Entropy Bounds in $R \times S^3$ Geometries

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Abstract

Exact calculations are given for the Casimir energy for various fields in $R \times S^3$ geometry. The Green’s function method naturally gives a result in a form convenient in the high-temperature limit, while the statistical mechanical approach gives a form convenient for low temperatures. The equivalence of these two representations is demonstrated. Some discrepancies with previous work are noted. In no case, even for $\mathcal{N} = 4$ SUSY, is the ratio of entropy to energy found to be bounded. This deviation, however, occurs for low temperature, where the equilibrium approach may not be relevant. The same methods are used to calculate the energy and free energy for the TE modes in a half-Einstein universe bounded by a perfectly conducting 2-sphere.
Introduction

The remarkable appearance of the holographic principle has fostered the understanding that some hitherto distant branches of theoretical physics may have a much deeper common origin that was expected. One bright example of this sort is the relation, suggested by Verlinde* between the Cardy entropy formula† and the Friedmann equation for the evolution of the scale factor of the universe. Moreover, the proposal that in the early universe there exists a holographic bound on the cosmological entropy associated with Casimir energy suggests that there should be a deeper relation between Friedmann cosmology and the Casimir effect.‡

*E. Verlinde, hep-th/00084140 v2.
There has much activity in the study of the Cardy-Verlinde formula and its various applications. Specifically, there has been much interest in studying the entropy and energy arising from quantum and thermal fluctuations in conformal field theories.* As we remarked, particularly interesting is the Verlinde bound for the ratio of the entropy to the thermal energy. Whether this bound can be realized in realistic situations is a matter for specific calculations. Previous computations have been limited to the specific regime of high temperatures, so they are unable to provide definitive results. Here we obtain exact results for various fields in the $\mathbb{R} \times S^3$ geometry, so the issues may be more definitively addressed. We also will consider electromagnetic modes in a half-Einstein universe, which give similar results using the same methods.

Conformally Coupled Scalar
A. Green’s Function Method

We can start from the formalism given in Kantowski and Milton.* The energy is given by the imaginary part of the Green’s function,

\[ U = V_3 \partial^0 \partial'^0 \Im G(x, y; x', y') \bigg|_{x=x', y=y'} \quad (1) \]

where the “external” coordinates \( x \) consist only of the time. We introduce a Fourier transform there

\[ G(t, y; t', y') = \int \frac{d\omega}{2\pi} e^{i\omega(t-t')} g(y, y'; \omega), \quad (2) \]

so

\[ U = -\frac{iV_3}{4\pi} \int_c d\omega \omega^2 g(y, y'; \omega), \quad (3) \]

where the contour \( c \) encircles the poles on the positive axis in a negative sense, and those

on the negative axis in a positive sense. The reduced Green’s function satisfies

$$\left( \nabla^2_3 + \omega^2 \right) g(y, y'; \omega) = -\delta(y - y').$$  \hspace{1cm} (4)$$

The eigenvectors and eigenvalues of the Laplacian on $S^3$ are

$$\nabla^2_3 Y_l^m(y) = -\frac{M_l^2}{a^2} Y_l^m, \quad M_l^2 = l(l + 2),$$  \hspace{1cm} (5)$$

which have degeneracy $D_l = (l + 1)^2$. The addition theorem is

$$\sum_m Y_l^m(y) Y_l^{m*}(y) = \frac{D_l}{V_3},$$  \hspace{1cm} (6)$$

so in view of the eigenfunction construction

$$g(y, y'; \omega) = \sum_{lm} \frac{Y_l^m(y) Y_l^{m*}(y')}{M_l^2/a^2 - \omega^2}$$  \hspace{1cm} (7)$$

the Casimir energy is

$$U = -\frac{i}{4\pi} \int_c d\omega \omega^2 \sum_l \frac{D_l}{M_l^2/a^2 - \omega^2}.$$  \hspace{1cm} (8)$$
Conformal coupling in incorporated by replacing the Laplacian in Eq. (4) by $\nabla_3^2 + \xi R$, which for the conformal value $\xi = 1/6$ amounts to the addition of 1 to the $M_l^2$ operator, that is, replacing $M_l^2 \rightarrow (l + 1)^2$.

