Classical Mechanics and Statistical/Thermodynamics

January 2009

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

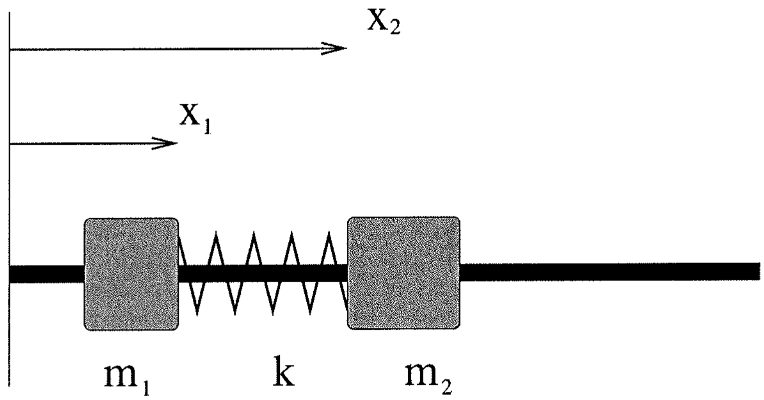
$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\begin{array}{cc} \zeta(1) = \infty & \zeta(-1) = 0.0833333 \\ \zeta(2) = 1.64493 & \zeta(-2) = 0 \\ \zeta(3) = 1.20206 & \zeta(-3) = 0.0083333 \\ \zeta(4) = 1.08232 & \zeta(-4) = 0 \end{array}$$

Classical Mechanics

1. Two masses, m_1 and m_2, are connected together by an ideal massless spring of spring constant k and equilibrium length l, but are otherwise free to slide on a straight frictionless rail. Their positions with respect to a fixed origin are denoted x_1 and x_2 respectively.



   (a) Determine the equation of motion for each mass using Newton's Laws of Motion. (Do not solve them yet.) [2 pts.]

   (b) Write the Lagrangian for this system and use it to derive the equation of motion for each mass. (Again, do not solve them yet.) [3 pts.]

   (c) Using either of your results, determine the frequency of oscillation of the two masses about their center of mass. [2 pts.]

   (d) Given the initial conditions, $x_1(0) = 0$, $v_1(0) = 0$, $x_2(0) = l$ and $v_2(0) = v_0$, solve for the subsequent motion. [3 pts.]

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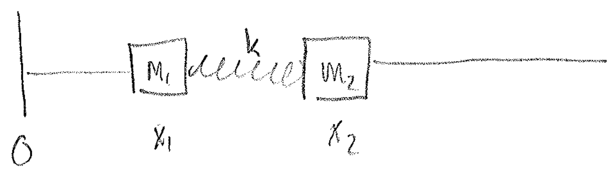
Classical #1



* Rail is frictionless

a) Determine equations of motion using Newtons Laws

$$F = k\Delta x = ma$$

$$\Rightarrow m_1 \ddot{x}_1 = k($$

2. A particle of mass m moves under the influence of a central force whose potential is given by $V(r) = K\,r^3$, where $K > 0$.

   (a) For what energy and angular momentum will the orbit be a circle of radius R_0 about the origin? (3 pts.)

   (b) What is the period of this circular orbit? (2 pts.)

   (c) If the particle is slightly displaced from the circular orbit, what will be the period for small oscillations about $r = R_0$? (3 pts.)

   (d) For the $1/r$ gravitational potential we know that Kepler's Second Law: "A line joining a planet and the sun sweeps out equal areas during equal intervals of time." Does this hold true for the cubic potential as well? Prove your answer. (2 pts.)

3. You are in a rocket ship in outer space, initially at rest. You have a nuclear reactor that supplies a constant power, P, and a large supply of iron pellets. The iron pellets comprise 99/100 of your ship's mass, m. You can use the power to eject the tiny iron beads out the back of your ship with an electromagnetic "gun". You can control the *rate* at which you fire them and their velocity, but you are limited by your power plant. (You can't fire an arbitrarily large mass at an arbitrarily large velocity.) As you fire off the beads, your ship moves in the opposite direction to conserve momentum. In addition, the mass of your ship decreases.

   (a) Calculate your final speed as a functional of $m(t)$ and $\dot{m}(t) \equiv dm/dt$. Your expression should take the form of an integral over time, $0 < t < t_f$. (2 pts.)

