Lecture 4
Moving Atmospheres
Formal Solution Along Characteristics
1D Gustafsson et al.
Formal Solution Along Characteristics

3D Cartoon

a

b
Mihalas Method
Want to work in the “Co-moving Frame”

The emissivity $\eta$ and opacity $\chi$ depend upon the angle as well as frequency in the inertial frame because of Doppler shifts, aberration, and advection induced by the motion of the material in the frame. The goal of this section is to rewrite equation (2.1) with all material and radiation-field quantities measured in the comoving frame; in that frame both the opacity and emissivity are isotropic, and can be related directly to proper variables that specify the thermodynamic state of the material. Furthermore, in that frame both the scattering properties of the material and the rate equations describing the mechanisms populating and depopulating its internal energy states are most easily defined. . .

In our analysis we shall, however, leave both the space and time variables in the inertial frame, as this is the only frame in which synchronism of clocks can be effected, and further this choice obviates the need to develop a metric for accelerated fluid frames (Castor 1972) which in general can only be done approximately. With this choice of frame we can write exact Lorentz transformations for all the material and radiation-field quantities and use these to develop a transfer equation that will remain valid for relativistic flow in the limit as $v/c \rightarrow 1$. . .
II. GENERAL EQUATIONS

a) Inertial-Frame Equation

The inertial-frame transfer equation for a spherically symmetric medium is

\[ \frac{1}{c} \frac{\partial I(\mu, v)}{\partial t} + \mu \frac{\partial I(\mu, v)}{\partial r} + \frac{(1 - \mu^2)}{r} \frac{\partial I(\mu, v)}{\partial \mu} = \eta(\mu, v) - \chi(\mu, v) I(\mu, v), \]

where \( r, t, \mu, \) and \( v \) are, respectively, the radial coordinate, time, the angle-cosine describing the direction of photon propagation, and frequency, all measured in the inertial frame, which we choose to be the frame in which the center of mass of the star and the observer at infinity are both at rest. The dependence of the specific intensity \( I \) upon \( r \) and \( t \) has been suppressed in the notation, but the \( (\mu, v) \) dependence will be written out explicitly to emphasize changes between the inertial and comoving frames. The emissivity \( \eta \) and opacity \( \chi \) depend upon angle as well as frequency in the inertial frame because of the effects of Doppler shifts, aberration, and advection induced by the motion of the material in the frame.

The goal of this section is to rewrite equation (2.1) with all material and radiation-field quantities measured in the comoving frame; in that frame both the opacity and emissivity are isotropic, and can be related directly to proper variables that specify the thermodynamic state of the material. Furthermore, in that frame both the scattering properties of the material and the rate equations describing the mechanisms populating and depopulating its internal energy states are most easily defined. Furthermore, in the comoving frame the frequency-bandwidth in line transfer calculations can be restricted to just the band required to cover the intrinsic width of the line profile measured by an observer at rest with respect to the material. (See Papers I, IV, and V for further discussion of these points.)

In our analysis we shall, however, leave both the space and time variables in the inertial frame, as this is the only frame in which synchronism of clocks can be effected, and further, this choice obviates the need to develop a metric for accelerated fluid frames (Castor 1972) which in general can be done only approximately. With this choice of frame we can write exact Lorentz transformations for all the material and radiation-field quantities and use these to develop a transfer equation that will remain valid for relativistic flow in the limit as \( v/c \to 1 \). The approach adopted here is conceptually identical to that used in special-relativistic formulations of the fluid equations in which material properties are expressed in the proper (comoving) frame and spacetime coordinates and flow velocities are measured in the inertial frame.
b) Lorentz Transformation of Material and Radiation-Field Properties

As was shown by Thomas (1930) (see also Mihalas 1978, pp. 493–496), the result of demanding Lorentz invariance of the transfer equation is that the specific intensity, opacity, and emissivity measured in the inertial frame are related to their comoving-frame counterparts by the expressions

\[ I(\mu, v) = (v/v_0)^3 I_0(\mu_0, v_0) , \]

\[ \chi(\mu, v) = (v_0/v) \chi_0(v_0) , \]

and

\[ \eta(\mu, v) = (v/v_0)^2 \eta_0(v_0) , \]

where all quantities with suffix zero are measured in the comoving frame, and we have noted the isotropy of \( \chi_0 \) and \( \eta_0 \) in that frame. The relationships between inertial- and comoving-frame angles and frequencies follow immediately from Lorentz transformation of the photon four-momentum:

\[ v_0 = v \gamma(1 - \beta \mu) , \]

\[ \mu_0 = \frac{\mu - \beta}{1 - \beta \mu} , \]

and

\[ (1 - \mu_0^2)^{1/2} = \frac{(1 - \mu^2)^{1/2}}{\gamma(1 - \beta \mu)} ; \]

or, reciprocally,

\[ v = v_0 \gamma(1 + \beta \mu_0) , \]

\[ \mu = \frac{\mu_0 + \beta}{1 + \beta \mu_0} , \]
\[(1 - \mu^2)^{1/2} = \frac{(1 - \mu_0^2)^{1/2}}{\gamma(1 + \beta \mu_0)}. \quad (2.6c)\]