Temperature dependence is incorporated by the replacement

$$\int_c \frac{d\omega}{2\pi} \rightarrow 4i \sum_{n=0}^{\infty}, \quad \omega^2 \rightarrow -\left(\frac{2\pi n}{\beta}\right)^2.$$  \hspace{1cm} (9)

The prime on the summation sign means that the $n = 0$ term is counted with half weight.

We carry out the sum on $l$ in Eq. (8) as follows: The general representation

$$\sum_{m=0}^{\infty} \frac{1}{m^2 - \alpha^2} = -\pi \cot \pi \alpha - \frac{1}{2\alpha^2}$$  \hspace{1cm} (10)

becomes here

$$\sum_{l=0}^{\infty} \frac{(l + 1)^2}{(l + 1)^2/a^2 - \omega^2} = \text{const.} + \frac{\pi i\omega a^3}{2} + \frac{\pi \omega a^3 i}{e^{-2\pi i\omega} - 1}. $$  \hspace{1cm} (11)
We then make the finite-temperature replacements (9) and obtain, after dropping the contact term arising from the constant ($\propto \zeta(-2) = 0$)

$$U = \frac{1}{a} \left( \frac{2\pi a}{\beta} \right)^4 \left[ \sum_{n=0}^{\infty} \frac{n^3}{e^{4\pi^2 an/\beta} - 1} + \frac{1}{240} \right],$$

(12)

which, since the summand vanishes at $n = 0$, gives only exponentially small corrections to Stefan’s law,

$$U \sim \frac{1}{a} \frac{(2\pi aT)^4}{240}, \quad aT \gg 1.$$

(13)

Here, and throughout the talk, we use the formulas for positive integer $n$

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}, \quad \zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|,$$

(14)

$B_{2n}$ being the Bernoulli numbers.
B. Statistical-Mechanical Approach

We recall the usual statistical mechanical expression for the free energy,

\begin{align}
F &= -kT \ln Z, \\
\ln Z &= -\sum_i \ln \left(1 - e^{-\beta p_i}\right) \\
&= -\sum_{n=0}^{\infty} (n + 1)^{d-2} \ln \left(1 - e^{-\beta(n+1)/a}\right),
\end{align}

for conformally coupled scalars in $S^{d-1}$. Here the zero-point energy has been subtracted. The latter, of course, is easily calculated,

\begin{align}
E_0 &= \sum_{n=0}^{\infty} (n + 1)^{d-2} \frac{n + 1}{2a} = \frac{1}{2a} \zeta(1 - d). \quad (17)
\end{align}

The specific results for two and four dimensions are

\begin{align}
d = 2 : \quad E_0 &= -\frac{1}{2a} \frac{1}{2} B_2 = -\frac{1}{24a}, \\
d = 4 : \quad E_0 &= -\frac{1}{2a} \frac{1}{4} B_4 = \frac{1}{240a}.
\end{align}
For the temperature dependence, we differentiate the partition function,

\[ E = -\frac{d}{d\beta} \ln Z = \frac{1}{a} \sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n\delta} - 1}, \quad \delta = \frac{\beta}{2\pi a}. \tag{20} \]

This is a very different representation from Eq. (12). Nevertheless, from it we may obtain the same result we found above if we use the Euler-Maclaurin sum formula,

\[
\sum_{n=0}^{\infty} f(n) = \int_{0}^{\infty} dn f(n) + \frac{1}{2}[f(\infty) + f(0)] + \sum_{k=1}^{\infty} \frac{1}{(2k)!} B_{2k} \left[ f^{(2k-1)}(\infty) - f^{(2k-1)}(0) \right].
\]

Doing so here yields for the integral

\[
\frac{1}{a} \int_{0}^{\infty} dn \frac{n^3}{e^{\beta n/a} - 1} = \frac{a^3 \pi^4}{15\beta^4}, \tag{21}
\]

while the third derivative term gives

\[
\frac{3 B_4}{a 4!} = -\frac{1}{240a}, \tag{22}
\]
so we exactly reproduce the results of Kutasov and Larsen, but derived by a much simpler and mathematically less suspect method,

\[ E = U - E_0 \sim \frac{1}{240a} \left( \frac{1}{\delta^4} - 1 \right), \quad \delta \ll 1, \quad (23) \]

coinciding with Eqs. (13) and (19).
Comparison with Usual Finite-Temperature Casimir Effect

