2. The isotropic harmonic oscillator.

   (a) Write the Lagrangian for a point mass m moving under the influence of an isotropic 3-dimensional harmonic oscillator potential

   $$V(x, y, z) = \frac{k}{2}(x^2 + y^2 + z^2).$$

   There is no external gravitational field. (1 point)

   (b) Using the Lagrange equations of motion show that angular momentum is conserved. i.e.,

   $$\frac{d}{dt}\mathbf{L} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = 0.$$

   Because the Lagrangian is invariant under rotations about the origin, you can choose coordinates so that motion is constrained to the x-y plane, i.e., the angular momentum points in the z direction. (3 points)

   (c) For 2-dimensional motion in the x-y plane choose cylindrical polar coordinates and proceed to solve the Lagrange equations of motion. You can leave the solution for $r(t)$ as an integral of the form $t = \int f(r)dr$. (Don't forget to use conservation of energy, E_0.) (3 points)

   (d) Compute the minimum and maximum values or the radial coordinate r as functions of the constants m, E_0, k, L^z. (3 points)

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Classical #2

a) Write Lagrangian for a point mass under the influence of the following potential:

$V(x,y,z) = \frac{k}{2}(x^2+y^2+z^2)$ * No external gravity

$$\mathcal{L} = T - V$$
$$= \frac{1}{2}mv^2 - \frac{1}{2}k(x^2+y^2+z^2)$$
$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{1}{2}k(x^2+y^2+z^2)$$

b) Use Lagrange equations of motion to show angular momentum conservation

i.e. $\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = 0$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{x}}$ $\quad$ $\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{y}}$ $\quad$ $\frac{\partial \mathcal{L}}{\partial z} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{z}}$

$\Rightarrow -kx = \frac{\partial}{\partial t}(m\dot{x})$ $\quad$ $\Rightarrow -ky = \frac{\partial}{\partial t}(m\dot{y})$ $\quad$ $\Rightarrow -kz = \frac{\partial}{\partial t}(m\dot{z})$

$-kx = m\ddot{x}$ $\quad$ $-ky = m\ddot{y}$ $\quad$ $-kz = m\ddot{z}$

$$\vec{r} \times m\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix} = \langle ym\dot{z} - zm\dot{y},\ xm\dot{z} - zm\dot{x},\ xm\dot{y} - ym\dot{x} \rangle$$

$$= m\langle y\dot{z} - z\dot{y}, x\dot{z} - z\dot{x}, x\dot{y} - y\dot{x} \rangle$$

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = m\langle \dot{y}\dot{z} + y\ddot{z} - (\dot{z}\dot{y} + z\ddot{y}), \dot{x}\dot{z} + x\ddot{z} - (\dot{z}\dot{x} + z\ddot{x}), \dot{x}\dot{y} + x\ddot{y} - (\dot{y}\dot{x} + y\ddot{x}) \rangle$$

$$= m\langle y\ddot{z} - z\ddot{y}, x\ddot{z} - z\ddot{x}, x\ddot{y} - y\ddot{x} \rangle$$

* substituting from above

$$= m\left(\frac{-k}{m}\right)\langle yz - zy, xz - zx, xy - yx \rangle$$

$$= -k\langle 0, 0, 0 \rangle$$

$$= 0$$

$\therefore \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d}{dt}\vec{L} = 0$ ✓

c) For 2-D motion in x-y plane, use cylindrical polar coordinates to solve Lagrange eqns of motion. Leave $r(t)$ as an integral of form $t = \int f(r)dr$

* Hint: use conservation of energy E_0

- In cylindrical polar: $x = r\cos\varphi$ $\quad \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\dot{\varphi}$

$y = r\sin\varphi$ $\quad \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\dot{\varphi}$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r\dot{\varphi}^2) - \tfrac{1}{2}kr^2 \quad (2\text{-D},\ z=0)$$

$$E_0 = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \tfrac{1}{2}kr^2$$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = 0$$

$$0 = kr + mr\dot{\varphi}^2 - m\ddot{r} \qquad 0 = -2mr\dot{r}\dot{\varphi} - mr^2\ddot{\varphi}$$

* Solving E_0 for $\dot{\varphi}$ yields

$$E_0 = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\varphi}^2 + \tfrac{1}{2}kr^2$$

$$\frac{2E_0}{m} = \dot{r}^2 + r^2\dot{\varphi}^2 + \frac{k}{m}r^2$$

$$\frac{2E_0}{m} - \frac{k}{m}r^2 - \dot{r}^2 = r^2\dot{\varphi}^2$$

$$\frac{2E_0}{mr^2} - \frac{k}{m} - \frac{\dot{r}^2}{r^2} = \dot{\varphi}^2$$

$$\Rightarrow 0 = kr + mr\left(\frac{2E_0}{mr^2} - \frac{k}{m} - \frac{\dot{r}^2}{r^2}\right) + \tfrac{1}{2}kr^2$$

$$= \cancel{kr} + \frac{2E_0}{r} - \cancel{kr} - \frac{m\dot{r}^2}{r} + \tfrac{1}{2}kr^2$$

$$0 = \frac{2E_0}{r} - \frac{m\dot{r}^2}{r} + \tfrac{1}{2}kr^2$$

$$\frac{m\dot{r}^2}{r} = \frac{2E_0}{r} + \tfrac{1}{2}kr^2$$

$$\dot{r}^2 = 2mE_0 + \tfrac{1}{2}\tfrac{k}{m}r^3$$

$$\Rightarrow t = \int \left(2mE_0 + \tfrac{1}{2}\tfrac{k}{m}r^3\right)^{1/2} dr$$