1. Using the series representation for Bessel functions,
\[ J_m(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(m+n)!} \left( \frac{t}{2} \right)^{m+2n}, \tag{1} \]
prove the relation
\[ J_m(t) = (-1)^m J_{-m}(t). \tag{2} \]
Hint: Break the sum on \( n \) into two parts. Note that the gamma function \( \Gamma(z) \), which generalizes the factorial,
\[ n! = \Gamma(n+1), \quad \Gamma(z + 1) = z\Gamma(z), \tag{3} \]
beyond positive integers, satisfies
\[ \frac{1}{\Gamma(-k)} = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots. \tag{4} \]

2. Use the integral representation of \( J_m(t) \),
\[ i^m J_m(t) = \int_{0}^{2\pi} \frac{d\alpha}{2\pi} e^{it \cos \alpha - im\alpha}, \tag{5} \]
to prove the recurrence relations
\[ 2\frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t), \tag{6a} \]
\[ 2\frac{m}{t} J_m(t) = J_{m-1}(t) + J_{m+1}(t). \tag{6b} \]

3. Using the recurrence relations of Eq. (6), show that
\[ \left( -\frac{d}{dt} + \frac{m-1}{t} \right) \left( \frac{d}{dt} + \frac{m}{t} \right) J_m(t) = \left( \frac{d}{dt} + \frac{m+1}{t} \right) \left( -\frac{d}{dt} + \frac{m}{t} \right) J_m(t) = J_m(t) \tag{7} \]
and from this derive the differential equation satisfied by \( J_m(t) \).
4. The modified Bessel functions, $I_m(t)$ and $K_m(t)$, satisfy the differential equation

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right]\left\{\frac{I_m(t)}{K_m(t)}\right\} = 0. \quad (8)$$

Derive the identity, for the Wronskian, (upto a constant $C$)

$$I_m(t)K'_m(t) - K_m(t)I'_m(t) = -\frac{C}{t}, \quad (9)$$

where

$$I'_m(t) \equiv \frac{d}{dt}I_m(t) \quad \text{and} \quad K'_m(t) \equiv \frac{d}{dt}K_m(t). \quad (10)$$

Further, determine the value of the constant $C$ on the right hand side of Eq. (9) using the asymptotic forms for the modified Bessel functions:

$$I_m(t) \xrightarrow{t \gg 1} \frac{1}{\sqrt{2\pi}} e^{t}, \quad (11)$$

$$K_m(t) \xrightarrow{t \gg 1} \sqrt{\frac{\pi}{2}} e^{-t}. \quad (12)$$

5. Show that the integral representation for modified Bessel functions,

$$K_m(t) = \int_0^\infty d\theta \cosh m\theta e^{-t\cosh \theta}, \quad (13)$$

satisfies the differential equation for modified Bessel functions,

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right]K_m(t) = 0. \quad (14)$$

6. Verify by substitution that

$$g_m(\rho, \rho'; k) = I_m(k\rho_<)K_m(k\rho_>), \quad (15)$$

satisfies the differential equation

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{m^2}{\rho^2} + k^2\right]g_m(\rho, \rho'; k) = \frac{\delta(\rho - \rho')}{\rho}. \quad (16)$$

Hint: Rewrite $g_m(\rho, \rho'; k)$ in terms of sign function $\theta(x)$ and use the identity $d\theta(x)/dx = \delta(x).$