Classical Mechanics and Statistical/Thermodynamics

August 2011

Possibly Useful Information

Handy Integrals:

$$
\begin{aligned}
\int_0^\infty x^n e^{-\alpha x}\, dx &= \frac{n!}{\alpha^{n+1}} \\
\int_0^\infty e^{-\alpha x^2}\, dx &= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \\
\int_0^\infty x\ e^{-\alpha x^2}\, dx &= \frac{1}{2\alpha} \\
\int_0^\infty x^2\ e^{-\alpha x^2}\, dx &= \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}} \\
\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\, dx &= \sqrt{\frac{\pi}{b}}e^{-a^2/4b}
\end{aligned}
$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\ \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

Moments of Inertia:

$$
\begin{aligned}
I_{\text{hoop}} &= MR^2 \\
I_{\text{disk}} &= \tfrac{1}{2}MR^2 \\
I_{\text{sphericalshell}} &= \tfrac{2}{3}MR^2 \\
I_{\text{ball}} &= \tfrac{2}{5}MR^2
\end{aligned}
$$

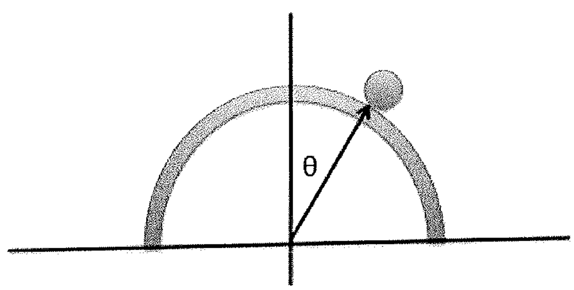
Classical Mechanics

1. A solid uniform marble with mass m and radius r starts from rest on top of a hemisphere with radius R. It will start to roll to the right, and eventually fly off the hemisphere.

   (a) Assume that the marble rolls without slipping at all times. Calculate θ_1, the angle with respect to the vertical at which the marble loses contact with the hemisphere. (3pts).

   (b) Where will the marble hit the ground, as measured from the center of the hemisphere? You may use the variable θ_1 in your answer. (If you do not solve part (a), you can still attempt this problem by writing your answer in terms of this variable.) (3pts).

   (c) Now assume that the force of friction between marble and the hemisphere is μN, where N is the normal force between the marble and the hemisphere. Calculate the angle θ_2 at which the marble will no longer roll without slipping. (4pts).



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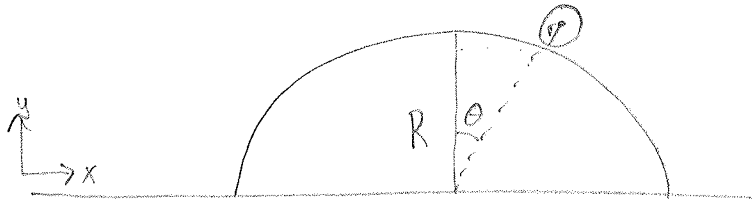
Classical #1



a) Assume the marble rolls w/o slipping. Find θ_1, where marble looses contact w/ the hemisphere

* Normal force is 0 when marble leaves surface

$$mg(R+r) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgy$$

$x = (R+r)\sin\theta$

$y = (R+r)\cos\theta$

$$mg(R+r) = \frac{1}{2}mv^2 + \frac{1}{5}mr^2\left(\frac{v}{r}\right)^2 + mg(R+r)\cos\theta_1$$

$$mg(R+r) = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 + mg(R+r)\cos\theta_1$$

