Chapter 11

Green's Functions

11.1 One-dimensional Helmholtz Equation

Suppose we have a string driven by an external force, periodic with frequency ω. The differential equation (here f is some prescribed function)

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) U(x,t) = f(x)\cos\omega t \tag{11.1}$$

represents the oscillatory motion of the string, with amplitude U, which is tied down at both ends (here l is the length of the string):

$$U(0,t) = U(l,t) = 0. \tag{11.2}$$

We seek a solution of the form (thus we are ignoring transients)

$$U(x,t) = u(x)\cos\omega t, \tag{11.3}$$

so $u(x)$ satisfies

$$\left(\frac{d^2}{dx^2} + k^2\right) u(x) = f(x), \quad k = \omega/c. \tag{11.4}$$

The solution to this *inhomogeneous Helmholtz equation* is expressed in terms of the Green's function $G_k(x,x')$ as

$$u(x) = \int_0^l dx'\, G_k(x,x') f(x'), \tag{11.5}$$

where the Green's function satisfies the differential equation

$$\left(\frac{d^2}{dx^2} + k^2\right) G_k(x,x') = \delta(x-x'). \tag{11.6}$$

As we saw in the previous chapter, the Green's function can be written down in terms of the eigenfunctions of d^2/dx^2, with the specified boundary conditions,

$$\left(\frac{d^2}{dx^2}-\lambda_n\right)u_n(x) = 0, \tag{11.7a}$$

$$u_n(0)=u_n(l) = 0. \tag{11.7b}$$

The normalized solutions to these equations are

$$u_n(x)=\sqrt{\frac{2}{l}}\sin\frac{n\pi x}{l}, \quad \lambda_n=-\left(\frac{n\pi}{l}\right)^2, \quad n=1,2,\dots. \tag{11.8}$$

The factor $\sqrt{2/l}$ is a normalization factor. From the general theorem about eigenfunctions of a Hermitian operator given in Sec. 10.5, we have

$$\frac{2}{l}\int_0^l dx\,\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}=\delta_{nm}. \tag{11.9}$$

Thus the Green's function for this problem is given by the eigenfunction expansion

$$G_k(x,x')=\sum_{n=1}^\infty\frac{\frac{2}{l}\sin\frac{n\pi x}{l}\sin\frac{n\pi x'}{l}}{k^2-\left(\frac{n\pi}{l}\right)^2}. \tag{11.10}$$

But this form is not usually very convenient for calculation.

Therefore we solve the differential equation (11.6) directly. When $x\neq x'$ the inhomogeneous term is zero. Since

$$G_k(0,x')=G_k(l,x')=0, \tag{11.11}$$

we must have

$$x<x': \qquad G_k(x,x')=a(x')\sin kx, \tag{11.12a}$$

$$x>x': \qquad G_k(x,x')=b(x')\sin k(x-l). \tag{11.12b}$$

We determine the unknown functions a and b by noting that the derivative of G must have a discontinuity at $x=x'$, which follows from the differential equation (11.6). Integrating that equation just over that discontinuity we find

$$\int_{x'-\epsilon}^{x'+\epsilon}dx\left(\frac{d^2}{dx^2}+k^2\right)G_k(x,x')=1, \tag{11.13}$$

or

$$\left.\frac{d}{dx}G_k(x,x')\right|_{x=x'-\epsilon}^{x=x'+\epsilon}=1, \tag{11.14}$$

because $2\epsilon G_k(x',x')\to 0$ as $\epsilon\to 0$. Although $\frac{d}{dx}G_k(x,x')$ is discontinuous at $x=x'$, $G(x,x')$ is continuous there:

$$G(x'+\epsilon,x')-G(x'-\epsilon,x') = \int_{x'-\epsilon}^{x'+\epsilon}dx\frac{d}{dx}G(x,x')$$

$$= \int_{x'-\epsilon}^{x'} dx \frac{d}{dx} G(x,x') \; + \int_{x'}^{x'+\epsilon} dx \frac{d}{dx} G(x,x')$$
$$= \epsilon \left[\frac{d}{dx} G(x,x') \bigg|_{x=x'-\xi} + \frac{d}{dx} G(x,x') \bigg|_{x=x'+\bar{\xi}} \right], \tag{11.15}$$

where by the mean value theorem, $0 < \xi \le \epsilon$, $0 < \bar{\xi} \le \epsilon$. Therefore

$$G(x,x') \bigg|_{x=x'-\epsilon}^{x=x'+\epsilon} = \mathcal{O}(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{11.16}$$

Now using the continuity of G and the discontinuity of G', we find two equations for the coefficient functions a and b:

$$a(x') \sin kx' \; = \; b(x') \sin k(x'-l), \tag{11.17a}$$
$$a(x') k \cos kx' + 1 \; = \; b(x') k \cos k(x'-l). \tag{11.17b}$$

It is easy to solve for a and b. The determinant of the coefficient matrix is

$$D = \begin{vmatrix} \sin kx' & -\sin k(x'-l) \\ k\cos kx' & -k\cos k(x'-l) \end{vmatrix} = -k \sin kl, \tag{11.18}$$

independent of x'. Then the solutions are

$$a(x') \; = \; \frac{1}{D} \begin{vmatrix} 0 & -\sin k(x'-l) \\ -1 & -k\cos k(x'-l) \end{vmatrix} = \frac{\sin k(x'-l)}{k \sin kl}, \tag{11.19a}$$
$$b(x') \; = \; \frac{1}{D} \begin{vmatrix} \sin kx' & 0 \\ k\cos kx' & -1 \end{vmatrix} = \frac{\sin kx'}{k \sin kl}. \tag{11.19b}$$

Thus we find a closed form for the Green's function in the two regions:

$$x < x' : \qquad G_k(x,x') = \frac{\sin k(x'-l) \sin kx}{k \sin kl}, \tag{11.20a}$$
$$x > x' : \qquad G_k(x,x') = \frac{\sin kx' \sin k(x-l)}{k \sin kl}, \tag{11.20b}$$

or compactly,

$$G_k(x,x') = \frac{1}{k \sin kl} \sin kx_< \sin k(x_> - l), \tag{11.21}$$

where we have introduced the notation

$$x_< \quad \text{is the lesser of } x, x',$$
$$x_> \quad \text{is the greater of } x, x'. \tag{11.22}$$

Note that $G_k(x,x') = G_k(x',x)$ as is demanded on general grounds, as a consequence of the reciprocity relation (10.110).