The same argument could be used for parallel plates, with $d$ transverse dimensions:

\[
F = TV \int \frac{d^d k}{(2\pi)^{d+1} a} \frac{\pi}{\beta} \sum_{n=-\infty}^{\infty} \ln \left( 1 - e^{-\beta \sqrt{k^2 + n^2 \pi^2 / a^2}} \right).
\]  

(24)

If we use the E-M formula, and replace the sum by an integral, and rescale variables, $\beta k \to k$, $\beta \pi n / a \to n$, we obtain

\[
F = TV \int \frac{d^d k}{(2\pi)^{d+1}} \ln \left( 1 - e^{-\sqrt{k^2 + n^2}} \right)
\]

\[
= TV \frac{A_{d+1}}{\beta^{d+1} (2\pi)^{d+1}} \int_0^{\infty} dk \, k^d \ln \left( 1 - e^{-k} \right)
\]

\[
= -VT^{d+2} \frac{\Gamma(d/2 + 1) \zeta(d + 2)}{\pi^{d/2+1}},
\]  

(25)

where $A_d = 2\pi^{d/2} / \Gamma(d/2)$ is the area of a $d$-dimensional sphere. This result is not the correct linear high-temperature $T$ dependence, but
is exactly the negative of the *low-temperature limit!* What is different about this case is that there is an inside (between the plates) and an outside, and the Stefan’s law term, being proportional to the volume, cancels between the two regions—it can give no force on the plates. Here there is no outside, so the volume term must be included. As explained in my monograph the high temperature dependence for parallel plates is linear,

\[ d = 2 : \quad F = -TV \frac{\zeta(3)}{16 \pi a^3}, \quad aT \gg 1, \quad (26) \]

while the low-temperature correction is quartic,

\[ d = 2 : \quad F = -\frac{\pi^2 V}{1440 a^4} + \frac{\pi^2 V}{90} T^4, \quad aT \ll 1. \quad (27) \]
Relation Between Representations

We have two representations for the Casimir energy, the one obtained from the Green’s function, Eq. (12), and the one obtained from the partition function, Eq. (20). The relation between the two can be found from the Poisson sum formula. If the Fourier transform of a function \( b(x) \) is defined by

\[
c(\alpha) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-i\alpha x} b(x),
\]

then

\[
\sum_{n=-\infty}^{\infty} b(n) = 2\pi \sum_{n=-\infty}^{\infty} c(2\pi n).
\]

This is just a rewriting of the elementary identity

\[
\sum_{n=-\infty}^{\infty} e^{-i2\pi nx} = \sum_{n=-\infty}^{\infty} \delta(x - n).
\]

So start from Eq. (20) and consider

\[
b(x) = \frac{1}{a} \begin{cases} 
\frac{x^3}{e^{\beta x/a} - 1}, & x \geq 0, \\
0, & x \leq 0.
\end{cases}
\]

It is easily seen that the Fourier transform is

\[
c(\alpha) = \frac{1}{2\pi a} \sum_{k=0}^{\infty} \frac{\Gamma(4)}{[\beta(k + 1)/a + i\alpha]^4}
\]

\[
= \frac{1}{2\pi a} \left(\frac{a}{\beta}\right)^4 \Gamma(4) \zeta(4, 1 + i\alpha a/\beta),
\]

in terms of the Hurwitz zeta function. Thus the energy is

\[
E = \frac{1}{a} \left(\frac{a}{\beta}\right)^4 \Gamma(4) \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(1 + k - i\beta 2\pi n/\beta)^4}.
\]

(33)

If we sum this on \(n\) first, we obtain the alternative expression

\[
E = \frac{1}{a} \Gamma(4) \frac{1}{(2\pi)^4} \left\{ \sum_{k=-\infty}^{\infty} \zeta(4, 1 - i\beta k/(2\pi a)) \right. \\
+ \zeta(4) \left[ \left(\frac{2\pi a}{\beta}\right)^4 - 1 \right] \left\}
\]

