(1) \[ \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \] \[ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} \] \[ \vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \] \[ \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} \] --- (i) --- (ii) --- (iii) --- (iv)

(2) Using (ii) and (iii) in (i) and (iv) we have the following two equations describing electromagnetism:

\[ -\nabla^2 \phi = \frac{\rho}{\varepsilon_0} + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) \] \[ (-\nabla^2 + \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}) \vec{A} = \mu_0 \vec{J} - \vec{\nabla} (\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t}) \]

(3) **Radiation gauge** (also called Coulomb gauge) \[ \vec{\nabla} \cdot \vec{A} = 0 \] \[ -\nabla^2 \phi = \frac{\rho}{\varepsilon_0} \] \[ (-\nabla^2 + \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}) \vec{A} = \mu_0 \vec{J} - \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{\nabla} \phi \] --- introduced by Lorentz, not Lorentz, in 1850's.

(4) **Lorenz gauge** \[ \vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t} = 0 \] \[ (-\nabla^2 + \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}) \phi = \frac{\rho}{\varepsilon_0} \] \[ (-\nabla^2 + \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}) \vec{A} = \mu_0 \vec{J} \] --- was not common in we until Hertz's experiment in lab 1860's.
3) If we choose to not introduce potentials, we have,

\(- \nabla^2 + \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\) \(\vec{E} = -\nabla \phi - \sigma \mu_0 \frac{\partial \vec{J}}{\partial t}\)

\(- \nabla^2 + \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\) \(\vec{B} = \mu_0 \nabla \times \vec{J}\)

This is derived from (ii) and (iv) of (i) using the vector identity,

\(\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\).

The implication of (i) is that the fields \(\vec{E}\) and \(\vec{B}\) satisfy the wave equation in regions outside the sources \(\phi\) and \(\vec{J}\). In particular, the wave has its speed given in terms of the property of vacuum, \(c_0\) and \(\mu_0\),

\[c^2 = \frac{1}{\varepsilon_0 \mu_0}\]

This motivated the concept of ether, and eventually to the Special Theory of Relativity.