Let us analyze the analytic structure of $G_k(x,x')$ as a function of k. We see that simple poles occur where

$$kl = n\pi, \quad n = \pm 1, \pm 2, \ldots. \tag{11.23}$$

There is no pole at $k=0$. For k near $n\pi/l$, we have

$$\sin kl = \sin n\pi + (kl - n\pi)\cos n\pi + \ldots = (kl - n\pi)(-1)^n. \tag{11.24}$$

If we simply sum over all the poles of G_k, we obtain

$$\begin{aligned} G_k(x,x') &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n \frac{\sin\frac{n\pi x_<}{l}\sin\frac{n\pi}{l}(x_> - l)}{\frac{n\pi}{l}(kl - n\pi)} \\ &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin\frac{n\pi x}{l}\sin\frac{n\pi x'}{l}}{n\pi\left(k - \frac{n\pi}{l}\right)} \\ &= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\sin\frac{n\pi x'}{l}\frac{1}{n\pi}\left[\frac{1}{k-\frac{n\pi}{l}} - \frac{1}{k+\frac{n\pi}{l}}\right] \\ &= \sum_{n=1}^{\infty} \frac{2}{l}\sin\frac{n\pi x}{l}\sin\frac{n\pi x'}{l}\frac{1}{k^2 - \left(\frac{n\pi}{l}\right)^2}. \end{aligned} \tag{11.25}$$

This is in fact equal to G_k, as seen in the eigenfunction expansion (11.10), because the difference is an entire function vanishing at infinity, which must be zero by Liouville's theorem, see Sec. 6.5.

11.2 Types of Boundary Conditions

Three types of second-order, homogeneous differential equations are commonly encountered in physics (the dimensionality of space is not important):

$$\text{Hyperbolic:} \quad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\mathbf{r},t) = 0, \tag{11.26a}$$

$$\text{Elliptic:} \quad \left(\nabla^2 + k^2\right) u(\mathbf{r}) = 0, \tag{11.26b}$$

$$\text{Parabolic:} \quad \left(\nabla^2 - \frac{1}{\kappa}\frac{\partial}{\partial t}\right) T(\mathbf{r},t) = 0. \tag{11.26c}$$

The first of these equations is the wave equation, the second is the Helmholtz equation, which includes Laplace's equation as a special case ($k=0$), and the third is the diffusion equation. The types of boundary conditions, specified on which kind of boundaries, necessary to uniquely specify a solution to these equations are given in Table 11.1. Here by *Cauchy boundary conditions* we means that both the function u and its normal derivative $\partial u/\partial n$ is specified on the boundary. Here

$$\frac{\partial u}{\partial n} = \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} u, \tag{11.27}$$

where $\hat{\mathbf{n}}$ is a(n outwardly directed) normal vector to the surface. As we have seen previously, Dirichlet boundary conditions refer to specifying the function u on the surface, Neumann boundary conditions refer to specifying the normal derivative $\partial u/\partial n$ on the surface, and mixed boundary conditions refer to

Type of Equation	Type of Boundary Condition	Type of Boundary
Hyperbolic	Cauchy	Open
Elliptic	Dirichlet, Neumann, or mixed	Closed
Parabolic	Dirichlet, Neumann, or mixed	Open

Table 11.1: Boundary conditions required for the three types of second-order differential equations. The boundary conditions referred to in the first and third cases are actually initial conditions.

specifying a linear combination, $\alpha u + \beta\partial u/\partial n$, on the surface. If the specified boundary values are zero, we say that the boundary conditions are *homogeneous*; otherwise, they are *inhomogeneous*.

Example.

To determine the vibrations of a string, described by

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u = 0, \tag{11.28}$$

we must specify

$$u(x,0), \quad \frac{\partial u}{\partial t}(x,0) \tag{11.29}$$

at some initial time ($t = 0$). The line $t = 0$ is an open surface in the (ct, x) plane.

11.3 Expression of Field in Terms of Green's Function

Typically, one determines the eigenfunctions of a differential operator subject to *homogeneous* boundary conditions. That means that the Green's functions obey the same conditions. See Sec. 10.8. But suppose we seek a solution of

$$(L - \lambda)\psi = S \tag{11.30}$$

subject to *inhomogeneous* boundary conditions. It cannot then be true that

$$\psi(\mathbf{r}) = \int_V (d\mathbf{r}')\, G(\mathbf{r},\mathbf{r}')S(\mathbf{r}'). \tag{11.31}$$

To see how to deal with this situation, let us consider the example of the three-dimensional Helmholtz equation,

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = S(\mathbf{r}). \tag{11.32}$$

We seek the solution $\psi(\mathbf{r})$ subject to arbitrary *inhomogeneous* Dirichlet, Neumann, or mixed boundary conditions on a surface Σ enclosing the volume V of interest. The Green's function G for this problem satisfies

$$(\nabla^2 + k^2)G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{11.33}$$

subject to *homogeneous* boundary conditions of the same type as ψ satisfies. Now multiply Eq. (11.32) by G, Eq. (11.33) by ψ, subtract, and integrate over the appropriate variables:

$$\begin{aligned}&\int_V (d\mathbf{r}')\left[G(\mathbf{r},\mathbf{r}')(\nabla'^2+k^2)\psi(\mathbf{r}') - \psi(\mathbf{r}')(\nabla'^2+k^2)G(\mathbf{r},\mathbf{r}')\right]\\ &= \int_V (d\mathbf{r}')\left[G(\mathbf{r},\mathbf{r}')S(\mathbf{r}') - \psi(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\right].\end{aligned} \tag{11.34}$$

Here we have interchanged $\mathbf{r}$ and $\mathbf{r}'$ in Eqs. (11.32) and (11.33), and have used the reciprocity relation,

$$G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r}). \tag{11.35}$$

(We have assumed that the eigenfunctions and hence the Green's function are real.) Now we use Green's theorem to establish

$$\begin{aligned}&-\int_\Sigma d\boldsymbol{\sigma}\cdot\left[G(\mathbf{r},\mathbf{r}')\boldsymbol{\nabla}'\psi(\mathbf{r}') - \psi(\mathbf{r}')\boldsymbol{\nabla}'G(\mathbf{r},\mathbf{r}')\right]\\ &+\int_V (d\mathbf{r}')\,G(\mathbf{r},\mathbf{r}')S(\mathbf{r}') = \begin{cases}\psi(\mathbf{r}), & \mathbf{r}\in V,\\ 0, & \mathbf{r}\notin V,\end{cases}\end{aligned} \tag{11.36}$$

where in the surface integral $d\boldsymbol{\sigma}$ is the outwardly directed surface element, and $\mathbf{r}'$ lies on the surface Σ. This generalizes the simple relation given in Eq. (11.31).