(34)

By comparing with the original expression (20) for \(E\) we can replace the sum over Hurwitz zeta
functions by the expression (12):

\[
E = \frac{1}{a} \left( \frac{2\pi a}{\beta} \right)^4 \sum_{n=1}^{\infty} \frac{n^3}{e^{4\pi^2 an/\beta} - 1} + \frac{1}{240a} \left[ (2\pi aT)^4 - 1 \right],
\]

(35)

to which must be added the zero-point energy, which cancels the \( T \)-independent term. (Thus the \( T = 0 \) term, which is the ZPE, arises from the sum, again as directly verified by the E-M formula, and in the low temperature limit the Stefan’s law term is cancelled.) Thus we again reproduce Eq. (12). The two representations are adapted explicitly for the two limits:

\[
U = \frac{1}{240a} + \frac{1}{a} \sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n \delta} - 1}
\]

(36)

\[
= \frac{1}{240a} (2\pi aT)^4 + \frac{1}{a} (2\pi aT)^4 \sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi na/\delta} - 1}.
\]

(37)
Because there is a subtle issue involving zero-modes here, it is useful to repeat the above calculation for $d = 2$. The partition function is given by

$$\ln Z = - \sum_{n=0}^{\infty} \ln \left(1 - e^{-\beta(n+1)/a}\right).$$  \hfill (38)

This immediately gives the low-temperature representation,

$$U = -\frac{1}{24a} + \frac{1}{a} \sum_{n=1}^{\infty} \frac{n}{e^{2\pi n\delta} - 1},$$  \hfill (39)

displaying an exponentially small correction to the zero-point energy if $\beta \gg 1$. Again, by use of the Euler-Maclaurin sum formula we can obtain the high-temperature limit,

$$U \sim \frac{1}{24a} (2\pi aT)^2 - \frac{1}{2}T, \quad aT \gg 1.$$  \hfill (40)

This again coincides with the result found in Kutasov and Larsen. However, the linear term
in $T$ is omitted in the analysis of Klemm et al. with an apparently erroneous remark that it only contributes when the saddle-point method breaks down, for a small central charge (the number of fields). So their derivation of the Cardy formula cannot be sustained.
It is easy to reproduce this result from the Green’s function method. After the finite-temperature substitutions, the expression is

\[ U = \frac{i}{4\pi} \frac{8i\pi}{\beta} \sum_{n=0}^{\infty} \left( \frac{2\pi n}{\beta} \right) \frac{a\pi}{2} \left[ 1 + \frac{2}{e^{4\pi^2 an/\beta} - 1} \right] \]

which gives the explicit exponential corrections to the high temperature limit (40). The low-temperature limit displayed in Eq. (39) may be easily obtained from this by using the Euler-Maclaurin sum formula.

A proof of the equivalence of the two representations (41) and (39) can be carried out along the lines sketched above. There is one complication, due to the presence of the zero-mode at \( n = 0 \). The Poisson sum formula (29), applied to a function \( b(x) \) which is positive for \( x \geq 0 \) and zero for \( x < 0 \) implies a
Fourier transform $c(\alpha)$ which has poles in the upper half plane and hence implies $b(x) = 0$ for $x \leq 0$. This contradiction at $x = 0$ is resolved by the following prescription: Write

$$\sum_{n=1}^{\infty} b(n) = \sum_{n=0}^{\infty} 'b(n) - \frac{1}{2}b(0), \quad (42)$$

replace the latter sum by the sum over the Fourier transform $c(2\pi n)$, and then when we interchange the sums and transform back the zero mode is not present. Thus only the term explicitly displayed above arises from the $n = 0$ term. In this way the equivalence between the two representations (41) and (39), with the $-\frac{1}{2}T$ term, is established.
The analysis proceeds similarly to that given above. For $S^{d-1}$ the degeneracy and eigenvalues are

