PROBLEM 1

Consider the step potential shown in the figure.

a) [1 pts] Consider a particle traveling from $x = -\infty$ to the right with an energy $E$. The appropriate wavefunction for this particle is given by

$$\phi = \begin{cases} e^{ik_L x} + Ae^{-ik_L x} & \text{for } x < 0 \\ Be^{ik_R x} & \text{for } x > 0 \end{cases}$$

Give expressions for $k_L$ and $k_R$ and define any undefined parameters/constants given in your expression.

b) [3 pts] For the case that $E > V_o$, use appropriate boundary conditions to find the coefficients $A$ and $B$.

c) [2 pts] For the case that $E > V_o$, find the probability that the particle will be reflected.

d) [2 pts] For the case that $E > V_o$, the probability that the particle will be transmitted is given by $T = 1 - R$. Determine and explain the physical meaning of the ratio $|B|^2/T$.

e) [2 pts] What is the probability for reflection when $E < V_o$?
PROBLEM 2

a) [2 pts] Calculate the energy eigenvalues for a particle of mass $m$ in the one-dimensional infinite well shown in Figure A.

b) [4 pts] For the time-independent Schrödinger Equation corresponding to potential (B), find a transcendental equation in $E$ giving the eigenenergies in terms of $V_o, L, m,$ and $\hbar$.

c) [4 pts] For the time-independent Schrödinger Equation corresponding to potential (B), what is the smallest value of $V_o$ that gives one bound state? What is the smallest value of $V_o$ that gives two bound states?
PROBLEM 3
Consider a quantum mechanical system that consists of two identical spin \( \frac{1}{2} \) particles that are fixed in space, separated by a distance \( d \). Particle 1 is at the origin \( \vec{r}_1 = \vec{0} \) whereas particle 2 is at \( \vec{r}_2 = d \hat{e}_z \). Each particle has a magnetic moment
\[
\vec{\mu}(j) = \frac{g\mu_B}{\hbar} \vec{S}(j), \quad j = 1, 2
\]
and a \( g \)-factor \( g = 2 \). \( \vec{S}(j) \) is the spin operator of the \( j^{th} \) particle. Throughout this problem we will use the basis states
\[
|1\rangle = |+, +\rangle, |2\rangle = |+, -\rangle, |3\rangle = |-, +\rangle, \text{ and } |4\rangle = |-, -\rangle,
\]
where these are the usual states written in terms of the \( z \)-components of the particles’ spins.

a) [2pts] First consider what happens if we place the particles in an external magnetic field \( \vec{B} = B\hat{e}_z \). Write the matrix representation for the Hamiltonian of the system
\[
\hat{H}_o = -\vec{\mu} \cdot \vec{B}
\]
in the \( |i\rangle, i = 1, 2, 3, 4 \) basis given above, considering only the interaction between the spins and the magnetic field. What are the energy eigenstates and eigenvalues for the system? Draw an energy-level diagram.

b) [3pts] We know, however, that the magnetic moment of each particle will create a magnetic field that the other particle will feel. The dipole field from particle 1 at particle 2 is (classically)
\[
\vec{B}_{21} = \frac{1}{d^3} (3\mu_z(1)\hat{e}_z - \vec{\mu}(1))
\]
so that the interaction Hamiltonian between the two particles is
\[
\hat{H}' = -\vec{\mu}_2 \cdot \vec{B}_{21} = \frac{g^2 \mu_B^2}{\hbar^2 d^3} \left( -3S_z(1)S_z(2) + \vec{S}(1) \cdot \vec{S}(2) \right).
\]
Compute the action of the interaction Hamiltonian on each of the basis states. In other words, calculate \( \hat{H}' |i\rangle \) for \( i = 1, 2, 3, 4 \). Hint: Use the usual angular momentum raising and lowering operators
\[
\hat{S}^\pm = \hat{S}_x(j) \pm i\hat{S}_y(j), \quad j = 1, 2
\]

c) [2pts] Write the total Hamiltonian, \( \hat{H} = \hat{H}_o + \hat{H}' \) as a matrix in the \( |i\rangle \) basis.

d) [3pts] Find the eigenstates and eigenvalues of this total Hamiltonian and draw the energy level diagram as a function of the magnetic field strength.
PROBLEM 4

Consider a two state system described by the time-dependent Hamiltonian

\[
H = \begin{pmatrix} 0 & \beta^* e^{-i\omega t} \\ \frac{\beta}{\hbar} e^{i\omega t} & \hbar \omega_1 \end{pmatrix}
\]

with

\[
\vec{v}(t) = \begin{pmatrix} v_0(t) \\ v_1(t) \end{pmatrix}.
\]

This is the Hamiltonian of a spin 1/2 particle in a strong magnetic field in the \( \hat{z} \) direction combined with a rotating magnetic field in the x-y plane and models many NMR experiments. To analyze this system, it is convenient to write \( \vec{v}(t) \) in terms of the time dependent vector \( \vec{s}(t) = \begin{pmatrix} s_0(t) \\ s_1(t) \end{pmatrix} \) so that

\[
\vec{v}(t) = \begin{pmatrix} s_0(t) \\ s_1(t)e^{-i\omega t} \end{pmatrix}.
\]