How do we use this result? We always suppose G satisfies homogeneous boundary conditions on Σ. If ψ satisfies the same conditions, then for $\mathbf{r}\in V$ Eq. (11.31) holds. But suppose ψ satisfies inhomogeneous Dirichlet boundary conditions on Σ,

$$\psi(\mathbf{r}')\big|_{\mathbf{r}'\in\Sigma} = \psi_0(\mathbf{r}'), \tag{11.37}$$

a specified function on the surface. Then we impose homogeneous Dirichlet conditions on G,

$$G(\mathbf{r},\mathbf{r}')\big|_{\mathbf{r}'\in\Sigma} = 0. \tag{11.38}$$

Then the first surface term in Eq. (11.36) is zero, but the second contributes. For example if $S(\mathbf{r}) = 0$ inside V, we have for $\mathbf{r}\in V$

$$\psi(\mathbf{r}) = \int_\Sigma d\boldsymbol{\sigma}\cdot[\boldsymbol{\nabla}'G(\mathbf{r},\mathbf{r}')]\psi_0(\mathbf{r}'), \tag{11.39}$$

which express ψ in terms of its boundary values.

If ψ satisfies inhomogeneous Neumann conditions on Σ,

$$\frac{\partial\psi}{\partial n'}(\mathbf{r}')\bigg|_{\mathbf{r}'\in\Sigma} = N(\mathbf{r}'), \tag{11.40}$$

a specified function, then we use the Green's function which respects homogeneous Neumann conditions,

$$\frac{\partial}{\partial n'} G(\mathbf{r}, \mathbf{r}')\bigg|_{\mathbf{r}' \in \Sigma} = 0, \tag{11.41}$$

so again if $S = 0$ inside V, we have within V

$$\psi(\mathbf{r}) = -\int_\Sigma d\sigma\, G(\mathbf{r}, \mathbf{r}') N(\mathbf{r}'). \tag{11.42}$$

Finally, if ψ satisfies inhomogeneous mixed boundary conditions,

$$\left[\frac{\partial}{\partial n'}\psi(\mathbf{r}') + \alpha(\mathbf{r}')\psi(\mathbf{r}')\right]\bigg|_{\mathbf{r}' \in \Sigma} = F(\mathbf{r}'), \tag{11.43}$$

then when G satisfies homogeneous boundary conditions of the same type

$$\left[\frac{\partial}{\partial n'} + \alpha(\mathbf{r}')\right] G(\mathbf{r}, \mathbf{r}')\bigg|_{\mathbf{r}' \in \Sigma} = 0, \tag{11.44}$$

we have for $\mathbf{r} \in V$

$$\psi(\mathbf{r}) = \int_V (d\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}') S(\mathbf{r}') - \int_\sigma d\sigma\, G(\mathbf{r}, \mathbf{r}') F(\mathbf{r}'). \tag{11.45}$$

11.4 Helmholtz Equation Inside a Sphere

Here we wish to find the Green's function for Helmholtz's equation, which satisfies

$$(\nabla^2 + k^2) G_k(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \tag{11.46}$$

in the interior of a spherical region of radius a, with homogeneous Dirichlet boundary conditions on the surface,

$$G_k(\mathbf{r}, \mathbf{r}')\bigg|_{|\mathbf{r}|=a} = 0. \tag{11.47}$$

We will use two methods.

11.4.1 Eigenfunction Method

We know that the eigenfunctions of the Laplacian are

$$j_l(kr) Y_l^m(\theta, \phi), \tag{11.48}$$

in spherical polar coordinates, r, θ, ϕ; that is,

$$(\nabla^2 + k^2) j_l(kr) Y_l^m(\theta, \phi) = 0. \tag{11.49}$$

Here j_l is the spherical Bessel function,

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x), \tag{11.50}$$

and the Y_l^m are the spherical harmonics,

$$Y_l^m(\theta, \phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta) e^{im\phi}, \tag{11.51}$$

where P_l^m is the associated Legrendre function. Here l is a nonnegative integer, and m is an integer in the range $-l \leq m \leq l$. For example, the first few spherical Bessel functions (which are simpler than the cylinder functions, the Bessel functions of integer order) are

$$j_0(x) = \frac{\sin x}{x}, \tag{11.52a}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \tag{11.52b}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x, \tag{11.52c}$$

and in general

$$j_l(x) = x^l \left(-\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}. \tag{11.53}$$

The associated Legrendre function is given by

$$P_l^m(\cos\theta) = (-1)^m \sin^m\theta \left(\frac{d}{d\cos\theta}\right)^{l+m} \frac{(\cos^2\theta - 1)^l}{2^l\, l!}. \tag{11.54}$$

For example, the first few spherical harmonics are

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \tag{11.55a}$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}, \tag{11.55b}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta, \tag{11.55c}$$

$$Y_1^1 = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}, \tag{11.55d}$$

$$Y_2^2 = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{2i\phi}, \tag{11.55e}$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \cos\theta \sin\theta\, e^{i\phi}, \tag{11.55f}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1), \tag{11.55g}$$

$$Y_2^{-1} = \sqrt{\frac{15}{8\pi}} \cos\theta \sin\theta \, e^{-i\phi}, \tag{11.55h}$$

$$Y_2^{-2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{-2i\phi}. \tag{11.55i}$$

The eigenfunctions must vanish ar $r = a$, so if β_{ln} is the nth zero of j_l,

$$j_l(\beta_{ln}) = 0, \quad n = 1, 2, 3, \dots, \tag{11.56}$$

the desired eigenfunctions are

$$\psi_{nlm}(r, \theta, \phi) = A_{nl} j_l \left(\beta_{ln} \frac{r}{a}\right) Y_l^m(\theta, \phi), \tag{11.57}$$

and the eigenvalues are

$$\lambda_{ln} = -k_{ln}^2 = -\left(\frac{\beta_{ln}}{a}\right)^2. \tag{11.58}$$

The normalization constant A_{nl} is determined by the requirement that

$$\int r^2 \, dr \, d\Omega \, |\psi_{nlm}(r, \theta, \phi)|^2 = 1, \tag{11.59}$$

where $d\Omega = \sin\theta \, d\theta \, d\phi$ is the element of solid angle. Since the spherical harmonics are normalized so that [$\Omega = (\theta, \phi)$ represents a point on the unit sphere]

$$\int d\Omega \, Y_{l'}^{m'*}(\Omega) Y_l^m(\Omega) = \delta_{ll'} \delta_{mm'}, \tag{11.60}$$

the normalization constant is determined by the requirement

$$|A_{nl}|^2 \int_0^a r^2 \, dr \left[j_l \left(\beta_{ln} \frac{r}{a}\right)\right]^2 = 1. \tag{11.61}$$

Now

$$\int_0^a r^2 \, dr \, j_l(\beta_{ln} r/a) j_l(\beta_{lm} r/a) = \delta_{nm} \frac{1}{2} a^3 j_{l+1}^2(\beta_{ln}), \tag{11.62}$$

which for $n \neq m$ follows from the orthogonality property (10.68). So

$$|A_{nl}| = \sqrt{\frac{2}{a^3}} \frac{1}{j_{l+1}(\beta_{ln})}, \tag{11.63}$$

and the Green's function has the eigenfunction expansion

$$G_k(\mathbf{r}, \mathbf{r}') = \sum_{nlm} \frac{2}{a^3} \frac{1}{j_{l+1}^2(\beta_{ln})} \frac{Y_l^m(\Omega) Y_l^{m*}(\Omega') j_l(\beta_{ln} r/a) j_l(\beta_{ln} r'/a)}{k^2 - (\beta_{ln}/a)^2}, \tag{11.64}$$

where $\Omega = (\theta, \phi)$, $\Omega' = (\theta', \phi')$.