$$D_l = \frac{2l \left( l + \frac{d}{2} - 1 \right) (l + d - 2)(l + d - 4)!}{(d - 3)(l + 1)!},$$

$$M^2_l = l(l + d - 2),$$

so for $d = 4$ if we again add the conformal coupling value $1$ to $M^2_l$ we obtain for the Green's function mode sum

$$\sum_{l=0}^{\infty} \frac{2l(l + 2)}{(l + 1)^2/a^2 - \omega^2} \to -\frac{1}{\omega^2} + i\pi a^2 \left( \omega a - \frac{1}{\omega a} \right) \left( 1 + \frac{2}{e^{-2\pi i\omega a} - 1} \right).$$

After making the finite temperature replacement, we carry out the sum on $n$, with the
result

\[
U = \frac{(2\pi aT)^4}{120a} - \frac{1}{12} \frac{(2\pi aT)^2}{a} + T
\]

\[
+ \frac{2(2\pi aT)^2}{a} \sum_{n=1}^{\infty} \left[ n + (2\pi aT)^2 n^3 \right] \frac{1}{e^{4\pi^2 aT n^2} - 1},
\]

where the \( T \) term comes from the \( n = 0 \) term in the sum. [Note that the leading term is exactly twice that found for a scalar field found in Eq. (13), as we would expect.] Since the remaining sum is exponentially small in the large \( T \) limit, this form is well-adapted for high temperature. (The \( T^4 \) and \( T^2 \) terms are as given in Kutasov and Larsen.) However, it is exact, and by using the Euler-Maclaurin sum formula it yields the low temperature limit,

\[
U \sim \frac{11}{120a}, \quad aT \ll 1,
\]

up to exponentially small corrections. The latter may be directly inferred from the partition function,

\[
\ln Z = - \sum_{l=1}^{\infty} 2l(l + 2) \ln \left(1 - e^{-\beta(l+1)/a}\right).
\]
By taking the negative derivative of this with respect to $\beta$ we obtain the alternative representation

$$ U = \frac{11}{120a} + \frac{2}{a} \sum_{l=1}^{\infty} \frac{l(l^2 - 1)}{e^{\beta l/a} - 1}. $$

(48)

The Euler-Maclaurin formula applied to this last sum yields the leading high-temperature term in Eq. (46), including the term linear in $T$, and the exact equivalence of the two expressions (48) and (46) again is demonstrated by the Poisson sum formula, including the zero-mode prescription enunciated above.
Weyl Fermions

The procedure is now routine, except for the complications due to fermions. The degeneracies and eigenvalues are

\[ D_l = 2(l + 2)(l + 1), \quad M_l^2 = l(l + 3), \quad (49) \]

so recalling the minus sign associated with a fermionic trace, and that the fermionic thermal Green’s functions must be antiperiodic, we
have the following expression for the energy,

\[
U = \frac{i}{4\pi} \frac{8\pi i}{\beta} 2a^2 \sum_{n=0}^{\infty} \frac{i\pi a}{2} \left[ \left( \frac{i2\pi(n + 1/2)}{\beta} \right)^3 - \frac{1}{3} \left( \frac{i2\pi(n + 1/2)}{\beta} \right) \right] \times \left( 1 + \frac{4}{e^{8\pi^2a(n+1/2)/\beta} - 1} - \frac{2}{e^{4\pi^2a(n+1/2)/\beta} - 1} \right)
\]

\[
= \frac{1}{a} \left\{ \frac{7}{960} \delta^{-4} - \frac{1}{96} \delta^{-2} - \frac{1}{4} \sum_{n=0}^{\infty} \left[ (2n + 1)^3 \delta^{-4} + (2n + 1) \delta^{-1} \right] \times \left( \frac{2}{e^{2\pi(2n+1)/\delta} - 1} - \frac{1}{e^{\pi(2n+1)/\delta} - 1} \right) \right\}.
\]  

(50)

The low-temperature limit (the zero-point energy) may be obtained from this by the Euler-Maclauin formula, and the exponential corrections in that limit may be obtained directly
from the partition function,

\[ \ln Z = \sum_{n=1}^{\infty} 2n(n + 1) \ln \left( 1 + e^{-\beta(2n+1)/2a} \right) . \]  