Here \(\beta \equiv v/c\), and \(\gamma \equiv (1 - \beta^2)^{-1/2}\) where \(v\) is the radial flow-velocity of the matter. Substituting equations (2.2)–(2.4) into (2.1), we find

\[
\left(\frac{v}{v_0}\right) \left[\frac{1}{c} \frac{\partial I_0(\mu_0, v_0)}{\partial t} + \mu \frac{\partial I_0(\mu_0, v_0)}{\partial r} + \frac{(1 - \mu^2)}{r} \frac{\partial I_0(\mu_0, v_0)}{\partial \mu}\right] - 3 \left(\frac{v}{v_0^2}\right) \left[\frac{1}{c} \frac{\partial v_0}{\partial t} + \mu \frac{\partial v_0}{\partial r} + \frac{(1 - \mu^2)}{r} \frac{\partial v_0}{\partial \mu}\right] I_0(\mu_0, v_0) \\
= \eta_0(v_0) - \chi_0(v_0) I_0(\mu_0, v_0). \quad (2.7)
\]

This equation is essentially equivalent to equation (7) of Prokof'ev (1962), who does not, however, reduce it to a useful form, but rather proceeds immediately to angle- and frequency-integrated moment equations (cf. \S\ IIe below). We now wish to rewrite equation (2.7) so that, with the exception of the inertial-frame spacetime coordinates, only comoving-frame quantities appear.

In equation (2.7), all derivatives that appear presume that the inertial-frame frequency \(\nu\) and (with the exception, of course, of \(\partial/\partial \mu\)) the inertial-frame angle-cosine \(\mu\) are held constant. Because the fluid velocity changes as a function of space and time, the corresponding comoving-frame quantities \(v_0\) and \(\mu_0\) are not constant, and their variations must be taken into account explicitly (as already seen by the appearance of the second square bracket in eq. [2.7]). The derivatives of \(v_0\) can be computed straightaway using equations (2.5) and (2.6). To calculate the derivatives of \(I_0(\mu_0, v_0)\) we apply the chain rules...
\[
\frac{\partial}{\partial t_{\mu\nu}} = \frac{\partial}{\partial t_{\mu_0\nu_0}} + \frac{\partial \mu_0}{\partial t_{\mu\nu}} \frac{\partial}{\partial \mu_0} + \frac{\partial v_0}{\partial t_{\mu\nu}} \frac{\partial}{\partial v_0},
\]  
(2.8a)

\[
\frac{\partial}{\partial r_{\mu\nu}} = \frac{\partial}{\partial r_{\mu_0\nu_0}} + \frac{\partial \mu_0}{\partial r_{\mu\nu}} \frac{\partial}{\partial \mu_0} + \frac{\partial v_0}{\partial r_{\mu\nu}} \frac{\partial}{\partial v_0},
\]  
(2.8b)

\[
\frac{\partial}{\partial \mu_{\nu\nu}} = \frac{\partial \mu_0}{\partial \mu_{\nu\nu}} \frac{\partial}{\partial \mu_0} + \frac{\partial v_0}{\partial \mu_{\nu\nu}} \frac{\partial}{\partial v_0},
\]  
(2.8c)
where, by repeated use of equations (2.5) and (2.6), one finds

\[
\frac{\partial \mu_0}{\partial t} = -\gamma^2 \frac{\partial \beta}{\partial t} (1 - \mu_0^2),
\]

(2.9a)

\[
\frac{\partial v_0}{\partial t} = -\gamma^2 \frac{\partial \beta}{\partial t} \mu_0 v_0,
\]

(2.9b)

\[
\frac{\partial \mu_0}{\partial r} = -\gamma^2 \frac{\partial \beta}{\partial r} (1 - \mu_0^2),
\]

(2.10a)

\[
\frac{\partial v_0}{\partial r} = -\gamma^2 \frac{\partial \beta}{\partial r} \mu_0 v_0,
\]

(2.10b)

and

\[
\frac{\partial \mu_0}{\partial \mu} = \gamma^2 (1 + \beta \mu_0)^2,
\]

(2.11a)

\[
\frac{\partial v_0}{\partial \mu} = -\beta \gamma^2 (1 + \beta \mu_0) v_0.
\]

(2.11b)
c) Comoving-Frame Transfer Equation

Using equations (2.5), (2.6) and (2.8)–(2.11) in equation (2.7), one finds, after a modest amount of straightforward algebra, the desired transfer equation:

\[
\frac{\gamma}{c} (1 + \beta \mu_0) \frac{\partial I_0(\mu_0, v_0)}{\partial t} + \gamma (\mu_0 + \beta) \frac{\partial I_0(\mu_0, v_0)}{\partial r}
\]

\[
+ \gamma (1 - \mu_0^2) \left[ \frac{(1 + \beta \mu_0)}{c} \frac{\gamma^2}{c} (1 + \beta \mu_0) \frac{\partial \beta}{\partial t} - \gamma^2 (\mu_0 + \beta) \frac{\partial \beta}{\partial r} \right] \frac{\partial I_0(\mu_0, v_0)}{\partial \mu_0}
\]