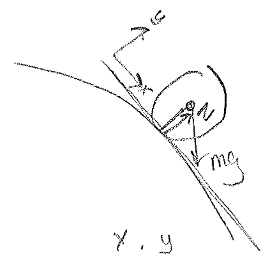
$$g(R+r) = \frac{3}{5}v^2 + g(R+r)\cos\theta_1$$

$$g(R+r) = \frac{3}{5}gR\cos\theta_1 + g(R+r)\cos\theta_1$$

$$\frac{g(R+r)}{\frac{3}{5}g + g(R+r)} = \cos\theta_1$$

$$\frac{R+r}{\frac{3}{5}+R+r} = \cos\theta_1$$

$$\Rightarrow \theta_1 = \cos^{-1}\left(\frac{R+r}{R+r+3/5}\right)$$



$x,\ y$

$\langle \parallel, \perp \rangle$

$F_N = \langle 0, N \rangle$

$F_G = mg\langle \sin\theta, -\cos\theta \rangle$

$\Rightarrow ma_x = mg\sin\theta$

$ma_y = N + mg\cos\theta$

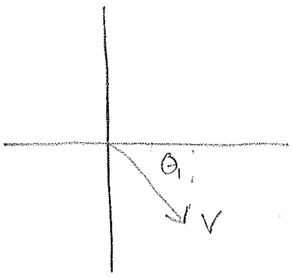
* at θ_1, $N = 0$, $a_y = \frac{v^2}{R}$

$ma_y = mg\cos\theta_1$

$\frac{v^2}{R} = g\cos\theta_1$

$\Rightarrow v = \sqrt{gR\cos\theta_1}$

b) Where will the marble hit the ground, measured from the center of the hemisphere?



$V_x = v\cos\theta$

$V_y = v\sin\theta$

$y_i = (R+r)\cos\theta_1 \qquad V_i = V$

$y_f = 0 \qquad a = g$

$$0 = -\tfrac{1}{2}gt^2 + vt + (R+r)\cos\theta_1$$

$$0 = -\tfrac{1}{2}gt^2 - \sqrt{gR\cos\theta_1}\,\sin\theta_1\, t + (R+r)\cos\theta_1$$

$$\Rightarrow \quad t = \frac{+\sqrt{gR\cos\theta_1}\,\sin\theta_1 \pm \sqrt{gR\cos\theta_1\sin^2\theta_1 - 4\left(-\tfrac{1}{2}g\right)(R+r)\cos\theta_1}}{-g}$$

$$= \frac{\sqrt{gR\cos\theta_1}\,\sin\theta_1 \pm \sqrt{gR\cos\theta_1(\sin^2\theta_1 + 2) + 2rg\cos\theta_1}}{-g}$$

$$= \frac{\sqrt{gR\cos\theta_1}\,\sin\theta_1 - \sqrt{gR\cos\theta_1(\sin^2\theta_1 + 2) + 2rg\cos\theta_1}}{-g}$$

(need (−) root to make overall time positive)

$x_i = (R+r)\sin\theta_1 \qquad v = \sqrt{gR\cos\theta_1}\,\cos\theta_1$

$x_f = ??$

$$x_f = vt + x_i$$

$$= gR\cos^{3/2}\theta_1 t + (R+r)\sin\theta_1$$

$$= gR\cos^{3/2}\theta_1\left(\frac{1}{-g}\left[\sqrt{gR\cos\theta_1}\,\sin\theta_1 - \sqrt{gR\cos\theta_1(\sin^2\theta_1 + 2) + 2rg\cos\theta_1}\right)\right) + (R+r)\sin\theta_1$$

2. Consider a point particle of mass m moving under the influence of a central force:

$$\vec{F}(\vec{r}) = -\frac{k}{r^n}\,\hat{r}$$

where n is an integer greater than one ($n = 2, 3, \ldots$), the variable r is the distance from the origin of the force ($r \equiv |\vec{r}|$) and $\hat{r}$ is a unit vector in the radial direction. In this problem, we will examine when circular orbits are stable for such a central force.