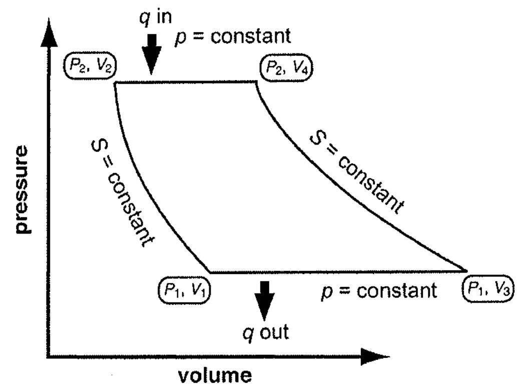
   (b) Find the function $m(t)$ that maximizes your final velocity after a time t_f. (4 pts.)

   (c) What is your final velocity? (2 pts.)

   (d) Prove that your answer in part (c) is larger than the velocity you would obtain by firing at a constant rate such that your pellets are used up by t_f. (2 pts.)

Statistical Mechanics

4. The gas turbine (jet engine) can be modeled as a Brayton cycle. Below is the P-V diagram for this process.



Assume that the working fluid is an ideal monatomic gas.

(a) Calculate the work done by the gas on each step in the cycle. (3 pts.)

(b) Find the heat for each step in the cycle. (3 pts.)

(c) Find the efficiency of this engine. Your answer should be in terms of the pressures (P_1 and P_2) and the volumes (V_1, V_2, V_3, and V_4). (3 pts.)

(d) To produce work, which way does the cycle operate? Clockwise or counter clockwise? (1 pt.)

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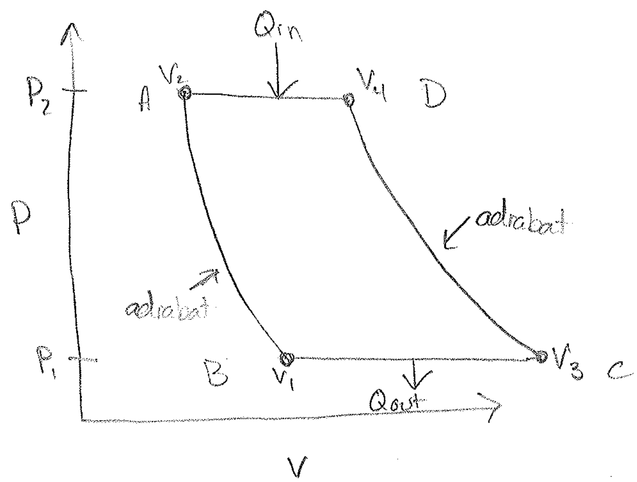
Stat Mech #1



* Ideal monatomic gas

$$\Rightarrow \begin{matrix} C_V = \frac{3}{2}R \\ C_P = \frac{5}{2}R \end{matrix} \Big\} \quad \gamma = \frac{5}{3}$$

$P_A = P_2$ $\qquad P_B = P_1$ $\qquad P_C = P_1$ $\qquad P_D = P_2$

$V_A = V_2$ $\qquad V_B = V_1$ $\qquad V_C = V_3$ $\qquad V_D = V_4$

$T_A = \frac{nR}{P_2V_2}$ $\qquad T_B = \frac{nR}{P_1V_1}$ $\qquad T_C = \frac{nR}{P_1V_3}$ $\qquad T_D = \frac{nR}{P_2V_4}$

a) Calculate the work done in each step.

$$W_{A\to D} = P\Delta V = P(V_4 - V_2)$$

$$W_{D\to C} = \frac{P_C V_C - P_D V_D}{1-\gamma} = \frac{P_1V_3 - P_2V_4}{1 - 5/3} = -\frac{3}{2}(P_1V_3 - P_2V_4)$$

$$W_{C\to B} = P\Delta V = P_1(V_1 - V_3)$$

$$W_{B\to A} = \frac{P_A V_A - P_B V_B}{1-\gamma} = \frac{P_2V_2 - P_1V_1}{1 - 5/3} = -\frac{3}{2}(P_2V_2 - P_1V_1)$$

b) Find the heat for each step

$$Q_{A\to D} = nC_P\Delta T = n\frac{5}{2}R\left(\frac{P_2V_4}{nR} - \frac{P_2V_2}{nR}\right) = \frac{5}{2}P_2(V_4 - V_2)$$