This result can be simplified by carrying out the sum on m, using the addition theorem for spherical harmonics,

$$\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^{m*}(\Omega')Y_l^m(\Omega) = P_l(\cos\gamma), \tag{11.65}$$

where $P_l(\cos\gamma) = P_l^0(\cos\gamma)$ is Legendre's polynomial, and γ is the angle between the directions represented by Ω and Ω', or

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'). \tag{11.66}$$

Then we obtain

$$G_k(\mathbf{r},\mathbf{r}') = \frac{2}{a^3}\sum_{nl}\frac{2l+1}{4\pi}P_l(\cos\gamma)\frac{1}{j_{l+1}^2(\beta_{ln})}\frac{j_l(\beta_{ln}r/a)j_l(\beta_{ln}r'/a)}{k^2-(\beta_{ln}/a)^2}. \tag{11.67}$$

This leads us to the second method.

11.4.2 Discontinuity (Direct) Method

Let us adopt the angular dependence found above:

$$G_k(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\cos\gamma)g_l(r,r'), \tag{11.68}$$

where we will call g_l the reduced Green's function. Because Y_l^m is an eigenfunction of the angular part of the Laplacian operator,

$$\nabla^2 Y_l^m(\Omega) = -\frac{l(l+1)}{r^2}Y_l^m(\Omega), \tag{11.69}$$

and the delta function can be written as

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{rr'}\delta(r-r')\delta(\Omega-\Omega'), \tag{11.70}$$

we see that, because of the orthonormality of the spherical harmonics, Eq. (11.60), the Green's function equation (11.46) corresponds to the following equation satisfied by the reduced Green's function, the inhomogeneous "spherical Bessel equation,"

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}+k^2\right]g_l(r,r') = \frac{1}{rr'}\delta(r-r'). \tag{11.71}$$

We solve this equation directly. For $(0 < r' < a)$

$$0 \le r < r': \qquad g_l(r,r') = a(r')j_l(kr), \tag{11.72a}$$

$$r' < r \le a: \qquad g_l(r,r') = b(r')j_l(kr) + c(r')n_l(kr). \tag{11.72b}$$

Only j_l appears in the first form because the solution must be finite at $r=0$, and the second solution to the spherical Bessel equation,

$$n_l(x)=\sqrt{\frac{\pi}{2x}}N_{l+1/2}(x), \tag{11.73}$$

where N_ν is the Neumann function, is singular at $x=0$. For example,

$$n_0(x)=-\frac{\cos x}{x}, \tag{11.74}$$

and in general

$$n_l(x)=-x^l\left(-\frac{1}{x}\frac{d}{dx}\right)^l\frac{\cos x}{x}. \tag{11.75}$$

To determine the functions a, b, and c, we proceed as follows. The boundary condition at $r=a$, $g_l(a,r')=0$, implies

$$0=b(r')j_l(ka)+c(r')n_l(ka), \tag{11.76}$$

or

$$\frac{b(r')}{c(r')}=-\frac{n_l(ka)}{j_l(ka)}. \tag{11.77}$$

Thus we can write in the outer region,

$$a\geq r>r':\quad g_l(r,r')=A(r')[j_l(kr)n_l(ka)-n_l(kr)j_l(ka)]. \tag{11.78}$$

The next condition we impose is that of the continuity of g_l at $r=r'$:

$$a(r')j_l(kr')=A(r')[j_l(kr')n_l(ka)-n_l(kr')j_l(ka)]. \tag{11.79}$$

On the other hand, the derivative of g_l is discontinuous at $r=r'$, as we may see by integrating Eq. (11.71) over a tiny interval around $r=r'$:

$$\frac{d}{dr}g_l(r,r')\bigg|_{r=r'-\epsilon}^{r=r'+\epsilon}=\frac{1}{r'^2}, \tag{11.80}$$

which implies

$$ka(r')j_l'(kr')-kA(r')[j_l'(kr')n_l(ka)-n_l'(kr')j_l(ka)]=-\frac{1}{r'^2}. \tag{11.81}$$

Now multiply Eq. (11.79) by $kj_l'(kr')$, and Eq. (11.81) by $j_l(kr')$, and subtract:

$$\frac{j_l(kr')}{r'^2}=-kA(r')j_l(ka)[j_l(kr')n_l'(kr')-n_l(kr')j_l'(kr')]. \tag{11.82}$$

Now j_l, n_l are the independent solutions of the spherical Bessel equation

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{l(l+1)}{r^2}+k^2\right]u=0, \tag{11.83}$$

the Wronskian of which,

$$\Delta(r) \equiv j_l(kr)n_l'(kr) - n_l(kr)j_l'(kr) \tag{11.84}$$

has the form

$$\Delta(r) = \frac{\text{const.}}{r^2}, \tag{11.85}$$

as we saw in Problem 4 of Assignment 8. We can determine the constant by considering the asymptotic forms of j_l, n_l,

$$j_l(kr) \sim \frac{\sin(kr - l\pi/2)}{kr}, \quad kr \gg 1, \tag{11.86a}$$

$$n_l(kr) \sim -\frac{\cos(kr - l\pi/2)}{kr}, \quad kr \gg 1, \tag{11.86b}$$

which imply

$$\begin{aligned}\Delta(r) &= \frac{1}{k^2r^2}[\sin^2(kr - l\pi/2) + \cos^2(kr - l\pi/2)] \\ &= \frac{1}{(kr)^2}.\end{aligned} \tag{11.87}$$

Thus since the right-hand side of Eq. (11.82) is proportional to the Wronskian, we find the function A:

$$A(r') = -k\frac{j_l(kr')}{j_l(ka)}, \tag{11.88}$$

and then from Eq. (11.79) we find the function a:

$$a(r') = -\frac{k}{j_l(ka)}[j_l(kr')n_l(ka) - n_l(kr')j_l(ka)]. \tag{11.89}$$

Hence the Green's function is explicitly

$$r < r': \qquad g_l(r,r') = -k\frac{j_l(kr)}{j_l(ka)}[j_l(kr')n_l(ka) - n_l(kr')j_l(ka)], \tag{11.90a}$$

$$r > r': \qquad g_l(r,r') = -k\frac{j_l(kr')}{j_l(ka)}[j_l(kr)n_l(ka) - n_l(kr)j_l(ka)], \tag{11.90b}$$

or

$$g_l(r,r') = -kj_l(kr_<)j_l(kr_>)\left[\frac{n_l(ka)}{j_l(ka)} - \frac{n_l(kr_>)}{j_l(kr_>)}\right], \tag{11.91}$$

where $r_<$ is the lesser of r, r', and $r_>$ is the greater.

