

6. Consider a free, non-interacting spin zero Bose gas in two dimensions. The energy of each particle is given by:

$$\mathcal{E}(\vec{k}) = \hbar^2 k^2 / 2m$$

where m is the mass of the boson. Assume your system is confined to a square region of length L on a side.

- (a) Write down an expression for the grand canonical free energy $\mathcal{G}(T, V, \mu)$ as a sum over \vec{k} states. Do not evaluate the sum. (1 pt.)
- (b) Calculate the number of particles in the system as a function of T , V and μ . (3 pts.)
- (c) Analyze your expression for $N(T, V, \mu)$ in the limit $T \rightarrow 0$. What does it imply about the possibility of a Bose-Einstein transition in this system? (3 pts.)
- (d) Prove that the pressure is equal to the energy density, so that $PV = U$. (Hint: you do not have to do any sums over states - you need only prove that this holds using analytic expressions for P and U in this particular system). (3 pts.)

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Stat Mech #3

* Consider a free, non-interacting spin 0 Bose gas in 2-D, where the energy of each particle is: $E(\mathbf{k}) = \frac{\hbar^2 k^2}{2m}$

- Assume m is mass of boson and the system is confined to a square region of side length L .

a) Write down the expression for the grand canonical free energy \mathcal{J} .

$$\begin{aligned}\mathcal{J} &= -PV \\ &= -kT \ln(\mathcal{Z}) \\ \text{with } \mathcal{Z} &= \prod_j [1 - \exp(\beta(\mu - \epsilon_j))]^{-1} \\ &= -kT \sum_j \ln(1 - \exp[\beta(\mu - \epsilon_j)]) \\ &= -kT \sum_{\mathbf{k}} \ln(1 - \exp[\beta(\mu - \frac{\hbar^2 k^2}{2m})])\end{aligned}$$

b) Calculate the # of particles in the system as a function of T , V , and μ .

$$\begin{aligned}N &= \left(\frac{\partial \mathcal{J}}{\partial \mu}\right)_{T, V} \\ &= \frac{\partial}{\partial \mu} kT \sum_{\mathbf{k}} \ln(1 - \exp[\beta(\mu - \frac{\hbar^2 k^2}{2m})]) \\ &= -kT \sum_{\mathbf{k}} \frac{-\beta \exp[\beta(\mu - \frac{\hbar^2 k^2}{2m})]}{1 - \exp[\beta(\mu - \frac{\hbar^2 k^2}{2m})]} \\ &= \sum_{\mathbf{k}} [\exp[-\beta(\mu - \frac{\hbar^2 k^2}{2m})] - 1] \\ &\approx \int\end{aligned}$$