$Q_{D\to C} = 0$ b/c adiabatic

$$Q_{C\to B} = nC_P\Delta T = n\frac{5}{2}R\left(\frac{P_1V_3}{nR} - \frac{P_1V_1}{nR}\right) = \frac{5}{2}P_1(V_3 - V_1)$$

$Q_{B\to A} = 0$ b/c adiabatic

c) Find the efficiency of the engine

$$\eta = 1 - \frac{Q_{in}}{Q_{out}}$$

$$= 1 - \left(\frac{P_2 (V_4 - V_2)}{P_1 (V_3 - V_1)}\right)^{-1}$$

d) Which way does the cycle operate?

CW

5. By shining an intense laser beam on a semiconductor, one can create a metastable collection of electrons (charge $-e$ and effective mass m_e) and holes (charge $+e$ and effective mass m_h). These oppositely charged particles may pair up to form an *exciton*, or they may dissociate into a plasma. This problem considers a simple model of this process. In this problem the densities of electrons and holes are so low that you can ignore their fermionic nature and treat them as classical particles in three dimensions.

   (a) Calculate the free energy $F(T, V, N)$ of a gas of N_e electrons and N_h holes at temperature T, treating them as classical, non-interacting, ideal gas particles in a 3D volume V. (2 pts.)

   (b) By pairing into an exciton, each electron-hole pair lowers its energy by ΔE. Calculate the free energy of a gas of N_p excitons, treating them as classical, non-interacting, ideal gas particles. (2 pts.)

   (c) Calculate the chemical potentials μ_e, μ_h, and μ_p of the electrons, holes, and exciton pairs respectively. What is the condition of equilibrium between excitons and eletrons and holes? (3 pts.)

   (d) Consider the case where the numbers of electrons and holes are equal, so that $n_h = n_e \equiv n_0$. Determine the approximate density of excitons as a function of n_0 in the high temperature limit (when the exciton population is low). (3 pts.)

6. Consider a free, non-interacting spin zero Bose gas in two dimensions. The energy of each particle is given by:

$$\mathcal{E}(\vec{k}) = \hbar^2 k^2/2m$$

where m is the mass of the boson. Assume your system is confined to a square region of length L on a side.

(a) Write down an expression for the grand canonical free energy $\mathcal{G}(T, V, \mu)$ as a sum over $\vec{k}$ states. Do not evaluate the sum. (1 pt.)

(b) Calculate the number of particles in the system as a function of T, V and μ. (3 pts.)

(c) Analyze your expression for $N(T, V, \mu)$ in the limit $T \to 0$. What does it imply about the possibility of a Bose-Einstein transition in this system? (3 pts.)

(d) Prove that the pressure is equal to the energy density, so that $PV = U$. (Hint: you do not have to do any sums over states - you need only prove that this holds using analytic expressions for P and U in this particular system). (3 pts.)

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Stat Mech #3

* Consider a free, non-interacting spin 0 Bose gas in 2-D, where the energy of each particle is: $E(\vec{k}) = \frac{\hbar^2 k^2}{2m}$

- Assume m is mass of boson and the system is confined to a square region of side length L.

a) Write down the expression for the grand canonical free energy Ω

$$\Omega = -PV$$
$$= -kT \ln(Z_1)$$

*but $Z_1 = \prod_j [1 - \exp[\beta(\mu - \epsilon_j)]]^{-1}$

$$= -kT \sum_j \ln(1 - \exp[\beta(\mu - \epsilon_j)])$$
$$= -kT \sum_k \ln(1 - \exp[\beta(\mu - \frac{\hbar^2 k^2}{2m})])$$

b) Calculate the # of particles in the system as a function of T, V, and μ

$$\overline{N} = \left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V}$$
$$= \frac{\partial}{\partial \mu} kT \sum_k$$
$$= -kT \sum_k \frac{-\beta \exp[\beta(\mu - \frac{\hbar^2 k^2}{2m})]}{1 - \exp[\beta(\mu - \frac{\hbar^2 k^2}{2m})]}$$
$$= \sum_k \left[\exp[-\beta(\mu - \frac{\hbar^2 k^2}{2m})] - 1\right]$$
$$\approx \int$$