$$= \left[\frac{1}{h^3}\int dp^3 dq^3 e^{-\beta\left(\frac{1}{2m}[p_x^2+p_y^2+p_z^2]+mgz\right)}\right]^N$$

$$= \left[\frac{1}{h^3}\int dp_x dp_y dp_z\, dx dy dz\, e^{-\beta\left[\frac{1}{2m}[p_x^2+p_y^2+p_z^2]+mgz\right]}\right]^N$$

$$= \left[\frac{1}{h^3}\left(\int_0^\infty dp\, e^{-\beta p^2/2m}\right)^3 \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz\, e^{-\beta mgz}\right]^N$$

$$= \left[\frac{1}{h^3}\left(\frac{1}{2}\sqrt{\frac{\pi}{\beta/2m}}\right)^3 L_x L_y \left(\frac{1}{-\beta mg} e^{-\beta mgz}\Big|_0^{L_z}\right)\right]^N$$

$$= \left[\left(\frac{1}{\cancel{2}h}\sqrt{\frac{2\pi m}{\beta}}\right)^3 L_x L_y \frac{1}{-\beta mg}\left(e^{-\beta mgL_z} - 1\right)\right]^N$$

b) Determine the internal energy of the gas

$$U = -\frac{\partial}{\partial\beta}\ln(Z)$$

$$= \frac{-\partial}{\partial\beta} N \ln\left[\left(\frac{2\pi m}{h^2\beta}\right)^{3/2} L_x L_y \frac{1}{-\beta mg}\left(e^{-\beta mgL_z} - 1\right)\right]$$

$$= \frac{-\partial}{\partial\beta} N \ln\left[\left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{L_x L_y}{mg} \frac{1}{-\beta^{5/2}}\left(e^{-\beta mgL_z} - 1\right)\right]$$

$$= \frac{-\partial}{\partial\beta} N\left(\ln\left[\left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{L_x L_y}{mg}\right] + \ln\left[1 - e^{-\beta mgL_z}\right] - \ln\left(\beta^{5/2}\right)\right)$$

$$= -N\left[\left(1 - e^{-\beta mgL_z}\right)^{-1} + mgL_z e^{-\beta mgL_z} - \frac{5}{2\beta}\right]$$

c) Calculate the specific heat C_V

$$C_V = \frac{\partial U}{\partial T}$$

$$= \frac{\partial}{\partial T}\left(-N\left(\frac{mgL_z e^{-\beta mgL_z}}{1-e^{-\beta mgL_z}} - \frac{5}{2\beta}\right)\right)$$

$$= \frac{\partial}{\partial T}\left(\frac{5}{2}Nk_BT - \frac{mgL_z}{e^{-mgL_z/k_BT}-1}\right)$$

$$= \frac{5}{2}Nk_B + mgL_z\left(e^{-mgL_z/k_BT}-1\right)^{-2} \cdot -mgL_z/k_BT^2\, e^{-mgL_z/k_BT}$$

$$= \frac{5}{2}Nk_B - \frac{(mgL_z)^2 e^{-mgL_z/k_BT}}{T^2k_B\left(e^{-mgL_z/k_BT}-1\right)^2}$$

6. **Boson Magnetism** Consider a gas of non-interacting spin-1 bosons in 3D, each subject to the Hamiltonian

$$\begin{aligned} H(\vec{p}, s_z) &= \frac{p^2}{2m} - \mu_0 s B \\ &= \frac{\hbar^2 k^2}{2m} - \mu_0 s B \end{aligned}$$

where s takes on one of three possible states, $s \in (-1, 0, +1)$, and $\vec{k} \equiv \vec{p}/\hbar$. In this Hamiltonian B is the z-component of the magnetic field, m is the mass of a particle, and μ_0 is the Bohr magneton. (We will ignore the orbital effect (or Lorentz force) where the momentum $\vec{p}$ would have been replaced, $\vec{p} \to \vec{p} + e\vec{A}/c$).

(a) In a grand canonical ensemble of chemical potential μ (which is **not** to be confused with the Bohr magneton, μ_0, above) and temperature T, write down $n_s(\vec{k})$, the average occupation number of the state with wave vector $\vec{k}$ and spin s. (1 point).

(b) Show that the total number of particles in a given spin state s is given by

$$N_s = \frac{V}{\lambda^3} \pi^{3/2}\, g_{3/2}(z e^{\beta \mu_0 s B})$$

where z is the fugacity, $z = e^{\beta\mu}$, λ is the thermal de Broglie wavelength,

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

and $g_p(z)$ is defined on the formula section on page 2 above. (4 points)

(c) The magnetization for fixed μ and T is given by

$$M(T, \mu) = \mu_0 (N_{(+)} - N_{(-)})$$

Show that the zero field susceptibility, χ, is given by:

$$\chi \equiv \left. \frac{\partial M}{\partial B} \right|_{B=0} = \frac{2\mu_0^2}{k_B T} \pi^{3/2} \frac{V}{\lambda^3}\, g_{1/2}(z).$$

(5 points).

Classical Mechanics and Statistical/Thermodynamics

August 2008

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \equiv \zeta(p)$$

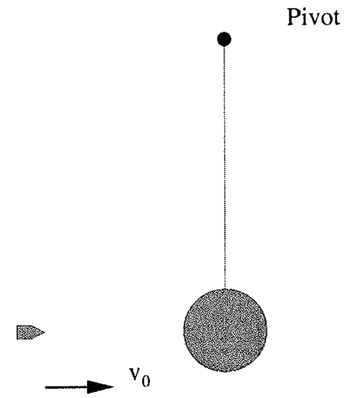
$$\sum_{n=1}^{\infty} \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^{\infty} (-1)^p\ \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\begin{array}{cc}
\zeta(1) = \infty & \zeta(-1) = 0.0833333 \\
\zeta(2) = 1.64493 & \zeta(-2) = 0 \\
\zeta(3) = 1.20206 & \zeta(-3) = 0.0083333 \\
\zeta(4) = 1.08232 & \zeta(-4) = 0
\end{array}$$

Classical Mechanics

1. **The ballistic pendulum:** Consider a pendulum with a bob of mass m connected to a frictionless pivot by an ideal massless rigid rod of length ℓ. A projectile of mass ϵm $(0 < \epsilon << 1)$ moving horizontally at speed v_0 hits the center of the bob, as shown. When it strikes, it becomes imbedded in the bob.



   (a) What is the minimum initial speed of the projectile such that the pendulum will make a full rotation? (2 points)

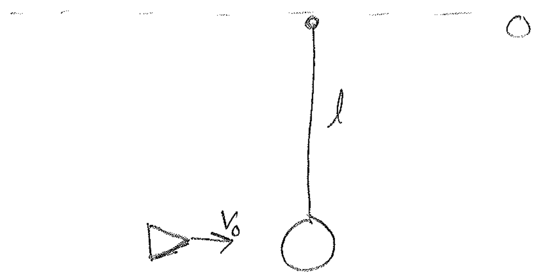
   (b) The rod is replaced by an ideal massless non-rigid string. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (3 points)

   (c) Now assume that projectile rebounds elastically from the bob in the horizontal direction. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (2 points)

   (d) Finally, assume that the projectile passes completely through the pendulum bob, in a time $t << \sqrt{\ell/g}$. After it exits, it carries with it some of the original mass of the bob, such that the exiting projectile now has a mass $2\epsilon m$ and moves at a speed $3v_0/4$. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (3 points)

Aug 2008

Classical #1



$m_b = \epsilon m$

$m_p = m$

a) * Collision (Inelastic)

$\cancel{\frac{1}{2} m\epsilon v_0^2 = \frac{1}{2} m(1+\epsilon)v^2}$

$\cancel{\frac{\epsilon}{1+\epsilon} v_0^2 = v^2}$

$\epsilon m v_0 = (1+\epsilon)mv$

$\frac{\epsilon}{1+\epsilon} v_0 = v$

* Conservation of energy

$\frac{1}{2} m(1+\epsilon)\left(\frac{\epsilon}{1+\epsilon} v_0\right)^2 - mgl = mgl + \frac{1}{2} m\cancel{(1+\epsilon)}v^2$

$\frac{1}{2} m \frac{\epsilon^2}{1+\epsilon} v_0^2 = 2mgl + \frac{1}{2} m\cancel{(1+\epsilon)}v^2$

$\frac{\epsilon^2}{1+\epsilon} v_0^2 = 4mgl$

2. The isotropic harmonic oscillator.

   (a) Write the Lagrangian for a point mass m moving under the influence of an isotropic 3-dimensional harmonic oscillator potential

$$V(x, y, z) = \frac{k}{2}(x^2 + y^2 + z^2).$$

   There is no external gravitational field. (1 point)

   (b) Using the Lagrange equations of motion show that angular momentum is conserved. i.e.,

$$\frac{d}{dt}\mathbf{L} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = 0.$$

   Because the Lagrangian is invariant under rotations about the origin, you can choose coordinates so that motion is constrained to the x-y plane, i.e., the angular momentum points in the z direction. (3 points)

   (c) For 2-dimensional motion in the x-y plane choose cylindrical polar coordinates and proceed to solve the Lagrange equations of motion. You can leave the solution for $r(t)$ as an integral of the form $t = \int f(r)dr$. (Don't forget to use conservation of energy, E_0.) (3 points)

   (d) Compute the minimum and maximum values or the radial coordinate r as functions of the constants m, E_0, k, L^z. (3 points)

Aug 2008

Classical #2

a) Write Lagrangian for a point mass under the influence of the following potential:

$V(x,y,z) = \frac{k}{2}(x^2+y^2+z^2)$ * No external gravity

$$\mathcal{L} = T - V$$
$$= \frac{1}{2}mv^2 - \frac{1}{2}k(x^2+y^2+z^2)$$
$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{1}{2}k(x^2+y^2+z^2)$$

b) Use Lagrange equations of motion to show angular momentum conservation

i.e. $\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r}\times m\vec{v}) = 0$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{y}} \qquad \frac{\partial \mathcal{L}}{\partial z} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$

$$\Rightarrow -kx = \frac{\partial}{\partial t}(m\dot{x}) \qquad \Rightarrow -ky = \frac{\partial}{\partial t}(m\dot{y}) \qquad \Rightarrow -kz = \frac{\partial}{\partial t}(m\dot{z})$$

$$-kx = m\ddot{x} \qquad -ky = m\ddot{y} \qquad -kz = m\ddot{z}$$

$$\vec{r}\times m\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix} = \langle ym\dot{z} - zm\dot{y},\ xm\dot{z} - zm\dot{x},\ xm\dot{y} - ym\dot{x} \rangle$$

$$= m\langle y\dot{z} - z\dot{y}, x\dot{z} - z\dot{x}, x\dot{y} - y\dot{x} \rangle$$

$$\frac{d}{dt}(\vec{r}\times m\vec{v}) = m\langle \dot{y}\dot{z} + y\ddot{z} - (\dot{z}\dot{y} + z\ddot{y}), \dot{x}\dot{z} + x\ddot{z} - (\dot{z}\dot{x} + z\ddot{x}), \dot{x}\dot{y} + x\ddot{y} - (\dot{y}\dot{x} + y\ddot{x}) \rangle$$

$$= m\langle y\ddot{z} - z\ddot{y}, x\ddot{z} - z\ddot{x}, x\ddot{y} - y\ddot{x} \rangle$$

* substituting from above

$$= m\left(\frac{-k}{m}\right)\langle yz - zy, xz - zx, xy - yx \rangle$$

$$= -k\langle 0, 0, 0 \rangle$$

$$= 0$$

$$\therefore \frac{d}{dt}(\vec{r}\times m\vec{v}) = \frac{d}{dt}\vec{L} = 0 \checkmark$$

c) For 2-D motion in x-y plane, use cylindrical polar coordinates to solve Lagrange eqns of motion. Leave $r(t)$ as an integral of form $t = \int f(r)dr$

* Hint: use conservation of energy E_0

- In cylindrical polar: $x = r\cos\varphi$ $\quad \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\dot{\varphi}$

$y = r\sin\varphi$ $\quad \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\dot{\varphi}$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r\dot{\varphi}^2) - \tfrac{1}{2}kr^2 \quad (2\text{-D},\ z=0)$$

$$E_0 = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \tfrac{1}{2}kr^2$$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = 0$$

$$0 = kr + mr\dot{\varphi}^2 - m\ddot{r} \qquad 0 = -2mr\dot{r}\dot{\varphi} - mr^2\ddot{\varphi}$$

* Solving E_0 for $\dot{\varphi}$ yields

$$E_0 = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\varphi}^2 + \tfrac{1}{2}kr^2$$

$$\frac{2E_0}{m} = \dot{r}^2 + r^2\dot{\varphi}^2 + \frac{k}{m}r^2$$

$$\frac{2E_0}{m} - \frac{k}{m}r^2 - \dot{r}^2 = r^2\dot{\varphi}^2$$

$$\frac{2E_0}{mr^2} - \frac{k}{m} - \frac{\dot{r}^2}{r^2} = \dot{\varphi}^2$$

$$\Rightarrow 0 = kr + mr\left(\frac{2E_0}{mr^2} - \frac{k}{m} - \frac{\dot{r}^2}{r^2}\right) + \tfrac{1}{2}kr^2$$

$$= \cancel{kr} + \frac{2E_0}{r} - \cancel{kr} - \frac{m\dot{r}^2}{r} + \tfrac{1}{2}kr^2$$

$$0 = \frac{2E_0}{r} - \frac{m\dot{r}^2}{r} + \tfrac{1}{2}kr^2$$

$$\frac{m\dot{r}^2}{r} = \frac{2E_0}{r} + \tfrac{1}{2}kr^2$$

$$\dot{r}^2 = 2mE_0 + \tfrac{1}{2}\tfrac{k}{m}r^3$$

$$\Rightarrow t = \int\left(2mE_0 + \tfrac{1}{2}\tfrac{k}{m}r^3\right)^{1/2}dr$$

3. Consider a particle attracted by a fixed gravitating body while also in a uniform gravitational field oriented along the z-axis. The potential energy is of the form:

$$V(r,z) = -m\left(\frac{k}{r} + g\,z\right)$$

where m is the particle's mass, k and g are constants, and r is the standard radial coordinate:

$$r \equiv \sqrt{x^2 + y^2 + z^2}$$

You are to examine the problem in *cylindrical parabolic coordinates* defined by

$$\begin{aligned}\zeta &\equiv r + z \\ \eta &\equiv r - z \\ \phi &\equiv \arctan y/x\end{aligned}$$

In these coordinates we may write the Cartesian coordinates as:

$$\begin{aligned}x &= \sqrt{\zeta\eta}\cos\phi \\ y &= \sqrt{\zeta\eta}\sin\phi \\ z &= \tfrac{1}{2}(\zeta - \eta)\end{aligned}$$

(a) Show that the kinetic energy, T, is given by:

$$T = \frac{m}{8}\left[\left(1+\frac{\zeta}{\eta}\right)\dot{\eta}^2 + \left(1+\frac{\eta}{\zeta}\right)\dot{\zeta}^2\right] + \frac{m}{2}\,\zeta\eta\,\dot{\phi}^2$$

in these coordinates. (2 points)

(b) What are the canonical momenta, p_ζ, p_η, and p_ϕ, expressed in cylindrical parabolic coordinates? (2 points)

(c) Use Hamilton-Jacobi theory to find the constants of the motion. *Hint:* While the total energy E does not separate in these coordinates, $E(\zeta + \eta)$ can be used to produce a quantity that **does** seperate. (3 points)

(d) What is Hamilton's characteristic function associated with ϕ? (1 point)

(e) Express Hamilton's characteristic functions associated with ζ, η as definite integrals. (2 points)

Aug 2002

Classical #3

For a particle attracted by a fixed gravitating body while also in a uniform gravitational field oriented along the z-axis $\left[V(r,z) = -m\left(\frac{k}{r}+gz\right)\right]$ where $r=\sqrt{x^2+y^2+z^2}$ and using cylindrical parabolic coordinates:

$$\xi = r+z \qquad x=\sqrt{\xi\eta}\cos\varphi$$
$$\eta = r-z \quad \Rightarrow \quad y=\sqrt{\xi\eta}\sin\varphi$$
$$\varphi = \arctan(y/x) \qquad z=\tfrac{1}{2}(\xi-\eta)$$

a) Show that the kinetic energy, T, is: $T=\frac{m}{8}\left[\left(1+\frac{\xi}{\eta}\right)\dot{\eta}^2+\left(1+\frac{\eta}{\xi}\right)\dot{\xi}^2\right]+\frac{m}{2}\xi\eta\dot{\varphi}^2$

$$T=\tfrac{1}{2}mv^2$$
$$=\tfrac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

* from above,

$$\dot{x} = \tfrac{1}{2}\xi^{-1/2}\eta^{1/2}\cos\varphi\dot{\xi}+\tfrac{1}{2}\xi^{1/2}\eta^{-1/2}\cos\varphi\dot{\eta}-\xi^{1/2}\eta^{1/2}\sin\varphi\dot{\varphi}$$
$$\dot{x}^2=\tfrac{1}{4}\xi^{-1}\eta\cos^2\varphi\dot{\xi}^2+\tfrac{1}{4}\xi\eta^{-1}\cos^2\varphi\dot{\eta}^2+\xi\eta\sin^2\varphi\dot{\varphi}^2+\tfrac{1}{2}\cos^2\varphi\dot{\xi}\dot{\eta}-\eta\cos\varphi\sin\varphi\dot{\xi}\dot{\varphi}$$
$$-\xi\cos\varphi\sin\varphi\dot{\eta}\dot{\varphi}$$
$$\dot{y}=\tfrac{1}{2}\xi^{-1/2}\eta^{1/2}\sin\varphi\dot{\xi}+\tfrac{1}{2}\xi^{1/2}\eta^{-1/2}\sin\varphi\dot{\eta}+\xi^{1/2}\eta^{1/2}\cos\varphi\dot{\varphi}$$
$$\dot{y}^2=\tfrac{1}{4}\xi^{-1}\eta\sin^2\varphi\dot{\xi}^2+\tfrac{1}{4}\xi\eta^{-1}\sin^2\varphi\dot{\eta}^2+\xi\eta\cos^2\varphi\dot{\varphi}^2+\tfrac{1}{2}\sin^2\varphi\dot{\xi}\dot{\eta}+\eta\sin\varphi\cos\varphi\dot{\xi}\dot{\varphi}$$
$$+\xi\sin\varphi\cos\varphi\dot{\eta}\dot{\varphi}$$
$$\dot{z}=\tfrac{1}{2}(\dot{\xi}-\dot{\eta})$$
$$\dot{z}^2=\tfrac{1}{4}(\dot{\xi}^2-2\dot{\xi}\dot{\eta}+\dot{\eta}^2)$$

$$\Rightarrow \dot{x}^2+\dot{y}^2+\dot{z}^2 = \tfrac{1}{4}\xi^{-1}\eta\dot{\xi}^2+\tfrac{1}{4}\xi\eta^{-1}\dot{\eta}^2-\xi\eta\sin^2\varphi\dot{\varphi}^2+\xi\eta\cos^2\varphi\dot{\varphi}^2+\tfrac{1}{2}\dot{\xi}\dot{\eta}+\tfrac{1}{4}\dot{\xi}^2-\tfrac{1}{2}\dot{\xi}\dot{\eta}+\tfrac{1}{4}\dot{\eta}^2$$
$$=\tfrac{1}{4}\tfrac{\eta}{\xi}\dot{\xi}^2+\tfrac{1}{4}\tfrac{\xi}{\eta}\dot{\eta}^2-\xi\eta\sin^2\varphi\dot{\varphi}^2+\xi\eta\cos^2\varphi\dot{\varphi}^2+\tfrac{1}{4}\dot{\xi}^2+\tfrac{1}{4}\dot{\eta}^2$$
$$=\tfrac{1}{4}\left[\dot{\xi}^2\left(\tfrac{\eta}{\xi}+1\right)+\dot{\eta}^2\left(\tfrac{\xi}{\eta}+1\right)\right]+\xi\eta\dot{\varphi}^2\left(\cancel{\cos^2\varphi+\sin^2\varphi}\right)$$

$$\Rightarrow T=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$
$$=\tfrac{m}{8}\left[\left(1+\tfrac{\xi}{\eta}\right)\dot{\eta}^2+\left(1+\tfrac{\eta}{\xi}\right)\dot{\xi}^2\right]+\tfrac{m}{2}\xi\eta\dot{\varphi}^2$$

b) Find the canonical momenta in cylindrical parabolic coordinates

$$P_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$\Rightarrow \mathcal{L} = T - V$$

$$V = -m\left(\frac{k}{r} + gz\right), \quad r = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow r = \sqrt{\xi\eta\cos^2\varphi + \xi\eta\sin^2\varphi + \frac{1}{4}(\xi-\eta)^2}$$

$$= \sqrt{\xi\eta + \frac{1}{4}\xi^2 - \frac{1}{2}\xi\eta + \frac{1}{4}\eta^2}$$

$$= \sqrt{\frac{1}{4}\xi^2 + \frac{1}{2}\xi\eta + \frac{1}{4}\eta^2}$$

$$= \sqrt{\frac{1}{4}(\xi^2 + 2\xi\eta + \eta^2)}$$

$$= \frac{1}{2}(\xi+\eta)$$

$$\Rightarrow V = -m\left(\frac{2k}{\xi+\eta} + g\left(\frac{\xi+\eta}{2}\right)\right)$$

$$\Rightarrow \mathcal{L} = T - V$$

$$= \frac{m}{8}\left[\left(1+\frac{\xi}{\eta}\right)\dot{\eta}^2 + \left(1+\frac{\eta}{\xi}\right)\dot{\xi}^2\right] + \frac{m}{2}\xi\eta\dot{\varphi}^2 + m\left[\frac{2k}{\xi+\eta} + \frac{g(\xi+\eta)}{2}\right]$$

$$P_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

$$= m\xi\eta\dot{\varphi}$$

$$P_\eta = \frac{\partial \mathcal{L}}{\partial \dot{\eta}}$$

$$= \frac{m}{4}\left(1+\frac{\xi}{\eta}\right)\dot{\eta}$$

$$P_\xi = \frac{\partial \mathcal{L}}{\partial \dot{\xi}}$$

$$= \frac{m}{4}\left(1+\frac{\eta}{\xi}\right)\dot{\xi}$$

c) $H = \sum_i p_i \dot{q}_i - \mathcal{L}$

$$H = p_\varphi \dot{\varphi} + p_\xi \dot{\xi} + p_\eta \dot{\eta} - \mathcal{L}$$

$$= \frac{m}{2} \xi \eta \dot{\varphi}^2 + \frac{m}{8}\left(1 + \frac{\eta}{\xi}\right)\dot{\xi}^2 + \frac{m}{8}\left(1 + \frac{\xi}{\eta}\right)\dot{\eta}^2 - m\left[\frac{2k}{\xi + \eta} + \frac{g(\xi + \eta)}{2}\right]$$

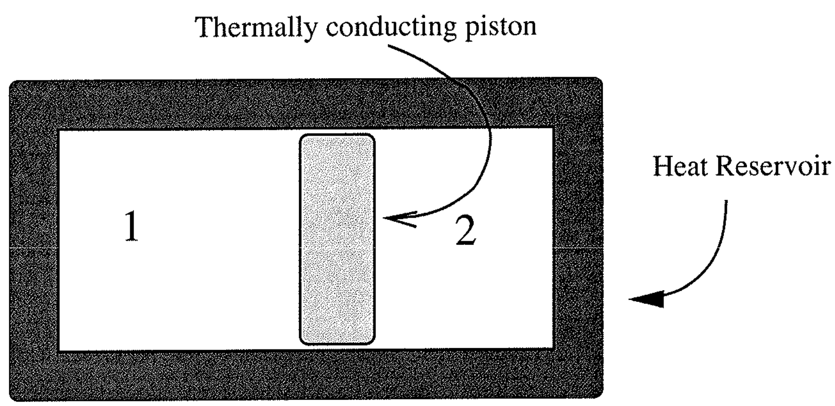
Statistical Mechanics

4. **Helmholtz Free Energy**: The Helmholtz free energy of an ideal monoatomic gas can be written as

$$F(T, V, N) = NkT\left\{A - \log\left[T^{3/2}\frac{V}{N}\right]\right\}$$

where N is the total number of gas atoms, V is the volume, T is temperature, k is Boltzmann's constant and A is a dimensionless constant.

Consider a piston separating a system into two parts, with equal numbers of particles on the left and the right hand side. The whole system is in good thermal contact with a reservoir at constant temperature T. Initially, $V_1 = 2V_2$. The total volume, $V_{\text{tot}} = V_1 + V_2$, is fixed for this whole problem.



(a) Calculate the equilibrium position of the piston, once it is released. You must prove your answer, and not simply assert it. (3 points)

(b) Calculate the maximum available work the system can perform as it changes from the initial condition to the equilibrium position. (3 points)

(c) Calculate the change in the internal energy, U of gas 1 and gas 2 in the process. (2 points)

(d) Given your answers above, explain the source of energy for the work done during the expansion. (2 points)

5. Consider a gas of N non-interacting **one dimensional** diatomic molecules enclosed in a box of "volume" L (actually, just a length) at temperature T.

   (a) The classical energy for a single molecule is:

   $$E(p_1, p_2, x_1, x_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}K(x_1 - x_2)^2$$

   where p_1 and p_2 are the classical momenta of the atoms in one diatomic molecule, x_1 and x_2 are their classical positions, and K is the spring constant. Calculate the specific heat for the gas. (You should assume that $KL^2/2 >> k_B T$, where k_B is Boltzmann's constant.) (4 points).

   (b) In the quantum limit the energy levels of the molecule are discrete. In a semiclassical approach we can write the energy of one molecule as:

   $$E(P, n) = \frac{P^2}{4m} + \hbar\omega(n + \frac{1}{2})$$

   where P is the momentum of the diatomic molecule (of mass $2m$), and ω is the natural frequency of the oscillator, and n is a non-negative integer ($n \geq 0$). Calculate the specific heat. (4 points).

