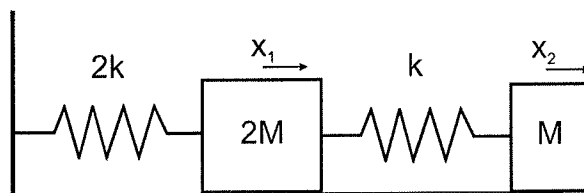


Mechanics and Statistical Mechanics Qualifying Exam
Fall 2010

Problem 1: (10 Points)

Two blocks are free to move in *one* dimension along a frictionless horizontal surface. The blocks of mass $2M$ and M are connected to each other and to a fixed wall by two springs with stiffness $2k$ and k as shown in the figure. Choose the dynamical coordinates of the system to be the position of block 1, x_1 , and block 2, x_2 , from their respective equilibrium positions. Consider only small oscillations so that the springs are linear. Neglect all damping.



- Write down the equations of motion for each mass. **(2 Points)**
- Show that the frequencies of the normal modes of the system are $\sqrt{2}\omega_0$ and $\omega_0/\sqrt{2}$ where $\omega_0 = \sqrt{k/m}$. **(2 Points)**
- Find the eigenvectors that describe the normal modes and sketch them. **(3 Points)**
- Suppose you grab mass M and push it slowly to the left by an amount A_0 . When mass $2M$ is in equilibrium show that it is $A_0/4$ from its equilibrium position. **(1 Point)**
- If you release the system from the starting position in (d.), what will be the displacement of the system as a function of time? Write an expression for the displacement of block 1 (mass $2M$) as a function of time from its original equilibrium position. **(2 Points)**

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Classical #1

Problem 2 (10 Points):

An isolated uniform sphere of mass m and radius R is rotating with angular velocity ω_0 about an axis running through the sphere. Through only internal forces, the radius increases linearly to $2R$ in a time τ , while maintaining uniform density and spherical symmetry.

- a. At time τ , what is the angular velocity of the sphere? **(2 Points)**
- b. Find an expression for the angular velocity as a function of time. **(1 Points)**
- c. When the system reaches $2R$ it immediately reverses and its radius linearly decreases to R over the period τ to 2τ . By what angle $\Delta\phi$ is the object behind in its rotation compared to a situation where the sphere does not expand between 0 and 2τ ? **(4 Points)**
- d. Consider the case where the radius of the sphere expands exponentially with some time constant τ_e . How much does the sphere rotate compared to the case where there is no expansion as $t \rightarrow \infty$? **(3 Points)**

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Classical #2

Initially: mass = m

radius = R

density: $\rho = \frac{m}{\frac{4}{3}\pi R^3}$

At $t = T$: mass: $8m$

radius = $2R$

density: $\rho = \frac{m}{\frac{4}{3}\pi (2R)^3}$

a) At time $t = T$, what is the angular velocity of the sphere

* Conservation of angular momentum

$$L = m\omega r^2 \quad (L = mvr, v = \omega r)$$

$$M\omega_0 R^2 = 8m\omega_f 2R^2$$

$$\omega_0 = \omega_f$$

b) Find an expression for $\omega(t)$

$$\begin{array}{ll} t=0 & t=T \\ r=R & r=2R \end{array} \Rightarrow r(t) = \frac{R}{T}t + R$$

$$MR^2\omega_0 = \left(m\left(\frac{R}{T}t + R\right)^2 \right) \omega_f$$

$$\omega_0 = \omega_f \left(\frac{t}{T} + 1\right)^2$$

$$\Rightarrow \omega_f = \frac{\omega_0}{\left(\frac{t}{T} + 1\right)^2}$$

c) If the expansion reverses at $r = 2R$ and returns to its initial state at $t = 2T$, by what angle $\Delta\phi$ is the sphere behind an identical sphere that didn't expand

$$2\omega_0 T = \left[\int_0^T \frac{\omega_0}{\left(\frac{t}{T} + 1\right)^2} dt + \int_T^{2T} \frac{\omega_0}{\left(\frac{t}{T} + 1\right)^2} dt \right]$$

