Classical Mechanics and Statistical/Thermodynamics

August 2008

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

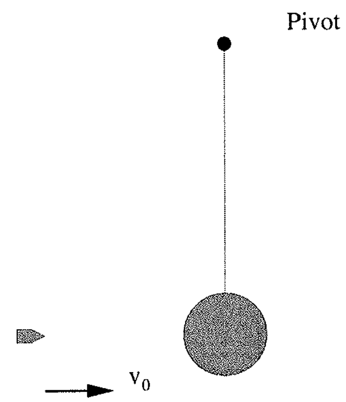
$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\ \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\begin{array}{rr} \zeta(1) = \infty & \zeta(-1) = 0.0833333 \\ \zeta(2) = 1.64493 & \zeta(-2) = 0 \\ \zeta(3) = 1.20206 & \zeta(-3) = 0.0083333 \\ \zeta(4) = 1.08232 & \zeta(-4) = 0 \end{array}$$

Classical Mechanics

1. **The ballistic pendulum:** Consider a pendulum with a bob of mass m connected to a frictionless pivot by an ideal massless rigid rod of length ℓ. A projectile of mass ϵm ($0 < \epsilon << 1$) moving horizontally at speed v_0 hits the center of the bob, as shown. When it strikes, it becomes imbedded in the bob.



   (a) What is the minimum initial speed of the projectile such that the pendulum will make a full rotation? (2 points)

   (b) The rod is replaced by an ideal massless non-rigid string. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (3 points)

   (c) Now assume that projectile rebounds elastically from the bob in the horizontal direction. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (2 points)

   (d) Finally, assume that the projectile passes completely through the pendulum bob, in a time $t << \sqrt{\ell/g}$. After it exits, it carries with it some of the original mass of the bob, such that the exiting projectile now has a mass $2\epsilon m$ and moves at a speed $3v_0/4$. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (3 points)

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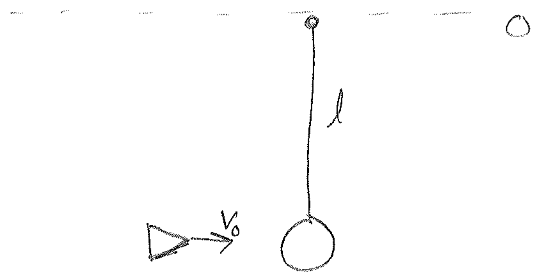
Classical #1



$m_b = \epsilon m$

$m_p = m$

a) * Collision (Inelastic)

$$\cancel{\tfrac{1}{2} m\epsilon v_0^2 = \tfrac{1}{2} m(1+\epsilon)v^2}$$

$$\cancel{\frac{\epsilon}{1+\epsilon} v_0^2 = v^2}$$

$$\epsilon m v_0 = (1+\epsilon)mv$$

$$\frac{\epsilon}{1+\epsilon} v_0 = v$$

* Conservation of energy

$$\frac{1}{2} m(1+\epsilon)\left(\frac{\epsilon}{1+\epsilon} v_0\right)^2 - mg\ell = mg\ell + \frac{1}{2} m\cancelto{0}{(1+\epsilon)}v^2$$

$$\frac{1}{2} m \frac{\epsilon^2}{1+\epsilon} v_0^2 = 2mg\ell + \frac{1}{2} m\cancelto{0}{(1+\epsilon)}v^2$$

$$\frac{\epsilon^2}{1+\epsilon} v_0^2 = 4mg\ell$$

2. The isotropic harmonic oscillator.

   (a) Write the Lagrangian for a point mass m moving under the influence of an isotropic 3-dimensional harmonic oscillator potential

   $$V(x, y, z) = \frac{k}{2}(x^2 + y^2 + z^2).$$

   There is no external gravitational field. (1 point)

   (b) Using the Lagrange equations of motion show that angular momentum is conserved. i.e.,

   $$\frac{d}{dt}\mathbf{L} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = 0.$$

   Because the Lagrangian is invariant under rotations about the origin, you can choose coordinates so that motion is constrained to the x-y plane, i.e., the angular momentum points in the z direction. (3 points)

   (c) For 2-dimensional motion in the x-y plane choose cylindrical polar coordinates and proceed to solve the Lagrange equations of motion. You can leave the solution for $r(t)$ as an integral of the form $t = \int f(r)dr$. (Don't forget to use conservation of energy, E_0.) (3 points)

   (d) Compute the minimum and maximum values or the radial coordinate r as functions of the constants m, E_0, k, L^z. (3 points)

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Classical #2

a) Write Lagrangian for a point mass under the influence of the following potential:

