Classical Mechanics and Statistical/Thermodynamics

January 2008

Possibly Useful Information

Handy Integrals:

$$\begin{aligned}
\int_0^\infty x^n e^{-\alpha x}\, dx &= \frac{n!}{\alpha^{n+1}} \\
\int_0^\infty e^{-\alpha x^2}\, dx &= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \\
\int_0^\infty x\, e^{-\alpha x^2}\, dx &= \frac{1}{2\alpha} \\
\int_0^\infty x^2\, e^{-\alpha x^2}\, dx &= \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}} \\
\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\, dx &= \sqrt{\frac{\pi}{b}} e^{-a^2/4b}
\end{aligned}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

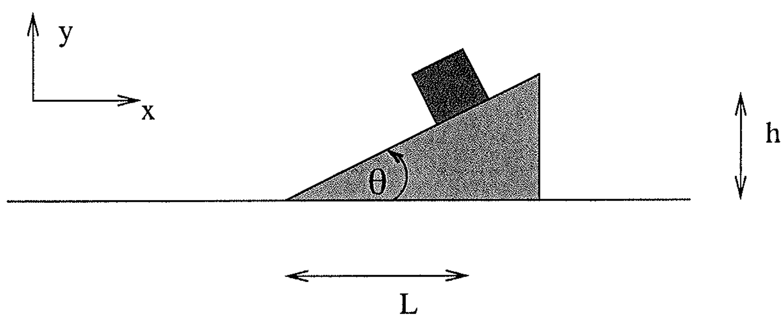
$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p \frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\begin{array}{cc}
\zeta(1) = \infty & \zeta(-1) = 0.0833333 \\
\zeta(2) = 1.64493 & \zeta(-2) = 0 \\
\zeta(3) = 1.20206 & \zeta(-3) = 0.0083333 \\
\zeta(4) = 1.08232 & \zeta(-4) = 0
\end{array}$$

Classical Mechanics

1. A block of mass m_1 sits atop a triangular wedge of mass m_2, which is itself on a frictionless plane, as shown. The two are initially at rest, and the block is a height h above the surface of the plane, a horizontal distance L from the bottom edge of the wedge. The wedge has an opening angle θ, as shown.



   (a) Assume that there is no friction between the block and the wedge. The block slides down the wedge. What are the velocities (measured with respect to the fixed inertial reference frame denoted by the x and y axes shown) of the block and wedge just as the block reaches the lower edge of the wedge? (3 points).

   (b) Now replace the block by a ball of radius R (and mass m_1). The ball rolls down the wedge without slipping. What are the velocities of the ball and wedge just as the ball reaches the lower edge of the wedge? (3 points).

   (c) Return to the block problem, but now assume that the coefficients of static and kinetic friction between the block and the wedge are μ (they have the same value). What is μ_{min}, the minimum value of μ for which the system is stable? (1 point).

   (d) If $\mu < \mu_{\text{min}}$, calculate the minimum **horizontal** force that can be applied to the wedge such that the block will not accelerate down the wedge. (3 points).

   Note: you can neglect the finite size of the block in your calculation, and you are asked for the velocities before the block or ball make contact with the frictionless plane.

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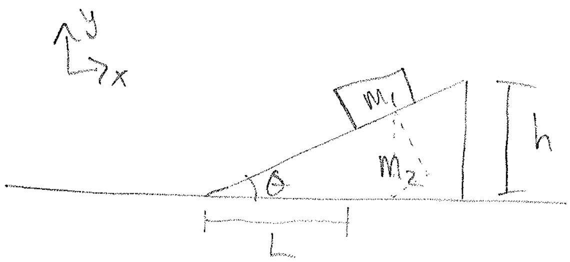
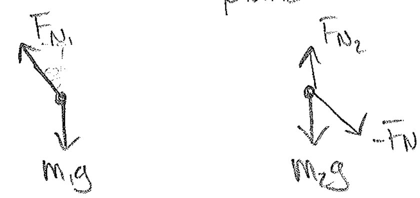
Classical #1