That is

\[ U = \frac{1}{a} \left[ \frac{17}{960} + \sum_{n=1}^{\infty} \frac{n(n + 1)(2n + 1)}{e^{\beta(2n+1)/2a} + 1} \right] . \]  

The equivalence between Eqs. (50) and (52) may be demonstrated again either though the Euler-Maclaurin sum formula applied to the latter, or exactly through the use of the Poisson sum formula.
Entropy Bounds

From the above results, thermodynamic information may extracted in terms of the free energy (with zero-point energy $E_0$ subtracted),

$$F = -T \ln Z,$$  \hspace{1cm} (53)

in terms of which the energy

$$E \equiv U - E_0 = -\frac{\partial}{\partial \beta} \ln Z = \frac{\partial}{\partial \delta} \delta F,$$  \hspace{1cm} (54)

and the entropy

$$S = 2\pi a \delta^2 \frac{\partial}{\partial \delta} F = \beta (E - F'),$$  \hspace{1cm} (55)

may be extracted.
A. Two-dimensional scalar

Klemm et al. ignore the linear $T$ term in the energy, and so have

$$E = \frac{1}{24a}(\delta^{-2} + 1), \quad (56)$$

$$F = -\frac{1}{24a}(\delta^{-2} - 1), \quad (57)$$

$$S = \frac{\pi}{6}\delta^{-1}. \quad (58)$$

These imply the Verlinde-Cardy formula

$$S = 4\pi a \sqrt{E_0(E + E_0)}, \quad (59)$$

and the entropy bound

$$\frac{S}{2\pi a E} = \frac{2 \delta}{\delta^2 + 1} \leq 1. \quad (60)$$

However, this result is not meaningful as it stands. Even in the high-temperature limit we must add the term linear in temperature to the energy, which implies instead from Eq. (40),
for $\delta \ll 1$, that

$$E = \frac{1}{24a} (\delta^{-2} + 1) - \frac{1}{4 \pi a \delta}, \quad (61)$$

$$F = -\frac{1}{24a} (\delta^{-2} - 1) - \frac{1}{4 \pi a \delta} \ln \delta, \quad (62)$$

$$S = \frac{\pi}{6} \delta^{-1} + \frac{1}{2} (\ln \delta - 1). \quad (63)$$

The ratio of $S$ to $E$ is then unbounded as $\delta \to \infty$. Yet this takes us to the low-temperature regime, where we must use the leading exponential corrections, for $\delta \gg 1$,

$$E \sim \frac{1}{a} e^{-\beta/a}, \quad (64)$$

$$F \sim -\frac{1}{\beta} e^{-\beta/a}, \quad (65)$$

$$S \sim \frac{\beta}{a} e^{-\beta/a}, \quad (66)$$

so the entropy-energy ratio is

$$\frac{S}{2 \pi a E} = \delta, \quad \delta \gg 1. \quad (67)$$

It is apparent that this latter result is universal because the energy always dominates the free energy in the low-temperature regime.
Entropy Bounds in Four Dimensions

In the following we will consider cases with $N_s$ conformal scalars, $N_v$ vectors, and $N_f$ Weyl fermions. For example, the $\mathcal{N} = 4$ supersymmetric multiplet has $N_s = 6$, $N_v = 1$, and $N_f = 4$.

In the high-temperature regime we may write the free energy, energy, and entropy as*:

\[
F \sim -\frac{1}{a}[a_4 \delta^{-4} + a_2 \delta^{-2} + a_1 \delta^{-1} \ln \delta + a_0],
\]

\[
E \sim \frac{1}{a}[3a_4 \delta^{-4} + a_2 \delta^{-2} - a_0 - a_1 \delta^{-1}],
\]

\[
S \sim 2\pi[4a_4 \delta^{-3} + 2a_2 \delta^{-1} - a_1 (1 - \ln \delta)].
\]

Here the coefficients were determined in the previous sections to be [see Eqs. (35), (46),

