Quantum Mechanics
Qualifying Exam–January 2006

Notes and Instructions:

• There are 6 problems and 7 pages.
• Be sure to write your alias at the top of every page.
• Number each page with the problem number, and page number of your solution (e.g. “Problem 3, p. 1/4” is the first page of a four page solution to problem 3).
• You must show all your work.

Possibly useful formulas:

Pauli spin matrices:

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

One-dimensional simple harmonic oscillator operators:

\[
X = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)
\]
\[
P = -i\sqrt{\frac{\hbar m\omega}{2}} (a - a^\dagger)
\]

Spherical Harmonics:

\[
Y^0_0(\theta, \varphi) = \frac{1}{\sqrt{4\pi}} \\
Y^2_0(\theta, \varphi) = \frac{5}{\sqrt{96\pi}} 3\sin^2 \theta e^{2i\varphi} \\
Y^1_0(\theta, \varphi) = -\frac{5}{\sqrt{24\pi}} 3\sin \theta \cos \theta e^{i\varphi} \\
Y^1_1(\theta, \varphi) = \frac{3}{\sqrt{8\pi}} \sin \theta e^{i\varphi} \\
Y^0_1(\theta, \varphi) = \frac{3}{\sqrt{4\pi}} \cos \theta \\
Y^{-1}_1(\theta, \varphi) = \frac{3}{\sqrt{8\pi}} \sin \theta e^{-i\varphi} \\
Y^{-2}_2(\theta, \varphi) = \frac{5}{\sqrt{96\pi}} 3\sin^2 \theta e^{-2i\varphi}
\]

Angular momentum raising and lowering operators:

\[
\hat{L}_\pm = (\hat{L}_x \pm i \hat{L}_y)
\]
PROBLEM 1: Eigenvalues and Eigenvectors

Suppose the Hamiltonian for a system is given by

\[ H = \hbar \omega_0 (\sigma_x + \sigma_y) \]

where \( \sigma_x \) and \( \sigma_y \) are two of the Pauli matrices.

(a). Calculate the eigenvalues and eigenvectors for this Hamiltonian. (5 points)

(b). In the Schrödinger picture, the state vector is

\[ |\psi(t)\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \]

At \( t = 0 \), the state of this system has \( \alpha(0) = 0 \) and \( \beta(0) = 1 \). Evaluate \( \alpha(t) \) and \( \beta(t) \) for \( t > 0 \). (5 points)
PROBLEM 2: One Dimensional Barrier

![Diagram of a one-dimensional rectangular barrier]

Figure 1: A one-dimensional rectangular barrier.

A stream of particles with kinetic energy $0 < E < V_0$ and mass $m$, traveling in the positive $x$ direction is incident on a barrier from the left. These incident particles have a wave function

$$\psi_{in}(x) = e^{ikx}$$

where $k = \sqrt{2mE}/\hbar$.

(a). Write down the wave function in each region, ($\psi_I(x)$, $\psi_{II}(x)$, and $\psi_{III}(x)$), in terms of the eigenfunctions in each region, with undetermined coefficients. (2 points)

(b). State the boundary conditions at $x = 0$, $x = a$, and $x = \infty$. (1 point)

(c). Sketch the wavefunction in a single figure, showing its form in all regions. (1 point)

(d). Find the transmission coefficient ($T$) for particles with

$$k = \sqrt{2mE}/\hbar$$

$$\lambda = \sqrt{2m(V_0 - E)/\hbar}.$$  

(4 points)

(e). If the barrier of transmission is small ($\lambda a \ll 1$), show that

$$T \approx \frac{16E}{V_0} \left(1 - \frac{E}{V_0}\right) \exp \left(-\frac{2a}{\hbar} \sqrt{2m(V_0 - E)}\right).$$

(2 points)
PROBLEM 3: Harmonic Oscillator in 3 Dimensions

Consider a particle subject to a 3-dimensional harmonic oscillator potential:

\[ H = H_x + H_y + H_z \]
\[ = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_x^2x^2 + \frac{p_y^2}{2m} + \frac{1}{2}m\omega_y^2y^2 + \frac{p_z^2}{2m} + \frac{1}{2}m\omega_z^2z^2 \]

where \( \omega_x = \omega_y = \omega \) and \( \omega_z = 2\omega \). The wave function is given by:

\[ \Psi_{n_x,n_y,n_z}(x,y,z) = \Phi_{n_x}(x)\Phi_{n_y}(y)\Phi_{n_z}(z) \]