$$2\omega_0 T = \left[\left[T\omega_0 \left(\frac{t}{T} + 1\right) \right]_0^T + \left[-T\omega_0 \left(\frac{t}{T} + 1\right) \right]_T^{2T} \right]$$

$$2\omega_0 T = \left[T\omega_0 \left(\frac{1}{2} - 1\right) - T\omega_0 (-1 - 0) \right]$$

$$2\omega_0 T = \left[-\frac{T\omega_0}{2} + \frac{T\omega_0}{1} \right]$$

$$2\omega_0 T = \boxed{\frac{T\omega_0}{2}}$$

$$\Rightarrow \Delta\phi = \frac{3T\omega_0}{2}$$

d) What if the sphere expands exponentially w/ time constant τ_e . How much does sphere rotate compared to the case where there is no expansion ($t \rightarrow \infty$)

* Now $r = R e^{t/\tau_e}$

$$4\pi R^2 \omega_0 = 4\pi (R e^{t/\tau_e})^2 \omega_f$$

$$\Rightarrow \omega_f = \omega_0 e^{-2t/\tau_e}$$

$$2\omega_0 t = \int_0^t \omega_0 e^{-2t/\tau_e} dt$$

$$2\omega_0 t = \left[-\frac{\tau_e}{2} \omega_0 e^{-2t/\tau_e} \right]_0^t$$

$$2\omega_0 t = -\frac{\tau_e}{2} \omega_0 e^{-2t/\tau_e} + \frac{\tau_e}{2} \omega_0$$

$$2\omega_0 t = \frac{\tau_e}{2} \omega_0 (e^{-2t/\tau_e} - 1) = \Delta \phi$$

Problem 3 (10 Points):

Consider the following Lagrangian

$$L = \left(\frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \right) e^{2\gamma t}$$

assuming that $\omega > \gamma$ for the questions that follow.

- a. Determine the Hamiltonian associated with this Lagrangian. **(3 Points)**
- b. Find a transformation to new phase space variables that make H independent of time and show that these form a canonical transformation by determining a generating function of the form $F_2(q, P, t)$. **(4 Points)**
- c. Using the equations of motion for the transformed Hamiltonian $K(Q, P, t)$, solve for $Q(t)$ and transform back to get $q(t)$. **(3 Points)**

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Classical #3

Given: $\mathcal{L} = (\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2)e^{2\gamma t}$

a) Determine the Hamiltonian associated w/ the above Lagrangian

$$H = \sum_i p_i \dot{q}_i - \mathcal{L}$$

$$p_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$= e^{2\gamma t} m \dot{q}$$

$$\Rightarrow \dot{q} = \frac{p}{m} e^{-2\gamma t}$$

$$\Rightarrow H = m\dot{q}^2 e^{2\gamma t} = e^{2\gamma t} (\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2)$$

$$= e^{2\gamma t} (\frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2)$$

$$= e^{2\gamma t} (\frac{1}{2}m\frac{p^2}{m^2} e^{-4\gamma t} + \frac{1}{2}m\omega^2 q^2)$$

$$= \frac{1}{2}m\frac{p^2}{m} e^{-2\gamma t} + \frac{1}{2}m\omega^2 q^2 e^{2\gamma t}$$

b) Find a transformation to new phase space variables that make H time independent and show that these form a canonical transformation by determining a $F_2(q, P, t)$ generating function.

* For an $F_2(q, P, t)$: $P = \frac{\partial F_2(q, P)}{\partial q}$

$$Q = \frac{\partial F_2(q, P)}{\partial P}$$

\Rightarrow Want: $H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2$

$$\Rightarrow Q = qe^{\gamma t}$$

$$P = pe^{-\gamma t}$$

$$\Rightarrow F_2(q, P, t) = Pqe^{\gamma t}$$

$$\Rightarrow K = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2$$

c) Use the Hamiltonian eqns. of motion for K to solve for Q and transform it back to $q(t)$

$$\dot{P} = \frac{\partial K}{\partial Q} = -m\omega^2 Q$$

$$\dot{Q} = \frac{\partial K}{\partial P} = \frac{P}{m}$$

$$\ddot{P} = m\ddot{Q}$$

$$\Rightarrow m\ddot{Q} = -\omega^2 mQ$$

$$\ddot{Q} = -\omega^2 Q \Rightarrow Q = Ae^{-i\omega t} + Be^{i\omega t}$$

$$\Rightarrow q = e^{\gamma t} Q = e^{\gamma t} (Ae^{-i\omega t} + Be^{i\omega t})$$