From this closed form we may extract the eigenvalues and eigenfunctions of the spherical Bessel differential operator appearing in Eq. (11.83). The poles of g_l occur where $j_l(ka)$ has zeroes, all of which are real, at $ka = \beta_{ln}$, the nth zero of j_l, or

$$k^2 = \left(\frac{\beta_{ln}}{a}\right)^2. \tag{11.92}$$

In the neighborhood of this zero,

$$j_l(ka) = (ka - \beta_{ln}) j_l'(\beta_{ln}). \tag{11.93}$$

But at the zero the Wronskian is

$$\frac{1}{(\beta_{ln})^2} = -n_l(\beta_{ln}) j_l'(\beta_{ln}). \tag{11.94}$$

Now from the recursion relation

$$J_\lambda'(z) = \frac{\lambda}{z} J_\lambda(z) - J_{\lambda+1}(z), \tag{11.95}$$

we see that the derivative of the spherical Bessel function (11.50) satisfies, at the zero,

$$j_l'(\beta_{ln}) = -j_{l+1}(\beta_{ln}). \tag{11.96}$$

Thus the residue of the pole of g_l at $k = \beta_{ln}/a$ is

$$\frac{1}{a^2 \beta_{ln}} \frac{j_l(\beta_{ln} r_</a) j_l(\beta_{ln} r_>/a)}{j_{l+1}^2(\beta_{ln})}. \tag{11.97}$$

Now j_l is an even or odd function of z depending on whether n is even or odd. So if β_{ln} is a zero of j_l, so is $-\beta_{ln}$, and hence if we add the contributions of these two poles, we get the corresponding contribution to g_l:

$$g_l(r, r') \sim \frac{1}{a^2 \beta_{ln}} \frac{j_l(\beta_{ln} r/a) j_l(\beta_{ln} r'/a)}{[j_{l+1}(\beta_{ln})]^2} \left(\frac{1}{k - \beta_{ln}/a} - \frac{1}{k + \beta_{ln}/a} \right). \tag{11.98}$$

Summing up the contribution of all such pairs of poles, we obtain

$$g_l(r, r') = \frac{2}{a^3} \sum_{n=1}^{\infty} \frac{j_l(\beta_{ln} r/a) j_l(\beta_{ln} r'/a)}{[j_{l+1}(\beta_{ln})]^2} \frac{1}{k^2 - (\beta_{ln}/a)^2}, \tag{11.99}$$

which is the eigenfunction expansion displayed in Eq. (11.67).

11.5 Helmholtz Equation in Unbounded Space

Again we are solving the equation

$$(\nabla^2 + k^2) G_k(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \tag{11.100}$$

but now in unbounded space. The solution to this equation is an outgoing spherical wave:

$$G_k(\mathbf{r}, \mathbf{r}') = G_k(\mathbf{r} - \mathbf{r}') = -\frac{1}{4\pi} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}. \tag{11.101}$$

This may be directly verified. Consider a small sphere S, of radius ϵ, centered on $\mathbf{r}'$:

$$\begin{aligned}\int_S (d\mathbf{r})(\nabla^2+k^2)G_k(\mathbf{r}-\mathbf{r}') &\approx \int_S (d\boldsymbol{\rho})\nabla^2_\rho\left(-\frac{1}{4\pi}\frac{e^{ik\rho}}{\rho}\right)\\ &= \int d\Omega\,\rho^2\frac{d}{d\rho}\left(-\frac{1}{4\pi}\frac{e^{ik\rho}}{\rho}\right)\bigg|_{\rho=\epsilon}\\ &\to 1, \end{aligned}\tag{11.102}$$

as $\epsilon \to 0$. Evidently, for $\mathbf{r}\neq\mathbf{r}'$, G_k satisfies the Helmholtz equation, $(\nabla^2+k^2)G_k=0$.

Alternatively, we may construct G_k from the eigenfunction expansion (10.109),

$$G_k(\mathbf{r}-\mathbf{r}') = \sum_n \frac{\psi_n^*(\mathbf{r}')\psi_n(\mathbf{r})}{\lambda_n-\lambda}\tag{11.103}$$

where $\lambda=-k^2$, $\lambda_n=-k'^2$, where the eigenfunctions are solutions of

$$(\nabla^2+k'^2)\psi_{\mathbf{k}'}(\mathbf{r})=0,\tag{11.104}$$

that is, they are plane waves,

$$\psi_{\mathbf{k}'}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}e^{i\mathbf{k}'\cdot\mathbf{r}},\tag{11.105}$$

Here the $(2\pi)^{-3/2}$ factor is for normalization:

$$\int (d\mathbf{k}')\,\psi_{\mathbf{k}'}(\mathbf{r})^*\psi_{\mathbf{k}'}(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'),\tag{11.106a}$$

$$\int (d\mathbf{r})\,\psi_{\mathbf{k}'}(\mathbf{r})^*\psi_{\mathbf{k}}(\mathbf{r}) = \delta(\mathbf{k}-\mathbf{k}'),\tag{11.106b}$$

where we have noted that the spectrum of eigenvalues is continuous,

$$\sum_n \to \int (d\mathbf{k}).\tag{11.107}$$

Thus the eigenfunction expansion for the Green's function has the form

$$G_k(\mathbf{r}-\mathbf{r}') = \int \frac{(d\mathbf{k}')}{(2\pi)^3}\frac{e^{-i\mathbf{k}'\cdot\mathbf{r}'}e^{i\mathbf{k}'\cdot\mathbf{r}}}{k^2-k'^2}.\tag{11.108}$$

Let us evaluate this integral in spherical coordinates, where we write

$$(d\mathbf{k}') = k'^2\,dk'\,d\phi'\,d\mu',\quad \mu'=\cos\theta',\tag{11.109}$$

where we have chosen the z axis to lie along the direction of $\mathbf{r}-\mathbf{r}'$. The integration over the angles is easy:

$$\begin{aligned}G_k(\mathbf{r}-\mathbf{r}') &= \frac{1}{(2\pi)^3}\int_0^\infty dk'\,k'^2\int_0^{2\pi}d\phi'\int_{-1}^1 d\mu'\,\frac{e^{ik'|\mathbf{r}-\mathbf{r}'|\mu'}}{k^2-k'^2}\\ &= \frac{1}{(2\pi)^2}\frac{1}{2}\int_{-\infty}^\infty \frac{dk'\,k'^2}{k^2-k'^2}\frac{1}{ik'\rho}\left(e^{ik'\rho}-e^{-ik'\rho}\right),\end{aligned}\tag{11.110}$$

