4. A heat engine is made from N atoms of an ideal mono-atomic gas starting at an initial temperature T_1, and volume V_1. Call this state "1." It is initially heated isochorically (at constant volume) to a state "2" with a temperature $T_2 = 4T_1$. It then undergoes an adiabatic expansion to state "3" where it has returned to its original pressure. Finally it is then cooled isobarically (at constant pressure) until it returns to its original condition.

   (a) Draw the thermodynamic cycle in the PV plane. (1 point).

   (b) Calculate the volume and temperature at states 2 and 3 in terms of V_1, T_1 and N. (1 point).

   (c) Calculate the work done by the gas in each step of the cycle. (3 points).

   (d) Calculate the heat in (or out) of the gas during each step. (3 points)

   (e) What is the efficiency of this engine? (2 points)

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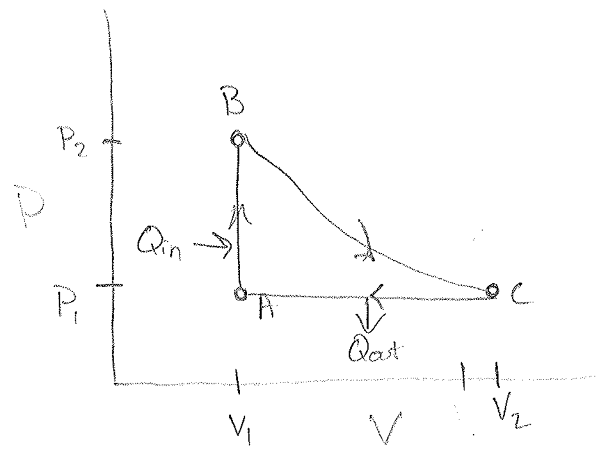
Stat Mech #1

A heat engine made of N atoms of an ideal monatomic gas starting at initial temp T_1 and volume V_1. It is heated isochorically to $T_2 = 4T_1$. It then expands adiabatically to state C where it has returned to its initial pressure. It is then isobarically cooled to its initial state.

a) Draw the cycle in the P-V plane.



b) Calculate the temperature at states B and C in terms of V_1, T_1, and N

$P_A = \frac{N k_B T_1}{V_1}$ $\qquad$ $P_B = \frac{4 N k_B T_1}{V_1}$ $\qquad$ $P_C = \frac{N k_B T_1}{V_1}$

$4^3 = 64$

$V_A = V_1$ $\qquad$ $V_B = V_1$ $\qquad$ $V_C = 4^{3/5} V_1$

$T_A = T_1$ $\qquad$ $\boxed{T_B = 4T_1}$ $\qquad$ $T_C = 4^{3/5} T_1$

$P_B V_B^{\gamma} = P_C V_C^{\gamma}$

$\frac{4 N \cancel{k_B} T_1}{\cancel{V_1}} V_1^{5/3} = \frac{N \cancel{k_B} T_1}{\cancel{V_1}} V_C^{5/3}$

$4^{3/5} V_1 = V_C$

$\frac{V_A}{T_A} = \frac{V_C}{T_C}$

$\Rightarrow T_C = \frac{V_C}{V_A} T_A$

$= \frac{4^{3/5} V_1}{V_1} T_1$

$= 4^{3/5} T_1$

c) Calculate the work done by the gas in each step.

$$W_{A\to B} = 0 \quad b/c \text{ isochoric}$$

$$W_{B\to C} = \frac{P_C V_C - P_B V_C}{1-\gamma}$$

$$= \frac{3}{2}\left(4^{3/5} N k_B T - 4 N k_B T\right)$$

$$= -\frac{3}{2} N k_B T \left(4^{3/5} - 4\right)$$

$$W_{C\to A} = P\Delta V$$

$$= \frac{N k_B T}{\cancel{V_1}}\left(\cancel{V_1} - 4^{3/5}\cancel{V_1}\right)$$

$$= N k_B T \left(1 - 4^{3/5}\right)$$

d) Calculate the heat during each step

$$Q_{A\to B} = n C_V \Delta T$$

$$= \frac{N}{n_{av}}\left(\frac{3}{2}R\right)\left(4T_1 - T_1\right)$$

$$= \frac{9 R N T_1}{2(6.02\cdot 10^{23})}$$

$$Q_{B\to C} = 0 \quad b/c \text{ adiabatic}$$

$$Q_{C\to A} = n C_p \Delta T$$

$$= \frac{N}{n_{av}}\left(\frac{5}{2}R\right)\left(T_1 - 4^{3/5}T_1\right)$$

$$= \frac{5 N R T_1 \left(1 - 4^{3/5}\right)}{2\cdot(6.02\cdot 10^{23})}$$

e) What is the efficiency of the engine?

$$\eta = 1 - \left|\frac{Q_{out}}{Q_{in}}\right|$$

$$= 1 - \left|\frac{\frac{5 N R T_1 (1-4^{3/5})}{2\cdot(6.02\cdot 10^{23})}}{\frac{9 N R T_1}{2\cdot(6.02\cdot 10^{23})}}\right| = 1 - \left|\frac{5(1-4^{3/5})}{9}\right|$$