\[
- \gamma \left[ \frac{\beta (1 - \mu_0^2)}{r} + \frac{\gamma^2}{c} \mu_0 (1 + \beta \mu_0) \frac{\partial \beta}{\partial t} + \gamma^2 \mu_0 (\mu_0 + \beta) \frac{\partial \beta}{\partial r} \right] \frac{\partial I_0(\mu_0, v_0)}{\partial v_0}
\]

\[
+ 3\gamma \left[ \frac{\beta (1 - \mu_0^2)}{r} + \frac{\gamma^2 \mu_0}{c} (1 + \beta \mu_0) \frac{\partial \beta}{\partial t} + \gamma^2 \mu_0 (\mu_0 + \beta) \frac{\partial \beta}{\partial r} \right] I_0(\mu_0, v_0)
\]

\[= \eta_0(v_0) - \chi_0(v_0) I_0(\mu_0, v_0). \quad (2.12)\]

If we take the limit \((1/r) \to 0\), we recover the corresponding equation for planar geometry. If we consider the limit \((v/c) \ll 1, \gamma \equiv 1, (\partial \beta/\partial t) \equiv 0\), and time intervals appropriate to fluid-flow, i.e., \(\Delta t \sim \Delta r/v\), then to \(O(v/c)\) we recover Castor’s (1972) equation (21) when we recall that his \((\partial/\partial t)\) denotes the Lagrangian derivative \((\partial/\partial t + v \partial/\partial r)\).
Characteristic Equations

\[ \frac{dr}{ds_M} = \gamma (\mu + \beta), \]
\[ \frac{d\mu}{ds_M} = \gamma (1 - \mu^2) \left[ 1 + \mu \frac{\beta}{r} - \gamma^2 (\mu + \beta) \frac{d\beta}{dr} \right]. \]
Characteristics are Curved

Fig. 1.—Characteristic rays of equations (3.4)—(3.7) for an expanding envelope with core radius $s_c = 1$ (inner semicircle) and surface radius $R = 11$ (outer semicircle). The velocity law is linear (left panels) or quadratic (right panels) as given by eq. (4.2). Each figure is labeled with the value of $\frac{V}{C_{\text{max}}} = \frac{v}{c_{\text{max}}}$, of the maximum expansion velocity at the outer boundary. These results, which follow from exact equations, may be compared with those in Fig. 1 of Paper III, which were obtained from equations that are correct only to $O(c)$; the earlier results are at least qualitatively correct for $\frac{V}{C_{\text{max}}} \leq 0.5$, but yield incorrect ray forms at higher speeds.
3.2. Comoving-frame transport equation

Equations (19) and (20) are the main results of this problem. We recall that the invariant transport equation is

$$e_i^a p^b J_{i}^a - \Omega_{b c}^a p^b \frac{\partial J^a}{\partial p^c} = e - a J.$$  \hspace{3cm} (21)

In working out the partial derivatives of $J$ with respect to the momentum components, we note that $J$ can be considered to be a function of three of them, since $J$ is defined only on the null surface in momentum space. We choose the three space-like tetrad components, so the momentum derivative comes out in terms of $\Omega_{b c}^a$. The derivative of $v_0$ itself can be expressed using $\Omega_{b c}^a$.

In terms of ordinary variables the transport equation becomes

$$\frac{dt}{ds} \frac{\partial J^0}{\partial t} + \frac{dr}{ds} \cdot \nabla \cdot J^0 + \frac{d\mathbf{p}}{ds} \cdot \nabla p^0 - \frac{2}{v_0} \frac{dv_0}{ds} f^0 = f^0 - k^0 l^0.$$  \hspace{3cm} (22)

The coefficients $dt/ds$ and $dr/ds$ are the time and space components of $p^\mu = e_i^a p^a$, divided by $v_0 = a / k^0$ for convenience, so they are

$$\frac{dt}{ds} = \gamma_u \left( 1 + u \cdot n_0 / c \right) / c \quad \text{and} \quad \frac{dr}{ds} = \gamma_u u / c + n_0 + (\gamma_u - 1) u \cdot n_0 u / u^2.$$  \hspace{3cm} (23)

The coefficients $d\mathbf{p}/ds$ are $-(1/v_0)\Omega_{b c}^a p^b p^c$. These are given by

$$\frac{d\mathbf{p}}{ds} = -\frac{v_0}{c} \left[ \gamma_u \frac{\gamma_u - 1}{u^2} \left\{ -(a + u \cdot \nabla u) \cdot n_0 u + (a + u \cdot \nabla u) u \cdot n_0 \right\} 
+ \frac{\gamma u (\gamma_u - 1)}{u^2} \left[ (u \cdot \nabla u) u + (u \cdot \nabla u) (u \cdot n_0) \right] 
+ \frac{\gamma^2 u (\gamma_u - 1)}{u^2} \left[ (n_0 \cdot \nabla u) u + (u \cdot \nabla u) (u \cdot n_0) \right] 
+ \frac{\gamma^2 u (\gamma_u - 1)}{u^2} \left[ (u \cdot \nabla u) u + (u \cdot \nabla u) (u \cdot n_0) \right] 
+ \frac{\gamma u}{u^2} \left\{ -\gamma_u u (n_0 \cdot a) (u \cdot n_0) / c + \gamma_u a (u \cdot n_0)^2 / c 
- c a (n_0 \cdot \nabla u) (u \cdot n_0) + c n_0 \cdot \nabla u (u \cdot n_0) 
+ \frac{\gamma u}{u^2} \left\{ -c (u \cdot \nabla u) (u \cdot n_0) + c (u \cdot \nabla u) (u \cdot n_0)^2 \right\} \right\} \right].$$  \hspace{3cm} (24)
The value of $d
u_0/ds$ is $-(1/\nu_0)\Omega_{bc}^0 p^b p^c$, and in view of equation (19) this is