   (c) Calculate the high and low temperature limits of your result in (b), and explain how they relate to the result of (a). (2 points)

6. Fermions:

(a) Show that for any non-interacting spin 1/2 fermionic system with chemical potential μ, the probability of occupying a single particle state with energy $\mu + \delta$ is the same as finding a state vacant at an energy $\mu - \delta$. (2 points)

(b) Consider non-interacting fermions that come in two types of energy states:

$$E_{\pm}(\vec{k}) = \pm\sqrt{m^2c^4 + \hbar^2k^2c^2}$$

At zero temperature all the states with negative energy (all states with energy $E_-(\vec{k})$) are occupied[1] and all positive energy states are empty, and that $\mu(T = 0) = 0$. Show that the result of part (a) above means that the chemical potential must remain at zero for all temperatures if particle number is to be conserved. (2 points)

(c) Using the results of (a) and (b) above, show that the average excitation energy, the change in the energy of the system from it's energy at $T = 0$ in three dimensions is given by:

$$\Delta E \equiv E(T) - E(0) = 4V \int \frac{d\vec{k}}{(2\pi)^3} E_+(\vec{k}) \frac{1}{1 + e^{\beta E_+(\vec{k})}}$$

(2 points)

(d) Evaluate the integral above for massless ($m = 0$) particles. (2 points)

(e) Calculate the heat capacity of such particles. (2 points)

[1]Technically this means the total energy of the system diverges. If this bothers you, you can assume some large cut-off to the wavevectors, $\hbar k_{\max} c >> kT$, which will have no effect on your final answers.

Classical Mechanics and Statistical/Thermodynamics

January 2009

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\, dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

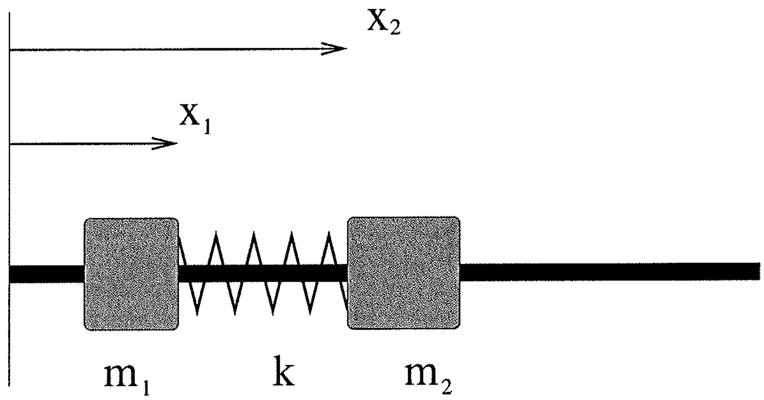
$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\ \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\begin{array}{cc} \zeta(1) = \infty & \zeta(-1) = 0.0833333 \\ \zeta(2) = 1.64493 & \zeta(-2) = 0 \\ \zeta(3) = 1.20206 & \zeta(-3) = 0.0083333 \\ \zeta(4) = 1.08232 & \zeta(-4) = 0 \end{array}$$

Classical Mechanics

1. Two masses, m_1 and m_2, are connected together by an ideal massless spring of spring constant k and equilibrium length l, but are otherwise free to slide on a straight frictionless rail. Their positions with respect to a fixed origin are denoted x_1 and x_2 respectively.



(a) Determine the equation of motion for each mass using Newton's Laws of Motion. (Do not solve them yet.) [2 pts.]

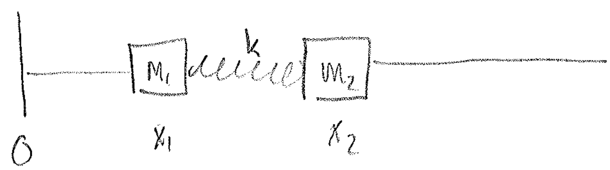
(b) Write the Lagrangian for this system and use it to derive the equation of motion for each mass. (Again, do not solve them yet.) [3 pts.]

(c) Using either of your results, determine the frequency of oscillation of the two masses about their center of mass. [2 pts.]

(d) Given the initial conditions, $x_1(0) = 0$, $v_1(0) = 0$, $x_2(0) = l$ and $v_2(0) = v_0$, solve for the subsequent motion. [3 pts.]

Jan 2009

Classical #1



* Rail is frictionless

a) Determine equations of motion using Newtons Laws

$$F = k\Delta x = ma$$

$$\Rightarrow m_1 \ddot{x}_1 = k($$

2. A particle of mass m moves under the influence of a central force whose potential is given by $V(r) = K\,r^3$, where $K > 0$.

   (a) For what energy and angular momentum will the orbit be a circle of radius R_0 about the origin? (3 pts.)

   (b) What is the period of this circular orbit? (2 pts.)

   (c) If the particle is slightly displaced from the circular orbit, what will be the period for small oscillations about $r = R_0$? (3 pts.)

   (d) For the $1/r$ gravitational potential we know that Kepler's Second Law: "A line joining a planet and the sun sweeps out equal areas during equal intervals of time." Does this hold true for the cubic potential as well? Prove your answer. (2 pts.)

3. You are in a rocket ship in outer space, initially at rest. You have a nuclear reactor that supplies a constant power, P, and a large supply of iron pellets. The iron pellets comprise 99/100 of your ship's mass, m. You can use the power to eject the tiny iron beads out the back of your ship with an electromagnetic "gun". You can control the *rate* at which you fire them and their velocity, but you are limited by your power plant. (You can't fire an arbitrarily large mass at an arbitrarily large velocity.) As you fire off the beads, your ship moves in the opposite direction to conserve momentum. In addition, the mass of your ship decreases.

   (a) Calculate your final speed as a functional of $m(t)$ and $\dot{m}(t) \equiv dm/dt$. Your expression should take the form of an integral over time, $0 < t < t_f$. (2 pts.)

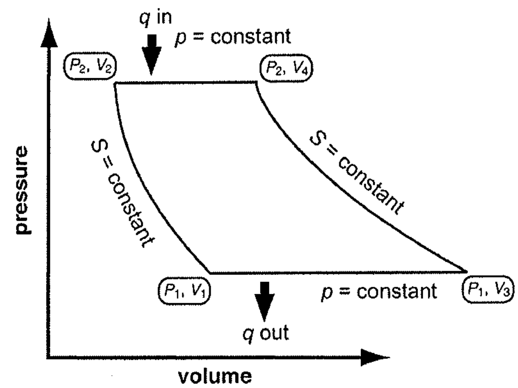
   (b) Find the function $m(t)$ that maximizes your final velocity after a time t_f. (4 pts.)

   (c) What is your final velocity? (2 pts.)

   (d) Prove that your answer in part (c) is larger than the velocity you would obtain by firing at a constant rate such that your pellets are used up by t_f. (2 pts.)

Statistical Mechanics

4. The gas turbine (jet engine) can be modeled as a Brayton cycle. Below is the P-V diagram for this process.



Assume that the working fluid is an ideal monatomic gas.

(a) Calculate the work done by the gas on each step in the cycle. (3 pts.)

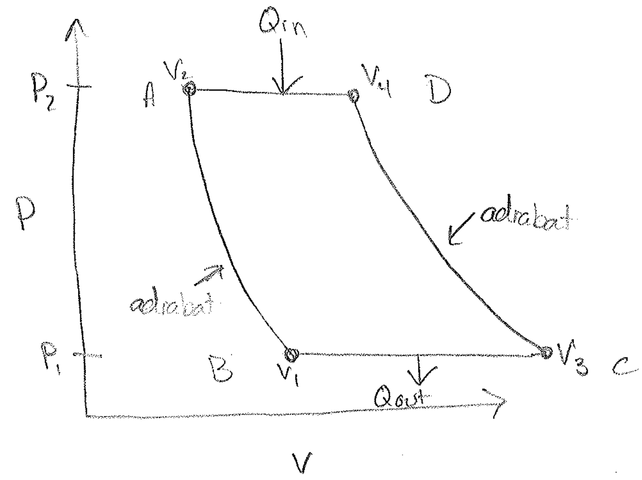
(b) Find the heat for each step in the cycle. (3 pts.)

(c) Find the efficiency of this engine. Your answer should be in terms of the pressures (P_1 and P_2) and the volumes (V_1, V_2, V_3, and V_4). (3 pts.)

(d) To produce work, which way does the cycle operate? Clockwise or counter clockwise? (1 pt.)

Jan 2009

Stat Mech # 1



* Ideal monatomic gas

$$\Rightarrow \begin{matrix} C_V = \frac{3}{2} R \\ C_P = \frac{5}{2} R \end{matrix} \Big\rangle \ \gamma = \frac{5}{3}$$

$P_A = P_2$ $\quad$ $P_B = P_1$ $\quad$ $P_C = P_1$ $\quad$ $P_D = P_2$

$V_A = V_2$ $\quad$ $V_B = V_1$ $\quad$ $V_C = V_3$ $\quad$ $V_D = V_4$

$T_A = \frac{nR}{P_2 V_2}$ $\quad$ $T_B = \frac{nR}{P_1 V_1}$ $\quad$ $T_C = \frac{nR}{P_1 V_3}$ $\quad$ $T_D = \frac{nR}{P_2 V_4}$

a) Calculate the work done in each step.

$$W_{A\to D} = P\Delta V$$
$$= P(V_4 - V_2)$$

$$W_{D\to C} = \frac{P_C V_C - P_D V_D}{1-\gamma}$$
$$= \frac{P_1 V_3 - P_2 V_4}{1 - 5/3}$$
$$= -\frac{3}{2}(P_1 V_3 - P_2 V_4)$$

$$W_{C\to B} = P\Delta V$$
$$= P_1 (V_1 - V_3)$$

$$W_{B\to A} = \frac{P_A V_A - P_B V_B}{1-\gamma}$$
$$= \frac{P_2 V_2 - P_1 V_1}{1 - 5/3}$$
$$= -\frac{3}{2}(P_2 V_2 - P_1 V_1)$$

b) Find the heat for each step

$$Q_{A\to D} = n C_p \Delta T$$
$$= n \tfrac{5}{2} R \left(\frac{P_2 V_4}{nR} - \frac{P_2 V_2}{nR} \right)$$
$$= \tfrac{5}{2} P_2 (V_4 - V_2)$$

$Q_{D\to C} = 0$ b/c adiabatic

$$Q_{C\to B} = n C_p \Delta T$$
$$= n \tfrac{5}{2} R \left(\frac{P_1 V_3}{nR} - \frac{P_1 V_1}{nR} \right)$$
$$= \tfrac{5}{2} P_1 (V_3 - V_1)$$

$Q_{B\to A} = 0$ b/c adiabatic

c) Find the efficiency of the engine

$$\eta = 1 - \frac{Q_{in}}{Q_{out}}$$

$$= 1 - \left(\frac{P_2 (V_4 - V_2)}{P_1 (V_3 - V_1)}\right)^{-1}$$

d) Which way does the cycle operate?

CW

5. By shining an intense laser beam on a semiconductor, one can create a metastable collection of electrons (charge $-e$ and effective mass m_e) and holes (charge $+e$ and effective mass m_h). These oppositely charged particles may pair up to form an *exciton*, or they may dissociate into a plasma. This problem considers a simple model of this process. In this problem the densities of electrons and holes are so low that you can ignore their fermionic nature and treat them as classical particles in three dimensions.

   (a) Calculate the free energy $F(T, V, N)$ of a gas of N_e electrons and N_h holes at temperature T, treating them as classical, non-interacting, ideal gas particles in a 3D volume V. (2 pts.)

   (b) By pairing into an exciton, each electron-hole pair lowers its energy by ΔE. Calculate the free energy of a gas of N_p excitons, treating them as classical, non-interacting, ideal gas particles. (2 pts.)

   (c) Calculate the chemical potentials μ_e, μ_h, and μ_p of the electrons, holes, and exciton pairs respectively. What is the condition of equilibrium between excitons and eletrons and holes? (3 pts.)

   (d) Consider the case where the numbers of electrons and holes are equal, so that $n_h = n_e \equiv n_0$. Determine the approximate density of excitons as a function of n_0 in the high temperature limit (when the exciton population is low). (3 pts.)

6. Consider a free, non-interacting spin zero Bose gas in two dimensions. The energy of each particle is given by:

$$\mathcal{E}(\vec{k}) = \hbar^2 k^2/2m$$

where m is the mass of the boson. Assume your system is confined to a square region of length L on a side.

(a) Write down an expression for the grand canonical free energy $\mathcal{G}(T, V, \mu)$ as a sum over $\vec{k}$ states. Do not evaluate the sum. (1 pt.)

(b) Calculate the number of particles in the system as a function of T, V and μ. (3 pts.)

(c) Analyze your expression for $N(T, V, \mu)$ in the limit $T \to 0$. What does it imply about the possibility of a Bose-Einstein transition in this system? (3 pts.)

(d) Prove that the pressure is equal to the energy density, so that $PV = U$. (Hint: you do not have to do any sums over states - you need only prove that this holds using analytic expressions for P and U in this particular system). (3 pts.)

Jan 2009

Stat Mech #3

* Consider a free, non-interacting spin 0 Bose gas in 2-D, where the energy of each particle is: $E(\vec{k}) = \frac{\hbar^2 k^2}{2m}$

 - Assume m is mass of boson and the system is confined to a square region of side length L.

a) Write down the expression for the grand canonical free energy Ω

$$\Omega = -PV$$
$$= -kT \ln(\mathcal{Z})$$
$$\text{*but } \mathcal{Z} = \prod_j \left[1 - \exp[\beta(\mu - \epsilon_j)]\right]^{-1}$$
$$= -kT \sum_j \ln\left(1 - \exp[\beta(\mu - \epsilon_j)]\right)$$
$$= -kT \sum_k \ln\left(1 - \exp\left[\beta\left(\mu - \frac{\hbar^2 k^2}{2m}\right)\right]\right)$$

b) Calculate the # of particles in the system as a function of T, V, and μ

$$\overline{N} = \left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V}$$
$$= \frac{\partial}{\partial \mu} kT \sum_k$$
$$= -kT \sum_k \frac{-\beta \exp\left[\beta\left(\mu - \frac{\hbar^2 k^2}{2m}\right)\right]}{1 - \exp\left[\beta\left(\mu - \frac{\hbar^2 k^2}{2m}\right)\right]}$$
$$= \sum_k \left[\exp\left[-\beta\left(\mu - \frac{\hbar^2 k^2}{2m}\right)\right] - 1\right]$$
$$\approx \int$$

Classical Mechanics and Statistical/Thermodynamics

August 2009

Possibly Useful Information

Handy Integrals:

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$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\, dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b}$$

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$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

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$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

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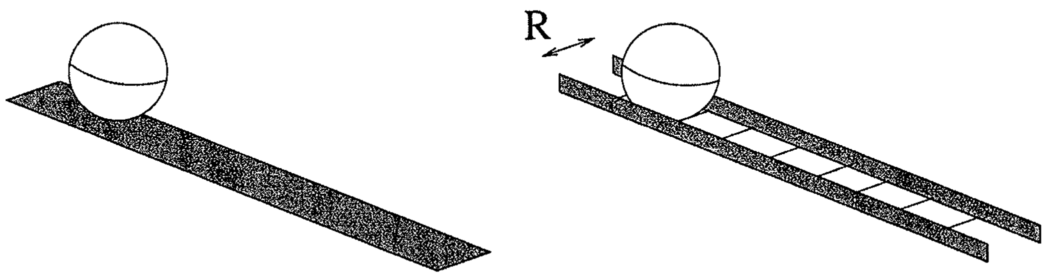
Classical Mechanics

1. Rolling spheres: Given that the moment of inertia of a sphere of mass m and radius R is $(2/5)mR^2$, please answer the following.



   (a) A sphere of radius R and mass m rolls without slipping down an inclined plane on to a horizontal table (left figure). The condition of "rolling without slipping" forces a relationship between v, the speed of the center of mass of the sphere, and ω, the angular velocity of the sphere about its center of mass. What is this relationship? (0.5 pt.)

   (b) Calculate the speed of the sphere at the bottom of the ramp if the center of mass of the sphere has dropped a distance h when it just touches the table. Assume that $h >> R$. (1.5 pt.)

   (c) The ramp is now replaced by two narrow rails separated by a distance R (right figure). Again the ball rolls downward without slipping, supported by the two rails. In this case, what the relationship between v and ω? (1 pt.)

   (d) In this second case, calculate the speed of the sphere at the bottom if the center of mass has dropped a distance h. (2 pts).

   (e) After the ball reaches the bottom of the rails (part b) it continues to move on the horizontal table. It will either be rolling too fast or too slow to roll without slipping. Which will it be? You must prove your result. (1 pt.)

   (f) Friction between the sphere and the plane will adjust the speed of the sphere until it can again roll without slipping. If the magnitude of the force of friction between the sphere and the plane is

μmg, determine the speed of the ball when it again rolls without slipping. (If you did not solve part (b) above, assume the sphere is moving at speed v_0 without rolling and determine its speed when it rolls without slipping). (4 pts.)

Aug 2009

Classical #1

a) $v = \omega r$

b) $E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgh$

$$\Rightarrow mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

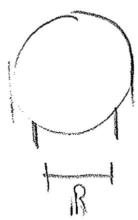
$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{5}mv^2$$

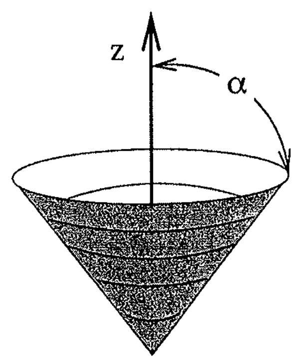
$$gh = \frac{7}{10}v^2$$

$$\Rightarrow v = \sqrt{\frac{10gh}{7}}$$

c)



2. A point particle of mass m travels on the frictionless inner surface of an inverted cone. The cone is oriented so its symmetry axis is parallel to the z-axis, with an opening angle α between the z-axis and the surface of the cone. The force of gravity points in the negative z-direction.



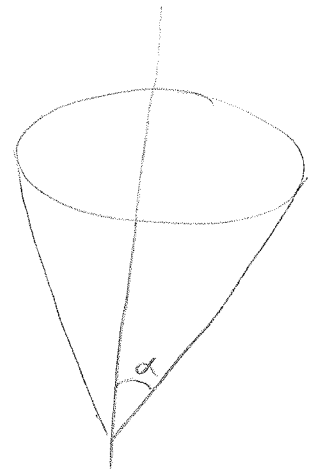
(a) Write the Lagrangian for the problem in cylindrical coordinates. (1 pt.)

(b) Assume the particle is moving in a uniform circular orbit at distance d from the cone tip, measured along the surface of the cone. Determine the angular frequency of the system. (3 pts.)

(c) The opening angle of the cone is abruptly decreased by $\Delta\alpha << \alpha$. This is done in a manner that does **not** impart an impulse or do work on the particle. (Imagine that the cone is instantaneously stretched so that its tip moves slightly downward, but the particle is not displaced during the stretching). Describe the subsequent motion of the particle in this limit. Express your answer in terms of ρ_0, the original radius of the circular orbit, m, α, $\Delta\alpha$, and g. Explain any approximations you are making in deriving your result. (6 pts.)

Aug 2009

Classical #2



* Cone is frictionless
* Particle travels on inner surface of cone
* Gravity points in $-\hat{z}$ direction

a) Write the Lagrangian in cylindrical coordinates

* In cylindrical coordinates: $x = r\cos\varphi \qquad \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\dot{\varphi}$

$y = r\sin\varphi \qquad \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\dot{\varphi}$

$z = z \qquad \dot{z} = \dot{z}$

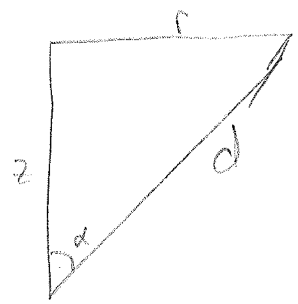
$$\mathcal{L} = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$= \tfrac{1}{2}m\left[(\dot{r}\cos\varphi - r\sin\varphi\dot{\varphi})^2 + (\dot{r}\sin\varphi + r\cos\varphi\dot{\varphi})^2 + \dot{z}^2\right] - mgz$$

$$= \tfrac{1}{2}m\left[\dot{r}^2\cancel{\cos^2\varphi} - 2r\cancel{\cos\varphi\sin\varphi}\dot{r}\dot{\varphi} + r^2\cancel{\sin^2\varphi}\dot{\varphi}^2 + \dot{r}^2\cancel{\sin^2\varphi} + 2r\cancel{\sin\varphi\cos\varphi}\dot{r}\dot{\varphi} + r^2\cancel{\cos^2\varphi}\dot{\varphi}^2 + \dot{z}^2\right] - mgz$$

$$= \tfrac{1}{2}m\left[\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right] - mgz$$

b) Assuming a uniform circular orbit at a distance d from the tip of the cone, determine the angular frequency of the system



$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m\left[\dot{d}^2\cos^2\alpha + d\cos^2\alpha\,\dot{\varphi}^2 + \dot{d}^2\sin^2\alpha\right] - mgd\sin\alpha$$

$$= \tfrac{1}{2}m\left[\dot{d}^2 + d\cos^2\alpha\,\dot{\varphi}^2\right] - mgd\sin\alpha$$

$r = d\cos\alpha \qquad \dot{r} = \dot{d}\cos\alpha$

$z = d\sin\alpha \qquad \dot{z} = \dot{d}\sin\alpha$

↑

α is const.

b) $\frac{\partial \mathcal{L}}{\partial d} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{d}}$

$\frac{1}{2} m\cos^2\alpha\,\dot{\varphi}^2 - mg\sin\alpha = \frac{\partial}{\partial t}[m\dot{d}]$

$\frac{1}{2} m\cos^2\alpha\,\dot{\varphi}^2 - mg\sin\alpha = m\ddot{d}$

$\frac{1}{2}\cos^2\alpha\,\dot{\varphi}^2 - g\sin\alpha = \ddot{d}$

$\Rightarrow \dot{\varphi} = \left(\frac{\ddot{d} + g\sin\alpha}{\frac{1}{2}\cos^2\alpha}\right)^{1/2}$

* but b/c of circular motion, $\ddot{d} = 0$

$\Rightarrow \dot{\varphi} = \left(2g\frac{\sin\alpha}{\cos^2\alpha}\right)^{1/2}$

$\dot{\varphi} = (2g\tan\alpha\sec\alpha)^{1/2}$

$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$

$0 = \frac{\partial}{\partial t}[md\cos^2\alpha\,\dot{\varphi}]$

$0 = m\cos^2\alpha\,(\dot{d}\dot{\varphi} + d\ddot{\varphi})$

$\dot{d}\dot{\varphi} = -d\ddot{\varphi}$

$\dot{\varphi} = -\frac{d\ddot{\varphi}}{\dot{d}}$

c) How does orbit charge as $\alpha \rightarrow \alpha - \Delta\alpha$

$\mathcal{L} = \frac{1}{2}m$

3. Consider the Lagrangian for a 1D system with generalized coordinate q:

$$L(q, \dot{q}, t) = e^{\lambda t/m}\left[\frac{m}{2}\dot{q}^2 - \frac{m\omega_0^2}{2}q^2\right] \tag{1}$$

In the above expression, m is a mass, ω_0 is a frequency, and λ is a positive and dimensionless constant.

(a) Derive the equation of motion for the system. (1 pt.)

(b) What is the canonical momentum, p? (1 pt.)

(c) Calculate the Hamiltonian. (3 pts.)

(d) We wish to make a canonical transformation $(q, p) \to (Q, P)$ using the generating function

$$F_2(q, P, t) = e^{\lambda t/2m} q\, P$$

What is the new coordinate and canonical momentum in terms of the old? (2 pts.)

(e) Show that the canonically transformed Hamiltonian is not time dependent. (3 pts.)

Aug 2009

Classical #3

$$\mathcal{L}(q,\dot q,t) = e^{\lambda t/m}\left[\frac{m}{2}\dot q^2 - \frac{m\omega_0^2}{2}q^2\right]$$

m is mass
ω_0 is a frequency
λ is positive, dimensionless const.

a) Derive equation of motion

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot q}$$

$$-e^{\lambda t/m} m\omega_0^2 q = \frac{\partial}{\partial t}\left[e^{\lambda t/m} m\dot q\right]$$

$$-m\omega_0^2 q \cancel{e^{\lambda t/m}} = m\ddot q e^{\lambda t/m} + m\dot q \frac{\lambda}{m} e^{\lambda t/m}$$

$$-m\omega_0^2 q = m\ddot q + \lambda \dot q$$

$$\Rightarrow 0 = m\omega_0^2 q + \lambda\dot q + m\ddot q$$

b) Find the canonical momentum

$$P_q = \frac{\partial \mathcal{L}}{\partial \dot q}$$

$$= m\dot q e^{\lambda t/m}$$

c) Find the Hamiltonian

$$H = \Sigma_i p\dot q - \mathcal{L}$$

$$= m\dot q^2 e^{\lambda t/m} - e^{\lambda t/m}\left[\frac{m}{2}\dot q^2 - \frac{m\omega_0^2}{2}q^2\right]$$

$$= \frac{m}{2}\dot q^2 e^{\lambda t/m} + \frac{m\omega_0^2}{2}q^2 e^{\lambda t/m}$$

d)

Statistical Mechanics

4. It can be shown that the Helmholtz free energy for a photon gas is given by:

$$F(T,V,N) = -\frac{1}{3}\sigma V T^4$$

where σ is the Stefan-Boltzmann constant. Using this relation, answer the following:

(a) What are the equations of state (that is, P, S, and μ as functions of T, V and N)? (3pts.)

(b) Consider a Carnot cycle using a photon gas as its working fluid. The cycle is driven by one hot and one cold temperature reservoir, with temperatures T_h and T_c respectively. Draw the cycle in the P-V plane. **Caution:** This is **not** an ideal gas! Think carefully about the steps in a Carnot cycle and use your results from above to determine what the cycle will look like. (2pts.)

(c) Solve for the heat exchanged in each leg of your Carnot cycle. Your answer may depend upon T_h, T_c, and any other variables you might choose in defining your cycle. (2pts.)

(d) Using these values for the heat exchanged, calculate the efficiency of a Carnot cycle that uses a photon gas as its working fluid. If you cannot calculate it, devise a careful argument for its value. (3pts.)