$V(x,y,z) = \frac{k}{2}(x^2+y^2+z^2)$ * No external gravity

$$\mathcal{L} = T - V$$

$$= \frac{1}{2} m v^2 - \frac{1}{2} k (x^2+y^2+z^2)$$

$$= \frac{1}{2} m (\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{1}{2} k (x^2+y^2+z^2)$$

b) Use Lagrange equations of motion to show angular momentum conservation

i.e. $\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = 0$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{y}} \qquad \frac{\partial \mathcal{L}}{\partial z} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$

$$\Rightarrow -kx = \frac{\partial}{\partial t}(m\dot{x}) \qquad \Rightarrow -ky = \frac{\partial}{\partial t}(m\dot{y}) \qquad \Rightarrow -kz = \frac{\partial}{\partial t}(m\dot{z})$$

$$-kx = m\ddot{x} \qquad -ky = m\ddot{y} \qquad -kz = m\ddot{z}$$

$$\vec{r} \times m\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix} = \langle ym\dot{z} - zm\dot{y}, xm\dot{z} - zm\dot{x}, xm\dot{y} - ym\dot{x} \rangle$$

$$= m\langle y\dot{z} - z\dot{y}, x\dot{z} - z\dot{x}, x\dot{y} - y\dot{x} \rangle$$

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = m\langle \dot{y}\dot{z} + y\ddot{z} - (\dot{z}\dot{y} + z\ddot{y}), \dot{x}\dot{z} + x\ddot{z} - (\dot{z}\dot{x} + z\ddot{x}), \dot{x}\dot{y} + x\ddot{y} - (\dot{y}\dot{x} + y\ddot{x}) \rangle$$

$$= m\langle y\ddot{z} - z\ddot{y}, x\ddot{z} - z\ddot{x}, x\ddot{y} - y\ddot{x} \rangle$$

* substituting from above

$$= m\left(\frac{-k}{m}\right)\langle yz - zy, xz - zx, xy - yx \rangle$$

$$= -k\langle 0, 0, 0 \rangle$$

$$= 0$$

$$\therefore \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d}{dt}\vec{L} = 0 \checkmark$$

c) For 2-D motion in x-y plane, use cylindrical polar coordinates to solve Lagrange eqns of motion. Leave $r(t)$ as an integral of form $t = \int f(r) dr$

* Hint: use conservation of energy E_0

- In cylindrical polar: $x = r\cos\varphi$ $\quad \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\dot{\varphi}$

$y = r\sin\varphi$ $\quad \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\dot{\varphi}$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r\dot{\varphi}^2) - \tfrac{1}{2}kr^2 \quad (2\text{-D},\ z=0)$$

$$E_0 = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \tfrac{1}{2}kr^2$$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = 0$$

$$0 = kr + mr\dot{\varphi}^2 - m\ddot{r} \qquad 0 = -2mr\dot{r}\dot{\varphi} - mr^2\ddot{\varphi}$$

* Solving E_0 for $\dot{\varphi}$ yields

$$E_0 = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\varphi}^2 + \tfrac{1}{2}kr^2$$

$$\frac{2E_0}{m} = \dot{r}^2 + r^2\dot{\varphi}^2 + \frac{k}{m}r^2$$

$$\frac{2E_0}{m} - \frac{k}{m}r^2 - \dot{r}^2 = r^2\dot{\varphi}^2$$

$$\frac{2E_0}{mr^2} - \frac{k}{m} - \frac{\dot{r}^2}{r^2} = \dot{\varphi}^2$$

$$\Rightarrow 0 = kr + mr\left(\frac{2E_0}{mr^2} - \frac{k}{m} - \frac{\dot{r}^2}{r^2}\right) + \tfrac{1}{2}kr^2$$

$$= \cancel{kr} + \frac{2E_0}{r} - \cancel{kr} - \frac{m\dot{r}^2}{r} + \tfrac{1}{2}kr^2$$

$$0 = \frac{2E_0}{r} - \frac{m\dot{r}^2}{r} + \tfrac{1}{2}kr^2$$

$$\frac{m\dot{r}^2}{r} = \frac{2E_0}{r} + \tfrac{1}{2}kr^2$$

$$\dot{r}^2 = 2mE_0 + \tfrac{1}{2}\tfrac{k}{m}r^3$$

$$\Rightarrow t = \int (2mE_0 + \tfrac{1}{2}\tfrac{k}{m}r^3)^{1/2} dr$$

3. Consider a particle attracted by a fixed gravitating body while also in a uniform gravitational field oriented along the z-axis. The potential energy is of the form:

$$V(r,z) = -m\left(\frac{k}{r} + g\,z\right)$$

where m is the particle's mass, k and g are constants, and r is the standard radial coordinate:

$$r \equiv \sqrt{x^2+y^2+z^2}$$

You are to examine the problem in *cylindrical parabolic coordinates* defined by

$$\begin{aligned} \zeta &\equiv r+z \\ \eta &\equiv r-z \\ \phi &\equiv \arctan y/x \end{aligned}$$