* Frictionless plane



a) * Assume no friction b/w block + wedge

* Find velocities when block reaches bottom of wedge

- Forces

* Block $= \langle -F_{N_1}\sin\theta, -m_1 g + F_{N_1}\cos\theta \rangle$

* Wedge $= \langle F_N \sin\theta, -m_2 g - F_{N_1}\cos\theta + F_{N_2} \rangle$

- Energy

$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = mgh$

- Kinematics

$r_{1,f} = r_{1,i} + \frac{1}{2}a_i t^2$

$v_{1,f}^2 = 2a\Delta r \qquad v_1 = a_1 t$

$r_{1,f} = \langle x_{f,1}, 0 \rangle$ $\qquad$ * Note: $-m_2 g - F_{N_1}\cos\theta + F_{N_2} = 0$

$\Rightarrow \cancel{y_{1,f}}^{0} = y_{1,i} + \cancel{v_{1,i}t}^{0} + \frac{1}{2}a_{1,y}t^2$

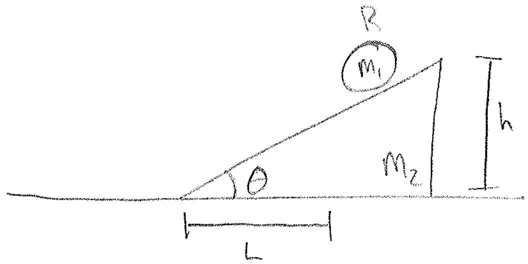
$0 = h + \frac{1}{2}a_{1,y}t^2$

$\left(\frac{-2h m_1}{F_{N_1}\cos\theta - m_1 g}\right)^{1/2} = t$

$\Rightarrow v_1 = at$

$v_1 = \left(\frac{-2h}{F_{N_1}\cos\theta - m_1 g}\right) \langle -F_{N_1}\sin\theta, F_{N_1}\cos\theta - m_1 g \rangle, \quad v_2 = \left(\frac{-2hm_1}{m_2(F_{N_1}\cos\theta - m_1 g)}\right)\langle F_{N_1}\sin\theta, 0 \rangle$

b) * Block is now ball rolling w/o slipping (ie. $V_{tangent} = V_{cm}$)



* Frictionless Plane

- Forces

  * Ball = $\langle -F_{N_1}\sin\theta,\ -m_1 g + F_{N_1}\cos\theta \rangle$

  * Wedge = $\langle F_{N_1}\sin\theta,\ -m_2 g - F_{N_1}\cos\theta + F_{N_2} \rangle$

- Energy

  $m_1 g h = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}m_2 v_2^2$, $I = \frac{2}{5}m_1 R^2$

- Kinematics

  $\vec{r}_f = \vec{r}_i + \frac{1}{2}\vec{a}t^2$ $\qquad v = \omega R$ $\qquad \vec{v}_f^{\,2} = 2\vec{a}\,\Delta\vec{r}$

  $\vec{\theta}_f = \vec{\theta}_i + \frac{1}{2}\vec{\alpha}t^2$ $\qquad a = \alpha R$ $\qquad \vec{v}_f = \vec{a}t$

2. Consider a point particle of mass m constrained to move on a parabola in the x-z plane, i.e.,

$$z = \frac{\alpha}{2} x^2.$$

Assume the constraint force is frictionless and gravity acts vertically ($F_z = -mg$).

(a) Use Lagrangian mechanics to write a second order differential equation for $x(t)$. (2 points)

(b) Find a first integral of this equation (any way you can) and evaluate the constant of integration using the maximum value x_{max} reached by x. (4 points)

(c) Assume that the particle is pulled a short distance from the origin and allowed to oscillate. Calculate the period in the limit of small oscillations, $\epsilon \equiv \alpha x_{\rm max} << 1$. (4 points)

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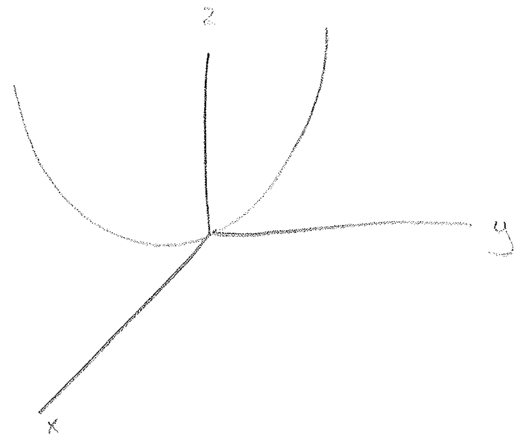
Classical # 2