5. A particular solid is made up of N distinguishable spin 1 atoms each on a fixed position in a lattice. The energy of each atom is given by:

$$E(\sigma_i) = -V_0\sigma_i^2 - \mu_0\sigma_i B$$

where V_0 arises from an internal field in the crystal, B is the applied external magnetic field and μ_0 is the Bohr magneton. The z-component of the spin of an atom can take on values $\sigma_i \in \{0, \pm 1\}$

(a) Calculate the free energy, $F(T, B, N)$. (2 pts.)

(b) Calculate the specific heat. (4 pts.)

(c) Calculate the magnetic susceptibility, $\chi(T, B, N)$ when $B = 0$. (4 pts.)

Mechanics and Statistical Mechanics Qualifying Exam
Spring 2010

Problem 1: (10 Points)

A child of mass m is playing on a swing hanging from a support by a uniform chain of length L and negligible mass. In this question, you will explore the behavior of the swing in a number of situations.

a. Determine the equation of motion for the system in polar coordinates. **(2 Points)**

b. Consider small oscillations about the equilibrium position. What is the equation of motion for these conditions? **(1 Points)**

c. What is the oscillation frequency for the conditions described in part (b.)? **(2 Points)**

d. By starting at a sufficiently large speed at the bottom of the swing ($\theta = 0°$) the child can go 'over the top' ($\theta = 180°$). If the chain remains maximally extended at the top of the loop, what is the minimum velocity the child must have at the bottom of the loop ($\theta = 0$)? θ is the angle that the chain forms with the vertical. **(2 Points)**

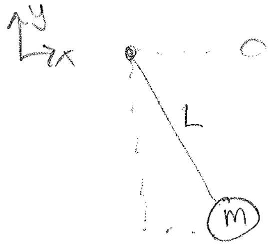
e. What is the minimum force applied to the child by the swing, that the child experiences at the bottom of the loop in part (d.)? **(1 Point)**

f. If the chain is replaced by a rigid rod of negligible mass, what is the minimum velocity of the child at the bottom required to go over the top? **(1 Point)**

g. What is the minimum force applied to the child by the swing, that the child experiences at the bottom of the loop in part (f.)? **(1 Point)**

Jan 2010

Classical #1



a) Determine equation of motion in polar coordinates

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$= \frac{1}{2}mL^2\dot{\varphi}^2 - mgl\sin\varphi$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$\frac{\partial \mathcal{L}}{\partial r} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

$$-mgL\cos\varphi = \frac{\partial}{\partial t}(mL^2\dot{\varphi})$$

$$-mgL\cos\varphi = mL^2\ddot{\varphi}$$

$$\Rightarrow \ddot{\varphi} = -\frac{g}{L}\cos\varphi$$

b) Consider small oscillations about equilibrium. What is the equation of motion?

* Make small angle approximation

$$\Rightarrow \mathcal{L} = \frac{1}{2}mL^2\dot{\varphi}^2 - mgl\varphi$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

$$-mg\cancel{L} = \frac{\partial}{\partial t}[mL^2\dot{\varphi}]$$

$$-\cancel{m}g\cancel{L} = \cancel{m}L^{\cancel{2}}\ddot{\varphi}$$

$$\ddot{\varphi} = -\frac{g}{L}$$

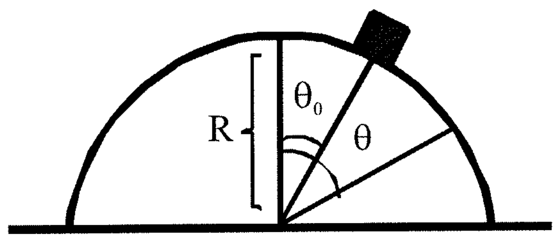
Lagrangian Mechanics

why constraint?

⇒ RADIAL + ANGULAR MOTION

Problem 2: (10 Points)

A hemisphere of radius R rests on the ground. A particle of mass m starts from rest on the sphere at an angle of θ_0 from the vertical that passes through the center of the sphere. Express answers in terms of R, θ_0 and the acceleration due to gravity near the surface of the earth, g.



a. The particle is released and slides without friction. At what angle, θ, measured relative to the vertical, does the particle leave the surface of the sphere? **(4 Points)**

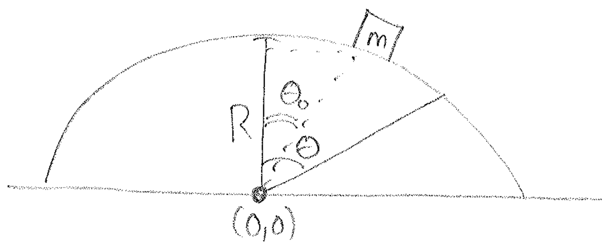
b. What is the angle θ when $\theta_0 = 0$? **(1 Points)**

c. Assume the particle was released with $\theta_0 = 0$. Once the particle leaves the sphere, how long does it take it to hit the ground? **(3 Points)**

d. How far from the center of the sphere is the particle when it hits the ground? **(2 Points)**

Jan 2010

Classical #2

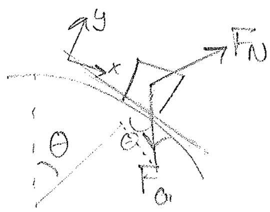


* Normal force 0 when block leaves surface of hemisphere

a) Block is released & slides w/o friction. At what θ (relative to vertical) does block leave the surface of the hemisphere?

$x = R\sin\theta$

$y = R\cos\theta$



$\langle \parallel, \perp \rangle$

$\Rightarrow F_N = \langle 0, F_N \rangle$

$F_G = mg\langle \sin\theta, -\cos\theta \rangle$

$\Rightarrow ma_x = mg\sin\theta$

$ma_y = F_N + mg\cos\theta$

* but $a_y = \frac{v^2}{R}$

$-m\frac{v^2}{R} = -mg\cos\theta_c$

$\frac{v^2}{gR} = \cos\theta_c$

or

$v = \sqrt{gr\cos\theta_c}$

$mgy = \frac{1}{2}mv^2 + mgy$

$mg\cos\theta_0 = \frac{1}{2}mv^2 + mg\cos\theta_c$

$\cancel{m}gR\cos\theta_0 = \frac{1}{2}\cancel{m}gR\cos\theta_c + \cancel{m}gR\cos\theta_c$

$\cos\theta_0 = \frac{3}{2}\cos\theta_c$

$\Rightarrow \theta_c = \cos^{-1}\left(\frac{2}{3}\cos\theta_0\right)$ ✓

b) What is θ_c if $\theta_0 = 0$?

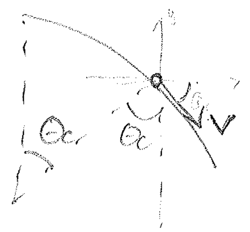
$$\theta_c = \cos^{-1}\left(\frac{2}{3}\cos\theta_0\right)$$

$$\theta_c = \cos^{-1}\left(\frac{2}{3}\right)$$

c) If particle released at $\theta = 0$, how long does it take to hit the ground after it leaves the surface of the hemisphere?

$y_i = R\cos\theta_c \qquad v_i = v\sin\theta_c \quad a = -g$

$y_f = 0 \qquad v_f = ?$



$$0 = -\frac{1}{2}gt^2 - v\sin\theta_c + R\cos\theta_c$$

$$\Rightarrow \quad t = \frac{v\sin\theta_c \pm \sqrt{v^2\sin^2\theta_c - 2gR\cos\theta_c}}{g}$$

$$t = \frac{v\sin\theta_c \pm \sqrt{v^2\sin^2\theta_c - 2v^2}}{g}$$

$$t = \frac{v\sin\theta_c \pm v\sqrt{\sin^2\theta_c - 2}}{g}$$

Problem 3 (10 Points):

A nonrelativistic electron of mass m and charge $-e$ moves between a wire of radius a at negative electric potential $-\phi_0$ and a concentric cylindrical conductor of radius R at zero potential. There is a uniform constant magnetic field B parallel to the axis. The electric scalar and magnetic vector potentials can be written as:

$$\phi = -\phi_0 \frac{\ln(r/R)}{\ln(a/R)} \qquad\qquad \vec{\mathbf{A}} = \frac{1}{2} B r \hat{\theta}$$

where $\hat{\theta}$ is a unit vector in the increasing θ direction.

a. Give the Lagrangian and Hamiltonian in cylindrical coordinates. Specify all the constants of motion for this system justifying your answer. *Recall that the electric and magnetic potentials can be given in terms of a velocity dependent potential* $U(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}) = q\phi - q\vec{\mathbf{A}} \cdot \dot{\vec{\mathbf{r}}}$, *where* q *is the charge.* **(5 Points)**

b. For an electron starting at rest on the inner wire ($r = a$), there is a value of the magnetic field B_c such that for $B \leq B_c$ the electron reaches the outer conductor and for $B > B_c$ it does not reach the outer conductor. Determine an expression for B_c in terms of the variables given; *you can assume that* $a \ll R$. **(5 Points)**

Jan 2010

Classical #3

Problem 4 (10 Points):

Assume that air obeys the ideal gas equation. Take M to be the molar mass, P the pressure, R the ideal gas constant, T the temperature, z the altitude, ρ the density, and g the acceleration due to gravity.

a. The density of our atmosphere decreases with increasing altitude. This is a consequence of hydrostatic equilibrium, where the pressure of the air at an altitude z, must balance the pressure from below and the weight of the column of air above. Given that air has a mass density $\rho = MP/RT$, find dP/dz. Assume that the atmosphere is isothermal. Neglect the curvature of the earth and the variation of g with altitude. **(4 Points)**

b. Using the model in part (a.), consider a volume of air that is moved adiabatically within the atmosphere and able to do work on its surroundings; that is, expand and contract to maintain the same pressure as the surrounding air. If this section is moved upwards, it will cool as it is lifted, thus increasing in density compared to the surrounding air, and tend to sink back to its original altitude. Find dT/dz, the adiabatic lapse rate for the air. Assume the air is composed of diatomic molecules (N_2). (Hint: first find dT/dP). **(4 Points)**

The significance of the adiabatic lapse rate is that it determines the stability of the atmosphere to convection. The temperature in the lower part of the real atmosphere (troposphere) is not isothermal, but decreases with increasing altitude because it is heated by the ground. If the temperature gradient in the atmosphere is greater than the lapse rate, convection can occur.

c. If the section of air was wet so that condensation can occur, how does the lapse rate change? Explain your reasoning. **(1 Points)**

d. A helium balloon ascends in the atmosphere, expanding adiabatically just as the section of air in (b.). Will the lapse rate of helium be higher, the same, or lower than air? Explain. **(1 Points)**

Problem 5 (10 Points):

A system consists of N identical non-interacting particles in equilibrium with a heat bath. The total number of individual states available to each particle is $2N$. Of these states, N are degenerate with energy 0 and N are degenerate with energy ϵ. It is found by observation that the total energy of the system is $N\epsilon/3$.

a. What is the average number of particles in the excited state? **(1.5 Points)**

Find the temperature of the system under the following three different assumptions.

b. The particles are bosons. **(2 Points)**

c. The particles are fermions. **(2 Points)**

d. The particles obey a Boltzmann distribution. **(2 Points)**

e. Are the temperatures you found in (b.), (c.) and (d.) the same? Why or why not? Explain your answer. **(2.5 Points)**

Problem 6 (10 Points):

A large flat surface is in contact with a mono-atomic gas above it. The volume of gas above the surface acts as an infinite reservoir of gas atoms, but does not otherwise enter into the problem. The suface consist of a square lattice of sites that gas atoms can occupy; denote the number of gas atoms on site i by n_i, where $n_i \in \{0,1\}$, and the total number of lattice sites by N_s. The energy of the system is given by:

$$E(\{n_i\}) = -\left[\sum_i n_i \,\epsilon + v_0 \sum_i \sum_{j \in n.n.} n_i n_j\right] \tag{1}$$

where ϵ is a binding energy of atom to the substrate, v_0 is an interaction between adjacent atoms, and the sum over j is restricted to the nearest neighbors of i.

a. Write down an expression for the grand canonical partition function $Z(T,\mu)$. Your answer should be in the form of a sum over states. **(2 Points)**

b. Calculate the grand canonical free energy, $\Omega(T,\mu,N_s)$ when $v_0 = 0$. **(2 Points)**

c. Calculate N, the number of gas atoms adsorbed to the surface, as a function of T, μ and N_s when $v_0 = 0$. **(2 Points)**

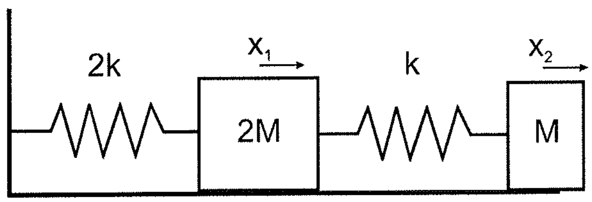
d. When $v_0 \neq 0$ the problem is in general more difficult. To simplify it, replace n_j in the above sum by $\bar{n}$, a constant that will be set equal to the average occupation of any site. Calculate the number of gas atoms adsorbed to the surface, N, as a function of T, μ, N_s and $\bar{n}$. **(2 Points)**

e. Discuss the possibility of a phase transition in $\bar{n}$ as a function of β. This can be done by graphically investigating the requirement that $N(T,\mu,\bar{n})/N_s = \bar{n}$, or by returning to the expression for the energy given in equation (1) and mapping it on to other well known problems in statistical mechanics. **(2 Points)**

Mechanics and Statistical Mechanics Qualifying Exam
Fall 2010

Problem 1: (10 Points)

Two blocks are free to move in *one* dimension along a frictionless horizontal surface. The blocks of mass 2M and M are connected to each other and to a fixed wall by two springs with stiffness 2k and k as shown in the figure. Choose the dynamical coordinates of the system to be the position of block 1, x_1, and block 2, x_2, from their respective equilibrium positions. Consider only small oscillations so that the springs are linear. Neglect all damping.



a. Write down the equations of motion for each mass. **(2 Points)**

b. Show that the frequencies of the normal modes of the system are $\sqrt{2}\omega_0$ and and $\omega_0/\sqrt{2}$ where $\omega_0 = \sqrt{k/m}$. **(2 Points)**

c. Find the eigenvectors that describe the normal modes and sketch them. **(3 Points)**

d. Suppose you grab mass M and push it slowly to the left by an amount A_0. When mass 2M is in equilibrium show that it is $A_0/4$ from its equilibrium position. **(1 Point)**

e. If you release the system from the starting position in (d.), what will be the displacement of the system as a function of time? Write an expression for the displacement of block 1 (mass 2M) as a function of time from its original equilibrium position. **(2 Points)**

Aug 2010

Classical #1

Problem 2 (10 Points):

An isolated uniform sphere of mass m and radius R is rotating with angular velocity ω_0 about an axis running through the sphere. Through only internal forces, the radius increases linearly to 2 R in a time τ, while maintaining uniform density and spherical symmetry.

a. At time τ, what is the angular velocity of the sphere? **(2 Points)**

b. Find an expression for the angular velocity as a function of time. **(1 Points)**

c. When the system reaches 2 R it immediately reverses and its radius linearly decreases to R over the period τ to 2τ. By what angle $\Delta\phi$ is the object behind in its rotation compared to a situation where the sphere does not expand between 0 and 2τ? **(4 Points)**

d. Consider the case where the radius of the sphere expands exponentially with some time constant τ_e. How much does the sphere rotate compared to the case where there is no expansion as t$\rightarrow \infty$? **(3 Points)**

Aug 2010

Classical #2

Initially: mass $= m$

radius $= R$

density: $\rho = \frac{m}{\frac{4}{3}\pi R^3}$

At $t = \tau$: mass: $8m$

radius $= 2R$

density: $\rho = \frac{m}{\frac{4}{3}\pi (2R)^3}$

a) At time $t = \tau$, what is the angular velocity of the sphere

* Conservation of angular momentum

$L = m\omega r^2 \quad (L = mvr,\ v = \omega r)$

$m\omega_0 R^2 = m\omega_f 2^2 R^2$

$\frac{\omega_0}{4} = \omega_f$

b) Find an expression for $\omega(t)$

$t = 0 \quad r = R$

$t = \tau \quad r = 2R$

$\Rightarrow r(t) = \frac{R}{\tau}t + R$

$mR^2\omega_0 = \left(m\left(\frac{R}{\tau}t + R\right)\right)^2 \omega_f$

$\omega_0 = \omega_f\left(\frac{t}{\tau}+1\right)^2$

$\Rightarrow \omega_f = \frac{\omega_0}{\left(\frac{t}{\tau}+1\right)^2}$

c) If the expansion reverses at $r = 2R$ and returns to its initial state at $t = 2\tau$, by what angle $\Delta\varphi$ is the sphere behind an identical sphere that didn't expand

$2\omega_0\tau - \left[\int_0^{\tau} \frac{\omega_0}{\left(\frac{t}{\tau}+1\right)^2} + \int_{\tau}^{2\tau} \frac{\omega_0}{\left(\frac{-t}{\tau}+1\right)^2}\right]$

$2\omega_0\tau - \left[\left(\tau\omega_0\left(\frac{t}{\tau}+1\right)^{-1}\right)\Big|_0^{\tau} + -\omega_0\left(\frac{-t}{\tau}+1\right)^{-1}\tau\Big|_{\tau}^{2\tau}\right]$

$2\omega_0\tau - \left[\tau\omega_0\left(\frac{1}{2}-1\right) - \tau\omega_0(-1-0)\right]$

$2\omega_0\tau - \left[-\frac{\tau\omega_0}{2} + \frac{\tau\omega_0}{1}\right]$

$2\omega_0\tau - \boxed{\frac{\tau\omega_0}{2}}$

$\Rightarrow \Delta\varphi = \frac{3\tau\omega_0}{2}$

d) What if the sphere exponentially w/ time constant τ_e. How much does sphere rotate compared to the case where there is no expansion ($t \to \infty$)

*Now $r = Re^{t/\tau_e}$

$$\cancel{m}R^2 w_0 = m\left(Re^{t/\tau_e}\right)^2 w_f$$

$$\Rightarrow w_f = w_0 e^{-2t/\tau_e}$$

$$2w_0 t - \left[\int_0^t w_0 e^{-2t/\tau_e} dt\right.$$

$$2w_0 t - \left[-\frac{\tau_e}{2} w_0 e^{-2t/\tau_e}\right|_0^t$$

$$2w_0 t - \frac{\tau_e}{2} w_0 e^{-2t/\tau} - \frac{\tau_e}{2} w_0$$

$$2w_0 t - \frac{\tau_e}{2} w_0 \left(e^{-2t/\tau} - 1\right) = \Delta\varphi$$

Problem 3 (10 Points):

Consider the following Lagrangian

$$L = \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right) e^{2\gamma t}$$

assuming that $\omega > \gamma$ for the questions that follow.

a. Determine the Hamiltonian associated with this Lagrangian. **(3 Points)**

b. Find a transformation to new phase space variables that make H independent of time and show that these form a canonical transformation by determining a generating function of the form $F_2(q, P, t)$. **(4 Points)**

c. Using the equations of motion for the transformed Hamiltonian $K(Q, P, t)$, solve for $Q(t)$ and transform back to get $q(t)$. **(3 Points)**

Aug 2010

Classical #3

Given: $\mathcal{L} = \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right)e^{2\gamma t}$

a) Determine the Hamiltonian associated w/ the above Lagrangian

$$H = \sum_i p_i \dot{q}_i - \mathcal{L}$$

$$P_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$= e^{2\gamma t} m\dot{q} \qquad \Rightarrow \quad \dot{q} = \frac{p}{m}e^{-2\gamma t}$$

$$\Rightarrow H = m\dot{q}^2 e^{2\gamma t} - e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right)$$

$$= e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2\right)$$

$$= e^{2\gamma t}\left(\frac{1}{2m}p^2 e^{-4\gamma t} + \frac{1}{2}m\omega^2 q^2\right)$$

$$= \frac{1}{2m}p^2 e^{-2\gamma t} + \frac{1}{2}m\omega^2 q^2 e^{2\gamma t}$$

b) Find a transformation to new phase space variables that make H time independent and show that these form a canonical transformation by determining a $F_2(q, P, t)$ generating function.

* For an $F_2(q, P, t)$: $\quad p = \frac{\partial F_2(q,P)}{\partial q} \qquad Q = \frac{\partial F_2(q,P)}{\partial P}$

$$\Rightarrow \text{Want: } H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q$$

$$\Rightarrow Q = qe^{\gamma t}$$

$$P = pe^{-\gamma t}$$

$$\Rightarrow F_2(q, P, t) = Pqe^{\gamma t}$$

$$\Rightarrow K = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2$$

c) Use the Hamiltonian eqns. of motion for K to solve for Q and transform it back to $q(t)$

$$\dot{P} = \frac{\partial K}{\partial Q} = m\omega^2 Q$$

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{P}{m}$$

$$\dot{P} = m\ddot{Q} \qquad \Rightarrow m\ddot{Q} = \omega^2 m Q$$

$$\ddot{Q} = \omega^2 Q \qquad \Rightarrow Q = Ae^{-i\omega t} + Be^{i\omega t}$$

$$\Rightarrow q = e^{-\gamma t}Q$$

$$= e^{-\gamma t}\left(Ae^{-i\omega t} + Be^{i\omega t}\right)$$

Problem 4 (10 Points):

The coffee purchased at rest stops is often too hot to drink. One way to cool off your coffee is to add ice, but how much ice should you add? Take the initial conditions for the coffee to be $T_0^{cof} = 80\,°\mathrm{C}$ and $V = 400\,\mathrm{ml}$. Take the initial conditions for the ice to be $T_0^{ice} = 0\,°\mathrm{C}$. The final temperature for the coffee and ice that you want to achieve is $T_f = 60\,°\mathrm{C}$. For the following questions assume that the coffee is pure water (a good assumption for most rest stop coffee) and the process is adiabatic with respect to the surroundings. Neglect volume changes of the coffee and ice and any temperature dependence of the heat capacity. The following thermodynamic properties of water may be useful:

$M = 18.0\,\mathrm{g\,mole^{-1}}$, molar mass
$\rho = 1.00\,\mathrm{g/cm^3}$, density
$\Delta H_{fus} = 6.00\,\mathrm{kJ\,mole^{-1}}$, heat of fusion
$C_p = 75.4\,\mathrm{J\,mole^{-1}\,K^{-1}}$, heat capacity of liquid

For parts (a.)-(c.) your answers should be in terms of the variables described here.

a. Find a general (algebraic solution) expression for the mass of ice, m, that is needed to cool the coffee to T_f? **(4 Points)**

b. Calculate, numerically, how many grams of ice you should add to your coffee to lower the temperature to $T_f = 60\,°\mathrm{C}$. **(1 Points)**

c. What is the entropy change of the system (coffee + ice)? Find an algebraic solution. **(3 Points)**

d. What is the entropy change of the surroundings? **(1 Points)**

e. Is this a thermodynamically reversible process? Explain. **(1 Points)**

Problem 5 (10 Points):

Consider a one dimensional ideal gas of electrons as a model for the conduction electrons in a one dimensional wire.

a. Determine the density of states g(E) for the one dimensional non-interacting electron system confined to a length, L. **(3 Points)**

b. What is the Fermi energy for this system? **(2 Points)**

c. What is the root mean square velocity of the electrons at $T = 0\,^\circ\mathrm{K}$? **(3 Points)**

d. What is the entropy of the electrons at $T = 0\,^\circ\mathrm{K}$? Justify your answer. **(2 Points)**

Problem 6 (10 Points):

The following questions refer to a stream of photons in equilibrium at temperature T (thermal light - say from a light bulb) incident on a perfect detector which detects (counts) all the particles that hit it. Your final answers should be in terms of the mean particle number.

a. Given $\bar{n}_s$ photons are counted on average in time t, calculate the variance in the photon number n_s, $\overline{(\Delta n_s)^2}$. **(2 Points)**

b. Calculate the fractional fluctuation of the detector signal defined as the square root of the variance divided by the mean photon number, $\bar{n}_s$, squared, $\sqrt{\overline{(\Delta n_s)^2}/\bar{n}_s^2}$. This is the inverse of the signal to noise ratio. **(2 Points)**

The following questions refer to a stream of electrons in equilibrium at temperature T incident on a detector which detects (counts) all the particles that hit it. Again, your final answers should be in terms of the mean particle number.

c. Given $\bar{n}_e$ electrons are counted on average in time t, calculate the variance in the electron number n_e, $\overline{(\Delta n_e)^2}$. **(2 Points)**

d. Calculate the fractional fluctuation of the detector signal defined as the square root of the variance divided by the mean electron number, $\bar{n}_e$, squared, $\sqrt{\overline{(\Delta n_e)^2}/\bar{n}_e^2}$. **(2 Points)**

e. Compare the two results. Are the results the same or different? Do the counts detected clump (bunch) or anti-clump (anti-bunch)? Why? **(2 Points)**

Classical Mechanics and Statistical/Thermodynamics

August 2011

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

Moments of Inertia:

$$\begin{aligned} I_{\text{hoop}} &= MR^2 \\ I_{\text{disk}} &= \tfrac{1}{2}MR^2 \\ I_{\text{sphericalshell}} &= \tfrac{2}{3}MR^2 \\ I_{\text{ball}} &= \tfrac{2}{5}MR^2 \end{aligned}$$

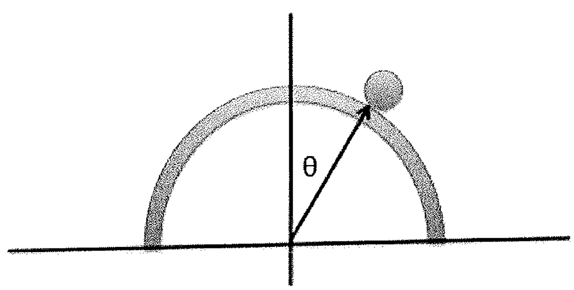
Classical Mechanics

1. A solid uniform marble with mass m and radius r starts from rest on top of a hemisphere with radius R. It will start to roll to the right, and eventually fly off the hemisphere.

   (a) Assume that the marble rolls without slipping at all times. Calculate θ_1, the angle with respect to the vertical at which the marble loses contact with the hemisphere. (3pts).