In these coordinates we may write the Cartesian coordinates as:

$$\begin{aligned} x &= \sqrt{\zeta\eta}\cos\phi \\ y &= \sqrt{\zeta\eta}\sin\phi \\ z &= \tfrac{1}{2}(\zeta-\eta) \end{aligned}$$

(a) Show that the kinetic energy, T, is given by:

$$T = \frac{m}{8}\left[\left(1+\frac{\zeta}{\eta}\right)\dot{\eta}^2 + \left(1+\frac{\eta}{\zeta}\right)\dot{\zeta}^2\right] + \frac{m}{2}\,\zeta\eta\,\dot{\phi}^2$$

in these coordinates. (2 points)

(b) What are the canonical momenta, p_ζ, p_η, and p_ϕ, expressed in cylindrical parabolic coordinates? (2 points)

(c) Use Hamilton-Jacobi theory to find the constants of the motion. *Hint:* While the total energy E does not separate in these coordinates, $E(\zeta+\eta)$ can be used to produce a quantity that **does** seperate. (3 points)

(d) What is Hamilton's characteristic function associated with ϕ? (1 point)

(e) Express Hamilton's characteristic functions associated with ζ, η as definite integrals. (2 points)

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Classical #3

For a particle attracted by a fixed gravitating body while also in a uniform gravitational field oriented along the z-axis $\left[V(r,z) = -m\left(\frac{k}{r} + gz\right)\right]$ where $r = \sqrt{x^2+y^2+z^2}$ and using cylindrical parabolic coordinates:

$$\xi = r + z \qquad\qquad x = \sqrt{\xi\eta}\,\cos\varphi$$
$$\eta = r - z \qquad \Rightarrow \qquad y = \sqrt{\xi\eta}\,\sin\varphi$$
$$\varphi = \arctan(y/x) \qquad\qquad z = \tfrac{1}{2}(\xi - \eta)$$

a) Show that the kinetic energy, T, is: $T = \frac{m}{8}\left[\left(1+\frac{\xi}{\eta}\right)\dot\eta^2 + \left(1+\frac{\eta}{\xi}\right)\dot\xi^2\right] + \frac{m}{2}\xi\eta\dot\varphi^2$

$$T = \tfrac{1}{2}m\dot v^2$$
$$= \tfrac{m}{2}(\dot x^2 + \dot y^2 + \dot z^2)$$

* from above,

$$\dot x = \tfrac{1}{2}\xi^{-1/2}\eta^{1/2}\cos\varphi\,\dot\xi + \tfrac{1}{2}\xi^{1/2}\eta^{-1/2}\cos\varphi\,\dot\eta - \xi^{1/2}\eta^{1/2}\sin\varphi\,\dot\varphi$$

$$\dot x^2 = \tfrac{1}{4}\xi^{-1}\eta\cos^2\varphi\,\dot\xi^2 + \tfrac{1}{4}\xi\eta^{-1}\cos^2\varphi\,\dot\eta^2 + \xi\eta\sin^2\varphi\,\dot\varphi^2 + \tfrac{1}{2}\cos^2\varphi\,\dot\xi\dot\eta - \eta\cos\varphi\sin\varphi\,\dot\xi\dot\varphi$$
$$- \xi\cos\varphi\sin\varphi\,\dot\eta\dot\varphi$$

$$\dot y = \tfrac{1}{2}\xi^{-1/2}\eta^{1/2}\sin\varphi\,\dot\xi + \tfrac{1}{2}\xi^{1/2}\eta^{-1/2}\sin\varphi\,\dot\eta + \xi^{1/2}\eta^{1/2}\cos\varphi\,\dot\varphi$$

$$\dot y^2 = \tfrac{1}{4}\xi^{-1}\eta\sin^2\varphi\,\dot\xi^2 + \tfrac{1}{4}\xi\eta^{-1}\sin^2\dot\eta^2 + \xi\eta\cos^2\varphi\,\dot\varphi^2 + \tfrac{1}{2}\sin^2\varphi\,\dot\xi\dot\eta + \eta\sin\varphi\cos\varphi\,\dot\xi\dot\varphi$$
$$+ \xi\sin\varphi\cos\varphi\,\dot\eta\dot\varphi$$

