Problem 3 (10 Points):

Consider the following Lagrangian

$$L = \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right) e^{2\gamma t}$$

assuming that $\omega > \gamma$ for the questions that follow.

a. Determine the Hamiltonian associated with this Lagrangian. **(3 Points)**

b. Find a transformation to new phase space variables that make H independent of time and show that these form a canonical transformation by determining a generating function of the form $F_2(q, P, t)$. **(4 Points)**

c. Using the equations of motion for the transformed Hamiltonian $K(Q, P, t)$, solve for $Q(t)$ and transform back to get $q(t)$. **(3 Points)**

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Classical #3

Given: $\mathcal{L} = (\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2)e^{2\gamma t}$

a) Determine the Hamiltonian associated w/ the above Lagrangian

$$H = \sum_i p_i \dot{q}_i - \mathcal{L}$$

$$p_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$= e^{2\gamma t} m\dot{q} \qquad \Rightarrow \quad \dot{q} = \frac{p}{m}e^{-2\gamma t}$$

$$\Rightarrow H = m\dot{q}^2 e^{2\gamma t} - e^{2\gamma t}(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2)$$

$$= e^{2\gamma t}(\frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2)$$

$$= e^{2\gamma t}(\frac{1}{2m}p^2 e^{-4\gamma t} + \frac{1}{2}m\omega^2 q^2)$$

$$= \frac{1}{2m}p^2 e^{-2\gamma t} + \frac{1}{2}m\omega^2 q^2 e^{2\gamma t}$$

b) Find a transformation to new phase space variables that make H time independent and show that these form a canonical transformation by determining a $F_2(q, P, t)$ generating function.

* For an $F_2(q, P, t)$: $\quad p = \frac{\partial F_2(q, P)}{\partial q} \qquad Q = \frac{\partial F_2(q, P)}{\partial P}$

$\Rightarrow$ Want: $H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2$

$$\Rightarrow Q = qe^{\gamma t}$$

$$P = pe^{-\gamma t}$$

$$\Rightarrow F_2(q, P, t) = Pqe^{\gamma t}$$

$$\Rightarrow K = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2$$

c) Use the Hamiltonian eqns. of motion for K to solve for Q and transform it back to $q(t)$

$$\dot{P} = \frac{\partial K}{\partial Q} = m\omega^2 Q$$

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{P}{m}$$

$$\dot{P} = m\ddot{Q} \qquad \Rightarrow m\ddot{Q} = \omega^2 m Q$$

$$\ddot{Q} = \omega^2 Q \qquad \Rightarrow Q = Ae^{-i\omega t} + Be^{i\omega t}$$

$$\Rightarrow q = e^{-\gamma t}Q$$

$$= e^{-\gamma t}(Ae^{-i\omega t} + Be^{i\omega t})$$