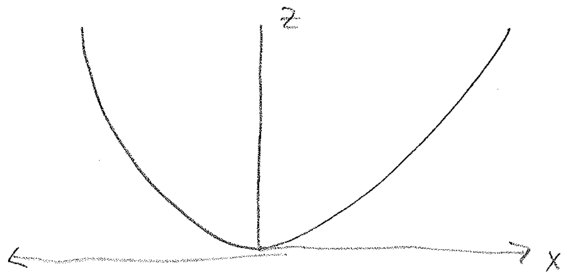


* Particle of mass m constrained to move on parabola in x-z plane

$\Rightarrow z = \frac{\alpha}{2}x^2$

* Constraint force is frictionless

$F_z = -mg$



a) $\mathcal{L} = T - U$

$T = \frac{1}{2}mv^2$ $\quad U = mgz$

$= \frac{1}{2}m(\dot{x}^2 + \dot{z}^2)$ $\quad = mg\frac{\alpha}{2}x^2$

$= \frac{1}{2}m(\dot{x}^2 + (\alpha\dot{x})^2)$

$= \frac{1}{2}m\dot{x}^2(1+\alpha)$

$\Rightarrow \mathcal{L} = \frac{1}{2}(m\dot{x}^2(1+\alpha) - mg\alpha x^2)$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

$$mg\alpha x = \frac{\partial}{\partial t}\left(m\dot{x}(1+\alpha)\right)$$

$$mg\alpha x = m(1+\alpha)\ddot{x}$$

$$g\alpha x = (1+\alpha)\ddot{x}$$

$$x = \frac{1+\alpha}{g\alpha}\ddot{x}$$

b)

3. **Angular momentum and the Rungé-Lenz vector:** Given a point particle of mass m, trajectory $\vec{r}(t)$, and momentum $\vec{p}(t)$, we can define the angular momentum

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

and the Rungé-Lenz vector

$$\vec{\mathcal{A}} = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

We consider the explicit case of a $1/r$ potential, so that

$$H = \frac{p^2}{2m} - \frac{1}{r}$$

(a) Prove that the Poisson bracket of H and $\vec{L}$ is zero, that is:

$$\left\{H, \vec{L}\right\} = 0.$$

(3 points).

(b) Prove that the Poisson bracket of H and $\vec{\mathcal{A}}$ is zero, that is:

$$\left\{H, \vec{\mathcal{A}}\right\} = 0.$$

(3 points)

(c) What do your results in parts (a) and (b) imply about the behavior of $\vec{\mathcal{A}}$ and $\vec{L}$? (1 point)

(d) Evaluate $\vec{r} \cdot \vec{\mathcal{A}} = r\mathcal{A}\cos\theta$, using the explicit form for $\vec{\mathcal{A}}$ above. Use this to calculate the orbital motion of the particle (that is, a relationship between r and θ as the particle moves about its orbit). (3 points)

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Classical #3

$\vec{L} = \vec{r} \times \vec{p}$

$\vec{A} = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$

$H = \frac{p^2}{2m} - \frac{1}{r}$

a) Prove Poisson bracket, $\{H, L\} = 0$

$$H = \frac{p^2}{2m} - \frac{1}{r}$$
$$= \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$L = \vec{r} \times p$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \langle yp_z - zp_y, -(xp_z - zp_x), xp_y - yp_x \rangle$$

$$\{H, L\} = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial L}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial L}{\partial q_i} \right)$$
$$= \frac{\partial H}{\partial x}\frac{\partial L}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial L}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial L}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial L}{\partial y} + \frac{\partial H}{\partial z}\frac{\partial L}{\partial p_z} - \frac{\partial H}{\partial p_z}\frac{\partial L}{\partial z}$$

$\frac{\partial H}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_x} = \frac{p_x}{m}$

$\frac{\partial L}{\partial p_x} = \langle 0, z, -y \rangle$ $\qquad$ $\frac{\partial L}{\partial x} = \langle 0, -p_z, p_y \rangle$

$\frac{\partial H}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_y} = \frac{p_y}{m}$

