2. Consider a point particle of mass m constrained to move on a parabola in the x-z plane, i.e.,

$$z = \frac{\alpha}{2} x^2.$$

Assume the constraint force is frictionless and gravity acts vertically ($F_z = -mg$).

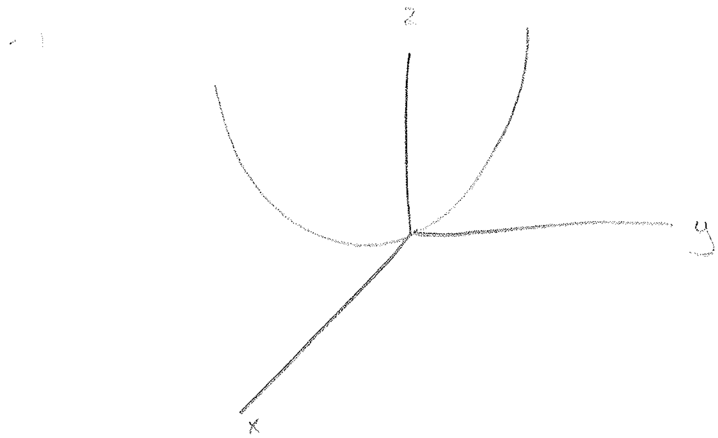
(a) Use Lagrangian mechanics to write a second order differential equation for $x(t)$. (2 points)

(b) Find a first integral of this equation (any way you can) and evaluate the constant of integration using the maximum value x_{max} reached by x. (4 points)

(c) Assume that the particle is pulled a short distance from the origin and allowed to oscillate. Calculate the period in the limit of small oscillations, $\epsilon \equiv \alpha x_{\mathrm{max}} << 1$. (4 points)

Jan 2008

Classical # 2

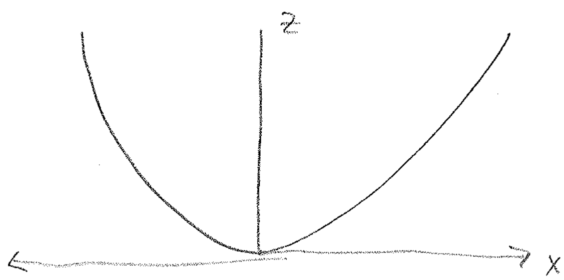


* Particle of mass m constrained to move on parabola in x-z plane

$\Rightarrow z = \frac{\alpha}{2} x^2$

* Constraint force is frictionless

$F_z = -mg$



a) $\mathcal{L} = T - U$

$T = \frac{1}{2} m v^2$ $\qquad U = mgz$

$= \frac{1}{2} m (\dot{x}^2 + \dot{z}^2)$ $\qquad = mg \frac{\alpha}{2} x^2$

$= \frac{1}{2} m (\dot{x}^2 + (\alpha \dot{x})^2)$

$= \frac{1}{2} m \dot{x}^2 (1 + \alpha)$

$\Rightarrow \mathcal{L} = \frac{1}{2}(m\dot{x}^2(1+\alpha) - mg\alpha x^2)$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

$$mg\alpha x = \frac{\partial}{\partial t}\left(m\dot{x}(1+\alpha)\right)$$

$$mg\alpha x = m(1+\alpha)\ddot{x}$$

$$g\alpha x = (1+\alpha)\ddot{x}$$

$$x = \frac{1+\alpha}{g\alpha}\ddot{x}$$

b)