$$\dot z = \tfrac{1}{2}(\dot\xi - \dot\eta)$$
$$\dot z^2 = \tfrac{1}{4}(\dot\xi^2 - 2\dot\xi\dot\eta + \dot\eta^2)$$

$$\Rightarrow \dot x^2 + \dot y^2 + \dot z^2 = \tfrac{1}{4}\xi^{-1}\eta\dot\xi + \tfrac{1}{4}\xi\eta^{-1}\dot\eta^2 - \xi\eta\sin^2\varphi\,\dot\varphi^2 + \xi\eta\cos^2\varphi\,\dot\varphi^2 + \tfrac{1}{2}\dot\xi\dot\eta + \tfrac{1}{4}\dot\xi^2 - \tfrac{1}{2}\dot\xi\dot\eta + \tfrac{1}{4}\dot\eta^2$$

$$= \tfrac{1}{4}\tfrac{\eta}{\xi}\dot\xi^2 + \tfrac{1}{4}\tfrac{\xi}{\eta}\dot\eta^2 - \xi\eta\sin^2\varphi\,\dot\varphi^2 + \xi\eta\cos^2\varphi\,\dot\varphi^2 + \tfrac{1}{4}\dot\xi^2 + \tfrac{1}{4}\dot\eta^2$$

$$= \tfrac{1}{4}\left[\dot\xi^2\left(\tfrac{\eta}{\xi}+1\right) + \dot\eta^2\left(\tfrac{\xi}{\eta}+1\right)\right] + \xi\eta\dot\varphi^2\left(\cancel{\cos^2\varphi + \sin^2\varphi}\right)$$

$$\Rightarrow T = \tfrac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2)$$
$$= \tfrac{m}{8}\left[\left(1+\tfrac{\xi}{\eta}\right)\dot\eta^2 + \left(1+\tfrac{\eta}{\xi}\right)\dot\xi^2\right] + \tfrac{m}{2}\xi\eta\dot\varphi^2$$

b) Find the canonical momenta in cylindrical parabolic coordinates

$$p_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$\Rightarrow \mathcal{L} = T - V$$

$$V = -m\left(\frac{k}{r} + gz\right), \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r = \sqrt{\xi\eta\cos^2\varphi + \xi\eta\sin^2\varphi + \frac{1}{4}(\xi - \eta)^2}$$

$$= \sqrt{\xi\eta + \frac{1}{4}\xi^2 - \frac{1}{2}\xi\eta + \frac{1}{4}\eta^2}$$

$$= \sqrt{\frac{1}{4}\xi^2 + \frac{1}{2}\xi\eta + \frac{1}{4}\eta^2}$$

$$= \sqrt{\frac{1}{4}(\xi^2 + 2\xi\eta + \eta^2)}$$

$$= \frac{1}{2}(\xi + \eta)$$

$$\Rightarrow V = -m\left(\frac{2k}{\xi + \eta} + g\left(\frac{\xi + \eta}{2}\right)\right)$$

$$\Rightarrow \mathcal{L} = T - V$$

$$= \frac{m}{8}\left[\left(1 + \frac{\xi}{\eta}\right)\dot{\eta}^2 + \left(1 + \frac{\eta}{\xi}\right)\dot{\xi}^2\right] + \frac{m}{2}\xi\eta\dot{\varphi}^2 + m\left[\frac{2k}{\xi + \eta} + \frac{g(\xi + \eta)}{2}\right]$$

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

$$= m\xi\eta\dot{\varphi}$$

$$p_\eta = \frac{\partial \mathcal{L}}{\partial \dot{\eta}}$$

$$= \frac{m}{4}\left(1 + \frac{\xi}{\eta}\right)\dot{\eta}$$

$$p_\xi = \frac{\partial \mathcal{L}}{\partial \dot{\xi}}$$

$$= \frac{m}{4}\left(1 + \frac{\eta}{\xi}\right)\dot{\xi}$$

c) $H = \sum_i p_i \dot{q}_i - \mathcal{L}$

$$H = p_\varphi \dot{\varphi} + p_z \dot{z} + p_x \dot{x} - \mathcal{L}$$

$$= \frac{m}{2} z x \dot{\varphi}^2 + \frac{m}{8}\left(1 + \frac{x}{z}\right)\dot{z}^2 + \frac{m}{8}\left(1 + \frac{z}{x}\right)\dot{x}^2 - m\left[\frac{2k}{z+x} + \frac{g(z+x)}{2}\right]$$

