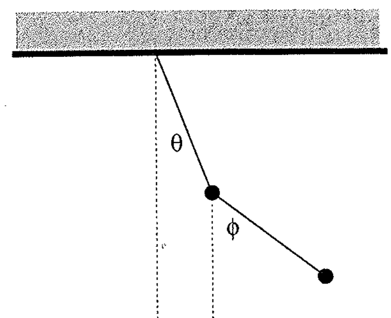
3. Consider a double pendulum, consisting of a mass m suspended from a point with a massless cord of length ℓ, with a second mass m suspended from the first with another massless cord of equal length ℓ. At a given instant, the first mass makes an angle θ with respect to the vertical, while the second mass makes an angle ϕ with respect to the vertical. A uniform gravitational field, with gravitational acceleration g, acts in the vertical direction.



(a) Starting from the description of kinetic and potential energy in Cartesian coordinates, obtain the Lagrangian in terms of the angles θ and ϕ and their time derivatives, $\dot{\theta}$ and $\dot{\phi}$. (2 points)

(b) Now simplify the Lagrangian to the situation when both angles are small, $\theta \ll 1$, $\phi \ll 1$, and obtain the form of two coupled harmonic oscillators. (1 point)

(c) For this system, obtain the mass matrix $\mathbf{M}$ and the spring-constant matrix $\mathbf{K}$, where the Lagragian is written:

$$L = \frac{1}{2}\begin{pmatrix} \dot{\theta}, & \dot{\phi} \end{pmatrix} \mathbf{M} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \theta, & \phi \end{pmatrix} \mathbf{K} \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

(2 points)

(d) Show that the normal modes satisfy

$$(\omega^2 \mathbf{M} - \mathbf{K}) \cdot \mathbf{Q} = 0.$$

(1 point)

(e) Determine the characteristic frequencies (eigenfrequencies) ω in terms of the quantity $\omega_0^2 = g/l$. (2 points)

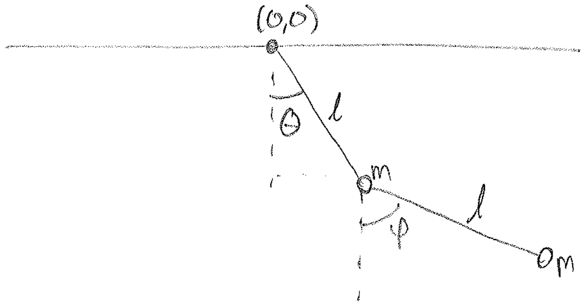
(f) If we write the normal mode vector as

$$\mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

determine the ratio q_2/q_1, which characterize the normal modes. (2 points)

Jan 2016

Classical #3



a) Starting w/ cartesian, obtain Lagrangian in terms of $\theta, \dot{\theta}, \varphi, \dot{\varphi}$

$$\mathcal{L} = T - U$$

$$= \frac{1}{2} m (v_1^2 + v_2^2) - mg(y_1 + y_2)$$

$x_1 = l \sin\theta$ $\qquad$ $y_1 = l\cos\theta$

$x_2 = l\sin\theta + l\sin\varphi$ $\qquad$ $y_2 = l\cos\theta + l\cos\varphi$

$\dot{x}_1 = l\cos\theta\,\dot{\theta}$ $\qquad$ $\dot{y}_1 = -l\sin\theta\,\dot{\theta}$

$\dot{x}_2 = l\cos\theta\,\dot{\theta} + l\cos\varphi\,\dot{\varphi}$ $\qquad$ $\dot{y}_2 = -l\sin\theta\,\dot{\theta} - l\sin\varphi\,\dot{\varphi}$

$$= \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) - mg(y_1 + y_2)$$

$$= \frac{1}{2} m l^2 [\cancel{\cos^2\theta}\,\dot{\theta}^2 + \cancel{\sin^2\theta}\,\dot{\theta}^2 + \cancel{\cos^2\theta}\,\dot{\theta}^2 + 2\cos\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \cancel{\cos^2\varphi}\,\dot{\varphi}^2 + \cancel{\sin^2\theta}\,\dot{\theta}^2 + 2\sin\theta\sin\varphi\,\dot{\theta}\dot{\varphi} + \cancel{\sin^2\varphi}\,\dot{\varphi}^2]$$

$$= \frac{1}{2} m l^2 [2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}(\cos\theta\cos\varphi + \sin\theta\sin\varphi)$$

$$= \frac{1}{2} m l^2 [2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}\cos(\theta - \varphi)] - mg(2l\cos\theta + l\cos\varphi)$$

b) Apply small angle approx. + get form of coupled harmonic oscillators

* In small angle: $\sin\theta \to \theta$
$\qquad\qquad\qquad\quad \cos\theta \to 1 - \frac{\theta^2}{2}$

$$\Rightarrow \mathcal{L} = \frac{1}{2} m l^2 [2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}(1 - \frac{(\theta-\varphi)^2}{2}) - mgl(2(1 - \frac{\theta^2}{2}) + (1 - \frac{\varphi^2}{2})$$

$$= \frac{1}{2} m l^2 [2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}(1 - \frac{1}{2}(\theta^2 - 2\theta\varphi + \varphi^2))] - mgl(2 - \theta^2 + 1 - \frac{1}{2}\varphi^2)$$

$$= \frac{1}{2} m l^2 [2\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\theta}\dot{\varphi}(2 - \theta^2 + 2\theta\varphi - \varphi^2] - mgl(3 - \theta^2 - \frac{1}{2}\varphi^2)$$

c) Find $\overleftrightarrow{M}$ and $\overleftrightarrow{K}$ when the Lagrangian is of the form:

$$\mathcal{L} = \frac{1}{2}(\dot{\theta}, \dot{\varphi})\overleftrightarrow{M}\begin{pmatrix}\dot{\theta}\\ \dot{\varphi}\end{pmatrix} - \frac{1}{2}(\theta, \varphi)\overleftrightarrow{K}\begin{pmatrix}\theta\\ \varphi\end{pmatrix}$$

$$\Rightarrow \overleftrightarrow{M} = \begin{bmatrix} ml^2 & \\ & \frac{1}{2}ml^2 \end{bmatrix}$$

$$\overleftrightarrow{K} = \Bigg[$$