$$
\frac{d\nu_0}{ds} = -\frac{\nu_0}{c} \left\{ \frac{\gamma_u^2}{c} (\mathbf{a} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{n}_0 + \frac{\gamma_u^2 (\gamma_u - 1)}{cu^2} \mathbf{u} \cdot (\mathbf{a} + \mathbf{u} \cdot \nabla \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}_0) \\
+ \frac{\gamma_u^2}{c^2} (\mathbf{n}_0 \cdot \mathbf{a}) (\mathbf{u} \cdot \mathbf{n}_0) + \frac{\gamma_u^2 (\gamma_u - 1)}{c^2 u^2} (\mathbf{u} \cdot \mathbf{a}) (\mathbf{u} \cdot \mathbf{n}_0)^2 + \gamma_u \mathbf{n}_0 \cdot \nabla \mathbf{u} \cdot \mathbf{n}_0 \\
+ \frac{\gamma_u (\gamma_u - 1)}{u^2} (\mathbf{n}_0 \cdot \nabla \mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}_0) + \frac{\gamma_u (\gamma_u - 1)}{u^2} (\mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{n}_0) (\mathbf{u} \cdot \mathbf{n}_0) \\
+ \frac{\gamma_u (\gamma_u - 1)^2}{u^4} (\mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}_0)^2 \right\}. \tag{25}
$$

A consistency condition is that

$$
\mathbf{n}_0 \cdot \frac{d\mathbf{p}}{ds} \equiv \frac{d\nu_0}{ds}, \tag{26}
$$
Formal Solution Along Characteristics
Affine Method (Chen et al. 2007)
The Boltzmann Equation is an integro-differential equation for the invariant photon distribution function $F(x, p)$ on the photon’s 7-dimensional phase-space $(x, p)$. This can be thought of as the photon “on-shell” subspace of a full 8 dimensional particle phase-space.

The number of photons $\Delta N$ found by observer $u(x)$ in a small 6-element $\Delta V_x \Delta P$ of phase-space at $(x, p)$ is measured by the 6-form $\delta N$, i.e., $\Delta N = \delta N(\Delta V_x, \Delta P)$ where

$$\delta N \equiv F(x, p) \delta V_6,$$

$$\delta V_6 \equiv -(u(x) \cdot p) \delta V_x \delta P.$$  \hspace{1cm} (1)

In the above, $u(x)$ is an arbitrary observer’s unit 4-velocity at spacetime point $x$,

$$-(u(x) \cdot p) = \frac{h}{\lambda}$$

is the magnitude of the photon’s 3-momentum as seen by observer $u(x)$, $\delta V_x$ is the observer dependent 3-dimensional volume element at
The Affine Approach II

\( x \), and \( \delta P \) is the covariant volume element on the photons’
3-dimensional momentum space at \((x, p)\).

The collisionless Boltzmann equation simply states that \( F[x(\xi), p(\xi)] \)
remains invariant (constant) along the Lagrangian flow of photons in
phase-space generated by their geodesic motion in spacetime.

Constancy of \( F[x(\xi), p(\xi)] \) is a natural consequence of Liouville’s
theorem, i.e., \( \delta V_6 \) is invariant under this flow, and the constancy of \( \Delta N \)
due to the absence of non-gravitational interactions of the photons.

Any lack of constancy of \( \Delta N \) in a finite volume \( \Delta V_6 \) is accounted for by
a collision term. To exhibit covariance, the Boltzmann equation with
collisions, is often written as a differential equation

\[
\frac{dF}{d\xi} = \frac{dx^\alpha}{d\xi} \frac{\partial F}{\partial x^\alpha} + \frac{dp^\alpha}{d\xi} \frac{\partial F}{\partial p^\alpha} = \left( \frac{dF}{d\xi} \right)_{\text{coll}},
\]

with \( F(x, p) \) explicitly given as a function of 8 variables (all 4
components of momentum are included but constrained by \( p \cdot p = 0 \)).
The collision term on the R.H.S. is a measure of the rate of change of
the number of photons $\Delta N$ in a $\Delta V_6$ transported along the would-be paths of non-interacting photons in phase-space. According to the geometrical optics approximation, photons travel on null spacetime geodesics independently of their wavelengths. Affine parameters, $\xi$, unique to each wavelength, can be chosen which generate the following orbits on phase-space:

\[
\frac{dx^\alpha}{d\xi} = p^\alpha, \quad \quad \quad (5) \\
\frac{dp^\alpha}{d\xi} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma, \quad \quad \quad (6)
\]

which reduces (4) to

\[
p^\alpha \frac{\partial F}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial F}{\partial p^\alpha} = \left( \frac{dF}{d\xi} \right)_{\text{coll}}. \quad \quad \quad (7)
\]
The Affine Approach IV

The R.H.S. is typically separated into absorption and emission terms

\[
\frac{dF}{d\xi}_{\text{coll}} = -fF + g, \tag{8}
\]

where \(f(x, p)\) and \(g(x, p)\) are identified respectively with the invariant absorptivity and emissivity.