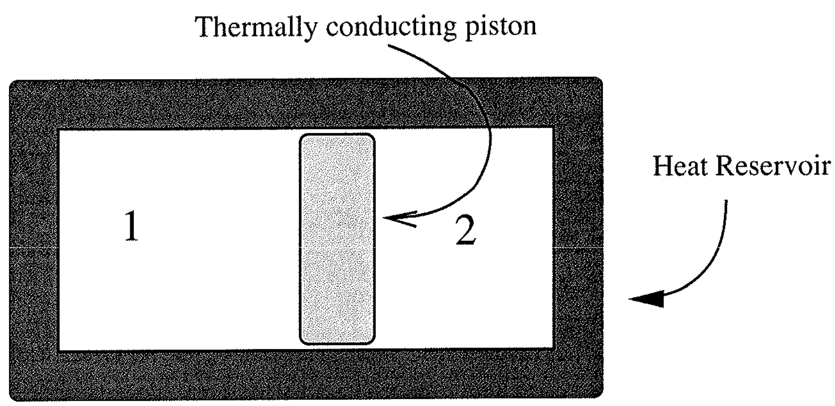
Statistical Mechanics

4. **Helmholtz Free Energy**: The Helmholtz free energy of an ideal monoatomic gas can be written as

$$F(T,V,N) = NkT\left\{A - \log\left[T^{3/2}\frac{V}{N}\right]\right\}$$

where N is the total number of gas atoms, V is the volume, T is temperature, k is Boltzmann's constant and A is a dimensionless constant.

Consider a piston separating a system into two parts, with equal numbers of particles on the left and the right hand side. The whole system is in good thermal contact with a reservoir at constant temperature T. Initially, $V_1 = 2V_2$. The total volume, $V_{\text{tot}} = V_1 + V_2$, is fixed for this whole problem.



(a) Calculate the equilibrium position of the piston, once it is released. You must prove your answer, and not simply assert it. (3 points)

(b) Calculate the maximum available work the system can perform as it changes from the initial condition to the equilibrium position. (3 points)

(c) Calculate the change in the internal energy, U of gas 1 and gas 2 in the process. (2 points)

(d) Given your answers above, explain the source of energy for the work done during the expansion. (2 points)

5. Consider a gas of N non-interacting **one dimensional** diatomic molecules enclosed in a box of "volume" L (actually, just a length) at temperature T.

   (a) The classical energy for a single molecule is:

   $$E(p_1, p_2, x_1, x_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}K(x_1 - x_2)^2$$

   where p_1 and p_2 are the classical momenta of the atoms in one diatomic molecule, x_1 and x_2 are their classical positions, and K is the spring constant. Calculate the specific heat for the gas. (You should assume that $KL^2/2 >> k_BT$, where k_B is Boltzmann's constant.) (4 points).

   (b) In the quantum limit the energy levels of the molecule are discrete. In a semiclassical approach we can write the energy of one molecule as:

   $$E(P, n) = \frac{P^2}{4m} + \hbar\omega(n + \frac{1}{2})$$

   where P is the momentum of the diatomic molecule (of mass $2m$), and ω is the natural frequency of the oscillator, and n is a non-negative integer ($n \geq 0$). Calculate the specific heat. (4 points).

   (c) Calculate the high and low temperature limits of your result in (b), and explain how they relate to the result of (a). (2 points)

6. Fermions:

(a) Show that for any non-interacting spin 1/2 fermionic system with chemical potential μ, the probability of occupying a single particle state with energy $\mu + \delta$ is the same as finding a state vacant at an energy $\mu - \delta$. (2 points)

(b) Consider non-interacting fermions that come in two types of energy states:

$$E_{\pm}(\vec{k}) = \pm\sqrt{m^2c^4 + \hbar^2k^2c^2}$$

At zero temperature all the states with negative energy (all states with energy $E_{-}(\vec{k})$) are occupied[1] and all positive energy states are empty, and that $\mu(T = 0) = 0$. Show that the result of part (a) above means that the chemical potential must remain at zero for all temperatures if particle number is to be conserved. (2 points)

(c) Using the results of (a) and (b) above, show that the average excitation energy, the change in the energy of the system from it's energy at $T = 0$ in three dimensions is given by:

$$\Delta E \equiv E(T) - E(0) = 4V \int \frac{d\vec{k}}{(2\pi)^3} E_{+}(\vec{k}) \frac{1}{1 + e^{\beta E_{+}(\vec{k})}}$$

(2 points)

(d) Evaluate the integral above for massless ($m = 0$) particles. (2 points)

(e) Calculate the heat capacity of such particles. (2 points)

[1] Technically this means the total energy of the system diverges. If this bothers you, you can assume some large cut-off to the wavevectors, $\hbar k_{\max} c >> kT$, which will have no effect on your final answers.