1. Consider the four-vector $x^\alpha = (ct, \mathbf{x})$. In terms of the proper time, that remains invariant under a Lorentz transformation,

$$-ds^2 = -c^2 dt^2 + d\mathbf{x} \cdot d\mathbf{x},$$

(1)

the energy $E$ and momentum $\mathbf{p}$ of a particle of mass $m$ is defined as

$$mc^2 \frac{dx^\alpha}{ds} = (E, \mathbf{p}).$$

(2)

Find the explicit expressions for $E$ and $\mathbf{p}$ in terms of $v = d\mathbf{x}/dt$, $c$, and $m$. Show that

$$\frac{dx^\alpha dx_\alpha}{ds} = -1,$$

(3)

and use this to derive $E^2 = p^2 c^2 + m^2 c^4$.

2. In terms of the four-vector potential

$$A^\mu = \left( \frac{1}{c} \phi, \mathbf{A} \right)$$

(4)

the Maxwell field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

(5)

and the corresponding dual tensor is defined as

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}.$$

(6)

Derive the following relations, which involve quantities that remain invariant under Lorentz transformations.

(a) $c^2 F^{\mu\nu} F_{\mu\nu} = 2(c^2 B^2 - E^2)$.
(b) $c^2 \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} = 2(E^2 - c^2 B^2)$.
(c) $c F^{\mu\nu} \tilde{F}_{\mu\nu} = -4 \mathbf{B} \cdot \mathbf{E}$. 