5. Consider a classical ideal gas in 3D that feels a linear gravitational potential,

$$V(z) = mgz$$

where m is the mass of a single gas atom and $0 < z < \infty$. This is not an interaction between gas atoms, it is simply their gravitational potential energy near the surface of the Earth.

The gas is in a box of dimensions L_x, L_y, and L_z, so that:

$$\begin{aligned} 0 < \ & z \ < L_z \\ 0 < \ & x \ < L_x \\ 0 < \ & y \ < L_y \end{aligned}$$

(a) Calculate the partition function in the canonical ensemble. (3 points)

(b) Determine the internal energy of the gas. (3 points)

(c) Calculate the specific heat c_v. (3 points)

(d) Explain the behavior of the specific heat when $\beta mgL_z >> 1$ and when $\beta mgL_z << 1$. (The approximation for the gravitational potential may or may not be valid for large L_z. Don't worry about that.) (1 point)

Stat Mech #2

*Consider a classical ideal gas in 3-D w/ linear gravitational potential $V(z) = mgz$. Note: m is mass of single atom, $0 < z < \infty$. Dimensions of box are: $0 < z < L_z$, $0 < x < L_x$, $0 < y < L_y$

a) Calculate the partition function in the classical ensemble

$$Z = \left[\frac{1}{h^3}\int dp^3 dq^3 e^{-\beta E}\right]^N$$

$$* \text{ let } E = \frac{P_x^2 + P_y^2 + P_z^2}{2m} + mgz$$

$$= \left[\frac{1}{h^3}\int dp^3 dq^3 e^{-\beta\left(\frac{1}{2m}[P_x^2 + P_y^2 + P_z^2] + mgz\right)}\right]^N$$

$$= \left[\frac{1}{h^3}\int dp_x dp_y dp_z\, dx dy dz\, e^{-\beta[\frac{1}{2m}(P_x^2 + P_y^2 + P_z^2) + mgz]}\right]^N$$

$$= \left[\frac{1}{h^3}\left(\int_0^\infty dp\, e^{-\beta p^2/2m}\right)^3 \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz\, e^{-\beta mgz}\right]^N$$

$$= \left[\frac{1}{h^3}\left(\frac{1}{2}\sqrt{\frac{\pi}{\beta/2m}}\right)^3 L_x L_y \left(\frac{1}{-\beta mg} e^{-\beta mgz}\Big|_0^{L_z}\right)\right]^N$$

$$= \left[\left(\frac{1}{\cancel{2}h}\sqrt{\frac{2\pi m}{\beta}}\right)^3 L_x L_y \frac{1}{-\beta mg}\left(e^{-\beta mgL_z} - 1\right)\right]^N$$

b) Determine the internal energy of the gas

$$U = -\frac{\partial}{\partial\beta}\ln(Z)$$

$$= \frac{-\partial}{\partial\beta} N \ln\left[\left(\frac{2\pi m}{h^2\beta}\right)^{3/2} L_x L_y \frac{1}{-\beta mg}\left(e^{-\beta mgL_z} - 1\right)\right]$$

$$= \frac{-\partial}{\partial\beta} N \ln\left[\left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{L_x L_y}{mg} \frac{1}{-\beta^{5/2}}\left(e^{-\beta mgL_z} - 1\right)\right]$$

$$= \frac{-\partial}{\partial\beta} N\left(\ln\left[\left(\frac{2\pi m}{h^2}\right)^{3/2}\frac{L_x L_y}{mg}\right] + \ln\left[1 - e^{-\beta mgL_z}\right] - \ln\left(\beta^{5/2}\right)\right)$$

$$= -N\left[\left(1 - e^{-\beta mgL_z}\right)^{-1} + mgL_z e^{-\beta mgL_z} - \frac{5}{2\beta}\right]$$

c) Calculate the specific heat C_V

$$C_V = \frac{\partial U}{\partial T}$$

$$= \frac{\partial}{\partial T}\left(-N\left(\frac{mgL_z e^{-\beta mgL_z}}{1-e^{-\beta mgL_z}} - \frac{5}{2\beta}\right)\right)$$

$$= \frac{\partial}{\partial T}\left(\frac{5}{2}Nk_BT - \frac{mgL_z}{e^{-mgL_z/k_BT}-1}\right)$$

$$= \frac{5}{2}Nk_B + mgL_z\left(e^{-mgL_z/k_BT}-1\right)^{-2} \cdot -mgL_z/k_BT^2\, e^{-mgL_z/k_BT}$$

$$= \frac{5}{2}Nk_B - \frac{(mgL_z)^2 e^{-mgL_z/k_BT}}{T^2k_B\left(e^{-mgL_z/k_BT}-1\right)^2}$$