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PROBLEM 6

1. (4 pts) Show that the formal solution of the plane-parallel radiative transfer equation can be written:

$$I_\nu(\tau_1, \mu) = I_\nu(\tau_2, \mu)e^{-(\tau_2-\tau_1)/\mu} + \frac{1}{\mu}\int_{\tau_1}^{\tau_2} S_\nu(t)e^{-(t-\tau_1)/\mu}\, d\cancel{\mu}\tau \qquad (7)$$

where $S_\nu(t)$ is the source function, $\tau_{1,2}$ are optical depth points in the atmosphere, and μ is the cosine of the angle of the ray.

2. (2 pts) Apply Eqn. 7 to an arbitrary point in the atmosphere of a semi-infinite slab to find:

$$I_\nu(\tau, \mu) = \int_\tau^\infty S_\nu(t)e^{-(t-\tau)/\mu}\, dt/\mu \quad \text{for } 0 \le \mu \le 1 \qquad (8)$$

$$I_\nu(\tau, \mu) = \int_0^\tau S_\nu(t)e^{-(\tau-t)/(+\mu)}\, dt/(+\mu) \quad \text{for} -1 \le \mu \le 0 \qquad (9)$$

3. (2pts) Integrate Eqns 8 and 9 over angle to find

$$J_\nu(\tau) = 1/2\left[\int_\tau^\infty dt\, S_\nu(t)\int_1^\infty dw\, e^{-w(t-\tau)}/w + \int_0^\tau dt\, S_\nu(t)\int_1^\infty dw\, e^{-w(\tau-t)}/w\right] \qquad (10)$$

These integrals are of standard form (the first exponential integral):

$$E_1(x) = \int_1^\infty e^{-xt}/t\, dt$$

4. (1 pt) Show that in terms of E_1, J may be written:

$$J_\nu(\tau) = 1/2\int_\tau^\infty dt\, S_\nu(t)E_1(|t-\tau|)$$

5. (1 pt) Explain the nature of this final operator.

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#6

a) $$\frac{dI}{ds} = \mu \frac{dI}{dz} = -\kappa_\nu I_\nu + \eta_\nu$$

$$\mu \frac{dI}{-\kappa_\nu dz} = +I_\nu - \frac{\eta_\nu}{\kappa_\nu}$$

$$\mu \frac{dI}{d\tau} = I - S$$

$$\mu \frac{dI}{d\tau} - I = -S$$

$$\frac{dI}{d\tau} - \frac{1}{\mu} I = -\frac{1}{\mu} S$$

$$\frac{dI}{d\tau} e^{-\tau/\mu} - \frac{I}{\mu} e^{-\tau/\mu} = -\frac{1}{\mu} S e^{-\tau/\mu}$$

$$\frac{d(I e^{-\tau/\mu})}{d\tau} = \frac{-1}{\mu} S e^{-\tau/\mu}$$

$$\int_{\tau_1}^{\tau_2} \frac{d(I e^{-\tau/\mu})}{d\tau} = \int_{\tau_1}^{\tau_2} \frac{-1}{\mu} S e^{-\tau/\mu}$$

$$I e^{-\tau/\mu} \Big|_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} \frac{-S}{\mu e^{\tau/\mu}}$$

$$I(\tau,\mu) e^{-\tau_2/\mu} - I(\tau,\mu) e^{-\tau_1/\mu} = \int_{\tau_1}^{\tau_2} \frac{-S}{\mu} e^{-\tau/\mu}$$

$$I(\tau_1,\mu) e^{-\tau_1/\mu} = I(\tau_2,\mu) e^{-\tau_2/\mu} + \int_{\tau_1}^{\tau_2} \frac{S}{\mu} e^{-\tau/\mu}$$

$$I(\tau_1,\mu) = I(\tau_2,\mu) \exp(-(\tau_2-\tau_1)/\mu) + \frac{1}{\mu}\int_{\tau_1}^{\tau_2} S \exp[-(t-\tau_1)/\mu\, dt$$

b) * At an arbitrary point, incoming rays ($\mu < 0$); outgoing rays ($\mu > 0$)

$\Rightarrow$ Outgoing rays $\tau_1 = \tau$, $\tau_2 = \infty$

$$I(\tau,\mu) = I \exp[\cancel{-(\infty-\tau)/\mu}]^{0} + \frac{1}{\mu}\int_{\tau}^{\infty} S \exp[-(t-\tau)/\mu]\,dt$$

$$= \frac{1}{\mu}\int_{\tau}^{\infty} S \exp[-(t-\tau)/\mu]\,d\tau$$

#6 (cont.)

b) $\Rightarrow$ for incoming rays $\tau_1 = 0 \quad \tau_2 = \tau$

$$I(\tau,\mu) = \int_0^{\tau} S(t)\exp[-(t-\tau)/-\mu]\frac{dt}{-\mu} + I_\nu(\infty,\cancel{-\mu})\cancel{e^{-(\infty-0)/\mu}} \to 0$$

c) $$J_\nu(\tau) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau,\mu)\,d\mu$$

$$= \frac{1}{2}\left[\int_{-1}^{0} I_\nu(\tau,\mu)d\mu + \int_0^1 I_\nu d\mu\right]$$

$$= \frac{1}{2}\left[\int_{-1}^{0}\int_0^{\tau} S_\nu(t)e^{-(\tau-t)/(-\mu)}dt\cdot\frac{1}{-\mu}d\mu + \int_0^1\int_\tau^{\infty} S_\nu e^{-(t-\tau)/\mu}\frac{dt}{\mu}d\mu\right]$$

* assuming an isotropic source function

$$= \frac{1}{2}\left[\int_\tau^{\infty}dt\,S_\nu\int_0^1\exp[-(t-\tau)/\mu]\frac{1}{\mu}d\mu + \int_0^{\tau}dt\,S_\nu\int_{-1}^{0}\exp[-(\tau-t)/+\mu]\frac{d\mu}{+\mu}\right]$$

* let $x = t-\tau \qquad\qquad x' = \tau - t$

$y = \frac{1}{\mu} \rightarrow dy = \frac{-1}{\mu^2}d\mu \rightarrow y^{-2}dy = d\mu \qquad y = \frac{-1}{\mu}$

d) $$= \frac{1}{2}\left[\int_\tau^{\infty}S_\nu dt\int_{\infty}^{1}-\exp[-xy]\frac{dy}{y} + \int_0^{\tau}dt\,S_\nu\int_{+1}^{+\infty}\exp[-xy]\frac{dy}{y}\right]$$

$$= \frac{1}{2}\left[\int_\tau^{\infty}dt\,S_\nu\int_1^{\infty}\exp[-xy]\frac{dy}{y} + \int_0^{\tau}dt\,S_\nu\int_1^{\infty}\exp[-xy]\frac{dy}{y}\right]$$

$$= \frac{1}{2}\left[\int_\tau^{\infty}dt\,S_\nu\,E_1(t-\tau) + \int_0^{\tau}dt\,S_\nu\,E_1(\tau-t)\right]$$

$$= \frac{1}{2}\int_0^{\tau_{max}}dt\,S_\nu\,E_1(|t-\tau|)$$

e) Known as Λ - operator