(a) Calculate potential energy of this force. Choose the zero of the potential to be at infinity ($r = \infty$). (1pt)

(b) Show that the angular momentum about the origin, L, is conserved. (You may use the Newtonian, Lagrangian, or Hamiltonian formulations of the problem). (2pts)

(c) Write an expression for the total energy of the particle E as a function of r, dr/dt, L, k, and n. (1pt)

(d) Assume the particle is moving in a circular orbit about the origin, so that $dr/dt = 0$. Calculate the radius of the orbit and the velocity of the particle as a function of the above variables. (3pts)

(e) When is this circular orbit stable? (Hint: look at dE/dr and d^2E/dr^2.) (3pts)

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Classical #2

* Consider a particle of mass m under the influence of a central force

$$\vec{F}(\vec{r}) = -\frac{k}{r^n}\hat{r} \quad \text{where} \quad n \in [\mathbb{Z} > 1].$$

a) Calculate the potential energy of this force

$$U(\vec{r}) = -\int \vec{F} \cdot d\vec{r}$$

$$= -\int_{\infty}^{r} -\frac{k}{r^n}\hat{r}\, d\vec{r}$$

$$= +\int_{\infty}^{r} \frac{k}{r^n}\, dr \qquad \text{(assuming spherical coordinates)}$$

$$= \int_{\infty}^{r} k r^{-(n)}\, dr$$

$$= -k\frac{1}{n+1} r^{-n+1}\Big|_{\infty}^{r}$$

$$= -\frac{k}{n+1} r^{-n+1}$$

b) Show that angular momentum about the origin is conserved

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{k}{n+1}r^{-n+1}, \qquad v = \langle \dot{r}, r\dot{\theta}, r\dot{\varphi}\sin\theta \rangle$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\varphi}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{n+1}r^{-n+1}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

$$0 = \frac{d}{dt}(mr^2\sin^2\theta\,\dot{\varphi})$$

$$\Rightarrow mr^2\sin^2\theta\,\dot{\varphi} = \text{const.} = L \quad \rightarrow L \text{ is conserved}$$

c) Write an expression for the total energy of the particle (E) as a function of: $r, \dot{r}, L, k, n$

$$E = T + U$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2\sin^2\theta} + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{n-1}r^{-n+1}$$

$$* \text{ if } \theta = \frac{\pi}{2}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{n-1}r^{-n+1}$$

d) Assume the particle is moving in a circular orbit about the origin. Find the radius of the orbit and the velocity of the particle as a function of the above variables

$$\mathcal{L} = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2 \sin^2\theta\, \dot{\varphi}^2 + \frac{1}{2} mr^2 \dot{\theta}^2 + \frac{k}{n+1} r^{-n+1} = \frac{L^2}{2mr^2} + \frac{k}{n-1} r^{-n+1}$$

$$\cancel{\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}}$$

0 b/c r is constant

$$\cancel{mr\sin^2\theta\,\dot{\varphi}^2 + mr\dot{\theta}^2 - kr^n = \frac{d}{dt}[m\dot{r}]}$$

$$mr\sin^2\theta\,\dot{\varphi}^2 + mr\dot{\theta}^2 - kr^n = 0$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{-L^2}{mr^3} + \frac{k}{r^n}$$

$$\Rightarrow r = \left(\frac{km}{L^2}\right)^{\frac{1}{n-3}}$$

$$\Rightarrow L = mr^2\dot{\varphi} \rightarrow \dot{\varphi} = \frac{L}{mr^2}$$

e) When is the orbit stable?

$$\frac{dE}{dr} = \frac{-L^2}{mr^3} + \frac{k}{r^n}$$

$$\frac{d^2E}{dr^2} = \frac{3L^2}{mr^4} + \frac{-nk}{r^{n+1}}$$

$$= \frac{3L^2}{m}\left(\frac{km}{L^2}\right)^{-4/(n-3)} + nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)}{n-3}} > 0$$

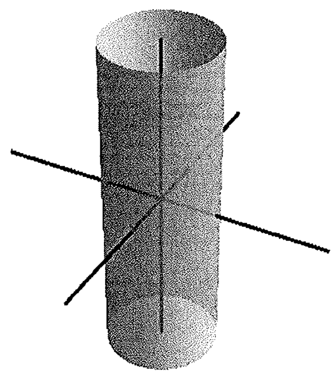
$$\frac{3L^2}{m}\left(\frac{km}{L^2}\right)^{\frac{-4}{n-3}} > +nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)}{n-3}}$$

$$\frac{3L^2}{m} > +nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)+4}{n-3}}$$

$$\frac{3\cancel{L^2}}{\cancel{m}} > +\frac{\cancel{L^2}nk}{\cancel{km}}$$

$$3 > +n$$

3. A particle of mass m is constrained to move on an infinitely long cylinder of radius a. The center of the cylinder is oriented along the z-axis, as shown. An attractive central potential, $U(r) = U(\sqrt{a^2 + z^2})$, is located at the origin, where r is the radius is spherical coordinates.



