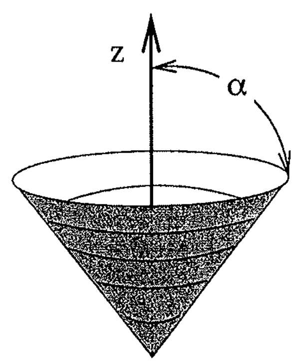
2. A point particle of mass m travels on the frictionless inner surface of an inverted cone. The cone is oriented so its symmetry axis is parallel to the z-axis, with an opening angle α between the z-axis and the surface of the cone. The force of gravity points in the negative z-direction.



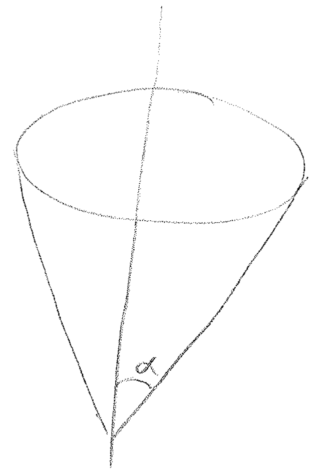
(a) Write the Lagrangian for the problem in cylindrical coordinates. (1 pt.)

(b) Assume the particle is moving in a uniform circular orbit at distance d from the cone tip, measured along the surface of the cone. Determine the angular frequency of the system. (3 pts.)

(c) The opening angle of the cone is abruptly decreased by $\Delta\alpha << \alpha$. This is done in a manner that does **not** impart an impulse or do work on the particle. (Imagine that the cone is instantaneously stretched so that its tip moves slightly downward, but the particle is not displaced during the stretching). Describe the subsequent motion of the particle in this limit. Express your answer in terms of ρ_0, the original radius of the circular orbit, m, α, $\Delta\alpha$, and g. Explain any approximations you are making in deriving your result. (6 pts.)

Aug 2009

Classical #2



* Cone is frictionless
* Particle travels on inner surface of cone
* Gravity points in $-\hat{z}$ direction

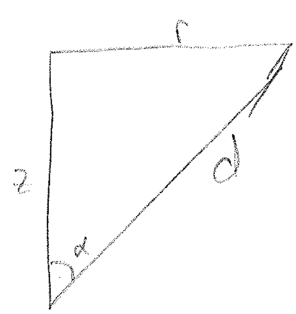
a) Write the Lagrangian in cylindrical coordinates

* In cylindrical coordinates:

$$x = r\cos\varphi \qquad \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\,\dot{\varphi}$$
$$y = r\sin\varphi \qquad \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\,\dot{\varphi}$$
$$z = z \qquad \dot{z} = \dot{z}$$

$$\mathcal{L} = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$
$$= \tfrac{1}{2}m\left[(\dot{r}\cos\varphi - r\sin\varphi\,\dot{\varphi})^2 + (\dot{r}\sin\varphi + r\cos\varphi\,\dot{\varphi})^2 + \dot{z}^2\right] - mgz$$
$$= \tfrac{1}{2}m\left[\dot{r}^2\cancel{\cos^2\varphi} - 2\cancel{r\cos\varphi\sin\varphi}\,\dot{r}\dot{\varphi} + r^2\cancel{\sin^2\varphi}\,\dot{\varphi}^2 + \dot{r}^2\cancel{\sin^2\varphi} + 2r\cancel{\sin\varphi\cos\varphi}\,\dot{r}\dot{\varphi} + r^2\cancel{\cos^2\varphi}\,\dot{\varphi}^2 + \dot{z}^2\right] - mgz$$
$$= \tfrac{1}{2}m\left[\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right] - mgz$$

b) Assuming a uniform circular orbit at a distance d from the tip of the cone, determine the angular frequency of the system



$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m\left[\dot{d}^2\cos^2\alpha + d\cos^2\alpha\,\dot{\varphi}^2 + \dot{d}^2\sin^2\alpha\right] - mgd\sin\alpha$$
$$= \tfrac{1}{2}m\left[\dot{d}^2 + d\cos^2\alpha\,\dot{\varphi}^2\right] - mgd\sin\alpha$$

$r = d\cos\alpha \qquad \dot{r} = \dot{d}\cos\alpha$

$z = d\sin\alpha \qquad \dot{z} = \dot{d}\sin\alpha$

$\uparrow$

$\underline{\alpha \text{ is const.}}$

b) $\frac{\partial \mathcal{L}}{\partial d} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{d}}$

$\frac{1}{2} m \cos^2\alpha \, \dot{\varphi}^2 - mg\sin\alpha = \frac{\partial}{\partial t}\left[m\dot{d}\right]$

$\frac{1}{2} m \cos^2\alpha \, \dot{\varphi}^2 - mg\sin\alpha = m\ddot{d}$

$\frac{1}{2} \cos^2\alpha \, \dot{\varphi}^2 - g\sin\alpha = \ddot{d}$

$\Rightarrow \dot{\varphi} = \left(\frac{\ddot{d} + g\sin\alpha}{\frac{1}{2}\cos^2\alpha}\right)^{1/2}$

* but b/c of circular motion, $\dot{d} = 0$

$\Rightarrow \dot{\varphi} = \left(2g\frac{\sin\alpha}{\cos^2\alpha}\right)^{1/2}$

$\dot{\varphi} = (2g\tan\alpha\sec\alpha)^{1/2}$

c) How does orbit change as $\alpha \to \alpha - \Delta\alpha$

$\mathcal{L} = \frac{1}{2} m$

$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$

$0 = \frac{\partial}{\partial t}\left[m d \cos^2\alpha \, \dot{\varphi}\right]$

$0 = m\cos^2\alpha\left(\dot{d}\dot{\varphi} + d\ddot{\varphi}\right)$

$\dot{d}\dot{\varphi} = -d\ddot{\varphi}$

$\dot{\varphi} = -\frac{d\ddot{\varphi}}{d}$