$\frac{\partial L}{\partial p_y} = \langle -z, 0, x \rangle$ $\qquad$ $\frac{\partial L}{\partial y} = \langle p_z, 0, -p_x \rangle$

$\frac{\partial H}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_z} = \frac{p_z}{m}$

$\frac{\partial L}{\partial p_z} = \langle y, -x, 0 \rangle$ $\qquad$ $\frac{\partial L}{\partial z} = \langle -p_y, p_x, 0 \rangle$

$$= \frac{x}{(\ )^{3/2}}\langle 0, z, -y \rangle - \frac{p_x}{m}\langle 0, -p_z, p_y \rangle +$$
$$\frac{y}{(\ )^{3/2}}\langle -z, 0, x \rangle - \frac{p_y}{m}\langle p_z, 0, -p_x \rangle +$$
$$\frac{z}{(\ )^{3/2}}\langle y, -x, 0 \rangle - \frac{p_z}{m}\langle -p_y, p_x, 0 \rangle$$

$\langle 0, 0, 0 \rangle$ ✓

b) Show $\{H, A\} = 0$

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$A = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

$$= \frac{1}{m}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ p_x & p_y & p_z \\ yp_z - zp_y & zp_x - xp_z & xp_y - yp_x \end{vmatrix} - \langle x, y, z \rangle \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{1}{m}\langle p_y(xp_y - yp_x) - p_z(zp_x - xp_z),\ p_z(yp_z - zp_y) - p_x(xp_y - yp_x),\ p_x(zp_x - xp_z) - p_y(yp_z - zp_y) \rangle - \langle x, y, z \rangle$$

$$\{H, \hat{A}\} = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial A}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial A}{\partial q_i}$$

$$= \frac{\partial H}{\partial x}\frac{\partial A}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial A}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial A}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial A}{\partial y} + \frac{\partial H}{\partial z}\frac{\partial A}{\partial p_z} - \frac{\partial H}{\partial p_z}\frac{\partial A}{\partial z}$$

$$\frac{\partial H}{\partial x} = -\left(\frac{-1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x\right) = \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

$$\frac{\partial A}{\partial x} = \frac{1}{m}\langle p_y^2 + p_z^2,\ -p_xp_y,\ -p_xp_z \rangle - \langle 1, 0, 0 \rangle$$

$$\frac{\partial A}{\partial p_x} = \frac{1}{m}\langle -yp_y - zp_z,\ -xp_y + 2yp_x,\ 2zp_x - xp_z \rangle$$

$$\frac{\partial H}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\frac{\partial A}{\partial y} = \frac{1}{m}\langle -p_xp_y,\ p_z^2 + p_x^2,\ -p_yp_z \rangle - \langle 0, 1, 0 \rangle$$

$$\frac{\partial A}{\partial p_y} = \frac{1}{m}\langle 2xp_y - yp_x,\ -zp_z - xp_x,\ -yp_z + 2zp_y \rangle$$

$$\frac{\partial H}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

$$\frac{\partial A}{\partial z} = \frac{1}{m}\langle -p_xp_z,\ -p_yp_z,\ p_x^2 + p_y^2 \rangle - \langle 0, 0, 1 \rangle$$

$$\frac{\partial A}{\partial p_z} = \frac{1}{m}\langle -zp_x + 2xp_z,\ 2yp_z - zp_y,\ -xp_x - yp_y \rangle$$

$$\Rightarrow \{H, \vec{A}\} = \frac{x}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle -yp_y - zp_z, -xp_y + 2yp_x, 2zp_x - xp_z \rangle$$

$$- \frac{p_x}{m}\left(\frac{1}{m} \langle p_y^2 + p_z^2, -p_xp_y, -p_xp_z \rangle - \langle 1, 0, 0 \rangle \right)$$

$$+ \frac{y}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle 2xp_y - yp_x, -zp_z - xp_x, -yp_z + 2zp_y \rangle$$

$$- \frac{p_y}{m}\left(\frac{1}{m} \langle -p_xp_y, p_z^2 + p_x^2, -p_yp_z \rangle - \langle 0, 1, 0 \rangle \right)$$

