3. **Angular momentum and the Rungé-Lenz vector:** Given a point particle of mass m, trajectory $\vec{r}(t)$, and momentum $\vec{p}(t)$, we can define the angular momentum

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

and the Rungé-Lenz vector

$$\vec{\mathcal{A}} = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

We consider the explicit case of a $1/r$ potential, so that

$$H = \frac{p^2}{2m} - \frac{1}{r}$$

(a) Prove that the Poisson bracket of H and $\vec{L}$ is zero, that is:

$$\left\{H, \vec{L}\right\} = 0.$$

(3 points).

(b) Prove that the Poisson bracket of H and $\vec{\mathcal{A}}$ is zero, that is:

$$\left\{H, \vec{\mathcal{A}}\right\} = 0.$$

(3 points)

(c) What do your results in parts (a) and (b) imply about the behavior of $\vec{\mathcal{A}}$ and $\vec{L}$? (1 point)

(d) Evaluate $\vec{r} \cdot \vec{\mathcal{A}} = r\mathcal{A}\cos\theta$, using the explicit form for $\vec{\mathcal{A}}$ above. Use this to calculate the orbital motion of the particle (that is, a relationship between r and θ as the particle moves about its orbit). (3 points)

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Classical #3

$\vec{L} = \vec{r} \times \vec{p}$

$\vec{A} = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$

$H = \frac{p^2}{2m} - \frac{1}{r}$

a) Prove Poisson bracket, $\{H, L\} = 0$

$$H = \frac{p^2}{2m} - \frac{1}{r}$$
$$= \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$L = \vec{r} \times p$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \langle y p_z - z p_y, -(x p_z - z p_x), x p_y - y p_x \rangle$$

$$\{H, L\} = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial L}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial L}{\partial q_i} \right)$$
$$= \frac{\partial H}{\partial x}\frac{\partial L}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial L}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial L}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial L}{\partial y} + \frac{\partial H}{\partial z}\frac{\partial L}{\partial p_z} - \frac{\partial H}{\partial p_z}\frac{\partial L}{\partial z}$$

$\frac{\partial H}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_x} = \frac{p_x}{m}$

$\frac{\partial L}{\partial p_x} = \langle 0, z, -y \rangle$ $\qquad$ $\frac{\partial L}{\partial x} = \langle 0, -p_z, p_y \rangle$

$\frac{\partial H}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_y} = \frac{p_y}{m}$

$\frac{\partial L}{\partial p_y} = \langle -z, 0, x \rangle$ $\qquad$ $\frac{\partial L}{\partial y} = \langle p_z, 0, -p_x \rangle$

$\frac{\partial H}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_z} = \frac{p_z}{m}$

$\frac{\partial L}{\partial p_z} = \langle y, -x, 0 \rangle$ $\qquad$ $\frac{\partial L}{\partial z} = \langle -p_y, p_x, 0 \rangle$

$$= \frac{x}{(\;)^{3/2}}\langle 0, z, -y \rangle - \frac{p_x}{m}\langle 0, -p_z, p_y \rangle +$$
$$\frac{y}{(\;)^{3/2}}\langle -z, 0, x \rangle - \frac{p_y}{m}\langle p_z, 0, -p_x \rangle +$$
$$\frac{z}{(\;)^{3/2}}\langle y, -x, 0 \rangle - \frac{p_z}{m}\langle -p_y, p_x, 0 \rangle$$

$\langle 0, 0, 0 \rangle$ ✓

b) Show $\{H, A\} = 0$

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$A = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

$$= \frac{1}{m}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ p_x & p_y & p_z \\ yp_z - zp_y & zp_x - xp_z & xp_y - yp_x \end{vmatrix} - \langle x, y, z \rangle \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{1}{m}\langle p_y(xp_y - yp_x) - p_z(zp_x - xp_z),\ p_z(yp_z - zp_y) - p_x(xp_y - yp_x),\ p_x(zp_x - xp_z) - p_y(yp_z - zp_y) \rangle - \langle x, y, z \rangle$$

$$\{H, \hat{A}\} = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial A}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial A}{\partial q_i}$$

$$= \frac{\partial H}{\partial x}\frac{\partial A}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial A}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial A}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial A}{\partial y} + \frac{\partial H}{\partial z}\frac{\partial A}{\partial p_z} - \frac{\partial H}{\partial p_z}\frac{\partial A}{\partial z}$$

$$\frac{\partial H}{\partial x} = -\left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x\right) = \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

$$\frac{\partial A}{\partial x} = \frac{1}{m}\langle p_y^2 + p_z^2, -p_xp_y, -p_xp_z \rangle - \langle 1, 0, 0 \rangle$$

$$\frac{\partial A}{\partial p_x} = \frac{1}{m}\langle -yp_y - zp_z, -xp_y + 2yp_x, 2zp_x - xp_z \rangle$$

$$\frac{\partial H}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\frac{\partial A}{\partial y} = \frac{1}{m}\langle -p_xp_y, p_z^2 + p_x^2, -p_yp_z \rangle - \langle 0, 1, 0 \rangle$$

