3. Consider a particle attracted by a fixed gravitating body while also in a uniform gravitational field oriented along the z-axis. The potential energy is of the form:

$$V(r, z) = -m\left(\frac{k}{r} + g\,z\right)$$

where m is the particle's mass, k and g are constants, and r is the standard radial coordinate:

$$r \equiv \sqrt{x^2 + y^2 + z^2}$$

You are to examine the problem in *cylindrical parabolic coordinates* defined by

$$\begin{aligned} \zeta &\equiv r + z \\ \eta &\equiv r - z \\ \phi &\equiv \arctan y/x \end{aligned}$$

In these coordinates we may write the Cartesian coordinates as:

$$\begin{aligned} x &= \sqrt{\zeta\eta}\cos\phi \\ y &= \sqrt{\zeta\eta}\sin\phi \\ z &= \tfrac{1}{2}(\zeta - \eta) \end{aligned}$$

(a) Show that the kinetic energy, T, is given by:

$$T = \frac{m}{8}\left[\left(1 + \frac{\zeta}{\eta}\right)\dot{\eta}^2 + \left(1 + \frac{\eta}{\zeta}\right)\dot{\zeta}^2\right] + \frac{m}{2}\,\zeta\eta\,\dot{\phi}^2$$

in these coordinates. (2 points)

(b) What are the canonical momenta, p_ζ, p_η, and p_ϕ, expressed in cylindrical parabolic coordinates? (2 points)

(c) Use Hamilton-Jacobi theory to find the constants of the motion. *Hint:* While the total energy E does not separate in these coordinates, $E(\zeta + \eta)$ can be used to produce a quantity that **does** seperate. (3 points)

(d) What is Hamilton's characteristic function associated with ϕ? (1 point)

(e) Express Hamilton's characteristic functions associated with ζ, η as definite integrals. (2 points)

Aug 2002

Classical #3

For a particle attracted by a fixed gravitating body while also in a uniform gravitational field oriented along the z-axis $\left[V(r,z) = -m\left(\frac{k}{r} + gz\right)\right]$ where $r = \sqrt{x^2+y^2+z^2}$ and using cylindrical parabolic coordinates:

$$\xi = r + z \qquad\qquad x = \sqrt{\xi\eta}\cos\varphi$$
$$\eta = r - z \qquad \Rightarrow \qquad y = \sqrt{\xi\eta}\sin\varphi$$
$$\varphi = \arctan(y/x) \qquad\qquad z = \tfrac{1}{2}(\xi - \eta)$$

a) Show that the kinetic energy, T, is: $T = \frac{m}{8}\left[\left(1+\frac{\xi}{\eta}\right)\dot\eta^2 + \left(1+\frac{\eta}{\xi}\right)\dot\xi^2\right] + \frac{m}{2}\xi\eta\dot\varphi^2$

$$T = \tfrac{1}{2}m\dot v^2$$
$$= \tfrac{m}{2}(\dot x^2 + \dot y^2 + \dot z^2)$$

* from above,

$$\dot x = \tfrac12 \xi^{-1/2}\eta^{1/2}\cos\varphi\,\dot\xi + \tfrac12 \xi^{1/2}\eta^{-1/2}\cos\varphi\,\dot\eta - \xi^{1/2}\eta^{1/2}\sin\varphi\,\dot\varphi$$
$$\dot x^2 = \tfrac14\xi^{-1}\eta\cos^2\varphi\,\dot\xi^2 + \tfrac14\xi\eta^{-1}\cos^2\varphi\,\dot\eta^2 + \xi\eta\sin^2\varphi\,\dot\varphi^2 + \tfrac12\cos^2\varphi\,\dot\xi\dot\eta - \eta\cos\varphi\sin\varphi\,\dot\xi\dot\varphi$$
$$- \xi\cos\varphi\sin\varphi\,\dot\eta\dot\varphi$$
$$\dot y = \tfrac12 \xi^{-1/2}\eta^{1/2}\sin\varphi\,\dot\xi + \tfrac12 \xi^{1/2}\eta^{-1/2}\sin\varphi\,\dot\eta + \xi^{1/2}\eta^{1/2}\cos\varphi\,\dot\varphi$$
$$\dot y^2 = \tfrac14\xi^{-1}\eta\sin^2\varphi\,\dot\xi^2 + \tfrac14\xi\eta^{-1}\sin^2\dot\eta^2 + \xi\eta\cos^2\varphi\,\dot\varphi^2 + \tfrac12\sin^2\varphi\,\dot\xi\dot\eta + \eta\sin\varphi\cos\varphi\,\dot\xi\dot\varphi$$
$$+ \xi\sin\varphi\cos\varphi\,\dot\eta\dot\varphi$$
$$\dot z = \tfrac12(\dot\xi - \dot\eta)$$
$$\dot z^2 = \tfrac14(\dot\xi^2 - 2\dot\xi\dot\eta + \dot\eta^2)$$