$$+ \frac{z}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle 2xp_z - zp_x, 2yp_z - zp_y, -xp_x - yp_y \rangle$$

$$- \frac{p_z}{m}\left(\frac{1}{m} \langle -p_xp_z, -p_yp_z, p_x^2 + p_y^2 \rangle - \langle 0, 0, 1 \rangle \right)$$

$$= \frac{1}{mr^3} \langle x(-yp_y - zp_z) + y(2xp_y - yp_x) + z(2xp_z - zp_x), x(-xp_y + 2yp_x) + y(-zp_z - xp_x) + z(2yp_z - zp_y)$$

$$, x(2zp_x - xp_z) + y(-yp_z + 2zp_y) + z(-xp_x - yp_y) \rangle$$

$$+ \frac{1}{m} \langle p_x, p_y, p_z \rangle - \frac{1}{m^2} \langle p_x(p_y^2 + p_z^2) + p_y(-p_xp_y) + p_z(-p_xp_z), p_x(-p_xp_y) + p_y(p_x^2 + p_z^2) + p_z(-p_yp_z)$$

$$, p_x(-p_xp_z) + p_y(-p_yp_z) + p_z(p_x^2 + p_y^2) \rangle$$

$$= \frac{1}{mr^3} \langle -xyp_y - xzp_z + \cancel{2}xyp_y - y^2p_x + 2xzp_z - z^2p_x, -x^2p_y + \cancel{2}yxp_x - \cancel{yzp_z} - \cancel{xyp_x} + \cancel{2}yzp_z - z^2p_y,$$

$$\cancel{2}xzp_x - x^2p_z - y^2p_z - \cancel{2yzp_y} - \cancel{xzp_x} - \cancel{yzp_y} \rangle$$

$$+ \frac{1}{m} \langle p_x, p_y, p_z \rangle - \frac{1}{m^2} \langle \cancel{p_xp_y^2} + \cancel{p_xp_z^2} - \cancel{p_xp_y^2} - \cancel{p_xp_z^2}, -\cancel{p_x^2p_y} + \cancel{p_x^2p_y} + \cancel{p_yp_z^2} - \cancel{p_yp_z^2},$$

$$, -\cancel{p_x^2p_z} - \cancel{p_y^2p_z} + \cancel{p_x^2p_z} + \cancel{p_y^2p_z} \rangle$$

$$= \frac{1}{mr^3} \langle xyp_y + xzp_z - y^2p_x - z^2p_x, xyp_x + yzp_z - x^2p_y - z^2p_y, xzp_x + yzp_y - x^2p_z - y^2p_z \rangle$$

$$+ \frac{1}{m} \langle p_x, p_y, p_z \rangle$$

c) If $\{H, L\}$ and $\{H, A\}$ are both 0, then both $\vec{L}$ and $\vec{A}$ are constants of the motion.

d) $\vec{r} \cdot \vec{A} = rA\cos\theta$

$$\vec{r} \cdot \left(\frac{1}{m}\vec{p} \times \vec{L} - \hat{r}\right) = \frac{1}{m}\left(\vec{r} \cdot (\vec{p} \times \vec{L})\right) - \vec{r} \cdot \hat{r}$$