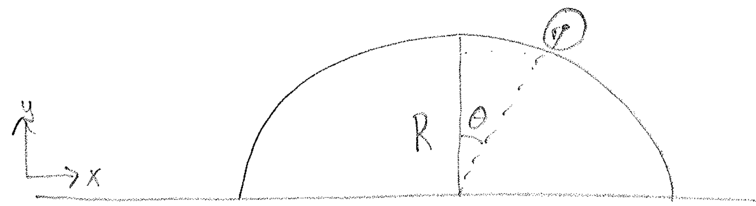
   (b) Where will the marble hit the ground, as measured from the center of the hemisphere? You may use the variable θ_1 in your answer. (If you do not solve part (a), you can still attempt this problem by writing your answer in terms of this variable.) (3pts).

   (c) Now assume that the force of friction between marble and the hemisphere is μN, where N is the normal force between the marble and the hemisphere. Calculate the angle θ_2 at which the marble will no longer roll without slipping. (4pts).



Aug 2011

Classical #1



a) Assume the marble rolls w/o slipping. Find θ_1, where marble looses contact w/ the hemisphere

* Normal force is 0 when marble leaves surface

$$mg(R+r) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgy$$

$x = (R+r)\sin\theta$

$y = (R+r)\cos\theta$

$$mg(R+r) = \frac{1}{2}mv^2 + \frac{1}{5}mr^2\left(\frac{v}{r}\right)^2 + mg(R+r)\cos\theta_1$$

$$mg(R+r) = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 + mg(R+r)\cos\theta_1$$

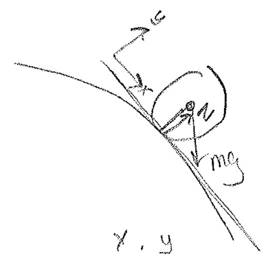
$$g(R+r) = \frac{3}{5}v^2 + g(R+r)\cos\theta_1$$

$$g(R+r) = \frac{3}{5}gR\cos\theta_1 + g(R+r)\cos\theta_1$$

$$\frac{g(R+r)}{\frac{3}{5}g + g(R+r)} = \cos\theta_1$$

$$\frac{R+r}{\frac{3}{5}+R+r} = \cos\theta_1$$

$$\Rightarrow \theta_1 = \cos^{-1}\left(\frac{R+r}{R+r+3/5}\right)$$



$x,\ y$

$\langle \parallel, \perp \rangle$

$F_N = \langle 0, N \rangle$

$F_G = mg\langle \sin\theta, -\cos\theta \rangle$

$\Rightarrow ma_x = mg\sin\theta$

$ma_y = N + mg\cos\theta$

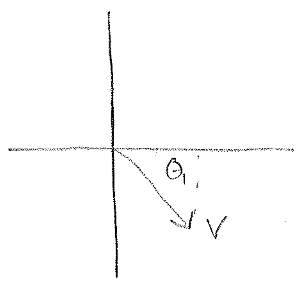
* at θ_1, $N = 0$, $a_y = \frac{v^2}{R}$

$ma_y = mg\cos\theta_1$

$\frac{v^2}{R} = g\cos\theta_1$

$\Rightarrow v = \sqrt{gR\cos\theta_1}$

b) Where will the marble hit the ground, measured from the center of the hemisphere?



$V_x = V\cos\theta$

$V_y = V\sin\theta$

$y_i = (R+r)\cos\theta_1$ $\qquad V_i = V$

$y_f = 0$ $\qquad a = g$

$$0 = -\tfrac{1}{2}gt^2 + vt + (R+r)\cos\theta_1$$

$$0 = -\tfrac{1}{2}gt^2 - \sqrt{gR\cos\theta_1}\,\sin\theta_1\, t + (R+r)\cos\theta_1$$

$$\Rightarrow \quad t = \frac{+\sqrt{gR\cos\theta_1}\,\sin\theta_1 \pm \sqrt{gR\cos\theta_1\sin^2\theta_1 - 4\left(-\tfrac{1}{2}g\right)(R+r)\cos\theta_1}}{-g}$$

$$= \frac{\sqrt{gR\cos\theta_1}\,\sin\theta_1 \pm \sqrt{gR\cos\theta_1\left(\sin^2\theta_1 + 2\right) + 2rg\cos\theta_1}}{-g}$$

$$= \frac{\sqrt{gR\cos\theta_1}\,\sin\theta_1 - \sqrt{gR\cos\theta_1\left(\sin^2\theta_1 + 2\right) + 2rg\cos\theta_1}}{-g}$$

(need (−) root to make overall time positive)

$x_i = (R+r)\sin\theta_1$ $\qquad v = \sqrt{gR\cos\theta_1}\,\cos\theta_1$

$x_f = ??$

$$x_f = vt + x_i$$

$$= gR\cos^{3/2}\theta_1\, t + (R+r)\sin\theta_1$$

$$= gR\cos^{3/2}\theta_1\left(\frac{1}{-g}\left[\sqrt{gR\cos\theta_1}\,\sin\theta_1 - \sqrt{gR\cos\theta_1\left(\sin^2\theta_1 + 2\right) + 2rg\cos\theta_1}\right]\right) + (R+r)\sin\theta_1$$

2. Consider a point particle of mass m moving under the influence of a central force:

$$\vec{F}(\vec{r}) = -\frac{k}{r^n}\,\hat{r}$$

where n is an integer greater than one ($n = 2, 3, \ldots$), the variable r is the distance from the origin of the force ($r \equiv |\vec{r}|$) and $\hat{r}$ is a unit vector in the radial direction. In this problem, we will examine when circular orbits are stable for such a central force.

(a) Calculate potential energy of this force. Choose the zero of the potential to be at infinity ($r = \infty$). (1pt)

(b) Show that the angular momentum about the origin, L, is conserved. (You may use the Newtonian, Lagrangian, or Hamiltonian formulations of the problem). (2pts)

(c) Write an expression for the total energy of the particle E as a function of r, dr/dt, L, k, and n. (1pt)

(d) Assume the particle is moving in a circular orbit about the origin, so that $dr/dt = 0$. Calculate the radius of the orbit and the velocity of the particle as a function of the above variables. (3pts)

(e) When is this circular orbit stable? (Hint: look at dE/dr and d^2E/dr^2.) (3pts)

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Classical #2

* Consider a particle of mass m under the influence of a central force
$$\vec{F}(\vec{r}) = -\frac{k}{r^n}\hat{r} \quad \text{where} \quad n \in [\mathbb{Z} > 1].$$

a) Calculate the potential energy of this force

$$U(\vec{r}) = -\int \vec{F} \cdot d\vec{r}$$
$$= -\int_{\infty}^{r} -\frac{k}{r^n}\hat{r}\, d\vec{r}$$
$$= +\int_{\infty}^{r} \frac{k}{r^n} dr \qquad \text{(assuming spherical coordinates)}$$
$$= \int_{\infty}^{r} k r^{-(n)} dr$$
$$= -k\frac{1}{n+1} r^{-n+1}\Big|_{\infty}^{r}$$
$$= -\frac{k}{n+1} r^{-n+1}$$

b) Show that angular momentum about the origin is conserved

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{k}{n+1}r^{-n+1}, \qquad v = \langle \dot{r}, r\dot{\theta}, r\dot{\varphi}\sin\theta \rangle$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\varphi}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{n+1}r^{-n+1}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$
$$0 = \frac{d}{dt}(mr^2\sin^2\theta\,\dot{\varphi})$$
$$\Rightarrow mr^2\sin^2\theta\,\dot{\varphi} = \text{const.} = L \quad \rightarrow L \text{ is conserved}$$

c) Write an expression for the total energy of the particle (E) as a function of:
$r, \dot{r}, L, k, n$

$$E = T + U$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2\sin^2\theta} + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{n-1}r^{-n+1}$$
$$* \text{ if } \theta = \frac{\pi}{2}$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{n-1}r^{-n+1}$$

d) Assume the particle is moving in a circular orbit about the origin. Find the radius of the orbit and the velocity of the particle as a function of the above variables

$$\mathcal{L} = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2 \sin^2\theta\,\dot{\varphi}^2 + \frac{1}{2} mr^2\dot{\theta}^2 + \frac{k}{n+1} r^{-n+1} = \frac{L^2}{2mr^2} + \frac{k}{n-1} r^{-n+1}$$

$$\cancel{\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}}$$

0 b/c r is constant

$$\cancel{mr\sin^2\theta\dot{\varphi}^2 + mr\dot{\theta}^2 - kr^n = \frac{d}{dt}[m\dot{r}]}$$

$$mr\sin^2\theta\dot{\varphi}^2 + mr\dot{\theta}^2 - kr^n = 0$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{-L^2}{mr^3} + \frac{k}{r^n}$$

$$\Rightarrow r = \left(\frac{km}{L^2}\right)^{\frac{1}{n-3}}$$

$$\Rightarrow L = mr^2\dot{\varphi} \rightarrow \dot{\varphi} = \frac{L}{mr^2}$$

e) When is the orbit stable?

$$\frac{dE}{dr} = \frac{-L^2}{mr^3} + \frac{k}{r^n}$$

$$\frac{d^2E}{dr^2} = \frac{3L^2}{mr^4} + \frac{-nk}{r^{n+1}}$$

$$= \frac{3L^2}{m}\left(\frac{km}{L^2}\right)^{-4/(n-3)} + nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)}{n-3}} > 0$$

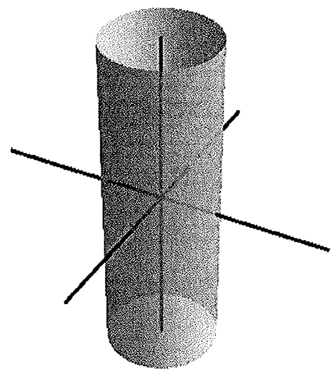
$$\frac{3L^2}{m}\left(\frac{km}{L^2}\right)^{\frac{-4}{n-3}} > +nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)}{n-3}}$$

$$\frac{3L^2}{m} > +nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)+4}{n-3}}$$

$$\frac{3\cancel{L^2}}{\cancel{m}} > +\frac{\cancel{L^2}nk}{\cancel{km}}$$

$$3 > +n$$

3. A particle of mass m is constrained to move on an infinitely long cylinder of radius a. The center of the cylinder is oriented along the z-axis, as shown. An attractive central potential, $U(r) = U(\sqrt{a^2 + z^2})$, is located at the origin, where r is the radius is spherical coordinates.



(a) Write down the Lagrangian for the problem. (1pt)

(b) From the Lagrangian, explicitly derive the Hamiltonian for the particle. (2pts)

(c) Is angular momentum about the z-axis conserved? Prove your answer. (2pts)

(d) Under what conditions is motion in the z-direction bounded? (2pts)

(e) Assume that the potential is $U(r) = \frac{1}{2}\alpha r^2$. Solve the equations of motion, and reduce the problem to quadrature. (3pts)

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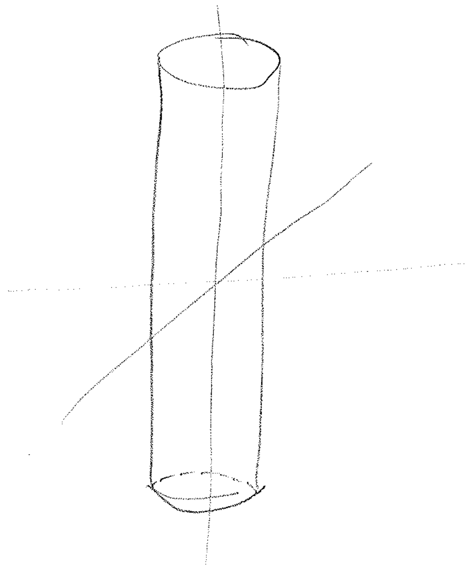
Classical #3



* Particle of mass m constained to move on infinetely long cylinder of radius a; cylinder oriented along z-axis

* Attractive central located at origin,

$U(r) = U(\sqrt{a^2+z^2})$, where r is radius in spherical

a) Find the Lagrangian

$x = \rho\cos\varphi \qquad \dot{x} = \dot{\rho}\cos\varphi - \rho\sin\varphi\dot{\varphi}$

$y = \rho\sin\varphi \qquad \dot{y} = \dot{\rho}\sin\varphi + \rho\cos\varphi\dot{\varphi}$

$$\mathcal{L} = T - U$$

$$= \tfrac{1}{2}mv^2 - U(\sqrt{a^2+z^2})$$

$$= \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - U(\sqrt{a^2+z^2})$$

$$= \tfrac{1}{2}m(\dot{\rho}^2\cos^2\varphi - 2\rho\cos\varphi\sin\varphi\dot{\rho}\dot{\varphi} + \rho^2\sin^2\varphi\dot{\varphi}^2 + \dot{\rho}^2\sin^2\varphi + 2\rho\sin\varphi\cos\varphi\dot{\rho}\dot{\varphi} + \rho^2\cos^2\varphi\dot{\varphi}^2 + \dot{z}^2) - U$$

$$= \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - U(\sqrt{a^2+z^2})$$

* Assume arbitrary central potential Ar^n

$$= \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - A(\sqrt{a^2+z^2})^n$$

b) From the Lagrangian, derive the Hamiltonian

$$H = \sum_i p_i\dot{q}_i - \mathcal{L}$$

$$P_\rho = \frac{\partial\mathcal{L}}{\partial\dot{\rho}} = m\dot{\rho}$$

$$P_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = m\rho^2\dot{\varphi}$$

$$P_z = \frac{\partial\mathcal{L}}{\partial\dot{z}} = m\dot{z}$$

$$\Rightarrow H = m\dot{\rho}^2 + m\rho^2\dot{\varphi}^2 + m\dot{z}^2 - \left[\tfrac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2) - A(\sqrt{a^2+z^2})^n\right]$$

$$= \tfrac{1}{2}m\dot{\rho}^2 + \tfrac{1}{2}m\rho^2\dot{\varphi}^2 + \tfrac{1}{2}m\dot{z}^2 + A(\sqrt{a^2+z^2})^n$$

c) Is angular momentum about z-axis conserved?

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right]$$

$$0 = \frac{d}{dt}\left[m\rho^2\dot{\varphi}\right]$$

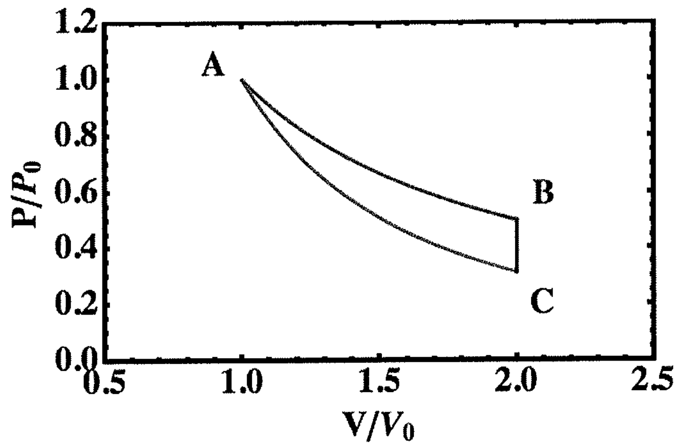
$$\Rightarrow m\rho^2\dot{\varphi} = \text{const.} = L \checkmark$$

d) Under what conditions is motion in the z direction bounded?

$\Rightarrow$ Motion is bounded when $E_T < U \rightarrow T < 0$

Statistical Mechanics

4. Consider an ideal monatomic gas used as the working fluid in a thermodynamic cycle. The number of particles is n_0. It follows a cycle consisting of one adiabat, one isochore and one isotherm, as shown below.



(a) Calculate the pressure, temperature, and volume at each corner of the cycle, A, B, and C, expressing your answer in terms of P_0, V_0, n_0 and perhaps R, the ideal gas constant. Note that point A the pressure is P_0 and the volume is V_0. (3pts)

(b) Calculate the work done on the system, the heat into the system and the change in the internal energy of the system for each process step. (4.5pts)

(c) What direction around the ycle must the system follow to be used as a functional heat engine? (1/2pt)

(d) What is the efficiency of the cycle, run as an engine? (1pt)

(e) What is the efficiency of an ideal Carnot engine run between reservoirs B and C? (1pt)

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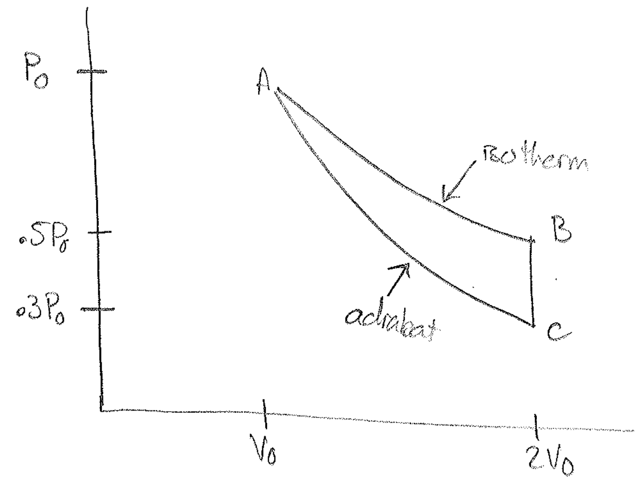
Stat Mech #1



* cycle consists of adiabat, isochore, and isotherm

* gas is ideal + monatomic

$$\Rightarrow \begin{matrix} C_V = \frac{3}{2}R \\ C_p = \frac{5}{2}R \end{matrix} \Big\} \gamma = 5/3$$

a) Find P, V, and T at each corner of the cycle in terms of P_0, V_0, n_0, and R

$P_A = P_0$	$P_B = .5P_0$	$P_C = .3P_0$
$V_A = V_0$	$V_B = 2V_0$	$V_C = 2V_0$
$T_A = \frac{P_0V_0}{nk_B}$	$T_B = \frac{P_0V_0}{nk_B}$	$T_C = \frac{.6P_0V_0}{nk_B}$

$PV = nk_BT$

$\Rightarrow T = \frac{PV}{nk_B}$

* For $A \to C$ (adiabat)

$$P_AV_A^\gamma = P_CV_C^\gamma$$

$$T_AV_A^{\gamma-1} = T_CV_C^{\gamma-1}$$

$$\frac{P_AV_A}{T_A} = \frac{P_CV_C}{T_C}$$

$$\frac{P_AV_A}{T_A} = \frac{\frac{P_AV_A^\gamma}{V_C^\gamma}V_C}{\frac{T_AV_A^{\gamma-1}}{V_C^{\gamma-1}}}$$

$$V_A = \frac{V_A^\gamma V_C^{1-\gamma}}{V_A^{\gamma-1}V_C^{1-\gamma}}$$

b) Find the work done on the system, the heat into the system, and the change in internal energy during each step of the cycle

$$W_{B\to C} = 0$$

$$Q_{B\to C} = nC_V\Delta T = \frac{A}{N_{av}}\left(\frac{3}{2}R\right)\left(\frac{2}{5}\frac{P_0V_0}{nk_B}\right) = \frac{3}{5}P_0V_0$$

$$\Delta E = Q = \frac{3}{5}P_0V_0$$

$$W_{C\to A} = \frac{P_fV_f - P_iV_i}{1-\gamma} = \frac{.6P_0V_0 - P_0V_0}{1-5/3} = \frac{2/5}{-2/3}P_0V_0 = -3/5\,P_0V_0$$

$$Q_{C\to A} = 0$$

$$\Delta E = -W = 3/5\,P_0V_0$$

b) $W_{A \to B} = n k_B T \ln\left(\frac{V_B}{V_A}\right)$

$= n k_B \frac{P_0 V_0}{n k_B} \ln\left(\frac{2V_0}{V_0}\right)$

$= P_0 V_0 \ln(2)$

$Q = W = P_0 V_0 \ln(2)$

$\Delta E = 0$ b/c isotherm

c) Which direction does the heat engine flow?

CW

d) What is the efficiency of the heat engine?

$\eta = 1 - \left|\frac{Q_{out}}{Q_{in}}\right|$

$= 1 - \frac{.6 P_0 V_0}{P_0 V_0 \ln(2)}$

$= 1 - \frac{.6}{\ln(2)}$

5. Consider the quantum mechanical linear rotator. It has energy levels

$$E_J = \frac{\hbar^2}{2I} J\,(J+1)$$

where I is the moment of inertia and J is the angular momentum quantum number, $J = 0, 1, 2, \ldots$. Each energy level is $(2J+1)$-fold degenerate.

(a) In the low temperature limit ($\hbar^2/2I \gg kT$) determine approximate expressions for:

i. The rotation partition function. (2pts)
ii. The internal energy. (1pt)
iii. The specific heat. (1pt)

(b) In the high temperature limit ($\hbar^2/2I \ll kT$) determine approximate expressions for:

i. The rotation partition function. (2pt)
ii. The internal energy. (1pt)
iii. The specific heat. (1pt)

(c) How do the quantum results compare with the equipartition theorem for a classical rotator with two transverse degrees of freedom? (2pts)

6. Consider the "bogon," a spin 5/2 fermion with the charge of an electron but with a dispersion relationship

$$E = cp^3.$$

where $p \equiv |\vec{p}|$ Assume that your bogons are confined in a three dimensional sample and are non-ineracting.

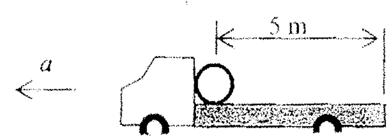
(a) Working in the grand canonical ensemble, determine the density, $\rho = \langle N \rangle / V$, as a function of the chemical potential, μ (or the fugacity, $z \equiv e^{\beta\mu}$), T, and V. (3pts)

(b) What is the bogonic Fermi energy (μ at $T = 0$) as a function of their density? (3pts) (*Hint:* This should not involve any complicated integrals).

(c) Derive a series expansion in z for the grand canonical free entropy, $\Xi = \frac{PV}{kT} = \log \mathcal{Z}$, where $\mathcal{Z}$ is the grand canonical partition function. (4pts)

*Spring 2012 Classical / Statistical Mechanics Qualifier

1. A section of steel pipe of radius 1 m and relatively thin wall is mounted as shown on a flat-bed truck. The driver of the truck, not realizing that the pipe has not been lashed in place, starts from rest and drives the truck forward with a constant acceleration of $0.5g$. As a result, the pipe rolls backward (relative to the truck bed) without slipping, and falls to the ground. The length of the truck bed is 5 m. (You can set $g = 10\text{m/s}^2$).



   (a) (2 pt) What is the relation between the acceleration of the center of mass of the pipe and the angular acceleration of rotation of the pipe about its center of mass?

   (b) (2 pt) What is the acceleration of the center of mass of the pipe as it rolls without slipping on the truckbed?

   (c) (2 pt) With what horizontal velocity does the pipe strike the ground?

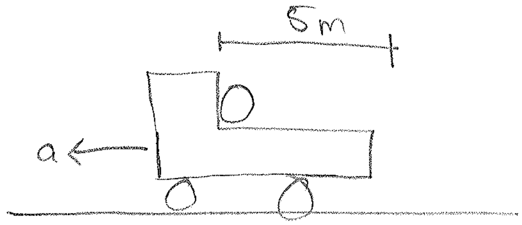
   (d) (1 pt) What is its angular velocity at this instant?

   (e) (2 pt) How far does it skid before beginning to roll without slipping, if the coefficient of friction between the pipe and ground is 0.3?

   (f) (1 pt) What is its linear velocity when its motion changes to rolling without slipping?

Spring 2012 CSM Qualifier

#1



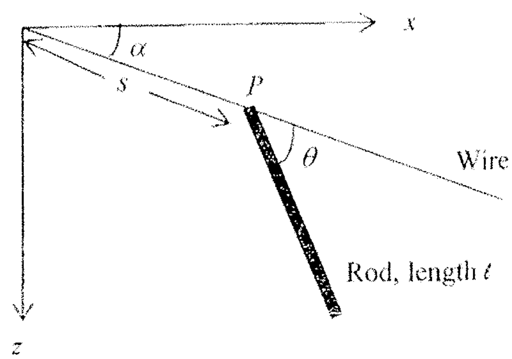
$a = 0.5g$, $g = 10 \frac{m}{s^2}$

a) $a = \alpha r$

b)

04

2. A homogeneous rod of length l and weight mg slides in the vertical x-z plane along a smooth frictionless wire which is inclined at an angle α as shown. The rod can pivot about its end (point P) in the x-z plane as it slides.



(a) (2 pt) Calculate the moment of inertia of the rod about the center of mass of the rod.

(b) (3 pt) Find the Lagrangian for this system in terms of the generalized coordinates θ and s.

(c) (3 pt) Determine the equations of motion in terms of the generalized coordinates θ and s.

(d) (2 pt) From these equations, determine if pure translational motion is possible (θ = constant) and, if so, for what values of θ.

#2

3. Consider the Lagrangian for a single particle described by the Lagrangian $L(\mathbf{r}, \dot{\mathbf{r}}, t)$.

(a) (1 pt) What is the canonical momentum $\mathbf{p}$ of the particle?

(b) (1 pt) What is the Hamiltonian of the particle? What are the natural or canonical variables of the Hamiltonian?

(c) (1 pt) What is the action for the system expressed in terms of trajectories for the particle, $\mathbf{r}(t)$, between two times t_1 and t_2.

(d) (2 pt) Consider an arbitrary infinitesimal variation in the particle trajectory, $\mathbf{r}(t) \to \mathbf{r}(t) + \delta\mathbf{r}(t)$, and require that the action W be stationary, $\delta W = 0$, provided endpoint variations vanish, $\delta\mathbf{r}(t_1) = \delta\mathbf{r}(t_2) = 0$. What is the resulting equation of motion of the particle?

(e) (1 pt) Suppose the endpoint variations are not zero, then what is δW?

(f) (1 pt) Suppose the system is invariant under a rigid coordinate translation:

$$\delta\mathbf{r} = \text{constant}: \quad \delta W = 0.$$

What do you then conclude about the momentum?

(g) (2 pt) Consider a time variation, δt_1, δt_2. Instead of changing the end times, we can change the time parameter of integration, $t \to t + \delta t(t)$, where $\delta t(t)$ is arbitrary but so chosen that $\delta t(t_{1,2}) = \delta t_{1,2}$. Then if we require δW changes only at the endpoints, $\delta W = G_2 - G_1$, where G_i depends only on dynamical variables at time t_i, $i = 1, 2$, what is the resulting equation of motion for dH/dt? (Hint: δW is stationary under interior variations of the trajectory.) Write the endpoint variation in terms of the Hamiltonian.