$$\Rightarrow \dot x^2 + \dot y^2 + \dot z^2 = \tfrac14\xi^{-1}\eta\dot\xi^2 + \tfrac14\xi\eta^{-1}\dot\eta^2 - \xi\eta\sin^2\varphi\,\dot\varphi^2 + \xi\eta\cos^2\varphi\,\dot\varphi^2 + \tfrac12\dot\xi\dot\eta + \tfrac14\dot\xi^2 - \tfrac12\dot\xi\dot\eta + \tfrac14\dot\eta^2$$
$$= \frac14\frac{\eta}{\xi}\dot\xi^2 + \frac14\frac{\xi}{\eta}\dot\eta^2 - \xi\eta\sin^2\varphi\,\dot\varphi^2 + \xi\eta\cos^2\varphi\,\dot\varphi^2 + \tfrac14\dot\xi^2 + \tfrac14\dot\eta^2$$
$$= \tfrac14\left[\dot\xi^2\left(\frac{\eta}{\xi}+1\right) + \dot\eta^2\left(\frac{\xi}{\eta}+1\right)\right] + \xi\eta\dot\varphi^2\left(\cancel{\cos^2\varphi + \sin^2\varphi}\right)$$

$$\Rightarrow T = \tfrac12 m(\dot x^2 + \dot y^2 + \dot z^2)$$
$$= \frac{m}{8}\left[\left(1+\frac{\xi}{\eta}\right)\dot\eta^2 + \left(1+\frac{\eta}{\xi}\right)\dot\xi^2\right] + \frac{m}{2}\xi\eta\dot\varphi^2$$

b) Find the canonical momenta in cylindrical parabolic coordinates

$$P_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$\Rightarrow \mathcal{L} = T - V$$

$$V = -m\left(\frac{k}{r} + gz\right), \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r = \sqrt{\xi\eta\cos^2\varphi + \xi\eta\sin^2\varphi + \frac{1}{4}(\xi - \eta)^2}$$

$$= \sqrt{\xi\eta + \frac{1}{4}\xi^2 - \frac{1}{2}\xi\eta + \frac{1}{4}\eta^2}$$

$$= \sqrt{\frac{1}{4}\xi^2 + \frac{1}{2}\xi\eta + \frac{1}{4}\eta^2}$$

$$= \sqrt{\frac{1}{4}(\xi^2 + 2\xi\eta + \eta^2)}$$

$$= \frac{1}{2}(\xi + \eta)$$

$$\Rightarrow V = -m\left(\frac{2k}{\xi + \eta} + g\left(\frac{\xi + \eta}{2}\right)\right)$$

$$\Rightarrow \mathcal{L} = T - V$$

$$= \frac{m}{8}\left[\left(1 + \frac{\xi}{\eta}\right)\dot{\eta}^2 + \left(1 + \frac{\eta}{\xi}\right)\dot{\xi}^2\right] + \frac{m}{2}\xi\eta\dot{\varphi}^2 + m\left[\frac{2k}{\xi + \eta} + \frac{g(\xi + \eta)}{2}\right]$$

$$P_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

$$= m\xi\eta\dot{\varphi}$$

$$P_\eta = \frac{\partial \mathcal{L}}{\partial \dot{\eta}}$$

$$= \frac{m}{4}\left(1 + \frac{\xi}{\eta}\right)\dot{\eta}$$

$$P_\xi = \frac{\partial \mathcal{L}}{\partial \dot{\xi}}$$

$$= \frac{m}{4}\left(1 + \frac{\eta}{\xi}\right)\dot{\xi}$$

c) $H = \sum_i p_i \dot{q}_i - \mathcal{L}$

$$H = p_\varphi \dot{\varphi} + p_\zeta \dot{\zeta} + p_\eta \dot{\eta} - \mathcal{L}$$

$$= \frac{m}{2}\zeta\eta\dot{\varphi}^2 + \frac{m}{8}\left(1+\frac{\eta}{\zeta}\right)\dot{\zeta}^2 + \frac{m}{8}\left(1+\frac{\zeta}{\eta}\right)\dot{\eta}^2 - m\left[\frac{2k}{\zeta+\eta} + \frac{g(\zeta+\eta)}{2}\right]$$