Classical Mechanics/Statistical Mechanics Qualifier

August 14, 2012

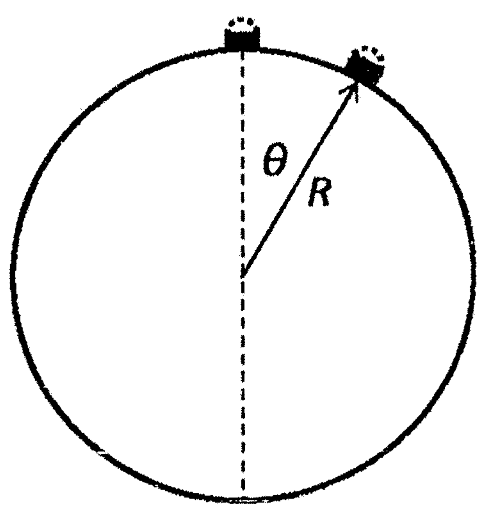
1. A very small object with mass m is placed on the top of a stationary large sphere with radius R (the upper hemisphere might represent the top of a grain silo). Choose the center of the large sphere to be the origin of the coordinate system, and the vertical axis as the z axis, and θ as the polar angle of the center of the small object relative to the coordinates defined here (see Figure). Assume the object is initially at rest on top of the large sphere, i.e., at $\theta = 0$, and that it begins to slide without friction down the spherical surface.

   (a) (2 pt) Obtain an expression for the velocity of the center of mass of the small sliding object as a function of θ.

   (b) (2 pt) Find the angle θ_c at which the sliding object loses contact with the spherical surface.

   (c) (4 pt) Now let the object be a small solid homogeneous sphere (that is, having uniform mass density) having a radius r and total mass m. Assume the small sphere rolls without slipping (but assume that there is no rolling friction) and that $r \ll R$. Again, obtain an expression for the velocity of the center of the rolling sphere as a function of θ.

   (d) (2 pt) Again, find the angle θ_c at which the small sphere loses contact with the large sphere. Is this larger or smaller than the angle found in part 1b? Give a physical reason for this.



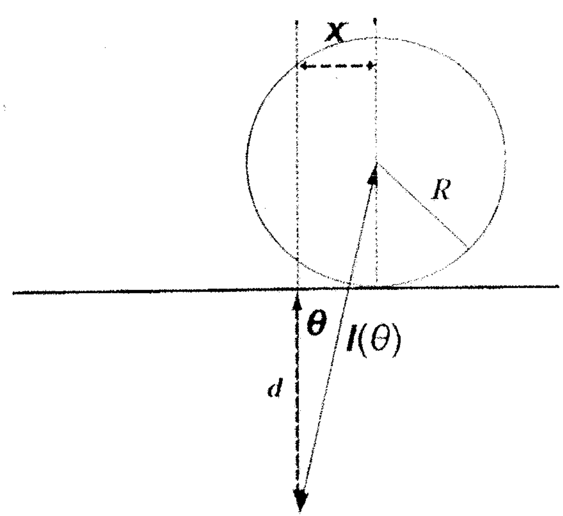
2. A homogeneous disk of radius R and mass M rolls without slipping on a horizontal surface. The disk's center is attracted to a point a distance d below the plane. See the figure. The force of attraction is proportional to the distance from the disk's center of mass, $l(\theta)$. This could be accomplished with a spring with spring constant k attached to the point d below the plane and the center of the disk.

   (a) (3 pt) Find the Lagrangian for the system in terms of x and $\dot{x}$.

   (b) (2 pt) Determine the equation of motion for x.

   (c) (3 pt) Solve the equation of motion.

   (d) (2 pt) What is the frequency of oscillation about the position of equilibrium?



3. Consider the bound motion of a particle in the non-central potential

$$V(\mathbf{r}) = \frac{\beta^2}{2mr^2}\sec^2\theta - \frac{k}{r},$$

where k and β are positive, real constants, and r, θ, and ϕ are spherical polar coordinates of the particle.

(a) (1 pt) Write down the Hamiltonian for this system in spherical coordinates.

(b) (2 pts) The characteristic function (sometimes called *Hamilton's* characteristic function) is separable in the form

$$W(r,\theta,\phi) = W_r(r) + W_\theta(\theta) + W_\phi(\phi).$$

Show that we may write $W_\phi(\phi) = \alpha_\phi\phi$, where α_ϕ is a constant, and calculate the azimuthal action

$$A_\phi = \oint_{\text{orbit}} d\phi \frac{\partial W_\phi}{\partial \phi}$$

associated with one orbit of the particle.