(h) (1 pt) If the system is translationally invariant in time, so under a rigid t translation, $\delta t = \text{constant}$, $\delta W = 0$, what do you conclude about the Hamiltonian?

#3

4. Consider the enthalpy $H = U + pV$ of a system of N particles of mass m attached to a reservoir with which it can exchange energy and particles.

(a) (2 pt) Use the thermodynamic identity for dH to derive the natural state variables of H.

(b) (2 pt) Derive expressions for the conjugate variables.

(c) (2 pt) Calculate the Maxwell relations.

(d) (2 pt) For an ideal gas, compute H in terms of its natural variables, recalling that for an ideal gas, the entropy is

$$S = kN\left(\frac{5}{2} + \frac{3}{2}\ln 2\pi mkT - \ln N/V\right),$$

in terms of the temperature T and the volume V.

(e) (2 pt) Show that, in this case, the relations found in part 4b are satisfied.

5. Imagine a system of N noninteracting spinless nonrelativistic bosons of mass m in a volume V in d spatial dimensions.

(a) (2 pt) If the possible single-particle energy levels are ε_j, what is the Bose-Einstein distribution function for the mean number of particles occupying the jth energy state, in terms of the temperature T and the chemical potential μ.

(b) (1 pt) What is the largest possible allowed value of the chemical potential?

(c) (1 pt) In d spatial dimensions, how many states are there in an element of phase-space $d^d p\, d^d x$?

(d) (1 pt) Derive a formula for the number of particles N in a macroscopic box of volume V in terms of the distribution given in part 5a. Use polar coordinates in momentum space to write

$$d^d p = p^{d-1}\, dp\, A(d), \quad A(d) = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

where the volume of a unit sphere embedded in d dimensions, $A(d)$, is given in terms of the gamma function.

(e) (2 pt) Show that in general as $T \to 0$ this formula cannot be satisfied, since the number of particles N is fixed, unless there is macroscopic occupation of the ground (lowest-energy) state. (This is Bose-Einstein condensation.)

(f) (2 pt) For $d = 3$ give a formula for the relative number of bosons in the ground state, in terms of the temperature T. (Hint:

$$\int_0^\infty dx \frac{x^{a-1}}{e^x - 1} = \Gamma(a)\zeta(a),$$

in terms of the Riemann zeta function $\zeta(a)$ defined by

$$\zeta(a) = \sum_{n=1}^{\infty} \frac{1}{n^a}.$$

(g) (1 pt) What happens when $d = 2$?

6. Consider a "lattice gas" of N_0 distinguishable atoms in a volume V that is split up into n cells of volume b. Each cell can either be empty, contain one atom, or contain two atoms bound into a molecule with binding energy ε. If there are N_1 singly occupied cells and N_2 doubly occupied cells, the energy of the system is $E = -N_2\varepsilon$.

   (a) (2 pt) Calculate the partition function $Z_n(N_1, N_2)$.

   (b) (2 pt) Calculate the Helmholtz free energy F when all the numbers N_1, N_2, and $n - N_1 - N_2$ are large. What are the natural (canonical) variables for F?

   (c) (2 pt) Calculate the chemical potentials of the atomic and molecular species.

   (d) (2 pt) Give an expression determining the volume fraction of molecules $f_2 = N_2/n$ at temperature T and density $f_0 = N_0/n$.

   (e) (2 pt) The interstellar density of hydrogen is about one hydrogen atom per cubic centimeter, and suppose this gas is at the temperature of the cosmic microwave background 2.73 K. (This is actually not realistic.) The binding energy for the hydrogen molecule is $\varepsilon = 4.5$ eV. Use this model to predict that a large fraction of the atoms will be bound into the molecular state. Why is this expected?

Classical Mechanics/Statistical Mechanics Qualifier

August 14, 2012

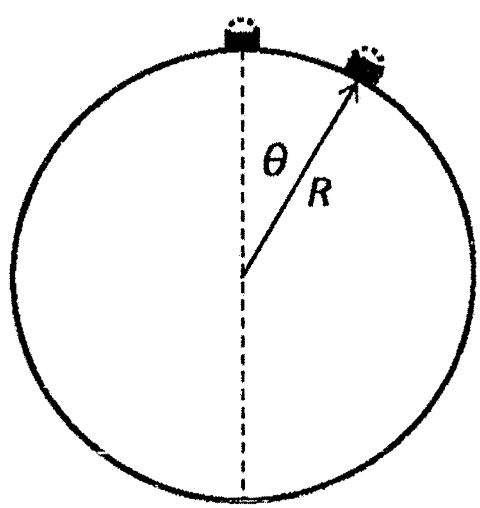
1. A very small object with mass m is placed on the top of a stationary large sphere with radius R (the upper hemisphere might represent the top of a grain silo). Choose the center of the large sphere to be the origin of the coordinate system, and the vertical axis as the z axis, and θ as the polar angle of the center of the small object relative to the coordinates defined here (see Figure). Assume the object is initially at rest on top of the large sphere, i.e., at $\theta = 0$, and that it begins to slide without friction down the spherical surface.

   (a) (2 pt) Obtain an expression for the velocity of the center of mass of the small sliding object as a function of θ.

   (b) (2 pt) Find the angle θ_c at which the sliding object loses contact with the spherical surface.

   (c) (4 pt) Now let the object be a small solid homogeneous sphere (that is, having uniform mass density) having a radius r and total mass m. Assume the small sphere rolls without slipping (but assume that there is no rolling friction) and that $r \ll R$. Again, obtain an expression for the velocity of the center of the rolling sphere as a function of θ.

   (d) (2 pt) Again, find the angle θ_c at which the small sphere loses contact with the large sphere. Is this larger or smaller than the angle found in part 1b? Give a physical reason for this.



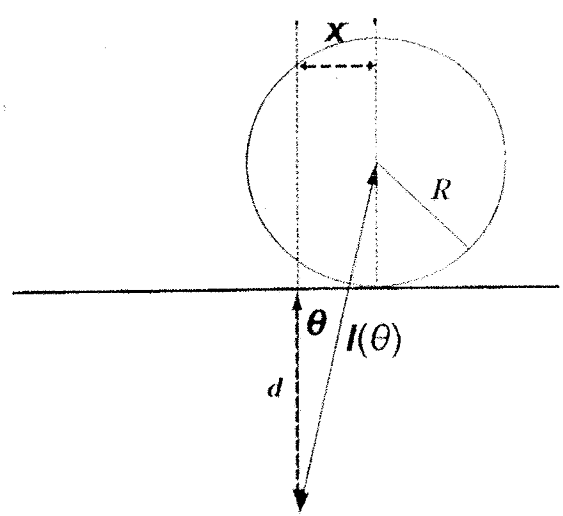
2. A homogeneous disk of radius R and mass M rolls without slipping on a horizontal surface. The disk's center is attracted to a point a distance d below the plane. See the figure. The force of attraction is proportional to the distance from the disk's center of mass, $l(\theta)$. This could be accomplished with a spring with spring constant k attached to the point d below the plane and the center of the disk.

(a) (3 pt) Find the Lagrangian for the system in terms of x and $\dot{x}$.

(b) (2 pt) Determine the equation of motion for x.

(c) (3 pt) Solve the equation of motion.

(d) (2 pt) What is the frequency of oscillation about the position of equilibrium?



3. Consider the bound motion of a particle in the non-central potential

$$V(\mathbf{r}) = \frac{\beta^2}{2mr^2}\sec^2\theta - \frac{k}{r},$$

where k and β are positive, real constants, and r, θ, and ϕ are spherical polar coordinates of the particle.

(a) (1 pt) Write down the Hamiltonian for this system in spherical coordinates.

(b) (2 pts) The characteristic function (sometimes called *Hamilton's* characteristic function) is separable in the form

$$W(r,\theta,\phi) = W_r(r) + W_\theta(\theta) + W_\phi(\phi).$$

Show that we may write $W_\phi(\phi) = \alpha_\phi\phi$, where α_ϕ is a constant, and calculate the azimuthal action

$$A_\phi = \oint_{\text{orbit}} d\phi \frac{\partial W_\phi}{\partial \phi}$$

associated with one orbit of the particle.

(c) (2 pts) Given the separable form above, show that $W_r(r)$ and $W_\theta(\theta)$ must satisfy

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \alpha_\phi^2\csc^2\theta + \beta^2\sec^2\theta = \alpha_\theta^2,$$

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} - \frac{2mk}{r} = 2mE,$$

where α_θ is a separation constant, and E is the energy.

(d) (3 pts) Show that the radial action

$$A_r = \oint_{\text{orbit}} dr \frac{\partial W_r}{\partial r}$$

associated with one orbit of the bound particle is given by (E is negative)

$$A_r = -2\pi\alpha_\theta + \sqrt{\frac{2\pi^2 mk^2}{-E}}.$$

[Hint: Evaluate the integral over the orbit by regarding it as a complex contour integral around a branch cut, and then use the residue theorem.]

(e) (2 pts) The action A_θ associated with motion in θ is difficult to calculate. However, it can be shown that the Hamiltonian can be written in terms of the action

$$H = -\frac{2\pi^2 mk^2}{(A_r + A_\phi + 2A_\theta + 2\pi\beta)^2}.$$

From this determine the elementary frequencies ν_a of the motion, using

$$\frac{\partial H}{\partial A_a} = \nu_a.$$

Are the orbits open or closed?

4. Consider a system of volume V in thermal equilibrium with a heat reservoir at temperature T, for which the canonical partition function is

$$Z = e^{aT^2V},$$

where a is a real, positive constant.

(a) (2 pts) Derive a formula for the Helmholtz free energy F.

(b) (2 pts) What are the normal (canonical) variables for F? Give an expression for the thermodynamic identity for dF and derive expressions for the conjugate variables.

(c) (2 pts) In terms of the normal variables from part 4b, derive an expression for the internal energy, U, and heat capacity at constant volume, c_v. How is U related to F?

(d) (2 pts) Does your expression for c_v agree with the prediction of the equipartition theorem? Explain your answer.

(e) (2 pts) Does your expression for c_v agree with the prediction of the Third Law of Thermodynamics? Explain your answer.

5. In this problem we will consider a lattice of N atoms, each with a spin of $1/2$. Because the atoms can be labeled by their position on the lattice, we will treat them as distinguishable particles, not as identical Fermions.

   In an external magnetic field, each atom can be in one of two possible energy states, $E_{\pm} = \pm\epsilon$.

   (a) (2 pts) The possible total energies for the spins can be written as $E_n = n\epsilon$, where n is an integer. What are the possible values for n? What is the number of distinct microstates, $\Omega(N, E_n) = \Omega(N, n)$, in terms of N and n, that have this energy? Remember, we are considering the atoms to be distinct particles.

   (b) (2 pts) Let us treat the number of microstates $\Omega(N, E_n)$ as the structure function for the microcanonical ensemble for the spin lattice. Therefore, we define the entropy as

   $$S(N, E) = k \ln \Omega(N, E),$$

   where k is Boltzmann's constant. Use this entropy to calculate the temperature T for the spin system. Show that there are energies where the temperature is negative. Explain the meaning of these negative temperatures.

   (c) (2 pts) What is the energy of the spin lattice as a function of temperature, $E(N, T)$? Show that your result makes physical sense in the limits $T \to 0$ and $T \to \infty$.

   (d) (2 pt) Write down the partition function $Z(N, T)$ in the canonical ensemble for the spin lattice.

   (e) (2 pts) Using the canonical ensemble, calculate the average energy for the spin lattice. Compare this result to what you found in part 5c for the microcanonical ensemble. Explain any differences.

6. This problem concerns Fermi gases in D spatial dimensions

(a) (2 pts) Consider a gas of noninteracting nonrelativistic Fermions in D dimensions. Show that the grand-canonical potential, $\psi = \ln \mathcal{Z}$, where $\mathcal{Z}$ is the grand partition function, has the form, ($\beta = 1/kT$)

$$\psi = V_D \gamma_D \int_0^\infty d\epsilon\, \epsilon^{D/2-1} \ln\left(1 + e^{-\beta(\epsilon-\mu)}\right),$$

where γ_D is a constant that is different for each dimension, μ is the chemical potential, and V_D is the D-dimensional volume.

(b) (2 pts) Write down an expression for the number of particles N and the total energy E of the ideal Fermi gas as an integral over the single-particle energy states ϵ.

(c) (2 pts) The grand-canonical potential is proportional to the pressure, because

$$p = kT\frac{\partial \psi}{\partial V},$$

so

$$\psi = \beta p V_D.$$

Using the expression above for ψ, show that

$$pV_D = \frac{2}{D}E.$$

(d) (2 pts) Show that the result from part 6c agrees with the ideal gas law in the $T \to \infty$ limit. You will need to consider the limit the of μ for large T, determined by the fact that the number of atoms in the gas is fixed.

(e) (2 pts) Finally, find an expression for the Fermi energy at $T = 0$ in D dimensions.

Mechanics and Statistical Mechanics Qualifying Exam
Spring 2013

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

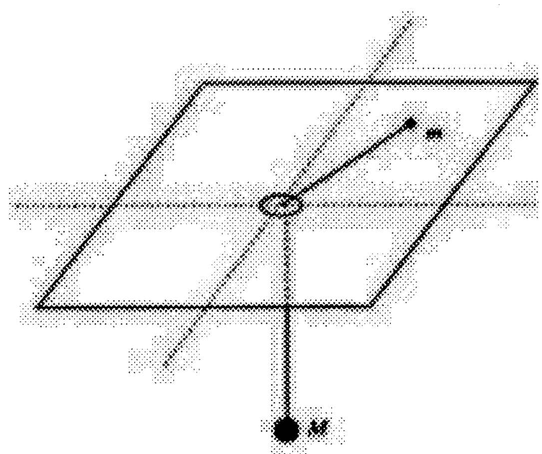
$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = 0.0833333$$
$$\zeta(2) = 1.64493 \qquad \zeta(-2) = 0$$
$$\zeta(3) = 1.20206 \qquad \zeta(-3) = 0.0083333$$
$$\zeta(4) = 1.08232 \qquad \zeta(-4) = 0$$

Problem 1: (10 Points)

A mass m moves on a frictionless table. It is tied to a string that runs through a hole in the table. A mass M hangs from the other end of the string and is acted upon by gravity. M is constrained to move vertically and the hole in table is small and smooth (frictionless).



a. For a mass m orbiting at radius r and velocity v with mass M stationary, determine an equation relating r and v. **(2 Points)**

b. Now imagine replacing the mass M with a force F provided by your hand. What happens if you pull the string to shorten r, what is conserved? How much work, ΔW, is done to change r by Δr? Put your answer in terms of r. **(2 Points)**

c. By pulling the string a distance $d < r$, how does the speed of mass m change? **(2 Points)**

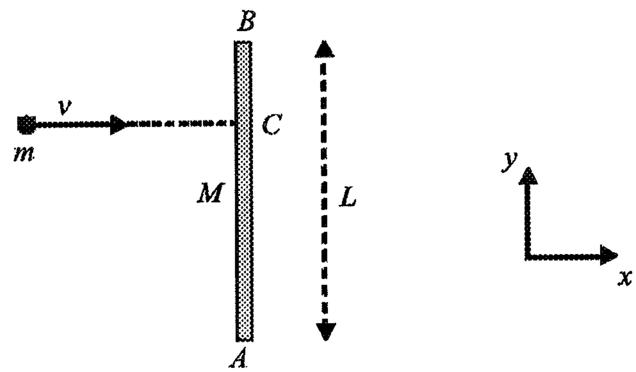
d. Using the expression for ΔW ,in terms of r and Δr from b.), how much work is done to change the orbital radius from r to $r/2$? **(2 Points)**

e. What is the change in angular frequency in part d.)? (Show this for the change from r to $r/2$) **(1 Points)**

f. For the change described in part d.), does the system obey the work energy theorem? **(1 Points)**

Problem 2 (10 Points):

A thin uniform rod of mass M and length $(\bar{AB}) = L$ lies on a horizontal frictionless surface aligned along the y direction as shown below. An object with mass m moving along the x direction with a speed of v collides with the rod at point C.



a. What is the moment of inertia of the rod about point A? **(1 Points)**

b. At what point should the object hit the rod so that immediately after the collision, the rod has pure rotation about the point A? Express your answer for $(\bar{AC})$ in terms of L. **(3 Points)**

c. Now assume the object with mass m collides with the rod at point C such that $(\bar{AC}) = 3L/4$ and the collision is elastic. After the collision, when the rod becomes aligned along the x direction for the first time, what is the distance the center of mass of the rod has moved? For part (c) and forward, assume that $m = M$ (which simplifies the problem) and express your answer in terms of L only. **(4 Points)**

d. At the same moment in time as (c), what is the distance the object with mass m has moved? **(2 Points)**

Problem 3 (10 Points):

A uniform ladder of length ℓ and mass m has one end on a smooth frictionless, horizontal floor and the other end against a smooth, frictionless vertical wall. The ladder is initially at rest making an angle θ_o with respect the horizontal.

a. Using the angle θ (with respect to the horizontal) as the only Lagrangian coordinate, derive an appropriate equation of motion for the time period before the ladder loses contact with the vertical wall. **(2 Points)**

b. Find the height of the upper end of the ladder when the ladder loses contact with the vertical wall. **(2 Points)**

c. Derive a new Lagrangian using the angle θ and the x coordinate of the top of the ladder. **(2 Points)**

d. Find the equations of motion for both coordinates using a Lagrange multiplier. **(2 Points)**

e. What physical quantity in the problem does the Lagrange multiplier represent? **(2 Points)**

f. Repeat part (b) using your new equations of motion. **(2 Points)**

Problem 4 (10 Points):

Consider a rubber band of length L which is being stretched by external force f.

a. Write down the thermodynamic identity (1st law of thermodynamics) relating the change in the internal energy dU to infinitesimal change in length dL, and to the heat TdS. **(2 Points)**

b. In one experiment the length of the band is fixed to $L = 1\,\text{m}$ and the temperature of the band $T = 300\,\text{K}$ is raised by a small amount $\Delta T = 3\,\text{K}$. This causes the force needed to maintain the length of the band to increase by the amount $\Delta f = 1.2\,\text{N}$. In another experiment, the band is stretched from L to $L + \Delta L$ at constant temperature T. As a result, the band exchanges heat with the environment.

1. Find a differential expression for dF, the free energy, in terms of the thermodynamic variables. **(2 Points)**
2. Using your result for the free energy, find the appropriate Maxwell relation for this process. **(2 Points)**

c. What is the amount of heat exchanged with the environment for $\Delta L = 2\,\text{cm}$? **(2 Points)**

d. Is the heat released or absorbed by the rubber band? **(2 Points)**

Problem 5 (10 Points):

A given solid state system consists of N spin 1 atoms, so that the projection of spin on a quantization axis $\sigma \in \{-1, 0, 1\}$. The energy of the i-th atom is

$$E(\sigma_i) = \epsilon\sigma_i^2 + h\sigma_i,$$

where ϵ and h are constants. In this problem you will calculate the partition function in different ensembles.

a. The canonical ensemble: Our goal is to calculate the free energy $F(T, h, N)$.

1. Calculate the partition function $Z(T, h, N)$ in the canonical ensemble. **(1 Points)**
2. From the result in 1., determine the free energy in the canonical ensemble, $F(T, h, N)$. **(1 Points)**
3. What is the magnetization in this ensemble, $M(T, h, N)$? **(2 Points)**

b. The microcanonical ensemble: Our goal is to calculate the entropy S in terms of the extensive quantities, which are the internal energy U, the magnetization M, and the number of atoms, N. Denote the number in each spin orientation as $n_{(-)}$, $n_{(0)}$ and $n_{(+)}$, respectively.

1. Calculate $\Omega(N, n_{(+)}, n_{(-)})$, the number of micro-states available to the system of N atoms for fixed values of $n_{(+)}$ and $n_{(-)}$. **(2 Points)**
2. The total magnetization of the system is given by

   $$M = \mu_0(n_{(+)} - n_{(-)})$$

   and the total internal energy is given by

   $$U(N, n_{(+)}, n_{(-)}) = \epsilon(n_{(+)} + n_{(-)}) + h(n_{(+)} - n_{(-)})$$

   Use these relations and your answer to the question above to determine the entropy in the microcanonical ensemble, $S(U, M, N)$. **(2 Points)**
3. What is the temperature in this ensemble, $T(U, M, N)$ (Hint: use Stirlings approximation)? **(2 Points)**

Problem 6 (10 Points):

A classical system of N distinguishable noninteracting particles each with a mass m is placed in a three-dimensional harmonic well:

$$U(r) = \frac{x^2 + y^2 + z^2}{2V^{2/3}}$$

a. Find the partition function. **(4 Points)**

b. Find the Helmholtz free energy. **(1 Point)**

c. Taking V as an external parameter, find the thermodynamic force $\tilde{P} = -\left(\frac{\partial F}{\partial V}\right)_T$ conjugate to this parameter, exerted by the system. **(1 Points)**

d. Express the equation of state in terms $\tilde{P}, V, T$. **(1 Point)**

e. Find the entropy, internal energy, and total heat capacity at constant volume. **(3 Points - 1 Point for each)**

You may need the following integration formula:

$$\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{(2n)!}{n! 2^{2n+1}} \sqrt{\frac{\pi}{a^{2n+1}}}$$

Mechanics and Statistical Mechanics Qualifying Exam
Fall 2013

Possibly Useful Information

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$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

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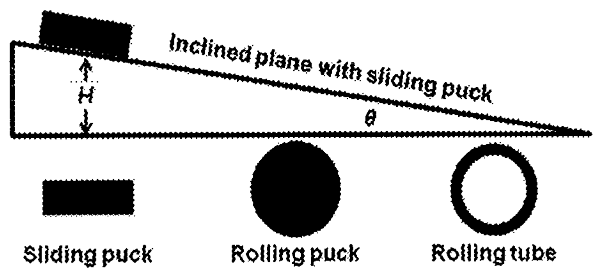
\end{array}$$

Problem 1: (10 Points)

Imagine a race between different objects rolling or sliding down a simple inclined plane. Let the mass of each object be M, the angle of the inclined plane be Θ, and the height be H, where the center of mass of all the objects change by H over a race. You will consider a cylinder with walls of negligible thickness and puck (short solid cylinder) in this problem. Assume the rolling objects roll without slipping.



a. Determine the velocity of a sliding puck (short cylinder) at the bottom of the inclined plane. **(3 Points)**

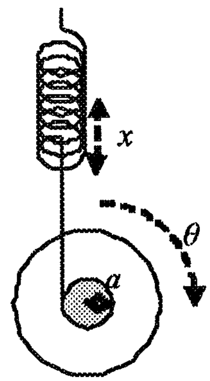
b. Determine the velocity of a round, symmetrical, smooth rolling object whose moment of inertia is $I = \alpha M R^2$, where α is a geometrical factor and R is the radius. **(3 Points)**

c. Show which wins, a sliding or rolling puck? **(2 Points)**

d. Discuss how your result depends on α, M and R. Use your answer to determine if a rolling puck or tube of mass M and negligible thickness would win a race down the incline. **(2 Points)**

Problem 2 (10 Points):

A yo-yo with a mass of m and moment of inertia I falls straight down and spins due to gravity. The string unwinds from the yo-yo around an axle of radius a. The other end of the string is attached to an ideal spring with spring constant k. Define x as the extension of the spring measured with respect to its unstretched length.



a. Using the generalized coordinates x and Θ write the Lagrangian for this system. **(2 Points)**

b. Derive the Lagrange equations of motion. **(2 Points)**

c. Derive a differential equation that describes the oscillation of the spring while the yo-yo is falling down and unwinding. **(2 Points)**

d. What is the oscillation frequency of the spring while the yo-yo is falling down and unwinding? **(2 Points)**

e. Consider the limit of a thin axle ($ma^2 \ll I$) and solve the differential equation found in (c) for the variable x. **(2 Points)**

f. Explain in words the motion described by the equation found in (e). **(1 Points)**

Problem 3 (10 Points):

A spherical pendulum consists of a particle of mass m that is in a gravitational field $\vec{g}$ and is constrained to move on the surface of a sphere of radius ℓ. Use the polar angle θ (measured from the downward vertical) and the azimuthal angle ϕ.

a. Derive the Lagrangian for this system. **(2 Points)**

b. Derive the Hamiltonian for this system. **(2 Points)**

c. Find the Hamiltonian equations of motion. **(1 Points)**

d. Consider the system is undergoing uniform circular motion in ϕ at constant polar angle θ_o. Assuming small variations in θ, expand the Hamiltonian in θ to second order around $\theta = \theta_o$. **(4 Points)**

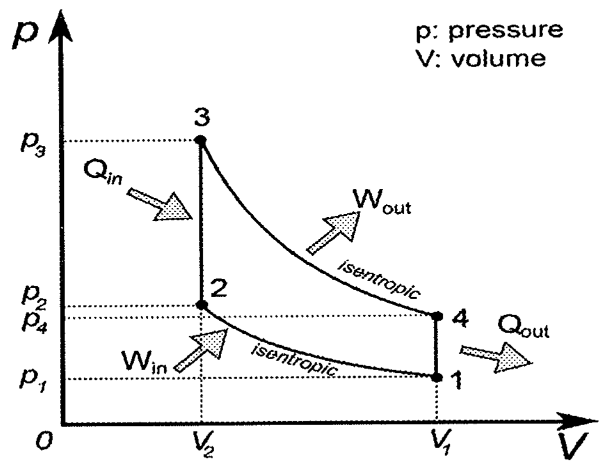
e. Show that the motion in θ is simple harmonic with angular frequency given by:

$$\omega^2 = \frac{g}{\ell \cos\theta_o}(1 + 3\cos^2\theta_o).$$

(1 Points)

Problem 4 (10 Points):

The diesel engine uses the Otto cycle. Below is the P-V diagram for this process. Assume a monatomic ideal gas.



