Problem 2 (10 Points):

An isolated uniform sphere of mass m and radius R is rotating with angular velocity ω_0 about an axis running through the sphere. Through only internal forces, the radius increases linearly to 2 R in a time τ, while maintaining uniform density and spherical symmetry.

a. At time τ, what is the angular velocity of the sphere? **(2 Points)**

b. Find an expression for the angular velocity as a function of time. **(1 Points)**

c. When the system reaches 2 R it immediately reverses and its radius linearly decreases to R over the period τ to 2τ. By what angle $\Delta\phi$ is the object behind in its rotation compared to a situation where the sphere does not expand between 0 and 2τ? **(4 Points)**

d. Consider the case where the radius of the sphere expands exponentially with some time constant τ_e. How much does the sphere rotate compared to the case where there is no expansion as t$\rightarrow \infty$? **(3 Points)**

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Classical #2

Initially: mass $= m$

radius $= R$

density: $\rho = \frac{m}{\frac{4}{3}\pi R^3}$

At $t = \tau$: mass: $8m$

radius $= 2R$

density: $\rho = \frac{m}{\frac{4}{3}\pi (2R)^3}$

a) At time $t = \tau$, what is the angular velocity of the sphere

* Conservation of angular momentum

$L = m\omega r^2$ ($L = mvr$, $v = \omega r$)

$m\omega_0 R^2 = \cancel{8} m \omega_f \, 2R^2$

$\frac{\omega_0}{4} = \omega_f$

b) Find an expression for $\omega(t)$

$t = 0$ $\quad$ $t = \tau$ $\quad \Rightarrow r(t) = \frac{R}{\tau}t + R$

$r = R$ $\quad$ $r = 2R$

$mR^2\omega_0 = \left(m\left(\frac{R}{\tau}t + R\right)^2 \omega_f \right.$

$\omega_0 = \omega_f \left(\frac{t}{\tau} + 1\right)^2$

$\Rightarrow \omega_f = \frac{\omega_0}{\left(\frac{t}{\tau}+1\right)^2}$

c) If the expansion reverses at $r = 2R$ and returns to its initial state at $t = 2\tau$, by what angle $\Delta\varphi$ is the sphere behind an identical sphere that didn't expand

$2\omega_0\tau - \left[\int_0^\tau \frac{\omega_0}{\left(\frac{t}{\tau}+1\right)^2} + \int_\tau^{2\tau} \frac{\omega_0}{\left(\frac{-t}{\tau}+1\right)^2}\right]$

$2\omega_0\tau - \left[\left(\tau\omega_0\left(\frac{t}{\tau}+1\right)^{-1}\right)\Big|_0^\tau + -\omega_0\left(\frac{-t}{\tau}+1\right)^{-1}\tau\Big|_\tau^{2\tau}\right]$

$2\omega_0\tau - \left[\tau\omega_0\left(\frac{1}{2} - 1\right) - \tau\omega_0(-1 - 0)\right]$

$2\omega_0\tau - \left[-\frac{\tau\omega_0}{2} + \frac{\tau\omega_0}{1}\right.$

$2\omega_0\tau - \boxed{\frac{\tau\omega_0}{2}}$

$\Rightarrow \Delta\varphi = \frac{3\tau\omega_0}{2}$

d) What if the sphere exponentially w/ time constant τ_e. How much does sphere rotate compared to the case where there is no expansion ($t \to \infty$)

*Now $r = Re^{t/\tau_e}$

$$\cancel{m}R^2 \omega_0 = \cancel{m}(Re^{t/\tau_e})^2 \omega_f$$

$$\Rightarrow \omega_f = \omega_0 e^{-2t/\tau_e}$$

$$2\omega_0 t - \left[\int_0^t \omega_0 e^{-2t/\tau_e} dt\right.$$

$$2\omega_0 t - \left[-\frac{\tau_e}{2}\omega_0 e^{-2t/\tau_e}\right|_0^t$$

$$2\omega_0 t - \frac{\tau_e}{2}\omega_0 e^{-2t/\tau} - \frac{\tau_e}{2}\omega_0$$

$$2\omega_0 t - \frac{\tau_e}{2}\omega_0 \left(e^{-2t/\tau} - 1\right) = \Delta\varphi$$