$$= \frac{1}{m}\vec{L} \cdot (\vec{r} \times \vec{p}) - r$$

$$= \frac{1}{m}L^2 - r$$

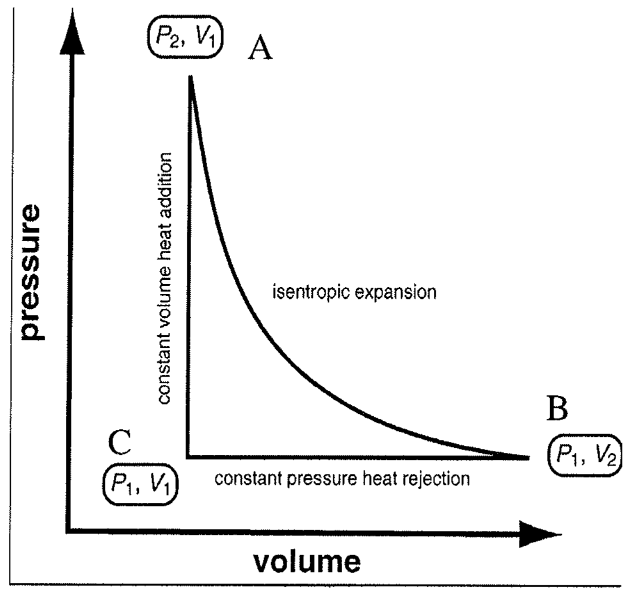
$$\frac{L^2}{m} - r = rA\cos\theta$$

$$\frac{L^2}{m} = r(A\cos\theta + 1)$$

$$\frac{L^2}{m} \cdot \frac{1}{A\cos\theta + 1} = \vec{r}$$

Statistical Mechanics

4. **Heat Engines:** A pulse jet operates under a Lenoir cycle. This consists of an adiabat, an isobar, and an isochore, as shown.



Assuming that the working fluid is an ideal 3D monoatomic gas of N particles:

(a) Find the work done in one complete cycle. (3 points)

(b) Find the heat exchanged in each step in the cycle. (3 points)

(c) Find the efficiency of the engine. Express your answer in terms of pressures and volutmes. (3 points)

(d) To produce work, should the engine cycle operate clockwise ($A \to B \to C \to A$) or counterclockwise ($A \to C \to B \to A$) ? (1 point)

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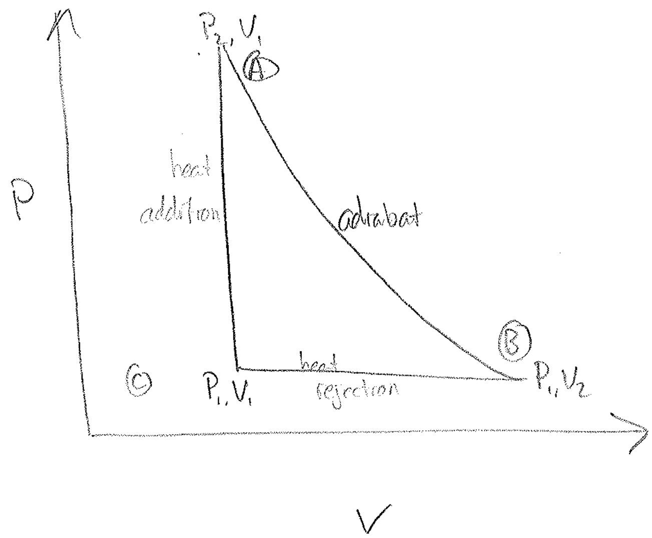
Stat Mech #1



* Assume 3-D monatomic gas of N particles

$\Rightarrow PV = Nk_B T$

a) Find work done in one complete cycle

$P_A = P_2 \qquad P_B = P_1 \qquad P_c = P_1$

$V_A = V_1 \qquad V_B = V_2 \qquad V_c = V_1$

$T_A = \frac{Nk_B}{P_2 V_1} \qquad T_B = \frac{Nk_B}{P_1 V_2} = \frac{Nk_B}{P_2 V_1} \qquad T_c = \frac{Nk_B}{V_2 P_2}$

$\frac{V_B}{T_B} = \frac{V_c}{T_c}$

$\Rightarrow T_c = \frac{V_c T_B}{V_B}$

$T_c = \frac{V_1}{V_2} \frac{Nk_B}{P_2 V_1}$

$= \frac{Nk_B}{V_2 P_2}$

$W = \int P dV$

$W_{A\to C} = 0, \quad dV = 0$

$W_{C\to B} = P_1(V_2 - V_1)$

$$W_{B\to A} = \frac{P_A V_A - P_B V_B}{1-\gamma}$$

$$= \frac{P_2 V_1 - P_1 V_2}{1 - 5/3}$$

$$= -\frac{3}{2}(P_2 V_1 - P_1 V_2)$$

$$W_{TOT} = 0 + P_1 V_2 - P_1 V_1 - \frac{3}{2}(P_2 V_1 - P_1 V_2)$$

$$= \frac{5}{2} P_1 V_2 - P_1 V_1 - \frac{3}{2} P_2 V_1$$

$$= \frac{5}{2} P_1 V_2 - V_1 \left(P_1 + \frac{3}{2} P_2\right)$$

b) Find the heat exchanged in each step of the cycle.