$$\frac{\partial A}{\partial p_y} = \frac{1}{m}\langle 2xp_y - yp_x, -zp_z - xp_x, -yp_z + 2zp_y \rangle$$

$$\frac{\partial H}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

$$\frac{\partial A}{\partial z} = \frac{1}{m}\langle -p_xp_z, -p_yp_z, p_x^2 + p_y^2 \rangle - \langle 0, 0, 1 \rangle$$

$$\frac{\partial A}{\partial p_z} = \frac{1}{m}\langle -zp_x + 2xp_z, 2yp_z - zp_y, -xp_x - yp_y \rangle$$

$$\Rightarrow \{H, A\}_3 = \frac{x}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle -yp_y - zp_z, \ -xp_y + 2yp_x, \ 2zp_x - xp_z \rangle$$

$$- \frac{p_x}{m} \left(\frac{1}{m} \langle p_y^2 + p_z^2, \ -p_xp_y, \ -p_xp_z \rangle - \langle 1, 0, 0 \rangle \right)$$

$$+ \frac{y}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle 2xp_y - yp_x, \ -zp_z - xp_x, \ -yp_z + 2zp_y \rangle$$

$$- \frac{p_y}{m} \left(\frac{1}{m} \langle -p_xp_y, \ p_z^2 + p_x^2, \ -p_yp_z \rangle - \langle 0, 1, 0 \rangle \right)$$

$$+ \frac{z}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle 2xp_z - zp_x, \ 2yp_z - zp_y, \ -xp_x - yp_y \rangle$$

$$- \frac{p_z}{m} \left(\frac{1}{m} \langle -p_xp_z, \ -p_yp_z, \ p_x^2 + p_y^2 \rangle - \langle 0, 0, 1 \rangle \right)$$

$$= \frac{1}{mr^3} \langle x(-yp_y - zp_z) + y(2xp_y - yp_x) + z(2xp_z - zp_x), \ x(-xp_y + 2yp_x) + y(-zp_z - xp_x) + z(2yp_z - zp_y),$$

$$x(2zp_x - xp_z) + y(-yp_z + 2zp_y) + z(-xp_x - yp_y) \rangle$$

$$+ \frac{1}{m} \langle p_x, p_y, p_z \rangle - \frac{1}{m^2} \langle p_x(p_y^2 + p_z^2) + p_y(-p_xp_y) + p_z(-p_xp_z), \ p_x(-p_xp_y) + p_y(p_x^2 + p_z^2) + p_z(-p_yp_z),$$

$$p_x(-p_xp_z) + p_y(-p_yp_z) + p_z(p_x^2 + p_y^2) \rangle$$

$$= \frac{1}{mr^3} \langle -xyp_y - xzp_z + 2xyp_y - y^2p_x + 2xzp_z - z^2p_x, \ -x^2p_y + \cancel{2}yxp_x - \cancel{yzp_z} - \cancel{xyp_x} + \cancel{2}yzp_z - z^2p_y,$$

$$\cancel{2}xzp_x - x^2p_z - y^2p_z - \cancel{2yzp_y} - \cancel{xzp_x} - \cancel{yzp_y} \rangle$$

$$+ \frac{1}{m} \langle p_x, p_y, p_z \rangle - \frac{1}{m^2} \langle \cancel{p_xp_y^2} + \cancel{p_xp_z^2} - \cancel{p_xp_y^2} - \cancel{p_xp_z^2}, \ -\cancel{p_x^2p_y} + \cancel{p_x^2p_y} + \cancel{p_yp_z^2} - \cancel{p_yp_z^2},$$

$$-\cancel{p_x^2p_z} - \cancel{p_y^2p_z} + \cancel{p_x^2p_z} + \cancel{p_y^2p_z} \rangle$$

$$= \frac{1}{mr^3} \langle xyp_y + xzp_z - y^2p_x - z^2p_x, \ xyp_x + yzp_z - x^2p_y - z^2p_y, \ xzp_x + yzp_y - x^2p_z - y^2p_z \rangle$$

$$+ \frac{1}{m} \langle p_x, p_y, p_z \rangle$$

c) If $\{H, L\}$ and $\{H, A\}$ are both 0, then both $\vec{L}$ and $\vec{A}$ are constants of the motion.

d) $\vec{r} \cdot \vec{A} = rA\cos\theta$

$$\vec{r} \cdot \left(\frac{1}{m}\vec{p} \times \vec{L} - \hat{r}\right) = \frac{1}{m}\left(\vec{r} \cdot (\vec{p} \times \vec{L})\right) - \vec{r} \cdot \hat{r}$$

$$= \frac{1}{m}\vec{L} \cdot (\vec{r} \times \vec{p}) - r$$

$$= \frac{1}{m}L^2 - r$$

$$\frac{L^2}{m} - r = rA\cos\theta$$

$$\frac{L^2}{m} = r(A\cos\theta + 1)$$

$$\frac{L^2}{m} \cdot \frac{1}{A\cos\theta + 1} = \vec{r}$$