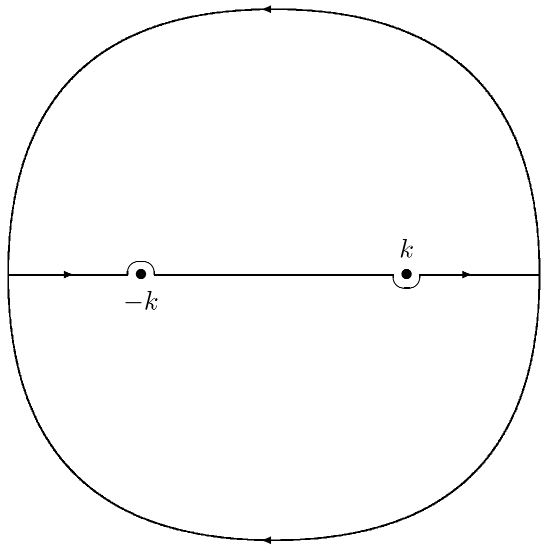

Figure 11.1: Contour in the k' plane used to evaluate the integral (11.110). The integral is closed in the upper (lower) half-plane if the exponent is positive (negative). The poles in the integrand are avoided by passing above the one on the left, and below the one on the right.

defining $\rho = |\mathbf{r} - \mathbf{r}'|$, where we have replaced $\int_0^\infty$ by $\frac{1}{2}\int_{-\infty}^\infty$ because the integrand is even in k'^2. We evaluate this integral by contour methods. Because now k can coincide with an eigenvalue k', we must choose the contour appropriately to define the Green's function. Suppose we choose the contour as shown in Fig. 11.1, passing below the pole at k and above the pole at $-k$. We close the contour in the upper half plane for the $e^{ik\rho}$ and in the lower half plane for the $e^{-ik\rho}$ term. Then by Jordan's lemma, we immediately evaluate the integral:

$$\begin{aligned} G_k(\mathbf{r} - \mathbf{r}') &= \frac{1}{(2\pi)^2}\frac{1}{2}\left[-\frac{2\pi i}{2k}\frac{k\,e^{ik\rho}}{i\rho} + \frac{2\pi i}{-2k}\frac{k\,e^{ik\rho}}{i\rho}\right] \\ &= -\frac{1}{4\pi}\frac{e^{ik\rho}}{\rho}, \end{aligned} \tag{11.111}$$

which coincides with Eq. (11.101). If a different contour defining the integral had been chosen, we would have obtained a different Green's function, not one corresponding to outgoing spherical waves. Boundary conditions uniquely determine the contour.

Note that

$$G_k(\mathbf{r}, \mathbf{r}') = G_k(\mathbf{r}', \mathbf{r}), \tag{11.112}$$

even though G_k is complex. The self-adjointness property (10.110) implied by the eigenfunction expansion is only formal, and is spoiled by the contour choice.

11.6 Green's Function for the Scalar Wave Equation

The inhomogeneous scalar wave equation,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(\mathbf{r},t) = \rho(\mathbf{r},t), \tag{11.113}$$

requires boundary and initial conditions. The boundary conditions may be Dirichlet, Neumann, or mixed. The initial conditions are Cauchy (see Sec. 11.2). Thus, we might specify at an initial time $t = t_0$ both $\psi(\mathbf{r},t_0)$ and $\frac{\partial}{\partial t}\psi(\mathbf{r},t_0)$ at every point $\mathbf{r}$ in the region being considered.

The corresponding Green's function $G(\mathbf{r},t;\mathbf{r}',t')$ satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{11.114}$$

It must satisfy the homogeneous form of the boundary conditions satisfied by ψ. Thus, if ψ has a specified value everywhere on the bounding surface, the corresponding Green's function must vanish on the surface. In classical physics it is customary to adopt as *initial conditions*

$$\left.\begin{array}{c} G(\mathbf{r},t;\mathbf{r}',t') \\ \frac{\partial G}{\partial t}(\mathbf{r},t;\mathbf{r}',t') \end{array}\right\} = 0 \quad \text{if} \quad t < t'. \tag{11.115}$$

These then define the so-called *retarded* Green's functions. They ensure that an effect occurs after its cause. (In fact, however, this time asymmetry of the Green's function, which is not present in the wave equation, is not necessary; and in fact it is impossible to maintain in relativistic quantum mechanics.)

With such a Green's function, what takes the place of the self-adjointness property given in Sec. 10.8? Since the second time derivative is invariant under $t \to -t$, we have in addition to the inhomogeneous equation (11.114)

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},-t;\mathbf{r}'',-t'') = \delta(\mathbf{r}-\mathbf{r}'')\delta(t-t''). \tag{11.116}$$

Multiply Eq. (11.116) by $G(\mathbf{r},t;\mathbf{r}',t)$, Eq. (11.114) by $G(\mathbf{r},-t;\mathbf{r}'',-t'')$, subtract, and integrate over the volume being considered, and over t from $-\infty$ to T, where $T > t', t''$:

$$\begin{aligned}
\int_{-\infty}^{T} dt \int_V (d\mathbf{r}) \Big\{ & G(\mathbf{r},t;\mathbf{r}',t')\nabla^2 G(\mathbf{r},-t;\mathbf{r}'',-t'') \\
& - G(\mathbf{r},-t;\mathbf{r}'',-t'')\nabla^2 G(\mathbf{r},t;\mathbf{r}',t') \\
& - G(\mathbf{r},t;\mathbf{r}',t')\frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(\mathbf{r},-t;\mathbf{r}'',-t'') \\
& + G(\mathbf{r},-t;\mathbf{r}'',-t'')\frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(\mathbf{r},t;\mathbf{r}',t') \Big\} \\
= -G(\mathbf{r}',-t';\mathbf{r}'',-t'') + G(\mathbf{r}'',t''&;\mathbf{r}',t').
\end{aligned} \tag{11.117}$$

Now use Green's theorem, together with the corresponding identity,

$$\frac{\partial}{\partial t}\left(A\frac{\partial}{\partial t}B - B\frac{\partial}{\partial t}A\right) = A\frac{\partial^2}{\partial t^2}B - B\frac{\partial^2}{\partial t^2}A, \tag{11.118}$$

to conclude that

$$\begin{aligned}
&G(\mathbf{r}'', t''; \mathbf{r}', t') - G(\mathbf{r}', -t'; \mathbf{r}'', -t'') \\
= &\int_{-\infty}^{T} dt \int_{\Sigma} d\boldsymbol{\sigma} \cdot \Big\{ G(\mathbf{r}, t; \mathbf{r}', t') \boldsymbol{\nabla} G(\mathbf{r}, -t; \mathbf{r}'', -t'') \\
&\quad - G(\mathbf{r}, -t; \mathbf{r}'', -t'') \boldsymbol{\nabla} G(\mathbf{r}, t; \mathbf{r}', t') \Big\} \\
&- \int_V (d\mathbf{r}) \frac{1}{c^2} \Big\{ G(\mathbf{r}, t; \mathbf{r}', t') \frac{\partial}{\partial t} G(\mathbf{r}, -t; \mathbf{r}'', -t'') \\
&\quad - G(\mathbf{r}, -t; \mathbf{r}'', -t'') \frac{\partial}{\partial t} G(\mathbf{r}, t; \mathbf{r}', t') \Big\} \Big|_{t=-\infty}^{t=T} .
\end{aligned} \tag{11.119}$$