$$Q_{A\to C} = nC_v \Delta T$$
$$= \frac{N}{6.02\cdot10^{23}}\cdot\frac{3}{2}R\cdot\left(\frac{nR}{V_2P_2}-\frac{nR}{P_2V_1}\right)$$
$$= \frac{3nR}{2}\cdot\frac{nR}{P_2}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)$$
$$= \frac{3n^2R^2}{2P_2}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)$$

$$Q_{C\to B} = nC_p\Delta T$$
$$= \frac{5nR}{2}\left(\frac{nR}{P_2V_1}-\frac{nR}{V_2P_2}\right)$$
$$= \frac{5n^2R^2}{2P_2}\left(\frac{1}{V_1}-\frac{1}{V_2}\right)$$

$Q_{B\to A} = 0$ b/c adiabatic

$$Q_{TOT} = \frac{3n^2R^2}{2P_2}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)+\frac{5n^2R^2}{2P_2}\left(\frac{1}{V_1}-\frac{1}{V_2}\right)$$
$$= \frac{n^2R^2}{P_2}\left(\frac{1}{V_1}-\frac{1}{V_2}\right)$$

c) Find the efficiency of the engine

$$\eta = \frac{W_{out}}{Q_{in}}$$

*Note: Q_{in} occurs in $A\to C$

$$\eta = \frac{\frac{5}{2}P_1V_2 - V_1(P_1+\frac{3}{2}P_2)}{\frac{3n^2R^2}{2P_2}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)}$$

$$= \frac{5P_1V_2 - 2V_1P_1 - 3V_1P_2}{\frac{3n^2R^2}{P_2}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)}$$

$$= \frac{5P_1P_2V_2 - 2P_1P_2V_1 - 3P_2^2V_1}{3n^2R^2\left(\frac{1}{V_2}-\frac{1}{V_1}\right)}$$

d) To produce work, does engine operate clockwise or counter-clockwise?

CW

5. Consider a classical ideal gas in 3D that feels a linear gravitational potential,

$$V(z) = mgz$$

where m is the mass of a single gas atom and $0 < z < \infty$. This is not an interaction between gas atoms, it is simply their gravitational potential energy near the surface of the Earth.

The gas is in a box of dimensions L_x, L_y, and L_z, so that:

$$\begin{aligned} 0 < & \; z \; < L_z \\ 0 < & \; x \; < L_x \\ 0 < & \; y \; < L_y \end{aligned}$$

(a) Calculate the partition function in the canonical ensemble. (3 points)

(b) Determine the internal energy of the gas. (3 points)

(c) Calculate the specific heat c_v. (3 points)

(d) Explain the behavior of the specific heat when $\beta mgL_z >> 1$ and when $\beta mgL_z << 1$. (The approximation for the gravitational potential may or may not be valid for large L_z. Don't worry about that.) (1 point)

Stat Mech #2

*Consider a classical ideal gas in 3-D w/ linear gravitational potential $V(z) = mgz$. Note: m is mass of single atom, $0 < z < \infty$. Dimensions of box are: $0 < z < L_z$, $0 < x < L_x$, $0 < y < L_y$

a) Calculate the partition function in the classical ensemble

$$Z = \left[\frac{1}{h^3}\int dp^3 dq^3 e^{-\beta E}\right]^N$$

$$* \text{ let } E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + mgz$$

$$= \left[\frac{1}{h^3}\int dp^3 dq^3 e^{-\beta\left(\frac{1}{2m}[p_x^2+p_y^2+p_z^2] + mgz\right)}\right]^N$$

$$= \left[\frac{1}{h^3}\int dp_x dp_y dp_z\, dx\,dy\,dz\, e^{-\beta[\frac{1}{2m}[p_x^2+p_y^2+p_z^2] + mgz]}\right]^N$$

$$= \left[\frac{1}{h^3}\left(\int_0^\infty dp\, e^{-\beta p^2/2m}\right)^3 \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz\, e^{-\beta mgz}\right]^N$$

$$= \left[\frac{1}{h^3}\left(\frac{1}{2}\sqrt{\frac{\pi}{\beta/2m}}\right)^3 L_x L_y \left(\frac{1}{-\beta mg} e^{-\beta mgz}\Big|_0^{L_z}\right)\right]^N$$