Problem 4 (10 Points):

The coffee purchased at rest stops is often too hot to drink. One way to cool off your coffee is to add ice, but how much ice should you add? Take the initial conditions for the coffee to be $T_0^{cof} = 80^\circ\text{C}$ and $V = 400\text{ ml}$. Take the initial conditions for the ice to be $T_0^{ice} = 0^\circ\text{C}$. The final temperature for the coffee and ice that you want to achieve is $T_f = 60^\circ\text{C}$. For the following questions assume that the coffee is pure water (a good assumption for most rest stop coffee) and the process is adiabatic with respect to the surroundings. Neglect volume changes of the coffee and ice and any temperature dependence of the heat capacity. The following thermodynamic properties of water may be useful:

$M = 18.0\text{ g mole}^{-1}$, molar mass

$\rho = 1.00\text{ g/cm}^3$, density

$\Delta H_{fus} = 6.00\text{ kJ mole}^{-1}$, heat of fusion

$C_p = 75.4\text{ J mole}^{-1}\text{ K}^{-1}$, heat capacity of liquid

For parts (a.)-(c.) your answers should be in terms of the variables described here.

- Find a general (algebraic solution) expression for the mass of ice, m , that is needed to cool the coffee to T_f ? **(4 Points)**
- Calculate, numerically, how many grams of ice you should add to your coffee to lower the temperature to $T_f = 60^\circ\text{C}$. **(1 Points)**
- What is the entropy change of the system (coffee + ice)? Find an algebraic solution. **(3 Points)**
- What is the entropy change of the surroundings? **(1 Points)**
- Is this a thermodynamically reversible process? Explain. **(1 Points)**

Problem 5 (10 Points):

Consider a one dimensional ideal gas of electrons as a model for the conduction electrons in a one dimensional wire.

- a. Determine the density of states $g(E)$ for the one dimensional non-interacting electron system confined to a length, L . **(3 Points)**
- b. What is the Fermi energy for this system? **(2 Points)**
- c. What is the root mean square velocity of the electrons at $T = 0^\circ\text{K}$? **(3 Points)**
- d. What is the entropy of the electrons at $T = 0^\circ\text{K}$? Justify your answer. **(2 Points)**

Problem 6 (10 Points):

The following questions refer to a stream of photons in equilibrium at temperature T (thermal light - say from a light bulb) incident on a perfect detector which detects (counts) all the particles that hit it. Your final answers should be in terms of the mean particle number.

- a. Given \bar{n}_s photons are counted on average in time t , calculate the variance in the photon number n_s , $\overline{(\Delta n_s)^2}$. **(2 Points)**
- b. Calculate the fractional fluctuation of the detector signal defined as the square root of the variance divided by the mean photon number, \bar{n}_s , squared, $\sqrt{\overline{(\Delta n_s)^2}}/\bar{n}_s^2$. This is the inverse of the signal to noise ratio. **(2 Points)**

The following questions refer to a stream of electrons in equilibrium at temperature T incident on a detector which detects (counts) all the particles that hit it. Again, your final answers should be in terms of the mean particle number.

- c. Given \bar{n}_e electrons are counted on average in time t , calculate the variance in the electron number n_e , $\overline{(\Delta n_e)^2}$. **(2 Points)**
- d. Calculate the fractional fluctuation of the detector signal defined as the square root of the variance divided by the mean electron number, \bar{n}_e , squared, $\sqrt{\overline{(\Delta n_e)^2}}/\bar{n}_e^2$. **(2 Points)**
- e. Compare the two results. Are the results the same or different? Do the counts detected clump (bunch) or anti-clump (anti-bunch)? Why? **(2 Points)**