a. Find the work done during each cycle. **(3 Points)**

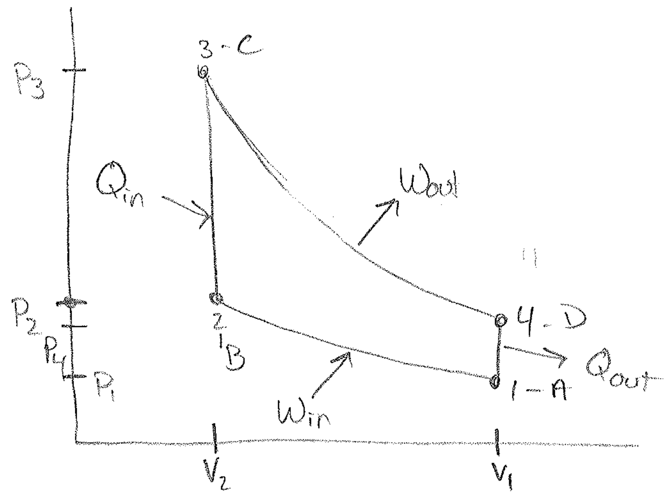
b. Find the heat exchanged each cycle. **(3 Points)**

c. What is the efficiency of this engine? **(3 Points)**

d. To produce work, which way does the cycle operate? Clockwise or counter clockwise in the diagram. **(1 Points)**

Aug 2013

Stat Mech #1



* Assume ideal monatomic gas

$\rightarrow C_V = \frac{3}{2}R$

$C_P = \frac{5}{2}R \quad > \gamma = 5/3$

$P_A = P_1$ $\quad$ $P_B = P_2$ $\quad$ $P_C = P_3$ $\quad$ $P_D = P_4$

$V_A = V_1$ $\quad$ $V_B = V_2$ $\quad$ $V_C = V_2$ $\quad$ $V_D = V_1$

$T_A = \frac{P_1V_1}{nR}$ $\quad$ $T_B = \frac{P_1V_1^{5/3}}{nRV_2^{2/3}}$ $\quad$ $T_C = \frac{P_1P_3V_1^{5/3}}{nRP_2V_2^{2/3}}$ $\quad$ $T_D = \frac{P_4V_1}{nR}$

$$T_B = T_A\left(\frac{V_A}{V_B}\right)^{\gamma-1} = \frac{P_1V_1}{nR}\left(\frac{V_1}{V_2}\right)^{2/3} = \frac{P_1V_1^{5/3}}{nRV_2^{2/3}}$$

$$T_C = T_B\left(\frac{P_C}{P_B}\right) = \frac{P_1V_1^{5/3}}{nRV_2^{2/3}}\left(\frac{P_3}{P_2}\right)$$

$$T_D = T_A\left(\frac{P_D}{P_A}\right) = \frac{P_1V_1}{nR}\left(\frac{P_4}{P_1}\right)$$

a) Find the work done during the cycle.

$$W_{A\to B} = \frac{P_BV_B - P_AV_A}{1-\gamma} = \frac{P_2V_2 - P_1V_1}{1-5/3} = \frac{-3}{2}(P_2V_2 - P_1V_1)$$

$$W_{C\to D} = \frac{P_DV_D - P_CV_C}{1-\gamma} = \frac{P_4V_1 - P_3V_2}{1-5/3} = \frac{-3}{2}(P_4V_1 - P_3V_2)$$

$W_{B\to C} = 0$ b/c isochoric

$W_{D\to A} = 0$ b/c isochoric

$$\Rightarrow W_{TOT} = \frac{3}{2}(P_1V_1 - P_2V_2) + \frac{3}{2}(P_3V_2 - P_4V_1)$$

$$= \frac{3}{2}\left(V_1[P_1 - P_4] + V_2[P_3 - P_2]\right)$$

b) Find the heat exchanged each cycle

$Q_{A\to B} = 0$ b/c adiabatic

$$Q_{B\to C} = nC_v\Delta T$$
$$= n\left(\tfrac{3}{2}R\right)\left(\frac{P_1P_3V_1^{5/3}}{nRP_2V_2^{2/3}} - \frac{P_1V_1^{5/3}}{nRV_2^{2/3}}\right)$$
$$= \frac{3}{2}\frac{P_1V_1^{5/3}}{V_2^{2/3}}\left(\frac{P_3}{P_2}-1\right)$$

$Q_{C\to D} = 0$ b/c adiabatic

$$Q_{D\to A} = nC_v\Delta T$$
$$= n\left(\tfrac{3}{2}R\right)\left(\frac{P_1V_1}{nR} - \frac{P_4V_1}{nR}\right)$$
$$= \frac{3}{2}(P_1 - P_4)$$

$$Q_{TOT} = \frac{3}{2}\left[\frac{P_1V_1^{5/3}}{V_2^{2/3}}\left(P_3/P_2 - 1\right) + P_1 - P_4\right]$$

c) What is the efficiency of the engine?

$$\eta = 1 - \left|\frac{Q_{out}}{Q_{in}}\right|$$
$$= 1 - \left|\frac{\frac{3}{2}(P_1 - P_4)}{\frac{3}{2}\frac{P_1V_1^{5/3}}{V_2^{2/3}}\left(\frac{P_3}{P_2}-1\right)}\right|$$
$$= 1 - \left|\frac{P_1 - P_4}{\frac{P_1V_1^{5/3}}{V_2^{2/3}}\left(\frac{P_3}{P_2}-1\right)}\right|$$

d) Which direction does the engine operate?

CW

Problem 5 (10 Points):

An electron confined to a 1D ring of radius R in a perpendicular magnetic field B has energy levels

$$\begin{aligned} E(m,\phi) &= \frac{\hbar^2}{2mR^2}\left(m - \frac{\phi}{\phi_0}\right)^2 \\ &= \epsilon\left(m - \frac{\phi}{\phi_0}\right)^2 \end{aligned}$$

where $\phi = \pi R^2 B$ is the magnetic flux through the ring, ϕ_0 is the magnetic flux quantum ($\phi_0 = e/\hbar c$) and m is the angular momentum quantum number, $m = 0, \pm 1, \pm 2, \ldots$. In this problem we will consider a set of N rings, and neglect the spin of the electron.

a. In the high temperature limit ($\epsilon \equiv \frac{\hbar^2}{2mR^2} << kT$) determine approximate expressions for:

1. The canonical partition function, $Z(T,N,B)$. **(1 Point)**
2. The internal energy, $U(T,N,B)$. **(1 Point)**
3. The magnetization, $\mathcal{M} \equiv \frac{\partial U}{\partial B}$. **(2 Points)**

b. In the low temperature ($\frac{\hbar^2}{2mR^2} >> kT$) and weak field ($-\phi_0/2 < \phi < \phi_0/2$) limit determine approximate expressions for:

1. The canonical partition function, $Z(T,N,B)$. **(1 Point)**
2. The internal energy, $U(T,N,B)$. **(1 Point)**
3. The magnetization, $\mathcal{M} \equiv \frac{\partial U}{\partial B}$. (If your result is quite complicated, make sure that you keep only the leading term in part (i) above. **(2 Points)**

c. Are your results similar or different? Explain either why they are similar or why they differ. **(2 Points)**

Problem 6 (10 Points):

Consider a system consisting of a large number N of distinguishable, noninteracting particles. Each particle has only two (nondegenerate) energy levels: 0 and $\epsilon > 0$. Let E/N denote the mean energy per particle in the thermodynamic limit $N \to \infty$.

a. What is the maximum possible value of E/N if the system is not necessarily in thermodynamic equilibrium? **(1 Point)**

b. What is the value of E/N if the system is in equilibrium at temperature T? **(4 Points)**

c. Explicitly take the low ($T \to 0$) and high ($T \to \infty$) limits of your result of part a). Sketch your results. **(2 Points)**

d. Find the entropy per particle $s = S/N$. **(2 Points)**

e. Explicitly take the $T \to 0$ and $T \to \infty$ limits of your result of part d). Explain. **(2 Points)**

Classical Mechanics and Statistical/Thermodynamics

January 2014

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\,e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\,e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{iax-bx^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

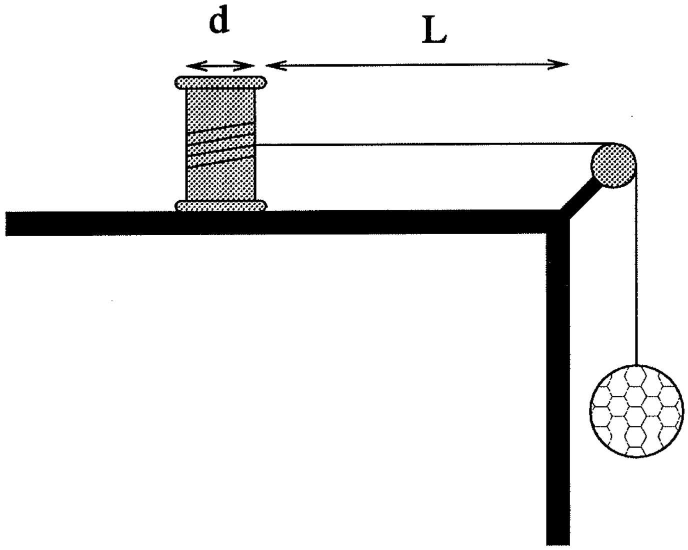
$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Physical Constants:

Coulomb constant $\mathrm{K} = 8.998 \times 10^9$ N-m$^2/C^2$
$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$
$\mu_0 = 4\pi \times 10^{-7}\mathrm{T\ m/A}$
electronic charge $e = 1.60 \times 10^{-19}\mathrm{C}$
electronic mass $m_e = 9.11 \times 10^{-31}\mathrm{kg}$
Density of pure water: $1.00\mathrm{gm/cm}^3$.
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}\mathrm{J/K}$
Planck's constant: $\hbar = 6.63 \times 10^{-34}\mathrm{m^2kg/s}$
speed of light: $c = 3.00 \times 10^8\mathrm{m/s}$

Classical Mechanics

1. A solid spool of uniform density has a mass m_s and diameter d. It rests on a frictionless table and is attached by a massless string to a hanging ball with mass m_b and radius r. The string runs over an ideal massless, frictionless pulley as shown. The system is released from rest with the spool a distance L from the edge of the table. When it is released, the spool starts to slide and rotate as it is pulled by the string. Denote the acceleration of gravity by the constant, g.



(a) Consider the spool to be a uniform cylinder and calculate the moment of inertia of the spool rotating about its center of mass. Do not simply state the result. (1 point)

(b) Find the constant acceleration of the spool as it moves to the right, in terms of the variables given, and g, the acceleration due to gravity. (3 points)

(c) What is the velocity of the ball when the spool has travelled a distance L and reaches the edge of the table? (3 points)

(d) What is the ratio of the total kinetic energy of the spool (translational and rotational) to the kinetic energy of the ball? (3 points)

2. *Wilberforce Pendulum:* A mass m is suspended from the ceiling by a long coiled spring, forming a *Wilberforce Pendulum.* The system can oscillate in the vertical direction (z) and twist about its vertical axis, (θ). The Lagrangian for the system is:

$$L = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}kz^2 - \frac{1}{2}\Omega\theta^2 - \frac{1}{2}\lambda z\theta$$

(a) Explain each term in the Lagrangian. (1 point).

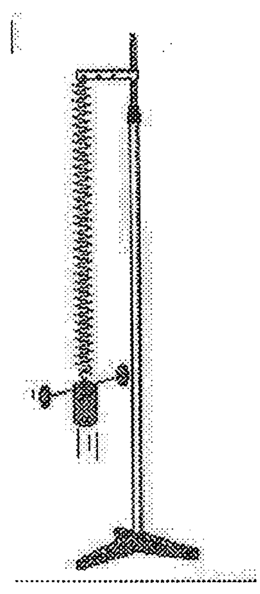
(b) Determine the equations of motion for the system. (3 points).

(c) Determine the frequencies of the normal modes. (3 points).

(d) At time $t = 0$ the ball is displaced upwards a distance z_0 from its equilibrium position, without any twist, and then released from rest. Determine the motion. You might find it helpful to define:

$$\begin{aligned} \omega_z &\equiv \sqrt{k/m} \\ \omega_\theta &\equiv \sqrt{\Omega/I} \end{aligned}$$

and/or other constants to simplify your algebra. (3 points)



3. Let $[F, G]$ denote the Poisson bracket of the quantities F and G.

   (a) Show that: (2 points)

      i. $[p_i, p_j] = 0$

      ii. $[p_i, q_j] = -\delta_{i,j}$

      where q_j is a co-ordinate and p_j is its canonical momentum.

   (b) If $\vec{L}$ is the angular momentum in three dimensions given by

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

      show that: (5 points)

      i. $[L_i, L_j] = L_k$, for i, j, k in cyclic order.

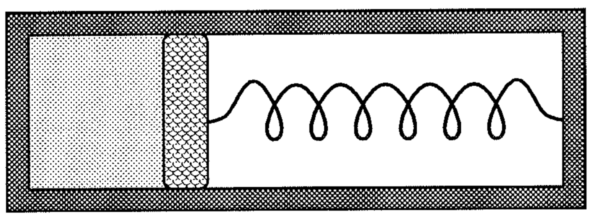
      ii. $[L^2, L_i] = 0$

   (c) Based on the above, can L_x, L_y, and L_z serve as a set of canonical momenta for some set of generalized coordinates in a central force problem? Why or why not? (3 points)

Statistical Mechanics

4. A thermodynamic system consists of n moles of an ideal mono-atomic gas confined in an insulating cylinder by a piston of cross-sectional area A. Initially the piston is locked into place so that the gas is in equilibrium with an initial volume V_0, temperature T_0, and pressure P_0. A spring of spring constant k is attached to the piston, but is initially neither stretched nor compressed. The volume occupied by the spring on the right hand side of the cylinder is a vacuum, (there is no gas to exert a pressure back on the piston.

   When the piston is released, it will compress the spring. Eventually, the system will come to equilibrium with some of the internal energy of the gas transferred to the potential energy stored in the spring which has been compressed a distance x. Denote the final volume, temperature and pressure by V_1, T_1 and P_1. You should neglect the heat capacities of the cylinder walls, the piston and the spring.



   It is trivially true that $(V_1 - V_0) = Ax$; in parts (a) and (b) you are asked to use physics to determine other relationships between x and the variables in the problem.

   (a) Find the relationship between the change in temperature of the gas, $T_0 - T_1$ and the final compression of the spring, x, in terms of the variables above. (1 point)

   (b) What is the relationship between the final pressure, P_1 and the final compression of the spring, x, in terms of the variables above? (1 point)

   (c) You are told that the when equilibrium is reached, the volume of the gas has doubled ($V_1 = 2V_0$). What will be the ratio of the final temperature to the initial temperature? Your answer should be dimensionless. (5 points)

   (d) If the above case, what will be the ratio of the final pressure to the initial pressure? Your answer should be dimensionless. (3 points).

5. Consider a system of N non-interactiong, distinguishable spin-1/2 particles in a magnetic field B at an initial temperature T. Each spin has energy $\pm g\mu_0 B/2$ depending on whether it is alligned $(-)$ or anti-alligned $(+)$ with the applied magnetic field. They have no other energy.

   (a) Show that the entropy is given by:

   $$S(N,T;B) = Nk_B \ln\left(2\cosh\frac{g\mu_0}{2k_BT}\right) - \frac{Ng\mu_0 B}{2T}\tanh\frac{g\mu_0 B}{2k_BT}.$$

   (3 points)

   (b) The magnetic field is reduced adiabatically. Show that if the field B is reduced to half its value, the temperature will also be reduced to half its value. (1 point)

   (c) Evaluate the entropy of the spin system in the limits $g\mu_0 B/k_BT \to \infty$ and $g\mu_0 B/k_BT \to 0$. Explain your answers. (1 point)

   (d) The above spin system is placed in thermal contact with an ideal mono-atomic gas of N particles in a volume V where the canonical partition function for a single gas atom is:

   $$Z_{\text{atom}} = CVT^{3/2}$$

   (the value of N is the same as the number of spins). What is the entropy of the ideal gas by itself? (2 points)

   (e) Initially the two systems are in equilibrium at temperature T_0 and $g\mu_0 B/k_BT_0 >> 1$. If we adiabatically reduce the magnetic field to zero and the two systems remain in thermal contact, what is the final temperature, T_1, of the gas? (3 points)

6. Consider a non-relativistic gas of N electrons confined on a two dimensional surface of area $A = L^2$.

   (a) Show that the density of states of the gas is

   $$g(\epsilon) = \frac{mA}{\pi\hbar^2}$$

   (3 points)

   (b) Find the Fermi energy E_F (in terms of N and A) **(2 points)**

   (c) Find the average energy per electron at $T = 0$ (in terms of E_F) **(2 points)**

   (d) Write down the expression for the total number of particles at temperature $T \neq 0$. Use this expression to find $\mu = \mu(T)$. **(3 points)**.

   (e) Take the $T \to \infty$ limit. Do you recover the classical limit? What about the $T \to 0$ limit? **(1 point)**

   (f) Calculate (in order of magnitude) E_F for a gas of density $\frac{N}{A} = 10^{16} m^{-2}$. Is this Fermi gas degenerate at room temperature and why or why not?**(1 point)**

Classical Mechanics and Statistical/Thermodynamics

August 2014

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{iax-bx^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

Physical constants:

$$\begin{aligned} \hbar &= 1.05457\times 10^{-34}\mathrm{m^2kgs^{-1}} \\ m_{\mathrm{electron}} &= 9.109\times 10^{-31}\mathrm{kg} \\ k_B &= 1.38\times 10^{-23}\mathrm{m^2kg\cdot s^{-2}K^{-1}} \end{aligned}$$

Classical Mechanics

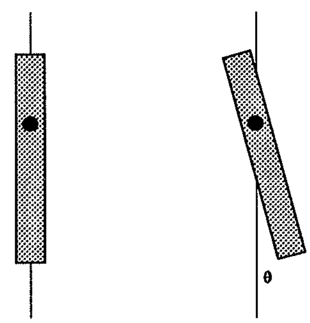
1. A bar of mass M and length L has a hole drilled in it one third along its length. This is used as a frictionless pivot point.

   (a) Calculate the moment of inertia of the system about the pivot point. You may assume that the rod is one-dimensional. (2 points).

   (b) In equilibrium, the bar hangs vertically ($\theta = 0$). The bar is rotated about its pivot, and released from rest when $\theta = \pi/2$. Caculate the angular velocity of the system when $\theta = 0$. (3 points)

   (c) Find the force exerted by the pivot in the above case just as θ passes through zero. (2 points)

   (d) Find an expression for the period of oscillation for small angles $\theta << 1$. Express you answer in terms of the quantities given, and g, the acceleration due to gravity. (3 points).



2. Consider the Earth as a frame rotating about its axis with frequency ω. In this frame of reference particles are subject to the Coriolis force given by:

$$\vec{F_c} = -2m(\vec{\omega} \times \vec{v_r})$$

where $\vec{v_r}$ is the velocity of the particle relative to the rotating frame and m is the mass of the particle. Choose the z-axis to be normal to the surface of the Earth and pointing outward from the center; the y-axis to point North, and the x-axis to point East. Let $z = 0$ be on the surface of the Earth. Assume you are in the Northern Hemisphere at a colatitude θ (that is, the angle between the z-axis and $\vec{\omega}$ is θ).

The goal of this problem is to calculate the "horizontal" deflection of an object thrown "straight up" in the rotating reference frame of a person on the surface of the Earth.

(a) A particle close to the surface of the Earth ($z << R$ where R is the radius of the Earth) moves under the influence of gravity and the Coriolis force. (You may neglect the centrifugal force in this non-inertial frame for this problem). Write the equations of motion in the x, y, and z directions, under the approximations that $|v_z| >> |v_x|$ and $|v_z| >> |v_y|$. (2 points)

(b) The particle starts from rest (in the rotating frame) and is dropped from a height h above the ground, and it arrives at the ground with a speed $\sqrt{2gh}$. Find the magnitude and direction of the deflection of its end point due to the Coriolis force. (3 points)

(c) The particle is now thrown vertically upward from the ground with an initial speed v_0 so that it reaches a maximum height h and then falls back to the ground. Find the magnitude and direction of the Coriolis deflection. (3 points)

(d) Compare your results from parts (b) and (c) above. Explain why they are the same or different. (1 point).

(e) In the above calculation of the deflection we neglected the centrifugal force; was that reasonable? Why? (1 point)

3. Consider a particle confined to one dimension subject to a force increasing linearly in time, $F = At$.

   (a) Find the Hamiltonian of the system. (2 points)

   (b) What is the corresponding Hamilton-Jacobi equation? (2 points)

   (c) Show that Hamilton's principal function, S can be written in the form:

   $$S = \frac{1}{2}At^2x + \alpha x - \phi(t),$$

   where α is a constant. (2 points)

   (d) (x pts) Solve the resulting equation for $\phi(t)$, finding the position and momentum as a function of time. (3 points)

   (e) Verify your solution by directly solving Newton's equations of motion for this force and compare this to your answer above. (1 point)

Statistical Mechanics

4. The latent heat of melting ice is L per unit mass. A bucket contains a mixture of water and ice at the melting point of the ice, T_0. We want to use an ideal, maximally efficient, cyclic (reversible) refrigerator to freeze a mass m amount more of the liquid water in the bucket into ice. Assume that this refrigerator is powered by some external source of work, and that it rejects all heat to a **finite** external reservoir of constant heat capacity C and initial temperature T_0.

   (a) What is the change in the entropy of:

      i. The ice and water mixture in the bucket, S_{bucket}, (2 points)

      ii. The external reservoir, S_{res}, (2 points)

      iii. The refrigerator aparatus S_{fridge} itself when run over several cycles, (1 point)

   during the process where a mass m of the water is turned into ice?

   (b) What is the change in the Gibbs free energy during the process? (1 point)

   (c) What is the minimum mechanical work required to run the refrigerator for this process? *Hint:* The most efficient process will have the smallest **total** entropy change. (4 points)

5. A *lattice gas* is a system of volume V divided into N_s cells of volume b so that $N_s = V/b$. Each cell can have either one or no atoms in it ($n_i \in \{0, 1\}$) and has c nearest neighbors. The energy of the system is:

$$E(\{n_i\}) = -\left[\frac{1}{2}\sum_i \sum_{j \in n.n.} w n_i n_j\right]$$

where w is an interaction between adjacent atoms, and the sum over j is restricted to the nearest neighbors of i and the factor of $1/2$ is present to avoid double counting. We will work in the grand canonical ensemble so that the total number of particles,

$$N = \sum_i n_i$$

is not fixed.

(a) Write down an expression for the grand canonical partition function $\mathcal{Z}(T, V, \mu)$ as a sum over the values of n_i. Because this expression depends on the product of n_i and n_j, you will not be able to evaluate it. (1 point)

(b) In the mean field approximation we rewrite the energy as

$$E_{mf}(\{n_i\}) = -\left[\frac{1}{2}\sum_i cw\bar{n}\, n_i\right]$$

where we have replaced n_j by it's average value, $\bar{n}$. This value must be determined self-consistently, so for the moment treat it as simply a constant. Calculate $\mathcal{Z}(T, V, \mu; \bar{n})$, performing the sum over n_i. (2 points)

(c) From your partition function calculate the average value of the number of atoms, $\langle N \rangle = N(T, V, \mu; \bar{n})$. (3 points)

(d) In order for this result to be self-consistent, we must have that

$$\frac{N(T, V, \mu; \bar{n})}{N_s} = \bar{n}$$

which can be thought of as the intersection of the function $N(\bar{n})/N_s$ with the "function" $f(\bar{n}) = \bar{n}$, (that is, a line of slope unity). Explain the behavior of this intersection as μ is varied from large negative to large positive values. (4 points).

6. Consider a photon gas enclosed in a volume V and in equilibrium at a temperature T. The photon is a massless, spinless particle so that $\epsilon(p) = pc = \hbar\omega$, where c is the speed of light. Photons can have two possible transverse polarizations.

   Throughout this problem you may reduce integrals to quadrature, i.e. definite integrals containing no physical parameters and thus equivalent to simple constants.

   (a) What is the chemical potential of the gas? (Recall that the number of photons is *not* conserved). **(1 point)**

   (b) Write down the grand thermodynamic potential Ω and replace the sum with an appropriate integration. (**2 points)**

   (c) Using your result of part a) and b) extract the temperature dependence of the free energy F. **(2 point)**

   (d) Using your result of part c) extract the temperature dependence of the pressure P and the entropy S. **(2 point)**

   (e) Using your previous results, find the relationship between the energy E and the free energy F. **(1 point)**

   (f) Write down the energy as $E = E(P, V)$. **(1 point)**

   (g) Find the temperature dependence of the number of photons N. **(1 point)**

Classical Mechanics and Statistical/Thermodynamics

January 2015

Possibly Useful Information

Handy Integrals:

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$$\int_0^\infty x\,e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

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$$\int_{-\infty}^\infty e^{i a x - b x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

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$$g_p(1) = \zeta(p) \qquad\qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad\qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad\qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad\qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad\qquad \zeta(-4) = 0$$

Classical Mechanics

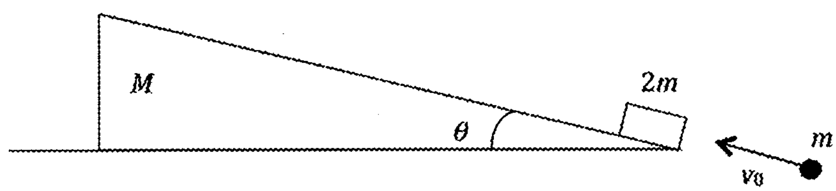
1. A block with mass $2m$ is initially at rest at the bottom edge of a wedge of mass M and angle θ. The wedge sits on a horizontal table. The table-wedge surface and block-wedge surface are frictionless. At $t = 0$ a bullet of mass m and velocity v_0 traveling parallel to the upper surface of the wedge collides with and embeds in the block.

   (a) What is the maximum height above the table reached by the block? (4 points)

   (b) At that time, what is the speed of the wedge? (2 point)

   (c) How much time does it take for the block to reach its maximum height? (2 points)

   (d) How far has the wedge moved along the table at that time? (2 points)



2. A thin rod of length ℓ and total mass m has a linear mass density (mass per unit length) given by $\lambda(x) = \alpha x$, where α is a constant. We will denote the end of the rod with vanishing mass density as $\mathcal{A}$, and the acceleration due to gravity by g.