(a) Write down the Lagrangian for the problem. (1pt)

(b) From the Lagrangian, explicitly derive the Hamiltonian for the particle. (2pts)

(c) Is angular momentum about the z-axis conserved? Prove your answer. (2pts)

(d) Under what conditions is motion in the z-direction bounded? (2pts)

(e) Assume that the potential is $U(r) = \frac{1}{2}\alpha r^2$. Solve the equations of motion, and reduce the problem to quadrature. (3pts)

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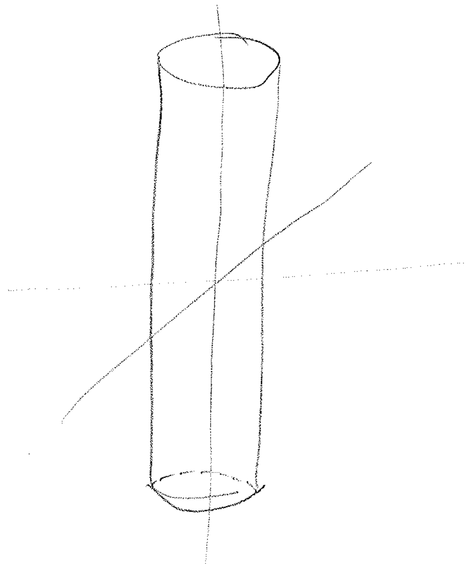
Classical #3



* Particle of mass m constained to move on infinetely long cylinder of radius a; cylinder oriented along z axis

* Attractive central located at origin,

$U(r) = U(\sqrt{a^2+z^2})$, where r is radius in spherical

a) Find the Lagrangian

$x = \rho\cos\varphi$ $\quad \dot{x} = \dot{\rho}\cos\varphi - \rho\sin\varphi\dot{\varphi}$

$y = \rho\sin\varphi$ $\quad \dot{y} = \dot{\rho}\sin\varphi + \rho\cos\varphi\dot{\varphi}$

$$\mathcal{L} = T - U$$

$$= \frac{1}{2}mv^2 - U(\sqrt{a^2+z^2})$$

$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - U(\sqrt{a^2+z^2})$$

$$= \frac{1}{2}m(\dot{\rho}^2\cos^2\varphi - 2\rho\cos\varphi\sin\varphi\dot{\rho}\dot{\varphi} + \rho^2\sin^2\varphi\dot{\varphi}^2 + \dot{\rho}^2\sin^2\varphi + 2\rho\sin\varphi\cos\varphi\dot{\rho}\dot{\varphi} + \rho^2\cos^2\varphi\dot{\varphi}^2 + \dot{z}^2) - U$$

$$= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - U(\sqrt{a^2+z^2})$$

* Assume arbitrary central potential Ar^n

$$= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - A(\sqrt{a^2+z^2})^n$$

b) From the Lagrangian, derive the Hamiltonian

$$H = \sum_i p_i\dot{q}_i - \mathcal{L}$$

$$P_\rho = \frac{\partial\mathcal{L}}{\partial\dot{\rho}} = m\dot{\rho}$$

$$P_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = m\rho^2\dot{\varphi}$$

$$P_z = \frac{\partial\mathcal{L}}{\partial\dot{z}} = m\dot{z}$$

$$\Rightarrow H = m\dot{\rho}^2 + m\rho^2\dot{\varphi}^2 + m\dot{z}^2 - \left[\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2) - A(\sqrt{a^2+z^2})^n\right]$$

$$= \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\varphi}^2 + \frac{1}{2}m\dot{z}^2 + A(\sqrt{a^2+z^2})^n$$

c) Is angular momentum about z-axis conserved?