The radiation transport equation is an integro-differential equation for the specific intensity \(I(x, p)\) which is equivalent to the Boltzmann equation (4) or (7) for \(F(x, p)\). Both are functions on the photon’s 7-dimensional phase-space \((x, p)\); however, \(I(x, p)\) depends on a choice of an observer at each point of spacetime through

\[
I_{\lambda}(x, p) = -\frac{c^2}{h}(u(x) \cdot p)^5 F(x, p). \tag{9}
\]

We have chosen to follow our group’s convention and use \(\lambda\) rather than \(\nu\) as often appears in the literature; however, it is straightforward
to change between $I_\lambda$ and $I_\nu$ using $\lambda I_\lambda = \nu I_\nu$. Once the observers are chosen, $I_\lambda(x, p)$ like $F(x, p)$, is a scalar. Defining a set of observers is equivalent to giving a unit time-like vector field on spacetime, $u(x)$, which appears in Eq. (9). Just as in equation (3), $-u(x) \cdot p$ is equal to the photon’s momentum as seen by observer $u(x)$. If $u(x)$ describes the material fluid with which the photons interact, $I_\lambda$ is called the comoving specific intensity and $\lambda$ the comoving wavelength. The transport equation for $I_\lambda(x, p)$ is obtained from equations (4) and (8) by substituting $I_\lambda(x, p)$ for $F(x, p)$ using Eq. (9)

$$\frac{dl_\lambda}{d\xi} = -(\chi_\lambda \frac{h}{\lambda} + \frac{5}{\lambda} \frac{d\lambda}{d\xi})I_\lambda + \eta_\lambda \frac{h}{\lambda},$$

(10)
where the observer dependent absorptivity $\chi_{\lambda}$ and emissivity $\eta_{\lambda}$ are related to $f$ and $g$ by

$$\chi_{\lambda} = -\frac{f}{(u(x) \cdot p)},$$

$$\eta_{\lambda} = \left(\frac{c^2}{h}\right) (u(x) \cdot p)^4 g.$$  \hspace{1cm} (11)

The other term ($\propto d\lambda/d\xi$) on the right in Eq. (10) is present because the definition of $I_\lambda$, Eq. (9), explicitly depends on comoving $\lambda$. If as is customary we divide the extinction into two parts: “true absorption” $\kappa_{\lambda}$ and “scattering” $\sigma_{\lambda}$, then $\chi_{\lambda} = \kappa_{\lambda} + \sigma_{\lambda}$. For a comoving observer we will also assume the emissivity is given by thermal emission (true absorption opacity $\kappa_{\lambda}$ times the Planck function $B_{\lambda}$) and that scattering is elastic and isotropic. For a comoving observer, $\chi_{\lambda}$ depends only on the magnitude of the momentum and not its direction (given isotropic sources), and consequently is a function of only $x$ and $u \cdot p$ in an arbitrary coordinate system.
If the energy density in the radiation field is written as $\epsilon_\lambda = \frac{4\pi J_\lambda}{c}$ the emissivity becomes

$$\eta_\lambda = \kappa_\lambda B_\lambda + \sigma_\lambda J_\lambda.$$  \hspace{1cm} (12)
Using the general formalism developed in Chen et al. we can derive the transfer equation in flat spacetime with arbitrary flows. We choose to work in spherical coordinates without loss of generality. The photon’s four-momentum can be written

\[ p^a \equiv \frac{dx^a}{d\xi} = \frac{h}{\lambda_\infty} (1, \hat{n}), \quad (13) \]

where \( h \) is the Planck constant, \( \xi \) is the affine parameter, \( \lambda_\infty \) is the rest frame wavelength, and \( \hat{n} \) is the 3-D direction of the photon as seen by a distant stationary observer. The four-velocity of the co-moving observer in an arbitrary flow can be written

\[ u^a = \gamma(r, t) [1, \vec{\beta}(r, t)], \quad (14) \]

and the co-moving wavelength \( \lambda \) can be obtained using

\[ \frac{h}{\lambda} = -(u \cdot p). \quad (15) \]
The 3-D geodesic in the flat spacetime can be parametrized as

$$r(s) = r_0 + \hat{n} s,$$

(16)

where $r_0$ is the starting point of the characteristics, and $s$ is the rest frame physical distance related to the affine parameter $\xi$ by

$$s \equiv \frac{h}{\lambda_\infty} \xi.$$

(17)