   (a) Find α in terms of m and ℓ. (1 point)

   (b) Calculate the distance between $\mathcal{A}$ and the center of mass and express it in terms of m and ℓ. (1 point)

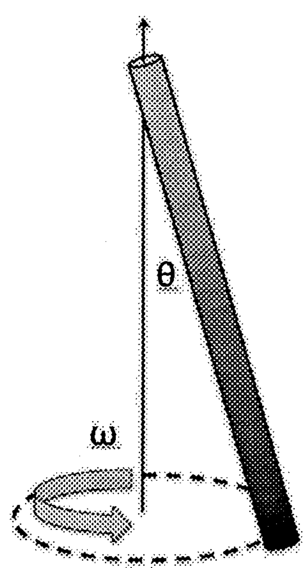
   (c) Find the moment of inertia of the rod about the end $\mathcal{A}$ about an axis perpendicular to the length of the rod and express it in terms of m and ℓ. (1 point)

   (d) The end $\mathcal{A}$ is attached to a fictionless pivot and then rotated azimuthally at constant angular frequency ω about the vertical line passing through the pivot. It is still free to rotate in the θ direction as well. Write down the Lagrangian of the system. (3 points)

   (e) Find the equation of motion for the system for θ. In the limit that $\omega^2 >> g/\ell$ there are two equilibrium solutions in which $\theta(t)$ is a constant.

      i. Show that $\theta(t) = 0$ is a solution and prove that it is unstable. (1 point).

      ii. There is a second solution in which the rod is nearly horizontal as it rotates. Derive an expression for the value of the equilibrium angle, θ_0 as a function of g, ℓ and ω and determine the frequency of small oscillations about this angle. (3 points)



3. **Angular momentum and the Rungé-Lenz[1] vector:** Given a point particle of mass m, trajectory $\vec{r}(t)$, and momentum $\vec{p}(t)$, we can define the angular momentum

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

and the Rungé-Lenz vector

$$\vec{\mathcal{A}} = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

We consider the explicit case of a $1/r$ potential, so that

$$H = \frac{p^2}{2m} - \frac{1}{r}$$

(a) Prove that the Poisson bracket of H and $\vec{L}$ is zero, that is:

$$\left\{H, \vec{L}\right\} = 0.$$

(3 points).

(b) Prove that the Poisson bracket of H and $\vec{\mathcal{A}}$ is zero, that is:

$$\left\{H, \vec{\mathcal{A}}\right\} = 0.$$

Hint: Expand the Poisson bracket of $\vec{\mathcal{A}}$ and use the fact that you know $\left\{H, \vec{L}\right\} = 0$. (3 points)

(c) What do your results in parts (a) and (b) imply about the behavior of $\vec{\mathcal{A}}$ and $\vec{L}$? (1 point)

(d) Evaluate $\vec{r} \cdot \vec{\mathcal{A}} = r\mathcal{A}\cos\theta$, using the explicit form for $\vec{\mathcal{A}}$ above. Use this and your answer to part (c) above to calculate the orbital motion of the particle (that is, a relationship between r and θ as the particle moves about its orbit). If you use the fact that

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right)$$

your answer should not involve much algebra. (3 points)

[1]This vector is commonly called the Rungé-Lenz vector, but they were not the first to discover it. Historically it may be more accurate to call this the "Hermann-Bernoulli vector," while it is also referred to as the "Laplace-Runge-Lenz vector." Others prefer to dodge the whole question of naming rights and simply refer to it as "the axial vector."[2]

[2]The above footnote has nothing to do with the solution of this problem.

Statistical Mechanics

4. A total of n moles of a spinless, mono-atomic, ideal gas is contained in a balloon of radius r and temperature T. The balloon is an ideal black body with emissivity of unity, and it exerts a constant pressure P_0 on the gas, independent of the size of the balloon. As the balloon radiates heat, it will cool, and shrink. You should assume that otherwise the balloon is in vacuum (there is no mechanism of heat loss other than radiation, and that there is no other force on the gas other than the constant balloon pressure).

   (a) Show that the rate of temperature change for the balloon due to radiation is given by:
   $$\frac{dT}{dt} = -\frac{8\pi\sigma r^2 T^4}{5nR}$$
   where σ is the Stefan-Boltzmann constant and R is the ideal gas constant. You should assume that the balloon does not absorb any heat from its surroundings. (4 points)

   (b) As the balloon shrinks, the elastic in the balloon compresses it and therefore does work. Derive an expression for dW/dt, the rate at which the balloon does work on the gas. Express your answer in terms of the variables V, n and P_0 as well as any numerical or physical constants you need. (3 points)

   (c) Given that the initial temperature of the gas is T_0, how long does it take for the gas in the balloon to lose half of its internal energy? Assume that the gas in the balloon has a uniform temperature, and ignore the specific heat of the elastic materal of the balloon itself. (4 points)

Note that you do not need to solve part (a) to solve parts (b) or (c).

5. Consider a lattice of N non-interacting, magnetic moments of magnitude μ, that are fixed in distinguishable locations. A magnetic field B is applied to the system such that each magnetic moment has two possible energy levels, $\pm\mu B$. Denote the number of moments in the higher energy state by the variable n.

   (a) Find an expression for the entropy in terms of N and n in the micro-canonical ensemble. (1 point)

   (b) Find the value of n for which the entropy is a maximum and sketch a graph of $S(n)$. (2 points)

   (c) Derive an expression for the energy of the system, U, in terms of N and n. In the micro-canonical ensemble this is an algebraic relationship, and not a result of ensemble averaging. (1 point)

   (d) Derive an expression for the temperature and show that it can be negative. (3 points)

   (e) If this system is put into thermal contact with a heat bath with positive temperature which way does the heat flow? Justify your answer. (3 points)

6. Consider an ideal gas of bosons confined in a two-dimensional surface of linear size L. An elementary criterion for the existence of Bose-Einstein condensation at a critical temperature T_c is

$$N_T \equiv \sum_{\epsilon_{\vec{k}} \neq 0} n(\epsilon_{\vec{k}}, \mu = 0, T = T_c) = N \tag{1}$$

where n is the Bose-Einstein distribution function and N is the total number of particles. (Assume periodic boundary conditions.)

First consider a gas which obeys the dispersion relation $\epsilon_{\vec{k}} = \hbar v k$, where v is a positive constant.

(a) Write down the Bose-Einstein condensation criterion explicitly. (1 point)

(b) Find the density of states. (2 points)

(c) Perform the sum in part a) by going to the continuum and calculate T_c in terms of N, L, v. (3 points)

(d) Now consider a gas which obeys the dispersion relation $\epsilon_{\vec{k}} = \hbar^2 k^2/2m$, where m is the particle mass. Repeat steps b) and c) and show that Bose-Einstein condensation can only happen at zero temperature. (4 points)

You may find it helpful to refer the table of formulae at the front of the exam in evaluating your sums.

Classical Mechanics and Statistical/Thermodynamics

August 2015

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i a x - b x^2}\, dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\mathrm{Li}_p(z) \equiv \sum_{n=1}^\infty \frac{z^n}{n^p}$$

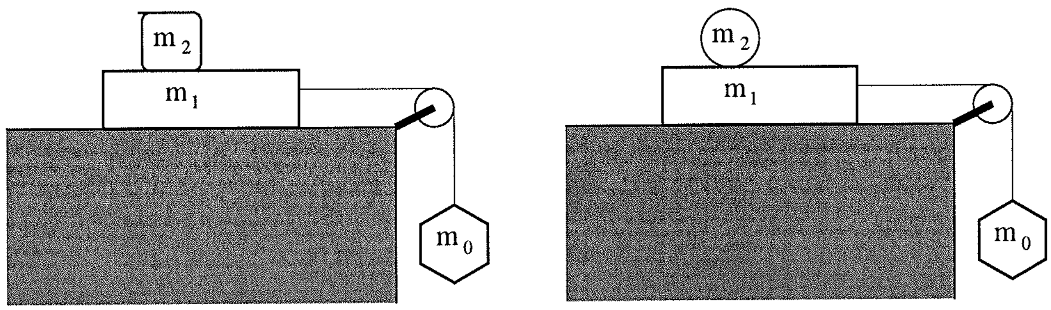
$$\mathrm{Li}_p(1) = \zeta(p)$$

Physical constants:

$$\begin{aligned}
\hbar &= 1.05457 \times 10^{-34} \mathrm{m^2 kg s^{-1}} \\
m_{\text{electron}} &= 9.109 \times 10^{-31} \mathrm{kg} \\
k_B &= 1.38 \times 10^{-23} \mathrm{m^2 kg \cdot s^{-2} K^{-1}}
\end{aligned}$$

Classical Mechanics

1. A block of mass m_1 sits on a frictionless horizontal table and is attahed to a hanging block with mass m_0 by an ideal massless rope draped over an ideal massless, frictionless pulley. We wish to put an object on top of the block, and will consider two cases.



   (a) Consider the case on the left, where we place a block of mass m_2 on top of the first block, and release the system from rest. Determine the minimum value of the coefficient of static friction, μ, between blocks 1 and 2 that will prevent block 2 to from slipping when block 1 accelerates to the right. (3 points)

   (b) Consider the case on the right, where we place a sphere of mass m_2 on top of the first block. Assume that the sphere at all times rolls without slipping.

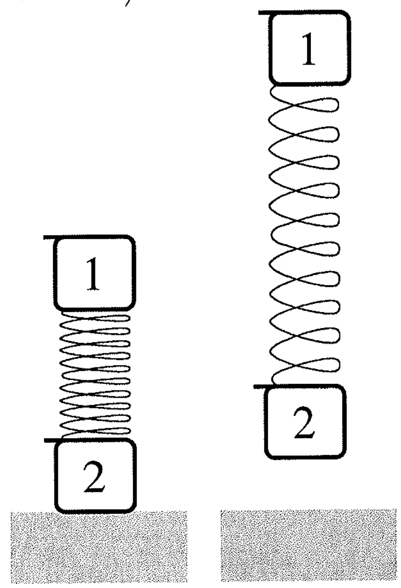
      i. Find the acceleraion of block 1 to the right. (5 points)

      ii. Find the acceleration of the sphere to the right. (1 point)

      iii. Find the rotational acceleration of the sphere. (1 point)

   The moment of inertia of a sphere and mass m and radius R is $(2/5)mR^2$.

2. Consider two identical blocks of mass m connected by an ideal massless spring of spring constant k and equilibrium length ℓ. (The equilibrium length is the length of the spring when there are no net external forces on it, such as when it is decoupled from the blocks) You should assume that $k\ell \gg mg$.



(a) The system is oriented vertically and the top block (labeled block "1") is depressed a distance y_0 and then released from rest. Find $y_0^{(\text{min})}$, the minimum value of y_0 such that the lower block (block "2") is barely lifted off the ground by block 1. Be sure to draw a co-ordinate system and indicate what direction is positive. (3 points)

(b) Now assume that block 1 is compressed an initial distance $y_0 = \ell/2 > y_0^{(\text{min})}$. Determine the trajectories $y_1(t)$ and $y_2(t)$ up until the time the second block hits the ground. You do not have to determine this final time. You will want to look at the trajectories over two different intervals:

   i. Before block 2 leaves the ground. (2 points)

   ii. After block 2 leaves the ground. (5 points)

   Again, be sure to draw a co-ordinate system and indicate what direction is positive.

Assume that all motion is in the vertical direction, and note that you do not have to solve part (a) to solve part (b).

3. Assume that we have generalized co-ordinates $\mathbf{q} = (q_1, q_2)$ and associated momenta $\mathbf{p} = (p_1, p_2)$ that satisfy Hamilton's equations of motion, so that $\dot{q}_i = \{q_i, H\}$ and $\dot{p}_i = \{p_i, H\}$, where the notation $\{x, y\}$ gives the Poisson bracket of x and y with respect to the variables $\mathbf{q}$ and $\mathbf{p}$. We wish to make a transformation to a new set of variables, $\mathbf{q}'(\mathbf{q}, \mathbf{p})$ and $\mathbf{p}'(\mathbf{q}, \mathbf{p})$.

   (a) Show that if

$$\begin{aligned} \left\{q'_i, p'_j\right\} &= \delta_{i,j} \\ \left\{q'_i, q'_j\right\} &= 0 \\ \left\{p'_i, p'_j\right\} &= 0 \end{aligned}$$

   then

$$\{F, G\} = \{F, G\}'$$

   for any quantities F and G where $\{x, y\}'$ is the Poisson bracket of x and y with respect to q' and p'. (4 points)

   (b) Consider the system described by the Hamiltonian:

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \cos\left(2q_1 + q_2\right)$$

   We would like to move to a new set of co-ordinates:

$$\begin{aligned} q'_1 &= 2q_1 + q_2 \\ q'_2 &= q_2 \end{aligned}$$

   which decouples the position co-ordinates. Find the p'_1 and p'_2 that make this transformation canonical, and express the new Hamiltonian in these new co-ordinates. (4 points)

   (c) Looking at the new Hamiltonian, it should be clear that there are two constants of the motion. One will be the Hamiltonian itself, because there is no explicit time dependence. What is the second constant of the motion, in terms of the original co-ordinates? (2 points)

Statistical Mechanics

4. Consider a thermally insulated vessel, divided into two parts by a partition. One side contains n_1 moles of nitrogen gas that occupies a volume V_1 at temperature T_1 and pressure P_1 and the other contains n_2 moles of argon gas that occupies a volume V_2 at T_2 and P_2. Assume nitrogen to be an ideal gas with $c_v = (5/2)R$ and argon to be an ideal gas with $c_v = (3/2)R$. The goal of this problem is to calculate the change in entropy of the system when the partition is removed and each gas expands freely through the container.

   Since entropy is a function of state, the change in entropy between an initial and final state of a system is independent of the path taken to get from one state to another. That means we can break this problem into separate segments of a path connecting the initial and final states such that the entropy change for each segment is more easily calculated.

   (a) First let the two parts of the system equilibrate thermally at constant volumes. Find the final temperature, T_f, and the entropy change of the system. (3 points)

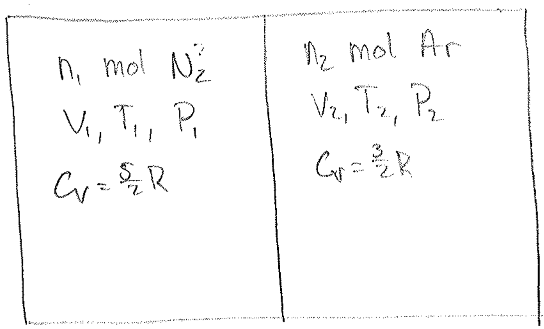
   (b) Second let the pressure of the two parts of the system equilibrate at this constant temperature (i.e., letting the partition between the chambers move). Find the entropy change of the system for this step. (3 points)

   (c) Finally, remove the partition and let the molecules of the gas mix. Find the entropy change for this step. (3 points)

   (d) What is the total entropy change in this process? (1 point)

Aug 2015

Stat Mech #1



* System is thermally isolated

$$S = \int \frac{dQ}{T}$$

a) Let the two parts of the system equilibrate thermally at constant V. Find the final temperature and the entropy change of the system

* for N_2

$$\frac{P_1}{T_1} = \frac{P_{f,1}}{T_f}$$

$$Q = mC_V \Delta T$$

* for Ar

$$\frac{P_2}{T_2} = \frac{P_{f,2}}{T_f}$$

$$Q = mC_V \Delta T$$

$$S = \int \frac{mC_V\,dT}{T}$$

$$\Delta S = mC_V \ln\left(\frac{T_f}{T_i}\right)$$

$$\Delta S_{N_2} = m_{N_2}$$

$$m_{N_2} \frac{5}{2} R (T_f - T_1) = m_{Ar} \frac{3}{2} R (T_f - T_2)$$

$$5m_{N_2}(T_f - T_1) = 3m_{Ar}(T_f - T_2)$$

$$5m_{N_2}T_f - 3m_{Ar}T_f = -3m_{Ar}T_2 + 5m_{N_2}T_i$$

$$T_f = \frac{5m_{N_2}T_1 - 3m_{Ar}T_2}{5m_{N_2} - 3m_{Ar}}$$

5. Consider a set of N distinguishable atoms that has an energy given by:

$$E = \sum_{i=1}^{N} \epsilon\, \sigma_i^2 + h\, \sigma_i$$

where $\sigma_i \in \{-1, 0, 1\}$. The quantity σ_i is the value of an atomic spin, ϵ is an internal crystal field, and h is an externally applied magnetic field. We will analyze this system in the canonical ensemble.

(a) What is the partition function for this system, $\mathcal{Z}(T, N)$? (1 point)

(b) What is the internal energy, $U(T, N)$? (2 points)

(c) Calculate the magnetic susceptibility,

$$\chi = \frac{\partial M}{\partial h}$$

where $M \equiv \langle \sum \sigma_i \rangle$ is the magnetization. (4 points)

(d) Show that $\chi(h)$ for large positive ϵ has a peak as a function of increasing βh, and explain physically why this is so. (3 points)

6. A *Dirac fermion* is a particle which obeys Fermi statistics and has an energy given by

$$E(\vec{k}) = \hbar v_0 \, |\vec{k}| = \hbar v_0 \, k$$

where v_0 is a characteristic velocity. In this problem we will work in the grand canonical ensemble and analyze Dirac fermions in a two dimensional system, similar to what is found in graphene.

(a) Calculate the density of states, $D(E)$ for spin-1/2 Dirac fermions in two dimensions. (3 points)

(b) What is the Fermi energy, E_f, as a function of the fermion density, N/A, where A is the area of the system? (3 points)

(c) Calculate the energy of the system as a function of T, A, and μ. (By this we mean that the *thermodynamic variable dependence* is on T, A, and μ. Your answer will involve other mathematical and physical constants such as $\hbar$ or v_0.) Your answer should involve the *Polylogarithm* function, $\mathrm{Li}_p(z)$, defined on page 2. (4 points)

Classical Mechanics and Statistical/Thermodynamics

January 2016

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\,e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\,e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad\qquad \sum_{n=1}^\infty (-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad\qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad\qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

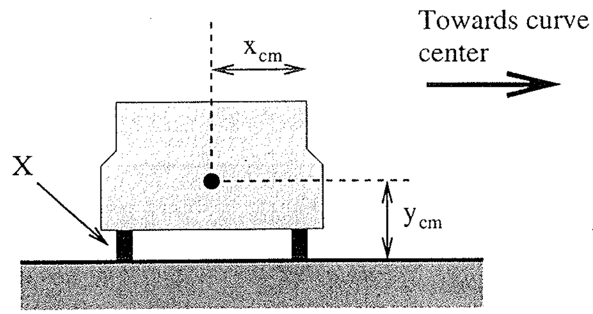
$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad\qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad\qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad\qquad \zeta(-4) = 0$$

Classical Mechanics

1. A car of mass m travels on a flat road at constant velocity v_0, and the coefficient of friction between the car tires and the road is μ. It comes to a circular curve of radius R. If the car is moving very fast, it may either slide off the road, (if μ is small) or even flip up on two wheels (if μ is large). In the figure below, the velocity of the car is into the page, and the center of the curve is to the right.



When the car is at rest, the center of mass of the car is located at a point y_{cm} above the road and a distance x_{cm} from either wheel. Treat the wheels as having a negligible width.

(a) Draw a free-body diagram for the car showing the forces that act on the car. (1 point).

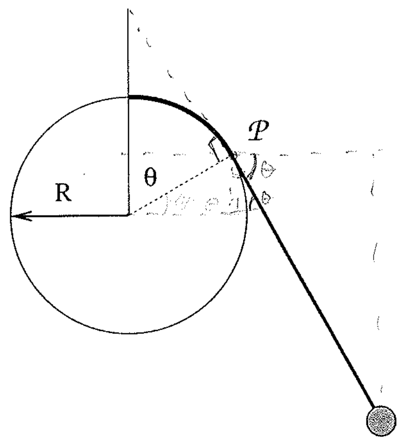
(b) Write down Newton's second law for the motion of the car tangential to the curve, perpendicular to the curve (outward) and for rotation of the car about the point marked "X". (2 point).

(c) If the coefficient of friction is small, the car will skid or slide off the road. At what velocity will that occur for a given value of m? (3 points)

(d) Assume that the car never skids or slides sideways. At what velocity will the car start to flip over? (3 points)

(e) Finally assume that the car does not skid, but exceeds the above speed and starts to flip. If it continues at the same speed, will it continue to flip, or will it achieve equilibrium at an angle θ_{eq} with the road? If the former, prove it, if the latter, calculate the equilibrium angle. (1 point)

2. A stationary disk of radius R is aligned vertically so that its axis is parallel to the ground. The disk is fixed and does not rotate. A string of length ℓ is attached to the top of the disk, and $\ell > \pi R$. A point mass m is attached to the end of the string and can swing in a vertical plane (left to right in the figure below). As the mass m swings, the point $\mathcal{P}$ where the string just contacts the disk will move. Assume that the string is always taut. The angle between $\mathcal{P}$ and the vertical is θ; it will be the generalized coordinate in this problem.



(a) Determine the x and y position of the point mass as a function of θ, R and ℓ. Use the center of the disk as the origin of your coordinate system. (Hint: Knowing the value of θ determines both the amount of string wrapped on the disk and the angle the straight length of string makes with the vertical.) (1 point)

(b) Treating $\theta(t)$ as the generalized coordinate, determine the kinetic energy of the point mass as a function of m, θ, $\dot{\theta}$, R, and ℓ. (2 points)

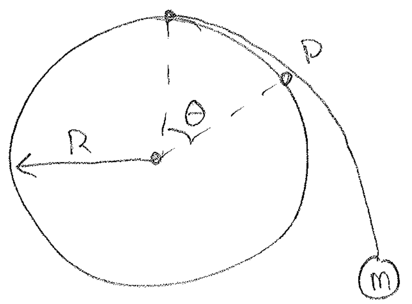
(c) What is the Lagrangian for the system in terms of this generalized coordinate? (2 points)

(d) What are the equations of motion? (1 point)

(e) Assume that the point mass makes small oscillations about some angle θ_0 (which might not be zero). Determine θ_0 and the angular frequency of these oscillations. (4 points).

Spring 2016

Classical #2



* Sting of length ℓ

a) $\vec{r}_m = \langle R\cos(90-\theta),\ R\sin(90-\theta) \rangle$

$= \langle R\sin\theta,\ R\cos\theta \rangle + \langle (\ell - R\theta)\cos\theta, -(\ell - R\theta)\sin\theta \rangle$

$= \langle R\sin\theta + (\ell - R\theta)\cos\theta;\ R\cos\theta - (\ell - R\theta)\sin\theta \rangle$

b) $T = \frac{1}{2} m v^2$

$= \frac{1}{2} m \dot{\vec{r}}^2$

$\dot{\vec{r}} = \langle R\cos\theta\,\dot{\theta} - \ell\sin\theta\,\dot{\theta} - [R\dot{\theta}\cos\theta - R\theta\sin\theta\,\dot{\theta}],\ -R\sin\theta\,\dot{\theta} - \ell\cos\theta\,\dot{\theta} + R\dot{\theta}\sin\theta + R\theta\cos\theta\,\dot{\theta} \rangle$

$\dot{\vec{r}} = \dot{\theta} \langle R\cos\theta - \ell\sin\theta - R\cos\theta + R\theta\sin\theta,\ -R\sin\theta - \ell\cos\theta + R\sin\theta + R\theta\cos\theta \rangle$

$= \dot{\theta} \langle -\ell\sin\theta + R\theta\sin\theta,\ R\theta\cos\theta - \ell\cos\theta \rangle$

$\dot{\vec{r}}^2 = \dot{\theta}^2 [\ell^2\sin^2\theta + R^2\theta^2\sin^2\theta - 2\ell R\theta\sin^2\theta + R^2\theta^2\cos^2\theta + \ell^2\cos^2\theta - 2R\theta\ell\cos^2\theta]$

$= \dot{\theta}^2 [\ell^2 + R^2\theta^2 - 2\ell R\theta] \Rightarrow T = \frac{1}{2} m \dot{\theta}^2 [\ell^2 + R^2\theta^2 - 2\ell R\theta]$

c) $\mathcal{L} = T - U$

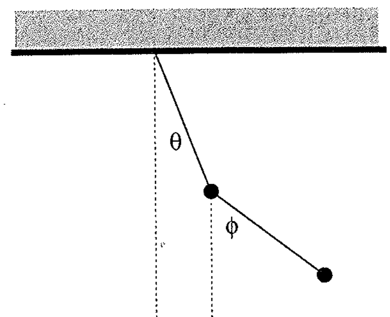
$= \frac{1}{2} m \dot{\theta}^2 [\ell^2 + R^2\theta^2 - 2\ell R\theta] - mg(R\cos\theta - (\ell - R\theta)\sin\theta)$

d) $\frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$

$\cancel{\frac{1}{2}} m\dot{\theta}^2 R^2\theta - 2\ell R + mgR\sin\theta + mg\ell\cos\theta - R\sin\theta - R\theta\cos\theta = \frac{\partial}{\partial t}[m\dot{\theta}(\ell^2 + R^2\theta^2 - 2\ell R\theta)]$

$m\dot{\theta}^2 R^2\theta - 2\ell R + R\sin\theta(mg - 1) + \cos\theta(mg\ell - R\theta) = m\ell^2\ddot{\theta} + R^2 m\ddot{\theta}\theta^2 + 2R^2 m\dot{\theta}^2\theta - 2\ell R(\ddot{\theta}\theta$

3. Consider a double pendulum, consisting of a mass m suspended from a point with a massless cord of length ℓ, with a second mass m suspended from the first with another massless cord of equal length ℓ. At a given instant, the first mass makes an angle θ with respect to the vertical, while the second mass makes an angle ϕ with respect to the vertical. A uniform gravitational field, with gravitational acceleration g, acts in the vertical direction.



