2. Consider a point particle of mass m moving under the influence of a central force:

$$\vec{F}(\vec{r}) = -\frac{k}{r^n}\,\hat{r}$$

where n is an integer greater than one ($n = 2, 3, \ldots$), the variable r is the distance from the origin of the force ($r \equiv |\vec{r}|$) and $\hat{r}$ is a unit vector in the radial direction. In this problem, we will examine when circular orbits are stable for such a central force.

(a) Calculate potential energy of this force. Choose the zero of the potential to be at infinity ($r = \infty$). (1pt)

(b) Show that the angular momentum about the origin, L, is conserved. (You may use the Newtonian, Lagrangian, or Hamiltonian formulations of the problem). (2pts)

(c) Write an expression for the total energy of the particle E as a function of r, dr/dt, L, k, and n. (1pt)

(d) Assume the particle is moving in a circular orbit about the origin, so that $dr/dt = 0$. Calculate the radius of the orbit and the velocity of the particle as a function of the above variables. (3pts)

(e) When is this circular orbit stable? (Hint: look at dE/dr and d^2E/dr^2.) (3pts)

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Classical #2

* Consider a particle of mass m under the influence of a central force
$$\vec{F}(\vec{r}) = -\frac{k}{r^n}\hat{r} \quad \text{where} \quad n \in [\mathbb{Z} > 1].$$

a) Calculate the potential energy of this force

$$U(\vec{r}) = -\int \vec{F}\cdot d\vec{r}$$
$$= -\int_{\infty}^{r} -\frac{k}{r^n}\hat{r}\, d\vec{r}$$
$$= +\int_{\infty}^{r} \frac{k}{r^n} dr \qquad \text{(assuming spherical coordinates)}$$
$$= \int_{\infty}^{r} k r^{-(n)} dr$$
$$= -k\frac{1}{n+1} r^{-n+1}\Big|_{\infty}^{r}$$
$$= -\frac{k}{n+1} r^{-n+1}$$

b) Show that angular momentum about the origin is conserved

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{k}{n+1}r^{-n+1}, \qquad v = \langle \dot{r}, r\dot{\theta}, r\dot{\varphi}\sin\theta \rangle$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\varphi}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{n+1}r^{-n+1}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$
$$0 = \frac{d}{dt}(mr^2\sin^2\theta\,\dot{\varphi})$$
$$\Rightarrow mr^2\sin^2\theta\,\dot{\varphi} = \text{const.} = L \quad \rightarrow L \text{ is conserved}$$

c) Write an expression for the total energy of the particle (E) as a function of:
$r, \dot{r}, L, k, n$

$$E = T + U$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2\sin^2\theta} + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{n-1}r^{-n+1}$$
$$* \text{ if } \theta = \frac{\pi}{2}$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{n-1}r^{-n+1}$$

d) Assume the particle is moving in a circular orbit about the origin. Find the radius of the orbit and the velocity of the particle as a function of the above variables

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\varphi}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{n+1}r^{-n+1} = \frac{L^2}{2mr^2} + \frac{k}{n-1}r^{-n+1}$$

$$\cancel{\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}}$$

0 b/c r is constant

$$\cancel{mr\sin^2\theta\,\dot{\varphi}^2 + mr\dot{\theta}^2 - kr^n = \frac{d}{dt}[m\dot{r}]}$$

$$mr\sin^2\theta\,\dot{\varphi}^2 + mr\dot{\theta}^2 - kr^n = 0$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{-L^2}{mr^3} + \frac{k}{r^n}$$

$$\Rightarrow r = \left(\frac{km}{L^2}\right)^{\frac{1}{n-3}}$$

$$\Rightarrow L = mr^2\dot{\varphi} \rightarrow \dot{\varphi} = \frac{L}{mr^2}$$

e) When is the orbit stable?

$$\frac{dE}{dr} = \frac{-L^2}{mr^3} + \frac{k}{r^n}$$

$$\frac{d^2E}{dr^2} = \frac{3L^2}{mr^4} + \frac{-nk}{r^{n+1}}$$

$$= \frac{3L^2}{m}\left(\frac{km}{L^2}\right)^{-4/(n-3)} + nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)}{n-3}} > 0$$

$$\frac{3L^2}{m}\left(\frac{km}{L^2}\right)^{\frac{-4}{n-3}} > +nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)}{n-3}}$$

$$\frac{3L^2}{m} > +nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)+4}{n-3}}$$

$$\frac{3\cancel{L^2}}{\cancel{m}} > +\frac{\cancel{L^2}nk}{\cancel{km}}$$

$$3 > +n$$