The radiative transfer equation can be written in terms of the affine parameter $\xi$ as (see Eq. (10) of Chen et al.):

$$\frac{\partial I_\lambda}{\partial \xi} + \frac{d\lambda}{d\xi} \frac{\partial I_\lambda}{\partial \lambda} = -\left(\chi_\lambda \frac{h}{\lambda} + \frac{5}{\lambda} \frac{d\lambda}{d\xi}\right) I_\lambda + \eta_\lambda \frac{h}{\lambda},$$

(18)

where $I_\lambda(r, t; \hat{n})$ is the specific intensity measured in a frame where wavelengths are measured by a co-moving observer and all other quantities are measured by an observer at infinity.
We can rewrite Eq. 18 as

$$\frac{d(ct)}{d\xi} \frac{1}{c} \frac{\partial I_\lambda}{\partial t} \bigg|_\lambda + \frac{d\vec{r}}{d\xi} \cdot \nabla I_\lambda + \frac{d\lambda}{d\xi} \frac{\partial I_\lambda}{\partial \lambda} = -(\chi_\lambda \frac{h}{\lambda} + \frac{5}{\lambda} \frac{d\lambda}{d\xi}) I_\lambda + \eta_\lambda \frac{h}{\lambda},$$  \hspace{1cm} (19)

The radiative transfer equation can be written in terms of physical distance parameter $s$ instead of the affine parameter $\xi$ as (see Eq. (18) of Chen et al.):

$$\frac{d(ct)}{d\xi} \frac{1}{c} \frac{\partial I_\lambda}{\partial t} \bigg|_\lambda + \frac{ds}{d\xi} \frac{d\vec{r}}{ds} \cdot \nabla I_\lambda + \frac{ds}{d\xi} \frac{d\lambda}{ds} \frac{\partial I_\lambda}{\partial \lambda} = -(\chi_\lambda \frac{h}{\lambda} + \frac{5}{\lambda} \frac{ds}{d\xi} \frac{d\lambda}{ds}) I_\lambda + \eta_\lambda \frac{h}{\lambda},$$  \hspace{1cm} (20)

Then using the definition of $s$ from Eq. 17 and the fact that

$$\frac{d(ct)}{d\xi} = cp^t = h/\lambda_\infty.$$
we find
\[
\frac{1}{c} \frac{\partial I_\lambda}{\partial t} \bigg|_\lambda + \frac{\partial I_\lambda}{\partial s} \bigg|_\lambda + \frac{d\lambda}{ds} \frac{\partial I_\lambda}{\partial \lambda} = -\left(\chi_\lambda \frac{\lambda_\infty}{\lambda} + \frac{5}{\lambda} \frac{d\lambda}{ds}\right) I_\lambda + \eta_\lambda \frac{\lambda_\infty}{\lambda}. \quad (21)
\]

Eq. 21 can be put into our standard form:
\[
\frac{1}{c} \frac{\partial I_\lambda}{\partial t} \bigg|_\lambda + \frac{\partial I_\lambda}{\partial s} + a(s) \frac{\partial}{\partial \lambda} (\lambda I_\lambda) + 4a(s)I_\lambda = -\chi_\lambda f(s) I_\lambda + \eta_\lambda f(s), \quad (22)
\]

where
\[
f(s) \equiv \frac{\lambda_\infty}{\lambda} = \gamma(r, t)[1 - \hat{n} \cdot \vec{\beta}(r, t)]. \quad (23)
\]
is simply the Doppler factor, and
\[
a(s) \equiv \frac{1}{\lambda} \frac{d\lambda}{ds}. \quad (24)
\]
Using Eqs. (16) and (23), \( a(s) \) is found to be

\[
a(s) = \frac{1}{1 - \hat{n} \cdot \vec{\beta}} \left[ \frac{d}{ds} (\hat{n} \cdot \vec{\beta}) - \gamma^2 \beta (1 - \hat{n} \cdot \vec{\beta}) \frac{d\beta}{ds} \right]
\]

(25)

where \( \beta \) is the magnitude of \( \vec{\beta} \), and

\[
\frac{d}{ds} = \frac{1}{c} \frac{\partial}{\partial t} + \hat{n} \cdot \nabla = \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial s}
\]

(26)

When we numerically integrate the radiation transfer equation, \( a(s) \) can be approximated as

\[
a(s) \approx \frac{\delta (\hat{n} \cdot \vec{\beta}) - \gamma^2 \beta (1 - \hat{n} \cdot \vec{\beta}) \delta \beta}{\delta s (1 - \hat{n} \cdot \vec{\beta})},
\]

(27)

where \( \delta s \) is the differential step size (physical distance) along the characteristics, \( \delta (\hat{n} \cdot \vec{\beta}) \) and \( \delta \beta \) are respectively the changes of \( \hat{n} \cdot \vec{\beta} \).
Applied Affine Method VI

and $\beta$ when we move one step forward which includes the changes induced by both time and spatial advances, say

$$\delta \beta = \beta(s_{i+1}, t_{i+1}) - \beta(s_i, t_i). \quad (28)$$

Since few numerical schemes will be able to provide the fully implicit derivative $\delta \beta$ will often be obtained for example by the backward difference

$$\delta \beta = \frac{1}{c}(\beta(s_i, t_i) - \beta(s_i, t_{i-1}) + (\beta(s_i, t_i) - \beta(s_{i-1}, t_i)). \quad (29)$$