(a). For the ground state wave function, determine, \( \Delta X, \Delta P_x \) and their product \( \Delta X \Delta P_x \) using Dirac notation and raising and lowering operators. (3 points)

(b). For the ground state, \( \Psi_{0,0,0} \), do you expect
   (i) \( \Delta Z \) to be larger or smaller than \( \Delta X \)?
   (ii) \( \Delta Z \Delta P_z \) to be larger or smaller than \( \Delta X \Delta P_x \)?
   Explain your reasoning in each case. (2 points)

(c). Assume that at \( t = 0 \) the particle is in the state:

\[ |\Psi(t = 0)\rangle = \frac{1}{\sqrt{6}}|\Psi_{1,0,0}\rangle + \frac{1}{\sqrt{3}}|\Psi_{2,1,0}\rangle + \frac{1}{2}|\Psi_{0,1,0}\rangle + \frac{1}{2}|\Psi_{1,0,1}\rangle \]

If one measures the total energy, \( E \), what is the probability of obtaining \( 5\hbar\omega \)? (2 points)

(d). Immediately after the measurement performed in part (c), what harmonic oscillator state (or superposition of states) is the system in? (1 point)

(e). Assume that at time \( t = 0 \) the measurement described in part (c) is performed and that the energy is found to be \( E = 5\hbar\omega \). If the observable, \( X \), is measured at a time \( t > 0 \), will its probability distribution be be time dependent or time independent? Explain your reasoning. (2 points)
PROBLEM 4: Angular Momentum

Consider the problem of a 3D rigid rotator. Let the rotator be far removed from any forces so its energy is purely kinetic.

\[
E = \frac{L^2}{2I}
\]

where \( I \) is the moment of inertia. The quantum mechanical Hamiltonian operator is

\[
H = \frac{\hat{L}^2}{2I}
\]

(a). What are the frequencies of photons emitted due to the energy decay between successive levels of the rotator? (3 points)

At a given instant, the rigid rotator is in the state

\[
\Psi(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sin \theta \sin \phi
\]

(b). What possible values of \( L_z \) will be found in a measurement, and with what probabilities? (3 points)

(c). What is \( \langle L_z \rangle \) for this state? (2 points)

(d). What is \( \langle L^2 \rangle \) for this state? (2 points)
PROBLEM 5: Variational Method

Let us consider an electron moving in a Coulomb potential:

\[ H = T + V = -\frac{\hbar^2}{2m} \nabla^2 - \frac{e^2}{r} \]

where \( m \) is the mass of the electron.

Let us choose

\[ \psi_\alpha(r) = e^{-\alpha r^2}, \quad \alpha > 0 \]

as a trial wave function for the ground state.

(a). Find \( \langle \psi_\alpha | \psi_\alpha \rangle \). \( (1 \text{ points}) \)

(b). Find the expectation value of the Hamiltonian \( \langle H \rangle \). \( (3 \text{ points}) \)

(c). Determine a bound on the ground state energy of this system using the variational method and this trial wavefunction. Express your answer in terms of

\[ \text{Ry} = \frac{me^4}{2\hbar^2}. \]

\( (5 \text{ points}) \)

(d). How does this compare to the actual ground state energy of the system? Be specific. \( (1 \text{ point}) \)
PROBLEM 6: Perturbation Theory

Consider a particle confined to a ring of unit radius. You are told that the Hamiltonian operator can be written as \( H = H_0 + H_1 \) where

\[
H_0 = (i \frac{\partial}{\partial \theta} + a)^2
\]

and

\[
H_1 = V_0 \cos \theta.
\]

The parameter \( a \) is a tunable constant in this toy model. (In this problem, units are chosen such that \( \hbar = 2m = 1 \))

(a). Find the complete set of eigenvalues and eigenfunctions of \( H_0 \) (2 points)

(b). Use perturbation theory to find the first order correction to the ground state energy of \( H_0 \) due to the perturbation \( H' \) for \( 0 < a < 1/2 \). (2 points)

(c). Use perturbation theory to find the second order corrections to the ground state energy of \( H_0 \) due to the perturbation \( H_1 \) for \( 0 < a < 1/2 \). (3 points)

(d). For \( a = 1/2 \), the ground state energy of \( H_0 \) is degenerate. Find the first order correction to the energy for this case. (3 points)