$$= \left[\left(\frac{1}{h}\sqrt{\frac{2\pi m}{\beta}}\right)^3 L_x L_y \frac{1}{-\beta mg}\left(e^{-\beta mgL_z} - 1\right)\right]^N$$

b) Determine the internal energy of the gas

$$U = -\frac{\partial}{\partial\beta}\ln(Z)$$

$$= \frac{-\partial}{\partial\beta} N \ln\left[\left(\frac{2\pi m}{h^2\beta}\right)^{3/2} L_x L_y \frac{1}{-\beta mg}\left(e^{-\beta mgL_z} - 1\right)\right]$$

$$= \frac{-\partial}{\partial\beta} N \ln\left[\left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{L_x L_y}{mg} \frac{1}{-\beta^{5/2}}\left(e^{-\beta mgL_z} - 1\right)\right]$$

$$= \frac{-\partial}{\partial\beta} N\left(\ln\left[\left(\frac{2\pi m}{h^2}\right)^{3/2}\frac{L_x L_y}{mg}\right] + \ln\left[1 - e^{-\beta mgL_z}\right] - \ln\left(\beta^{5/2}\right)\right)$$

$$= -N\left[\left(1 - e^{-\beta mgL_z}\right)^{-1} + mgL_z e^{-\beta mgL_z} - \frac{5}{2\beta}\right]$$

c) Calculate the specific heat C_V

$$C_V = \frac{\partial U}{\partial T}$$

$$= \frac{\partial}{\partial T}\left(-N\left(\frac{mgL_z e^{-\beta mgL_z}}{1-e^{-\beta mgL_z}} - \frac{5}{2\beta}\right)\right)$$

$$= \frac{\partial}{\partial T}\left(\frac{5}{2}Nk_BT - \frac{mgL_z}{e^{-mgL_z/k_BT}-1}\right)$$

$$= \frac{5}{2}Nk_B + mgL_z\left(e^{-mgL_z/k_BT}-1\right)^{-2} \cdot -mgL_z/k_BT^2\, e^{-mgL_z/k_BT}$$

$$= \frac{5}{2}Nk_B - \frac{(mgL_z)^2 e^{-mgL_z/k_BT}}{T^2 k_B\left(e^{-mgL_z/k_BT}-1\right)^2}$$

6. **Boson Magnetism** Consider a gas of non-interacting spin-1 bosons in 3D, each subject to the Hamiltonian

$$\begin{aligned} H(\vec{p}, s_z) &= \frac{p^2}{2m} - \mu_0 s B \\ &= \frac{\hbar^2 k^2}{2m} - \mu_0 s B \end{aligned}$$

where s takes on one of three possible states, $s \in (-1, 0, +1)$, and $\vec{k} \equiv \vec{p}/\hbar$. In this Hamiltonian B is the z-component of the magnetic field, m is the mass of a particle, and μ_0 is the Bohr magneton. (We will ignore the orbital effect (or Lorentz force) where the momentum $\vec{p}$ would have been replaced, $\vec{p} \to \vec{p} + e\vec{A}/c$).

(a) In a grand canonical ensemble of chemical potential μ (which is **not** to be confused with the Bohr magneton, μ_0, above) and temperature T, write down $n_s(\vec{k})$, the average occupation number of the state with wave vector $\vec{k}$ and spin s. (1 point).

(b) Show that the total number of particles in a given spin state s is given by

$$N_s = \frac{V}{\lambda^3}\pi^{3/2}\, g_{3/2}(z e^{\beta \mu_0 s B})$$

where z is the fugacity, $z = e^{\beta\mu}$, λ is the thermal de Broglie wavelength,

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

and $g_p(z)$ is defined on the formula section on page 2 above. (4 points)

(c) The magnetization for fixed μ and T is given by

$$M(T, \mu) = \mu_0(N_{(+)} - N_{(-)})$$

Show that the zero field susceptibility, χ, is given by:

$$\chi \equiv \left.\frac{\partial M}{\partial B}\right|_{B=0} = \frac{2\mu_0^2}{k_B T}\,\pi^{3/2}\,\frac{V}{\lambda^3}\, g_{1/2}(z).$$

(5 points).