In the stationary case, both $\beta$ and $f(s)$ are independent of time and specializing Eqs. (16) and (23) to that case, $a(s)$ becomes:

$$a(s) = \frac{1}{1 - \hat{n} \cdot \vec{\beta}} \left[ \frac{d}{ds}(\hat{n} \cdot \vec{\beta}) - \gamma^2 \beta (1 - \hat{n} \cdot \vec{\beta}) \frac{d\beta}{ds} \right]$$
Applied Affine Method VII

\[
\frac{\hat{n} \cdot \nabla (\hat{n} \cdot \vec{\beta}) - \gamma^2 \beta (1 - \hat{n} \cdot \vec{\beta}) (\hat{n} \cdot \nabla \beta)}{1 - \hat{n} \cdot \vec{\beta}} = -(\hat{n} \cdot \nabla) \ln (1 - \hat{n} \cdot \vec{\beta}) - \gamma^2 \beta (\hat{n} \cdot \nabla \beta),
\]

where \( \beta \) is the magnitude of \( \vec{\beta} \), and we have used the fact that along the characteristics, \( d/ds \) no longer contains the time derivative and is thus directional derivative operator, i.e., \( d/ds = \hat{n} \cdot \nabla \). When we numerically integrate the radiation transfer equation, \( a(s) \) can be approximated as

\[
a(s) \approx \frac{\delta(\hat{n} \cdot \vec{\beta}) - \gamma^2 \beta (1 - \hat{n} \cdot \vec{\beta}) \delta \beta}{\delta s (1 - \hat{n} \cdot \vec{\beta})},
\]

where \( \delta s \) is the differential step size along the characteristics, \( \delta(\hat{n} \cdot \vec{\beta}) \) and \( \delta \beta \) are respectively the changes of \( \hat{n} \cdot \vec{\beta} \) and \( \beta \) when we move one step forward.
Applied Affine Method VIII

In terms of its spherical components, $\vec{\beta}$ can be written

$$\vec{\beta} = \beta_r \hat{e}_r + \beta_\theta \hat{e}_\theta + \beta_\phi \hat{e}_\phi,$$

(32)

where $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ are the spherical orthonormal basis vectors at point $\mathbf{r}(r, \theta, \phi)$, i.e.,

$$\hat{e}_r = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),$$

$$\hat{e}_\theta = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta),$$

$$\hat{e}_\phi = (-\sin \phi, \cos \phi, 0),$$

(33)

and consequently

$$\hat{n} \cdot \vec{\beta} = \beta_r \hat{n} \cdot \hat{e}_r + \beta_\theta \hat{n} \cdot \hat{e}_\theta + \beta_\phi \hat{n} \cdot \hat{e}_\phi$$

$$\equiv \beta_r n_r + \beta_\theta n_\theta + \beta_\phi n_\phi$$

(34)

(Note that along the characteristics, $\hat{n}$ has constant Cartesian components, $n_x, n_y, n_z$, but changing spherical components, $n_r, n_\theta, n_\phi$.)
Applied Affine Method IX

\( n_r, n_\theta, n_\phi \). Writing \( \hat{n} = (n_x, n_y, n_z) \), the spherical components \( n_r, n_\theta, n_\phi \) can be easily computed using Eq. (33).

At first glance comparing Eq 22 with Eq. 2.12 of (Mihalas 1980) something seems amiss. Both we and Mihalas (1980) work in the frame where spatial coordinates and clocks are measured by an observer at rest. However, Mihalas’ time derivative contains a Doppler factor, whereas ours does not. Also, our terms on the right hand side contain Doppler factors, \( f(s) \), whereas those of Mihalas do not. The discrepancy has been noted in passing by us and is due to the fact that our \( s \) is a true distance measured in the Observer’s frame, whereas that of Mihalas, \( s_M \), contains an extra Doppler factor:

\[
\begin{align*}
s_M &= \frac{\lambda_\infty}{\lambda} s.
\end{align*}
\]

Thus, we can transform from \( s \) to \( s_M \) in Eq. 21 to find

\[
\begin{align*}
\left. \frac{\lambda}{\lambda_\infty} \frac{1}{c} \frac{\partial I_\lambda}{\partial t} \right|_\lambda + \left. \frac{\partial I_\lambda}{\partial s_M} \right|_\lambda + \frac{d\lambda}{ds_M} \frac{\partial I_\lambda}{\partial \lambda} &= -(\chi_\lambda + \frac{5}{\lambda} \frac{d\lambda}{ds_M}) I_\lambda + \eta_\lambda,
\end{align*}
\] (35)
which is very similar to the equation of Mihalas, except that the coefficient multiplying the time derivative term is the inverse Doppler factor \( f(s)^{-1} \), since we are working with \( I_\lambda \) instead of \( I_\nu \) as does Mihalas.
Comparison with Mihalas I