(a) Starting from the description of kinetic and potential energy in Cartesian coordinates, obtain the Lagrangian in terms of the angles θ and ϕ and their time derivatives, $\dot{\theta}$ and $\dot{\phi}$. (2 points)

(b) Now simplify the Lagrangian to the situation when both angles are small, $\theta \ll 1$, $\phi \ll 1$, and obtain the form of two coupled harmonic oscillators. (1 point)

(c) For this system, obtain the mass matrix $\mathbf{M}$ and the spring-constant matrix $\mathbf{K}$, where the Lagragian is written:

$$L = \frac{1}{2}\begin{pmatrix} \dot{\theta}, & \dot{\phi} \end{pmatrix} \mathbf{M} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \theta, & \phi \end{pmatrix} \mathbf{K} \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

(2 points)

(d) Show that the normal modes satisfy

$$(\omega^2 \mathbf{M} - \mathbf{K}) \cdot \mathbf{Q} = 0.$$

(1 point)

(e) Determine the characteristic frequencies (eigenfrequencies) ω in terms of the quantity $\omega_0^2 = g/l$. (2 points)

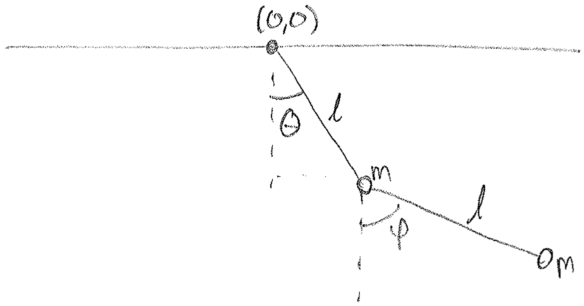
(f) If we write the normal mode vector as

$$\mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

determine the ratio q_2/q_1, which characterize the normal modes. (2 points)

Jan 2016

Classical #3



a) Starting w/ cartesian, obtain Lagrangian in terms of $\theta, \dot{\theta}, \varphi, \dot{\varphi}$

$$\mathcal{L} = T - U$$

$$= \frac{1}{2}m(v_1^2 + v_2^2) - mg(y_1 + y_2)$$

$x_1 = l\sin\theta \qquad\qquad y_1 = l\cos\theta$

$x_2 = l\sin\theta + l\sin\varphi \qquad\qquad y_2 = l\cos\theta + l\cos\varphi$

$\dot{x}_1 = l\cos\theta\,\dot{\theta} \qquad\qquad \dot{y}_1 = -l\sin\theta\,\dot{\theta}$

$\dot{x}_2 = l\cos\theta\,\dot{\theta} + l\cos\varphi\,\dot{\varphi} \qquad\qquad \dot{y}_2 = -l\sin\theta\,\dot{\theta} - l\sin\varphi\,\dot{\varphi}$

$$= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) - mg(y_1 + y_2)$$

$$= \frac{1}{2}ml^2[\cos^2\theta\,\dot{\theta}^2 + \cancel{\sin^2\theta}\,\dot{\theta}^2 + \cancel{\cos^2\theta}\,\dot{\theta}^2 + 2\cos\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \cancel{\cos^2\varphi}\,\dot{\varphi}^2 + \sin^2\theta\,\dot{\theta}^2 + 2\sin\theta\sin\varphi\,\dot{\theta}\dot{\varphi} + \cancel{\sin^2\varphi}\,\dot{\varphi}^2]$$

$$= \frac{1}{2}ml^2[2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}(\cos\theta\cos\varphi + \sin\theta\sin\varphi)$$

$$= \frac{1}{2}ml^2[2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}\cos(\theta - \varphi)] - mg(2l\cos\theta + l\cos\varphi)$$

b) Apply small angle approx. + get form of coupled harmonic oscillators

* In small angle: $\sin\theta \to \theta$
$\cos\theta \to 1 - \frac{\theta^2}{2}$

$$\Rightarrow \mathcal{L} = \frac{1}{2}ml^2[2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}(1 - \frac{(\theta-\varphi)^2}{2}) - mgl(2(1 - \frac{\theta^2}{2}) + (1 - \frac{\varphi^2}{2})$$

$$= \frac{1}{2}ml^2[2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}(1 - \frac{1}{2}(\theta^2 - 2\theta\varphi + \varphi^2))] - mgl(2 - \theta^2 + 1 - \frac{1}{2}\varphi^2)$$

$$= \frac{1}{2}ml^2[2\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\theta}\dot{\varphi}(2 - \theta^2 + 2\theta\varphi - \varphi^2] - mgl(3 - \theta^2 - \frac{1}{2}\varphi^2)$$

c) Find $\overleftrightarrow{M}$ and $\overleftrightarrow{K}$ when the Lagrangian is of the form:

$$\mathcal{L} = \frac{1}{2}(\dot{\theta}, \dot{\varphi})\overleftrightarrow{M}\begin{pmatrix}\dot{\theta}\\ \dot{\varphi}\end{pmatrix} - \frac{1}{2}(\theta, \varphi)\overleftrightarrow{K}\begin{pmatrix}\theta\\ \varphi\end{pmatrix}$$

$$\Rightarrow \overleftrightarrow{M} = \begin{bmatrix} ml^2 & \\ & \frac{1}{2}ml^2 \end{bmatrix}$$

$$\overleftrightarrow{K} = \Big[$$

Statistical Mechanics

4. A heat engine is made from N atoms of an ideal mono-atomic gas starting at an initial temperature T_1, and volume V_1. Call this state "1." It is initially heated isochorically (at constant volume) to a state "2" with a temperature $T_2 = 4T_1$. It then undergoes an adiabatic expansion to state "3" where it has returned to its original pressure. Finally it is then cooled isobarically (at constant pressure) until it returns to its original condition.

   (a) Draw the thermodynamic cycle in the PV plane. (1 point).

   (b) Calculate the volume and temperature at states 2 and 3 in terms of V_1, T_1 and N. (1 point).

   (c) Calculate the work done by the gas in each step of the cycle. (3 points).

   (d) Calculate the heat in (or out) of the gas during each step. (3 points)

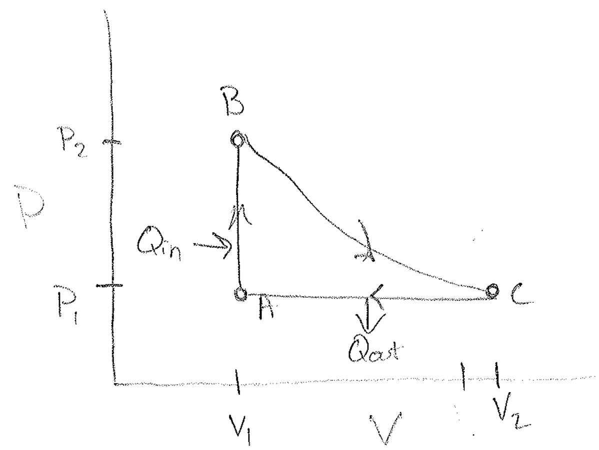
   (e) What is the efficiency of this engine? (2 points)

Jan 2016

Stat Mech #1

A heat engine made of N atoms of an ideal monatomic gas starting at initial temp T_1 and volume V_1. It is heated isochorically to $T_2 = 4T_1$. It then expands adiabatically to state C where it has returned to its initial pressure. It is then isobarically cooled to its initial state.

a) Draw the cycle in the P-V plane.



b) Calculate the temperature at states B and C in terms of V_1, T_1, and N

$P_A = \frac{N k_B T_1}{V_1}$ $\quad P_B = \frac{4N k_B T_1}{V_1}$ $\quad P_C = \frac{N k_B T_1}{V_1}$

$4^3 = 64$

$V_A = V_1$ $\quad V_B = V_1$ $\quad V_C = 4^{3/5} V_1$

$T_A = T_1$ $\quad \boxed{T_B = 4T_1}$ $\quad T_C = 4^{3/5} T_1$

$P_B V_B^{\gamma} = P_C V_C^{\gamma}$

$\frac{4N k_B T_1}{\cancel{V_1}} V_1^{5/3} = \frac{N k_B T_1}{\cancel{V_1}} V_C^{5/3}$

$4^{3/5} V_1 = V_C$

$\frac{V_A}{T_A} = \frac{V_C}{T_C}$

$\Rightarrow T_C = \frac{V_C}{V_A} T_A$

$= \frac{4^{3/5} V_1}{V_1} T_1$

$= 4^{3/5} T_1$

c) Calculate the work done by the gas in each step.

$$W_{A \to B} = 0 \quad b/c \text{ isochoric}$$

$$W_{B \to C} = \frac{P_C V_C - P_B V_C}{1 - \gamma}$$

$$= \frac{3}{2}\left(4^{3/5} N k_B T - 4 N k_B T\right)$$

$$= -\frac{3}{2} N k_B T \left(4^{3/5} - 4\right)$$

$$W_{C \to A} = P \Delta V$$

$$= \frac{N k_B T}{\cancel{V_1}}\left(\cancel{V_1} - 4^{3/5}\cancel{V_1}\right)$$

$$= N k_B T \left(1 - 4^{3/5}\right)$$

d) Calculate the heat during each step

$$Q_{A \to B} = n C_V \Delta T$$

$$= \frac{N}{n_{av}}\left(\frac{3}{2}R\right)\left(4T_1 - T_1\right)$$

$$= \frac{9 R N T_1}{2(6.02 \cdot 10^{23})}$$

$$Q_{B \to C} = 0 \quad b/c \text{ adiabatic}$$

$$Q_{C \to A} = n C_P \Delta T$$

$$= \frac{N}{n_{av}}\left(\frac{5}{2}R\right)\left(T_1 - 4^{3/5}T_1\right)$$

$$= \frac{5 N R T_1 \left(1 - 4^{3/5}\right)}{2 \cdot (6.02 \cdot 10^{23})}$$

e) What is the efficiency of the engine?

$$\eta = 1 - \left|\frac{Q_{out}}{Q_{in}}\right|$$

$$= 1 - \left|\frac{\frac{5 N R T_1 (1 - 4^{3/5})}{2 \cdot (6.02 \cdot 10^{23})}}{\frac{9 N R T_1}{2 \cdot (6.02 \cdot 10^{23})}}\right| = 1 - \left|\frac{5\left(1 - 4^{3/5}\right)}{9}\right|$$

5. Consider a system of N distinguishable particles with only 3 possible energy levels: 0, ϵ and 2ϵ. The system occupies a fixed volume V and is in thermal equilibrium with a reservoir at temperature T. Ignore interactions between particles and assume that Boltzmann statistics applies.

   (a) What is the partition function for a single particle in the system? (1 point).

   (b) What is the average energy per particle? (1 points).

   (c) What is probability that the 2ϵ level is occupied in the high temperature limit, $k_B T \gg \epsilon$? Explain your answer on physical grounds. (1 point).

   (d) What is the average energy per particle in the high temperature limit, $k_B T \gg \epsilon$? (1 point).

   (e) At what approximate temperature is the ground state 1.1 times as likely to be occupied as the 2ϵ level? (1 point).

   (f) Find the heat capacity of the system, c_v, analyze the low-T (when $k_B T \ll \epsilon$) and high-T ($k_B T \gg \epsilon$) limits, and sketch c_v as a function of T. Explain your answer on physical grounds. (5 points).

Jan 2016

Stat Mech #2

System: N distinguishable particles

3 possible energy levels $(0, \epsilon, 2\epsilon)$

V is a fixed volume

T is temperature of heat reservoir, in thermal equilibrium

* Ignore particle interactions, assume Boltzmann statistics

a) What is the partition function for a single particle?

$$Z = \sum_i e^{-\beta E_i}$$

$$= e^{-\beta 0} + e^{-\beta\epsilon} + e^{-\beta 2\epsilon}$$

$$= 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}$$

b) What is the avg. energy per particle

$$\langle E \rangle = \frac{-\partial}{\partial\beta} \ln(z)$$

$$= \frac{-\partial}{\partial\beta} \ln\left(1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}\right)$$

$$= \frac{-\epsilon e^{-\beta\epsilon} - 2\epsilon e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}$$

c) What is the probability that the 2ϵ energy level is occupied in the high T limit ($k_B T \gg \epsilon$)? Explain answer on physical grounds

$$P = \frac{1}{z} e^{-\beta E}$$

$$= \frac{e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}$$

$$= \frac{1}{e^{2\beta\epsilon} + e^{\beta\epsilon} + 1}$$

$$= \frac{1}{3}$$

d) What is the avg energy per particle in the high T limit?

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln(z)$$

$$= \frac{-\epsilon e^{-\beta\epsilon} - 2\epsilon e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}$$

$$= \frac{-\epsilon - 2\epsilon}{3}$$

$$= -\epsilon$$

e) At what approximate T is the ground state 1.1 times as likely to be occupied as the 2ϵ level?

$$\frac{P_0}{P_{2\epsilon}} = 1.1 = \frac{\frac{1}{z} e^{-\beta 0}}{\frac{1}{z} e^{-\beta 2\epsilon}}$$

$$1.1 = \frac{e^{-\beta 0}}{e^{-2\beta\epsilon}}$$

$$1.1 = \frac{1}{e^{-2\beta\epsilon}}$$

$$e^{2\beta\epsilon} = \frac{1}{1.1}$$

$$2\beta\epsilon = \ln\left(\frac{1}{1.1}\right)$$

$$\frac{1}{k_B T} = \frac{1}{2\epsilon} \ln\left(\frac{1}{1.1}\right)$$

$$T = \frac{2\epsilon}{k_B \ln\left(\frac{1}{1.1}\right)}$$

f) Find the heat capacity of the system in both the high and low T limits. Sketch C_v as a function of T. Explain your answer on physical grounds.

$$C_v = \left(\frac{\partial U}{\partial T}\right)_V$$

$$U = -\frac{\partial}{\partial \beta} \ln(Z)$$

$$= -\frac{\partial}{\partial \beta} N \ln\left(1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}\right)$$

$$= \frac{N(-\epsilon e^{-\beta\epsilon} - 2\epsilon e^{-2\beta\epsilon})}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}$$

$$C_v = \frac{\partial U}{\partial T}$$

$$= \frac{\partial}{\partial T}\left[\frac{-N\epsilon(e^{-\beta\epsilon} + 2e^{-2\beta\epsilon})}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}\right]$$

$$= \frac{\partial}{\partial T}\left[\frac{-N\epsilon(e^{\beta\epsilon} + 2)}{e^{2\beta\epsilon} + e^{\beta\epsilon} + 1}\right]$$

$$= \frac{\partial}{\partial T}\left[\frac{-N\epsilon(e^{\epsilon/kT} + 2)}{e^{2\epsilon/kT} + e^{\epsilon/kT} + 1}\right]$$

$$= \frac{-N\epsilon \frac{-\epsilon}{kT^2} e^{\epsilon/kT}}{e^{2\epsilon/kT} + e^{\epsilon/kT} + 1} + -N\epsilon(e^{\epsilon/kT} + 2)(e^{2\epsilon/kT} + e^{\epsilon/kT} + 1)^{-2} \cdot \left(-\frac{2\epsilon}{kT^2} e^{2\epsilon/kT} + -\frac{\epsilon}{kT^2} e^{\epsilon/kT}\right)$$

$$=$$

f) $$C_V = +N\epsilon \frac{\epsilon}{kT^2} e^{\epsilon/kT}\left(e^{2\epsilon/kT} + e^{\epsilon/kT} + 1\right)^{-1} - N\epsilon\left(e^{\epsilon/kT} + 2\right)\left(\frac{2\epsilon}{kT^2} e^{2\epsilon/kT} + \frac{\epsilon}{kT^2} e^{\epsilon/kT}\right)\left(e^{2\epsilon/kT} + e^{\epsilon/kT} + 1\right)^{-2}$$

$$= \frac{N\epsilon^2}{kT^2}\left[e^{\epsilon/kT}\left(e^{2\epsilon/kT} + e^{\epsilon/kT} + 1\right)^{-1} - \left(e^{\epsilon/kT} + 2\right)\left(e^{2\epsilon/kT} + e^{\epsilon/kT}\right)\left(e^{2\epsilon/kT} + e^{\epsilon/kT} + 1\right)^{-2}\right]$$

* in the high T limit

$C_V \to 0$

* in the low T limit

$C_V \to \infty$

6. Consider an ideal gas of bosons confined in a three-dimensional box of linear size L on each side. The gas obeys a dispersion relation $\epsilon_{\vec{k}} = \alpha k$, where α is a positive constant and $k \equiv |\vec{k}|$ and where

$$\vec{k} = \frac{2\pi}{L}\left(n_x\,\hat{\imath} + n_y\,\hat{\jmath} + n_z\,\hat{k}\right)$$

and where the n_x, n_y, and n_z are integers.

(a) Find the density of states. (2 points)

(b) Now assume that the bosons are photons with two possible polarizations. This fixes μ, the chemical potential, so that $\mu = 0$. Why? (1pt).

(c) Show that the average energy density in the box varies as T^4. (3 points)

(d) Calculate the pressure on the walls of the box and show that it is proportional to the average energy density. (4 points).

Classical Mechanics and Statistical/Thermodynamics

August 2016

		✓	?✓
Total:	Classical -		
	Stat Mech -		
	Total -		

Possibly Useful Information

Handy Integrals:

$$\begin{aligned}
\int_0^\infty x^n e^{-\alpha x}\, dx &= \frac{n!}{\alpha^{n+1}} \\
\int_0^\infty e^{-\alpha x^2}\, dx &= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \\
\int_0^\infty x\; e^{-\alpha x^2}\, dx &= \frac{1}{2\alpha} \\
\int_0^\infty x^2\; e^{-\alpha x^2}\, dx &= \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}} \\
\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\, dx &= \sqrt{\frac{\pi}{b}}e^{-a^2/4b}
\end{aligned}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\mathrm{Li}_p(z) \equiv \sum_{n=1}^\infty \frac{z^n}{n^p}$$

$$\mathrm{Li}_p(1) = \zeta(p)$$

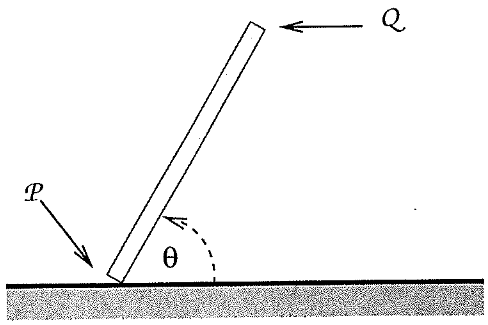
Physical constants:

$$\begin{aligned}
\hbar &= 1.05457 \times 10^{-34}\mathrm{m^2kgs^{-1}} \\
m_{\text{electron}} &= 9.109 \times 10^{-31}\mathrm{kg} \\
k_B &= 1.38 \times 10^{-23}\mathrm{m^2kg \cdot s^{-2}K^{-1}}
\end{aligned}$$

1 + 3 ?

Classical Mechanics

1. A uniform stick of length ℓ and mass m is initially held upright at an angle $\theta(t=0)=\theta_0$ to the horizontal with one end on the floor. The stick is released and falls to the floor so that θ changes in time. As it falls, it pivots around its contact point $\mathcal{P}$ with the floor without sliding, due to the coefficient of static friction μ, between the stick and the floor. You should ignore the thickness of the stick, and assume that μ is constant.



?√(a) Draw the free-body diagram for this problem. (1 point)

√(b) Calculate the moment of inertia of the stick with respect to the contact point, $\mathcal{P}$. (1 point)

?√(c) Determine the tangential velocity of the end of the stick (point $\mathcal{Q}$) as a function of θ, assuming that the pivot point does not slide. (2 points)

(d) Show that if $0<\theta_0<\pi/2$, the frictional force at the pivot changes sign (direction) as a function of θ, and calculate the angle at which it does so as a function of m, ℓ, g (the acceleration due to gravity) and/or θ_0. (3 points)

(e) Show that in the limit $\theta_0 \to \pi/2$ there is an angle θ_c at which the stick will always slide, for any finite value of μ, and calculate θ_c. (3 points)

@ 2:10 21 + 8?

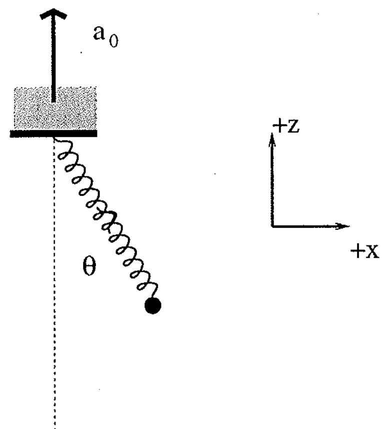
3:10 16 + 6 Classical

5 + 5 Stat Mech

21 + 11

8

2. Consider a pendulum that consists of a mass, m, suspended by a massless spring of equilibrium length ℓ_0 and spring constant k. The spring is located on Earth, in a uniform gravitational acceleration, g, pointing downward (the $-\hat{z}$ direction). The pendulum swings only in the x-z plane, and the point of support of the pendulum accelerates upward with a constant acceleration a_0.



(a) Find the Lagrangian in terms of the generalized coordinates consisting of the length of the spring, $\ell(t)$, and the angle $\theta(t)$ it makes with respect to the vertical. (2 points)

(b) Find the equations of motion for the generalized co-ordinates. (2 points)

(c) Find the Hamiltonian for the system from the Lagrangian. (2 points)

(d) Derive Hamilton's equations of motion for the system. (2 points)

(e) Assume that $\ell(t)$ and $\theta(t)$ undergo small, slow oscillations. What are the periods of the swinging motion and the oscillation of the spring? (2 points)

7 + 3?

3. Consider a relativistic particle of rest mass m_0 moving in a given potential V, described by the Hamiltonian

$$H = \sqrt{p^2c^2 + m_0^2c^4} + V(\mathbf{r}).$$

✓(a) Write down Hamilton's equations. (2 points)

✓(b) From these obtain an expression for the momentum in terms of the velocity. (2 points)

✓(c) Using the result of (b) above, obtain an explicit expression for the rate of change of the momentum in terms of the velocity $\mathbf{v}$ and the acceleration $\dot{\mathbf{v}}$. (2 points)

?✓(d) Derive the corresponding Lagrangian. Be sure to write the Lagrangian as a function of velocity and position. (2 points)

✓(e) What is the energy expressed in terms of the velocity rather than the momentum? (1 point)

?✓(f) If the potential is rotationally invariant, what are the corresponding constants of the motion? Is the Hamiltonian among these? Why or why not? (1 point)

Note that you may find it convenient to introduce the quantity $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

~~3+2?~~

4 + 3?

Statistical Mechanics

4. It can be shown that the Helmholtz free energy for a photon gas is given by:

$$F(T, V, N) = -\frac{1}{3}\sigma V T^4$$

where σ is the Stefan-Boltzmann constant. Using this relation, answer the following:

✓ (a) What are the equations of state (that is, P, S, and μ as a function of T, V and N)? (3 points)

?✓ (b) Consider a Carnot cycle using a photon gas as its working fluid. The cycle is driven by one hot and one cold temperature reservoir, with temperatures T_h and T_c respectively. Draw the cycle in the P-V plane. **Caution:** This is **not** an ideal gas! Think carefully about the steps in a Carnot cycle and use your results from above to determine what the cycle will look like. (2 points)

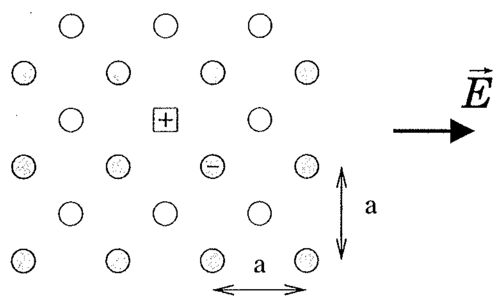
1✓ 2?✓

(c) Solve for the heat exchanged in each leg of your Carnot cycle. Your answer may depend upon T_h, T_c, and any other variables you might choose in defining your cycle. (3 points)

(d) Using these values for the heat exchanged, calculate the efficiency of a Carnot cycle that uses a photon gas as its working fluid. If you cannot calculate it, devise a careful argument for its value. (2 points)

5. Consider a two-dimensional model of a solid at absolute temperature T that contains a small number N of electron donor atoms, represented by a square in the figure below. These atoms replace a small fraction of the number of ordinary atoms of the solid, and rapidly ionize. The donated electron always sits on one of the four atoms immediately and diagonally adjacent to the donor atom. While the positively charged donor ion is fixed in place, the electron is free to move between any one of the four lattice sites surrounding the positive ion. The lattice spacing is a. Neglect any interaction between impurities on different sites, and assume that all donor atoms are ionized in this fashion.

In this problem a uniform electric field will be applied in the x-direction, polarizing the system.



(a) Calculate the mean electric polarization, i.e. mean electric dipole moment per unit volume, in the presence of a uniform electrical field applied along the x direction, so that $\vec{E} = E_0\,\hat{\imath}$. (2 points)

(b) Calculate the entropy per unit volume as a function of temperature. (3 points)

(c) What is the entropy per unit volume at very low temperature ($k_BT \ll eaE$)? Explain why this must be the case based on the physics of the problem. (If you cannot solve part (b) above, you can still determine the correct answer based on principles of symmetry). (2.5 points)

(d) What is the entropy per unit volume at very high temperature ($k_BT \gg eaE$)? Explain why this must be the case based on the physics of the problem. (If you cannot solve part (b) above, you can still determine the correct answer based on principles of symmetry). (2.5 points)

6. Consider a set of spinless free bosonic gas atoms each of mass m moving in three dimensions. The state of an atom is given by its momentum $\vec{p}$, and a variable σ which can be either 0 or 1. The energy for an atom is given by

$$E(\vec{p}, \sigma) = \frac{p^2}{2m} + \sigma \Delta$$

where $\Delta > 0$, and $\sigma \in \{0, 1\}$.

(a) If $\Delta = 0$, then we have a degeneracy of two for every energy eigenstate. What is the transition temperature for Bose-Einstein condensation for the system in this limit, and how does it compare to a similar gas without the σ degree of freedom? (4 points)

✓(b) Write a formal expression for the partition function in the grand canonical ensemble when $\Delta > 0$. Show that it factors into a product of a partition function for the ground state atoms, and a partition for the excited state atoms. (This expression will involve a product over states that you cannot simplify.) (1 point)

✓(c) Calculate $\bar{N} \equiv \langle N \rangle$ for this system. You should get an expression in terms of T, $z \equiv e^{\mu/k_B T}$ (or μ) and Δ. (2 points)

(d) Determine the critical density at which the transition occurs as a function of T, m, and Δ, and expand it to lowest order in $\Delta/k_B T$. Is the critical density increased or decreased as Δ is increased from zero? Why? (3 points)

$\Omega = \frac{-1}{\beta} \ln(Z)$

$U = \frac{-\partial}{\partial\beta} \ln(Z_1)$