The surface integral vanishes, since both Green's functions satisfy the same homogeneous boundary conditions on Σ. (The boundary conditions are time independent.) The second integral is also zero because from Eq. (11.115)

$$\left. \begin{array}{c} G(\mathbf{r}, -\infty; \mathbf{r}', t') \\ \frac{\partial G}{\partial t}(\mathbf{r}, -\infty; \mathbf{r}', t') \end{array} \right\} = 0, \tag{11.120a}$$

since $-\infty < t'$, and

$$\left. \begin{array}{c} G(\mathbf{r}, -T; \mathbf{r}'', -t'') \\ \frac{\partial G}{\partial t}(\mathbf{r}, -T; \mathbf{r}'', -t'') \end{array} \right\} = 0, \tag{11.120b}$$

since $-T < -t''$. Thus the reciprocity relation here is

$$G(\mathbf{r}, t; \mathbf{r}', t') = G(\mathbf{r}', -t'; \mathbf{r}, -t) \tag{11.121}$$

How do we express a solution to the wave equation (11.113) in terms of the Green's function? The procedure is the same as that given earlier. The field, and the Green's function, satisfy

$$\nabla'^2 \psi(\mathbf{r}', t) - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\psi(\mathbf{r}', t') = \rho(\mathbf{r}', t'), \tag{11.122a}$$

$$\nabla'^2 G(\mathbf{r}, t; \mathbf{r}', t') - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} G(\mathbf{r}, t; \mathbf{r}', t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \tag{11.122b}$$

Note that the differentiations on G are with respect to the second set of arguments (this equation follows from the reciprocity relation). Again multiply the first equation by $G(\mathbf{r}, t; \mathbf{r}', t')$, the second by $\psi(\mathbf{r}', t')$, subtract, integrate over the volume, and over t' from $t_0 < t$ to t^+, where t^+ means $t + \epsilon$, $\epsilon \to 0$ through positive values. Then for $r \in V$,

$$\int_{t_0}^{t^+} dt' \int_V (d\mathbf{r}') \bigg\{ G(\mathbf{r},t;\mathbf{r}',t')\nabla'^2\psi(\mathbf{r}',t') - \psi(\mathbf{r}',t')\nabla'^2 G(\mathbf{r},t;\mathbf{r}',t') - \frac{1}{c^2}\left[G(\mathbf{r},t;\mathbf{r}',t')\frac{\partial^2}{\partial t'^2}\psi(\mathbf{r}',t') - \psi(\mathbf{r}',t')\frac{\partial^2}{\partial t'^2}G(\mathbf{r},t;\mathbf{r}',t')\right]\bigg\} = -\psi(\mathbf{r},t) + \int_{t_0}^{t^+} dt' \int_V (d\mathbf{r}')\, G(\mathbf{r},t;\mathbf{r}',t')\rho(\mathbf{r}',t'). \tag{11.123}$$

Now we again use Green's theorem and the identity (11.118) to conclude

$$\psi(\mathbf{r},t) = \int_{t_0}^{t^+} dt' \int_V (d\mathbf{r}')\, G(\mathbf{r},t;\mathbf{r}',t')\rho(\mathbf{r}',t') - \int_{t_0}^{t^+} dt' \oint_\Sigma d\boldsymbol{\sigma}\cdot\left[G(\mathbf{r},t;\mathbf{r}',t')\boldsymbol{\nabla}'\psi(\mathbf{r}',t') - \psi(\mathbf{r}',t')\boldsymbol{\nabla}'G(\mathbf{r},t;\mathbf{r}',t')\right] - \frac{1}{c^2}\int_V (d\mathbf{r}')\left[G(\mathbf{r},t;\mathbf{r}',t_0)\frac{\partial}{\partial t_0}\psi(\mathbf{r}',t_0) - \psi(\mathbf{r}',t_0)\frac{\partial}{\partial t_0}G(\mathbf{r},t;\mathbf{r}',t_0)\right]. \tag{11.124}$$

This is our result. The interpretation is as follows:

1. The first integral represents the effect of the sources ρ distributed throughout the volume V.
2. The second integral represents the boundary conditions. If, for example, ψ satisfies inhomogeneous Neumann boundary conditions on Σ,
$$\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\psi\Big|_\Sigma = f(\mathbf{r}') \tag{11.125}$$
is specified, then we use homogeneous Neumann boundary conditions for G,
$$\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}G(\mathbf{r},t;\mathbf{r}',t')\Big|_\Sigma = 0. \tag{11.126}$$
Then the second integral reads
$$-\int_{t_0}^{t^+} dt' \oint_\Sigma d\boldsymbol{\sigma}\cdot G(\mathbf{r},t;\mathbf{r}',t)\boldsymbol{\nabla}'\psi(\mathbf{r}',t'). \tag{11.127}$$
That is, $-\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}'\psi(\mathbf{r}',t')$ represents a surface source distribution. Other types of boundary conditions are as discussed earlier.
3. The third integral represents the effect of the initial conditions, where
$$\psi(\mathbf{r}',t_0), \quad \frac{\partial}{\partial t_0}\psi(\mathbf{r}',t_0) \tag{11.128}$$

are specified. They correspond to impulsive sources at $t = t_0$:

$$\rho_{\text{init}}(\mathbf{r}', t') = -\frac{1}{c^2}\left[\frac{\partial}{\partial t_0}\psi(\mathbf{r}', t_0)\delta(t'-t_0) + \psi(\mathbf{r}', t_0)\delta'(t'-t_0)\right]. \quad (11.129)$$

We verify this statement by integrating by parts, and letting the lower limit of the t' integration be $t_0 - \epsilon$.