(c) (2 pts) Given the separable form above, show that $W_r(r)$ and $W_\theta(\theta)$ must satisfy

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \alpha_\phi^2\csc^2\theta + \beta^2\sec^2\theta = \alpha_\theta^2,$$

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} - \frac{2mk}{r} = 2mE,$$

where α_θ is a separation constant, and E is the energy.

(d) (3 pts) Show that the radial action

$$A_r = \oint_{\text{orbit}} dr \frac{\partial W_r}{\partial r}$$

associated with one orbit of the bound particle is given by (E is negative)

$$A_r = -2\pi\alpha_\theta + \sqrt{\frac{2\pi^2 mk^2}{-E}}.$$

[Hint: Evaluate the integral over the orbit by regarding it as a complex contour integral around a branch cut, and then use the residue theorem.]

(c) (2 pts) The action A_θ associated with motion in θ is difficult to calculate. However, it can be shown that the Hamiltonian can be written in terms of the action

$$H = -\frac{2\pi^2 mk^2}{(A_r + A_\phi + 2A_\theta + 2\pi\beta)^2}.$$

From this determine the elementary frequencies ν_a of the motion, using

$$\frac{\partial H}{\partial A_a} = \nu_a.$$

Are the orbits open or closed?

4. Consider a system of volume V in thermal equilibrium with a heat reservoir at temperature T, for which the canonical partition function is

$$Z = e^{aT^2V},$$

where a is a real, positive constant.

(a) (2 pts) Derive a formula for the Helmholtz free energy F.

(b) (2 pts) What are the normal (canonical) variables for F? Give an expression for the thermodynamic identity for dF and derive expressions for the conjugate variables.

(c) (2 pts) In terms of the normal variables from part 4b, derive an expression for the internal energy, U, and heat capacity at constant volume, c_v. How is U related to F?

(d) (2 pts) Does your expression for c_v agree with the prediction of the equipartition theorem? Explain your answer.

(e) (2 pts) Does your expression for c_v agree with the prediction of the Third Law of Thermodynamics? Explain your answer.

5. In this problem we will consider a lattice of N atoms, each with a spin of $1/2$. Because the atoms can be labeled by their position on the lattice, we will treat them as distinguishable particles, not as identical Fermions.

   In an external magnetic field, each atom can be in one of two possible energy states, $E_{\pm} = \pm\epsilon$.

   (a) (2 pts) The possible total energies for the spins can be written as $E_n = n\epsilon$, where n is an integer. What are the possible values for n? What is the number of distinct microstates, $\Omega(N, E_n) = \Omega(N, n)$, in terms of N and n, that have this energy? Remember, we are considering the atoms to be distinct particles.

   (b) (2 pts) Let us treat the number of microstates $\Omega(N, E_n)$ as the structure function for the microcanonical ensemble for the spin lattice. Therefore, we define the entropy as

   $$S(N, E) = k \ln \Omega(N, E),$$

   where k is Boltzmann's constant. Use this entropy to calculate the temperature T for the spin system. Show that there are energies where the temperature is negative. Explain the meaning of these negative temperatures.

   (c) (2 pts) What is the energy of the spin lattice as a function of temperature, $E(N, T)$? Show that your result makes physical sense in the limits $T \to 0$ and $T \to \infty$.

   (d) (2 pt) Write down the partition function $Z(N, T)$ in the canonical ensemble for the spin lattice.

   (e) (2 pts) Using the canonical ensemble, calculate the average energy for the spin lattice. Compare this result to what you found in part 5c for the microcanonical ensemble. Explain any differences.

6. This problem concerns Fermi gases in D spatial dimensions

(a) (2 pts) Consider a gas of noninteracting nonrelativistic Fermions in D dimensions. Show that the grand-canonical potential, $\psi = \ln \mathcal{Z}$, where $\mathcal{Z}$ is the grand partition function, has the form, ($\beta = 1/kT$)

$$\psi = V_D \gamma_D \int_0^\infty d\epsilon\, \epsilon^{D/2-1} \ln\left(1 + e^{-\beta(\epsilon-\mu)}\right),$$

where γ_D is a constant that is different for each dimension, μ is the chemical potential, and V_D is the D-dimensional volume.

(b) (2 pts) Write down an expression for the number of particles N and the total energy E of the ideal Fermi gas as an integral over the single-particle energy states ϵ.

(c) (2 pts) The grand-canonical potential is proportional to the pressure, because

$$p = kT\frac{\partial \psi}{\partial V},$$

so

$$\psi = \beta p V_D.$$

Using the expression above for ψ, show that

$$pV_D = \frac{2}{D}E.$$

(d) (2 pts) Show that the result from part 6c agrees with the ideal gas law in the $T \to \infty$ limit. You will need to consider the limit the of μ for large T, determined by the fact that the number of atoms in the gas is fixed.

(e) (2 pts) Finally, find an expression for the Fermi energy at $T = 0$ in D dimensions.