To derive Mihalas Eq. 2.12, starting from the Initial frame transfer equation 2.1

\[
\frac{1}{c} \frac{\partial I(\mu, \nu)}{\partial t} + \mu \frac{\partial I(\mu, \nu)}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I(\mu, \nu)}{\partial \mu} = \eta - \chi I(\mu, \nu). \tag{36}
\]

Note this equation can be easily derived using

\[
\frac{dr}{ds} = \mu, \quad \frac{d\mu}{ds} = \frac{d}{ds} \left( \frac{\hat{n} \cdot r}{r} \right) = \frac{1 - \mu^2}{r}. \tag{37}
\]

Applying Eq. 2.2, 2.3, 2.4 of Mihalas to change initial \( I, \eta, \chi \) to comoving \( I_0, \eta_0, \chi_0 \). Then change phase space coordinates from \( (t, r, \mu, \nu) \) to \( (t_0 = t, r_0 = r, \mu_0, \nu_0) \).

\[
t_0 = t \\
r_0 = r \\
\mu_0 = \frac{\mu - \beta}{1 - \mu\beta}
\]
Comparison with Mihalas II

\[ \nu_0 = \nu \gamma (1 - \beta \mu) \]  \hspace{1cm} (38)

note that \( \beta \) and \( \gamma \) are functions of \( r, t \).

Work out the Jacobian Matrix

\[ J = \frac{\partial (t_0, r_0, \mu_0, \nu_0)}{\partial (t, r, \mu, \nu)}, \]  \hspace{1cm} (39)

and change the coordinate vectors using

\[ \left( \frac{\partial}{\partial x_i} \right) = J^T \left( \frac{\partial}{\partial x_i^0} \right). \]  \hspace{1cm} (40)

The derivations are just straightforward algebra.

To derive Milalas’ result using our affine method, starting from (our inertial/comoving notation is exactly the opposite of that of Mihalas)

\[ \frac{dF}{d\xi} = -fF + g, \]  \hspace{1cm} (41)
Comparison with Mihalas III

using

\[ \chi_\lambda = \frac{f}{\nu}, \quad \eta_\lambda = \nu^4 g, \quad \chi_\lambda = \chi_\nu, \quad \eta_\lambda = \nu^2 \eta_\nu. \quad (42) \]

we obtain transfer equation in affine parameter

\[ \frac{dl_\nu}{d\xi} - \frac{3}{\nu} \frac{d\nu}{d\xi} l_\nu = \nu \left[ -\chi_\nu l_\nu + \eta_\nu \right] \quad (43) \]

Changing from affine parameter to physical distance

\[ \frac{\nu_0}{\nu} \left( \frac{dl_\nu}{ds} - \frac{3}{\nu} \frac{d\nu}{ds} l_\nu \right) = -\chi_\nu l_\nu + \eta_\nu \quad (44) \]

or

\[ \frac{\nu_0}{\nu} \left( \frac{1}{c} \frac{\partial l_\nu}{\partial t} + \mu_0 \frac{\partial l_\nu}{\partial r} + \frac{d\mu}{ds} \frac{\partial l_\nu}{\partial \mu} + \frac{d\nu}{ds} \frac{\partial l_\nu}{\partial \nu} - \frac{3}{\nu} \frac{d\nu}{ds} l_\nu \right) = -\chi_\nu l_\nu + \eta_\nu \quad (45) \]
Comparison with Mihalas IV

Using

\[
\mu(s) = \frac{\mu_0 - \beta}{1 - \beta \mu_0}, \quad \nu(s) = \nu_0 \gamma (1 - \beta \mu_0)
\] (46)

and

\[
\frac{d\mu}{ds} = \frac{\partial \mu}{\partial \mu_0} \frac{d\mu_0}{ds} + \frac{\partial \mu}{\partial \beta} \frac{d\beta}{ds}, \quad \mu_0 = \hat{n} \cdot \hat{r}, \quad \frac{d\mu_0}{ds} = 1 - \mu_0^2 \frac{1}{r}, \quad \frac{d\beta}{ds} = \frac{1}{c} \frac{\partial \beta}{\partial t} + \mu_0 \frac{\partial \beta}{\partial t}
\] (47)

we can easily work out the same transfer equation (2.12) in Mihalas.
Formal Solution Along Characteristics
Affine Method (Baron, Hauschildt, Chen 2009)

\[
\frac{\partial I_\lambda}{\partial s}|_\lambda + a(s)\lambda \frac{\partial I_\lambda}{\partial \lambda} = -[\chi_\lambda f(s) + 5a(s)]I_\lambda + \eta_\lambda f(s). \tag{48}
\]

Funny Frame

\( \mathbf{r} \) and \( \mathbf{n} \) in Observer’s Frame
\( \lambda \) is measured by a comoving observer
Scattering Problem in Co-moving Frame

Need $J_\lambda$ in co-moving frame (in momentum space)

So just integrate RTE equation over $d\Omega$ (where $d\Omega$ is measured by a comoving observer), but since $I_\lambda$ is known in the funny frame:

Use Chain Rule

\[
d\Omega = \left(\gamma [1 - \beta \cdot n]\right)^{-2} d\Omega_0 = f(s)^{-2} d\Omega_0.
\]

\[
J_\lambda = \Lambda S_\lambda
\]