11.7 Wave Equation in Unbounded Space

We now wish to solve Eq. (11.114)

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t; \mathbf{r}', t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \quad (11.130)$$

in unbounded space by noting that then G is a function of $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $T = t - t'$ only,

$$G(\mathbf{r}, t; \mathbf{r}', t') = G(\mathbf{r}-\mathbf{r}', t-t') = G(\mathbf{R}, T). \quad (11.131)$$

Then we can introduce a Fourier transform in space and time,

$$g(k, \omega) = \int (d\mathbf{R})\, dT e^{i\mathbf{k}\cdot\mathbf{R}} e^{-i\omega T} G(\mathbf{R}, T). \quad (11.132)$$

The Fourier transform of the Green's function equation is (we have set $c = 1$ temporarily for convenience),

$$(-k^2 + \omega^2) g(\mathbf{k}, \omega) = 1, \quad (11.133)$$

where we write $k^2 = \mathbf{k}\cdot\mathbf{k}$, which has the immediate solution

$$g(\mathbf{k}, \omega) = \frac{1}{\omega^2 - k^2}. \quad (11.134)$$

Thus the Green's function has the formal representation

$$G(\mathbf{R}, T) = \int \frac{(d\mathbf{k})}{(2\pi)^3}\frac{d\omega}{2\pi} e^{-i\mathbf{k}\cdot\mathbf{R}} e^{i\omega T}\frac{1}{\omega^2 - k^2}. \quad (11.135)$$

The ω integral here is not well defined until we impose the boundary condition (11.115)

$$G(\mathbf{R}, T) = 0 \quad \text{if} \quad T < 0. \quad (11.136)$$

This will be true if the poles are located above the real axis, as shown in Fig. 11.2. Here the contour is closed in the upper half plane if $T > 0$, and in the lower half plane if $T < 0$. In both cases, by Jordan's lemma, the infinite semicircle gives no contribution. We have

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega T}\frac{1}{(\omega - k)(\omega + k)} = \begin{cases} i\left(\frac{e^{ikT}}{2k} - \frac{e^{-ikT}}{2k}\right), & T > 0, \\ 0 & T < 0. \end{cases} \quad (11.137)$$

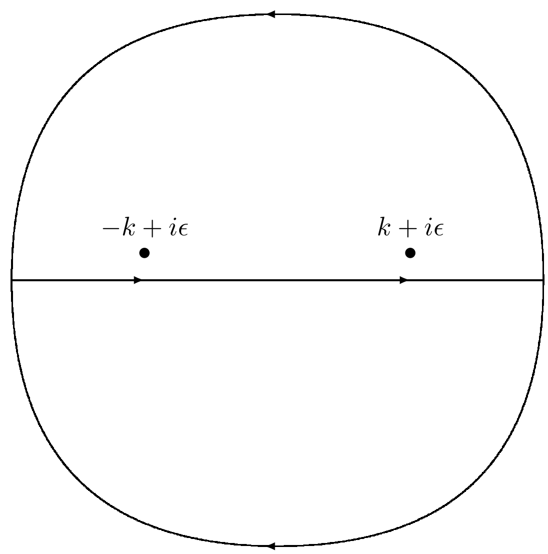

Figure 11.2: Contour in the ω plane used to evaluate the integral (11.135).

Thus, if $T > 0$,

$$\begin{aligned} G(\mathbf{R},T) &= \frac{1}{(2\pi)^3}\int_0^\infty k^2\,dk\,2\pi\int_{-1}^1 d\mu\,e^{-ikR\mu}\frac{i}{2k}\left(e^{ikT}-e^{-ikT}\right) \\ &= \frac{i}{(2\pi)^2}\int_0^\infty \frac{k\,dk}{2ikR}\left(e^{ikR}-e^{-ikR}\right)\left(e^{ikT}-e^{-ikT}\right) \\ &= \frac{1}{(2\pi)^2}\frac{1}{2R}\frac{1}{2}\int_{-\infty}^\infty dk\left(e^{ik(R+T)}+e^{-ik(R+T)}-e^{ik(R-T)}-e^{ik(T-R)}\right) \\ &= \frac{1}{2\pi}\frac{1}{2R}\left[\delta(R+T)-\delta(R-T)\right]. \end{aligned} \tag{11.138}$$

But R and T are both positive, so $R+T$ can never vanish. Thus we are left with

$$G(R,T) = -\frac{1}{4\pi}\frac{1}{R}\delta(R-T), \tag{11.139}$$

or restoring c,

$$G(\mathbf{r}-\mathbf{r}',t-t') = -\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(\frac{|\mathbf{r}-\mathbf{r}'|}{c}-(t-t')\right). \tag{11.140}$$

The effect at the observation point $\mathbf{r}$ at time t is due to the action at the source point $\mathbf{r}'$ at time

$$t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}. \tag{11.141}$$

Physically, this means that the "signal" propagates with speed c.

Let us make this more concrete by considering a simple example, a point "charge" moving with velocity $\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t)$,

$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{r}(t)). \tag{11.142}$$

There are no effects from the infinite surface, nor from the infinite past, so we have from Eq. (11.124)

$$\begin{aligned}\psi(\mathbf{r}, t) &= \int_{-\infty}^{t^+} dt' \int_V (d\mathbf{r}')\, G(\mathbf{r} - \mathbf{r}', t - t')\rho(\mathbf{r}', t') \\ &= -\frac{q}{4\pi}\int_{-\infty}^{t^+} dt' \frac{1}{|\mathbf{r} - \mathbf{r}(t')|}\delta\left(\frac{|\mathbf{r} - \mathbf{r}(t')|}{c} - (t - t')\right). \end{aligned} \tag{11.143}$$

If we let $R(t') = \mathbf{r} - \mathbf{r}(t')$, the distance from the source to the observation point at time $t' = t - \frac{R(t')}{c}$, we write this as

$$\psi(\mathbf{r}, t) = -\frac{q}{4\pi}\int_{-\infty}^{t^+} dt' \frac{1}{R(t')}\delta\left(\frac{R(t')}{c} - (t - t')\right). \tag{11.144}$$

Let $\tau = R(t')/c + t'$, where $\tau = t$ determines the "retarded time" t' so

$$d\tau = dt'\left(1 + \frac{1}{c}\frac{dR}{dt'}\right), \tag{11.145}$$

where

$$\frac{dR}{dt'} = \frac{1}{2R}\frac{d}{dt'}\mathbf{R}\cdot\mathbf{R} = -\frac{\mathbf{R}\cdot\mathbf{v}}{R}, \tag{11.146}$$

that is

$$d\tau = dt'\left(1 - \frac{\mathbf{R}\cdot\mathbf{v}}{Rc}\right). \tag{11.147}$$

Thus the field is evaluated as

$$\begin{aligned}\psi(\mathbf{r}, t) &= -\frac{q}{4\pi}\int_{-\infty}^{t+R(t)/c} \frac{d\tau}{R(\tau)}\frac{1}{\left(1 - \frac{\mathbf{R}\cdot\mathbf{v}}{Rc}\right)(\tau)}\delta(\tau - t) \\ &= -\frac{q}{4\pi}\frac{1}{R(t) - \mathbf{R}(t)\cdot\mathbf{v}(t)/c}. \end{aligned} \tag{11.148}$$