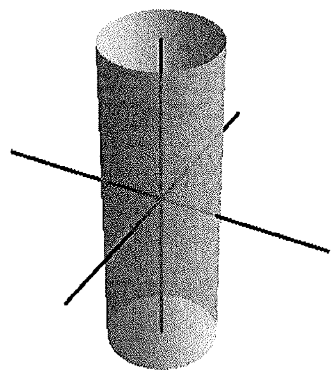
3. A particle of mass m is constrained to move on an infinitely long cylinder of radius a. The center of the cylinder is oriented along the z-axis, as shown. An attractive central potential, $U(r) = U(\sqrt{a^2 + z^2})$, is located at the origin, where r is the radius is spherical coordinates.



(a) Write down the Lagrangian for the problem. (1pt)

(b) From the Lagrangian, explicitly derive the Hamiltonian for the particle. (2pts)

(c) Is angular momentum about the z-axis conserved? Prove your answer. (2pts)

(d) Under what conditions is motion in the z-direction bounded? (2pts)

(e) Assume that the potential is $U(r) = \frac{1}{2}\alpha r^2$. Solve the equations of motion, and reduce the problem to quadrature. (3pts)

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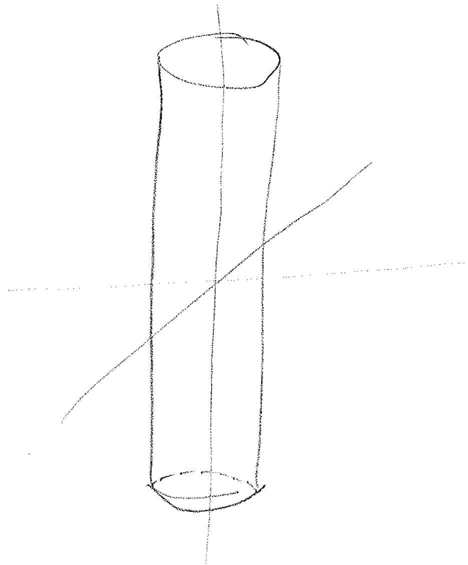
Classical #3



* Particle of mass m constained to move on infinetely long cylinder of radius a; cylinder oriented along z axis

* Attractive central located at origin,

$$U(r) = U(\sqrt{a^2+z^2}), \text{ where r is radius in spherical}$$

a) Find the Lagrangian

$$x = \rho\cos\varphi \qquad \dot{x} = \dot{\rho}\cos\varphi - \rho\sin\varphi\,\dot{\varphi}$$

$$y = \rho\sin\varphi \qquad \dot{y} = \dot{\rho}\sin\varphi + \rho\cos\varphi\,\dot{\varphi}$$

$$\mathcal{L} = T - U$$

$$= \tfrac{1}{2}mv^2 - U(\sqrt{a^2+z^2})$$

$$= \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - U(\sqrt{a^2+z^2})$$

$$= \tfrac{1}{2}m(\dot{\rho}^2\cos^2\varphi - 2\rho\cos\varphi\sin\varphi\,\dot{\rho}\dot{\varphi} + \rho^2\sin^2\varphi\,\dot{\varphi}^2 + \dot{\rho}^2\sin^2\varphi + 2\rho\sin\varphi\cos\varphi\,\dot{\rho}\dot{\varphi} + \rho^2\cos^2\varphi\,\dot{\varphi}^2 + \dot{z}^2) - U$$

$$= \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - U(\sqrt{a^2+z^2})$$

* Assume arbitrary central potential Ar^n

$$= \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - A(\sqrt{a^2+z^2})^n$$

b) From the Lagrangian, derive the Hamiltonian

$$H = \sum_i p_i\dot{q}_i - \mathcal{L}$$

$$P_\rho = \frac{\partial\mathcal{L}}{\partial\dot{\rho}} = m\dot{\rho}$$

$$P_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = m\rho^2\dot{\varphi}$$

$$P_z = \frac{\partial\mathcal{L}}{\partial\dot{z}} = m\dot{z}$$

$$\Rightarrow H = m\dot{\rho}^2 + m\rho^2\dot{\varphi}^2 + m\dot{z}^2 - \left[\tfrac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2) - A(\sqrt{a^2+z^2})^n\right]$$

$$= \tfrac{1}{2}m\dot{\rho}^2 + \tfrac{1}{2}m\rho^2\dot{\varphi}^2 + \tfrac{1}{2}m\dot{z}^2 + A(\sqrt{a^2+z^2})^n$$

c) Is angular momentum about z-axis conserved?

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right]$$

$$0 = \frac{d}{dt}\left[m\rho^2\dot{\varphi}\right]$$

$$\Rightarrow m\rho^2\dot{\varphi} = \text{const.} = L \checkmark$$

d) Under what conditions is motion in the z direction bounded?

$\Rightarrow$ Motion is bounded when $E_T < U \rightarrow T < 0$