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right]$$

$$0 = \frac{d}{dt}\left[m\rho^2 \dot{\varphi}\right]$$

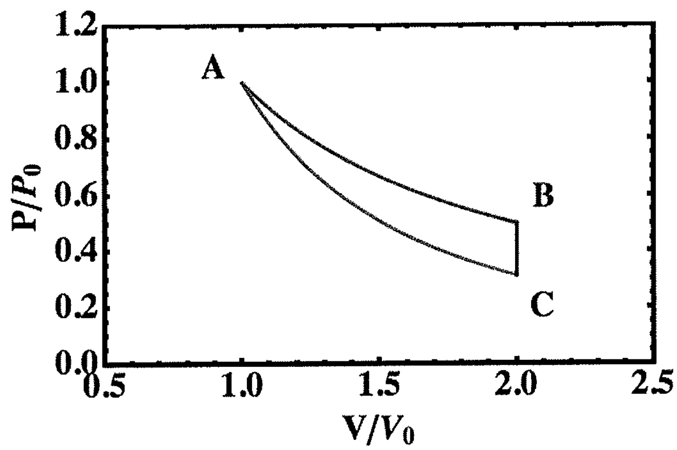
$$\Rightarrow m\rho^2\dot{\varphi} = \text{const.} = L \checkmark$$

d) Under what conditions is motion in the z direction bounded?

$\Rightarrow$ Motion is bounded when $E_T < U \rightarrow T < 0$

Statistical Mechanics

4. Consider an ideal monatomic gas used as the working fluid in a thermodynamic cycle. The number of particles is n_0. It follows a cycle consisting of one adiabat, one isochore and one isotherm, as shown below.



(a) Calculate the pressure, temperature, and volume at each corner of the cycle, A, B, and C, expressing your answer in terms of P_0, V_0, n_0 and perhaps R, the ideal gas constant. Note that point A the pressure is P_0 and the volume is V_0. (3pts)

(b) Calculate the work done on the system, the heat into the system and the change in the internal energy of the system for each process step. (4.5pts)

(c) What direction around the ycle must the system follow to be used as a functional heat engine? (1/2pt)

(d) What is the efficiency of the cycle, run as an engine? (1pt)

(e) What is the efficiency of an ideal Carnot engine run between reservoirs B and C? (1pt)

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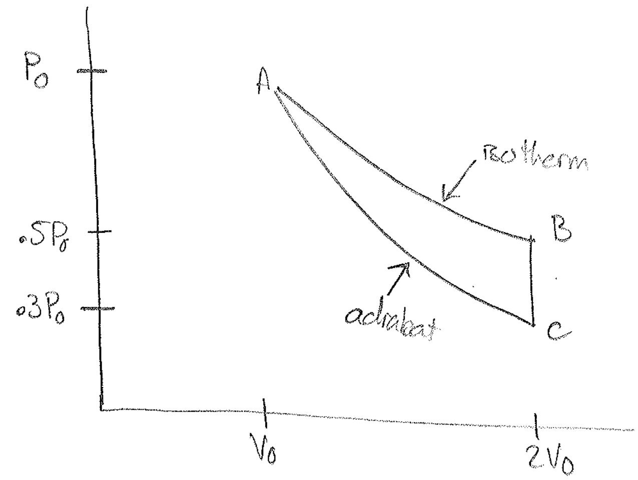
Stat Mech #1



* cycle consists of adiabat, isochore, and isotherm

* gas is ideal + monatomic

$\Rightarrow$ $C_V = \frac{3}{2}R$, $C_P = \frac{5}{2}R$ $\rangle$ $\gamma = 5/3$

a) Find P, V, and T at each corner of the cycle in terms of P_0, V_0, n_0, and R

$P_A = P_0$	$P_B = .5P_0$	$P_C = .3P_0$
$V_A = V_0$	$V_B = 2V_0$	$V_C = 2V_0$
$T_A = \frac{P_0V_0}{nk_B}$	$T_B = \frac{P_0V_0}{nk_B}$	$T_C = \frac{.6P_0V_0}{nk_B}$