For the case that \( \beta = 0 \) and \( \omega = \omega_1 \) (no rotating magnetic field), \( s_0(t) \) and \( s_1(t) \) are constant. The time dependence of \( s_0(t) \) and \( s_1(t) \) allows us to determine the probability that the rotating magnetic field induces a spin flip.

a) [1pt] Show that for \( \beta = 0 \), \( \vec{v}(t) \) satisfies the time-dependent Schrödinger equation

\[
H \vec{v}(t) = i\hbar \frac{\partial \vec{v}(t)}{\partial t}.
\]

when when \( s_0(t) \) and \( s_1(t) \) are constant and \( \omega = \omega_1 \).

b) [3pts] For the case that \( \beta \) is a nonzero constant, use Schrödinger’s equation for \( \vec{v}(t) \) to show that \( \vec{s}(t) \) evolves according to the effective Hamiltonian \( H' \) with

\[
H' = \begin{pmatrix} 0 & \beta^* \frac{i}{\hbar} \\ \frac{\beta}{\hbar} & \hbar \Delta \omega \end{pmatrix}
\]

and

\[
\Delta \omega = \omega_1 - \omega.
\]

c) [3pts] Assuming the system starts in the state \( \vec{s}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) at \( t = 0 \), find \( \vec{s}(t) \).

d) [3pts] Assuming the system starts in the state \( \vec{s}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) at \( t = 0 \), find the probability of finding the system in the state \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) as a function of time.
A particle of mass $m$ is confined to a two-dimensional plane. The potential energy of the particle is

$$V(\rho) = \begin{cases} 0 & \rho < \rho_o \\ \infty & \rho \geq \rho_o \end{cases},$$

where $\rho$ is the radial coordinate of plane polar coordinate $(\rho, \varphi)$. This potential confines the particle to the region of space $\rho \leq \rho_o$. The particle in this “circular square well” is the quantum analog of a marble on the head of a drum. The stationary-state Hamiltonian eigenfunctions of the particle are $\Psi_{n,m,\ell}(\rho, \varphi)$ with eigenenergies $E$.

a) [4pts] Write down a second-order differential equation for the radial function $R_{n,m,\ell}(\rho)$ in the bound-state Hamiltonian eigen functions

$$\psi_{n,m,\ell}(\rho, \varphi) = R_{n,m,\ell}(\rho) \Phi_{m,\ell}(\varphi),$$

where $\Phi_{m,\ell}(\varphi)$ is an eigenfunction of the orbital angular momentum operator $\hat{L} = -i\hbar \partial/\partial \varphi$. Write down and justify the boundary conditions that physically admissible solutions to your differential equation must satisfy, and write down the normalization integral for the radial functions.

b) [2pts] What, if anything, can you conclude from your differential equation about the degree of degeneracy of the bound-state energies $E_{n,m,\ell}$. Justify your answer.

c) [2pts] Derive an equation for the bound-state energies $E_{n,m,\ell}$ in terms of the zeros $\zeta_{n,\nu}$ of the cylindrical Bessel function of the first kind, $J_{\pm \nu}(z)$. (See the hint below.)

d) [2pts] Estimate the energies of the lowest three bound states of the cylindrical square well. Express your answers in terms of fundamental constants, the mass $m$, and the well radius $\rho_o$.

Hint: The cylindrical Bessel functions are solutions of Bessel’s equation

$$\left[ z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} + (z^2 - \nu^2) \right] J_{\pm \nu}(z) = 0$$

The so-called cylindrical Bessel functions of the first kind, $J_{\pm \nu}(z)$, are regular at the origin and normalizable. These functions oscillate with increasing $z$ and have an infinite number of nodes, i.e., values for which $z = \zeta_{n,\nu} > 0$ at which $J_{\pm \nu}(z) = 0$; these nodes are indexed by $n = 1, 2, \ldots$. The figure shows the first four cylindrical Bessel functions.
First four cylindrical Bessel functions of the first kind (for use in problem 5.)
PROBLEM 6

Consider an ensemble of identical particles whose state space is spanned by the basis

\[ |e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

Assume that the Hamiltonian \( H \) and an observable \( A \) are represented by

\[ H = \hbar \omega_o \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

The eigenvalues of \( H \) are \( \hbar \omega, 2\hbar \omega, \) and \(-\hbar \omega\) with eigenvectors given by

\[ |\hbar \omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad |2\hbar \omega\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and } |\hbar \omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \]

The eigenvalues of \( A \) are \(-1, 1, \text{and } 1\) with eigenvectors given by

\[ |a_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad |a_{1,1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad |a_{1,2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

For all times \( t < 0 \), the particles are in a state given by

\[ |\psi_o\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix}. \]

a) [1pt] Write down an expression for the time evolution operator \( U(t, t_o = 0) \) in Dirac notation.

b) [2 pt] Determine \( |\psi(t)\rangle \), the state vector at an arbitrary time.

c) [2 pt] What is the probability that a measurement of \( A \) at a time \( t = 0 \) yields \( a = -1 \)?

d) [2 pt] What is the probability that a measurement of \( A \) at an arbitrary time \( t \) yields a value \( a = -1 \)?

e) [3 pt] Assume that at \( t = 0 \) the operator \( A \) is observed to be 1. What is the probability that a short time later \((0 < t << 1/\omega)\), the eigenenergy of the system is observed to be \(-\hbar \omega\)?