$PV = nk_BT$

$\Rightarrow T = \frac{PV}{nk_B}$

* For $A \to C$ (adiabat)

$P_AV_A^\gamma = P_CV_C^\gamma$

$T_AV_A^{\gamma-1} = T_CV_C^{\gamma-1}$

$\frac{P_AV_A}{T_A} = \frac{P_CV_C}{T_C}$

$\frac{P_AV_A}{T_A} = \frac{\frac{P_AV_A^\gamma}{V_C^\gamma}V_C}{\frac{T_AV_A^{\gamma-1}}{V_C^{\gamma-1}}}$

$V_A = \frac{V_A^\gamma V_C^{1-\gamma}}{V_A^{\gamma-1}V_C^{1-\gamma}}$

b) Find the work done on the system, the heat into the system, and the change in internal energy during each step of the cycle

$W_{B\to C} = 0$

$Q_{B\to C} = nC_V\Delta T$

$= \frac{N}{N_{av}}\left(\frac{3}{2}R\right)\left(\frac{2}{5}\frac{P_0V_0}{nk_B}\right)$

$= \frac{3}{5}P_0V_0$

$\Delta E = Q = \frac{3}{5}P_0V_0$

$W_{C\to A} = \frac{P_fV_f - P_iV_i}{1-\gamma}$

$= \frac{.6P_0V_0 - P_0V_0}{1 - 5/3}$

$= \frac{-2/5}{-2/3}P_0V_0$

$= -3/5\,P_0V_0$

$Q_{C\to A} = 0$

$\Delta E = -W = 3/5\,P_0V_0$

b) $W_{A \to B} = n k_B T \ln\left(\frac{V_B}{V_A}\right)$

$= n k_B \frac{P_0 V_0}{n k_B} \ln\left(\frac{2V_0}{V_0}\right)$

$= P_0 V_0 \ln(2)$

$Q = W = P_0 V_0 \ln(2)$

$\Delta E = 0$ b/c isotherm

c) Which direction does the heat engine flow?

CW

d) What is the efficiency of the heat engine?

$\eta = 1 - \left|\frac{Q_{out}}{Q_{in}}\right|$

$= 1 - \frac{.6 P_0 V_0}{P_0 V_0 \ln(2)}$

$= 1 - \frac{.6}{\ln(2)}$

5. Consider the quantum mechanical linear rotator. It has energy levels

$$E_J = \frac{\hbar^2}{2I} J\,(J+1)$$

where I is the moment of inertia and J is the angular momentum quantum number, $J = 0, 1, 2, \ldots$. Each energy level is $(2J+1)$-fold degenerate.

(a) In the low temperature limit ($\hbar^2/2I \gg kT$) determine approximate expressions for:

i. The rotation partition function. (2pts)

ii. The internal energy. (1pt)

iii. The specific heat. (1pt)

(b) In the high temperature limit ($\hbar^2/2I \ll kT$) determine approximate expressions for:

i. The rotation partition function. (2pt)

ii. The internal energy. (1pt)

iii. The specific heat. (1pt)

(c) How do the quantum results compare with the equipartition theorem for a classical rotator with two transverse degrees of freedom? (2pts)

6. Consider the "bogon," a spin 5/2 fermion with the charge of an electron but with a dispersion relationship

$$E = cp^3.$$

where $p \equiv |\vec{p}|$ Assume that your bogons are confined in a three dimensional sample and are non-ineracting.

(a) Working in the grand canonical ensemble, determine the density, $\rho = \langle N \rangle / V$, as a function of the chemical potential, μ (or the fugacity, $z \equiv e^{\beta\mu}$), T, and V. (3pts)

(b) What is the bogonic Fermi energy (μ at $T = 0$) as a function of their density? (3pts) (*Hint:* This should not involve any complicated integrals).

(c) Derive a series expansion in z for the grand canonical free entropy, $\Xi = \frac{PV}{kT} = \log \mathcal{Z}$, where $\mathcal{Z}$ is the grand canonical partition function. (4pts)