*What is called the Cardy formula is simply the observation that the leading behavior of $S$ is the geometric mean of the leading and subleading terms in $E$. The term "Casimir energy" for the latter is misleading in other than $1+1$ dimensions.
and (50)]

\[
\begin{align*}
    a_4 &= \frac{N_s}{720} + \frac{N_v}{360} + \frac{7N_f}{2880}, \\
    a_2 &= \frac{N_v}{12} - \frac{N_f}{96}, \\
    a_0 &= 3a_4 - a_2, \\
    a_1 &= -\frac{N_v}{2\pi}.
\end{align*}
\]

(69)  
(70)  
(71)  
(72)

Even ignoring the \(a_1\) term, Klemm et al. note that no entropy bound is possible, unless special choices are made for the field multiplicities. For the \(N = 4\) case the first three coefficients are

\[
\begin{align*}
    a_4 &= \frac{1}{48}, & a_2 &= -\frac{1}{8}, & a_0 &= \frac{3}{16},
\end{align*}
\]

(73)

and the entropy-energy ratio becomes

\[
\frac{S}{2\pi aE} = \frac{1 - \ln \delta + \frac{\pi}{6}\delta^{-3}(1 - 3\delta^2)}{\delta^{-1} + \frac{\pi}{8}\delta^{-4}(1 + \delta^2)(1 - 3\delta^2)}. 
\]

(74)

If the \(a_1\) terms here were omitted, the zero in both the energy and entropy at \(\delta^2 = 1/3\)
would cancel, and we would have the limit given Klemm et al.: 

\[
\frac{S}{2\pi a E} = \frac{4}{3} \frac{\delta}{1 + \delta^2} \leq \frac{4}{3}
\]  

(75)
in the high temperature regime. But \(a_1 \neq 0\), and the ratio (74) diverges as \(\delta \rightarrow \infty\). Of course that limit is the low-temperature one, but the argument given above then applies and shows that 

\[
\frac{S}{2\pi a E} \sim \delta, \quad \delta \rightarrow \infty.
\]  

(76)

Although in this limit both the entropy and the subtracted energy are exponentially small, their ratio is unbounded.
The reader should note that we are not in formal disagreement with previous studies. The interest there was restricted to high temperature, which is presumably all that is relevant to the early history of the universe. In that case, only the leading terms in $1/\delta$ are relevant, and the ratio of entropy to energy is always of order $\delta$. The previous authors were perfectly aware of the limitation of their formulas to the high-temperature regime, $\delta \ll 1$, and although they wrote down formulas such as Eq. (75), they regarded such results as having only heuristic value. It is not surprising that it is an unreliable guide to the moderate and low temperature regimes.

Another point, which is more closely connected with physics, is that it is permissible to make use of the thermodynamical formalism for fluctuating quasi-classical systems only when the
temperature $T$ is sufficiently high. As discussed recently by Das et al.*, one must have

$$T \gg \frac{1}{\tau}, \quad (77)$$

where $\tau$ is the relaxation time, in order to use thermodynamics. Now, in our case it is presumably legitimate to estimate $\tau$ to be of the same order of magnitude as the transit time for light across a distance of order $a$, i.e., $\tau \sim a$. This leads to the condition $T \gg 1/a$, which actually means

$$\delta \ll 1. \quad (78)$$

In other words, the physical condition for using Eq. (68) above seems to be that the linear term, which is of order $\delta^3$ relative to the first term, is negligible! It seems that we must be careful in not assigning too much physical significance to the subleading corrections.

On the half Einstein universe

The metric of the static Einstein universe has continuously attracted interest, both because it is easily tractable analytically and also because it is conformally equivalent to all the closed Robertson-Walker metrics. The Einstein metric can be written as

$$ds^2 = -dt^2 + a^2\left[d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)\right],$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. In the case of the full Einstein universe, $\chi \in [0, \pi]$. The energy density $\rho$ consists of two parts, one a matter (dust) part, $\rho_0$, and one a vacuum part, $\Lambda/8\pi G$, $\Lambda$ being the cosmological constant. The pressure is $p = -\Lambda/8\pi G$. From the Friedmann equations, $\rho_0 = \Lambda/4\pi G$, and the scale factor becomes $a = \Lambda^{-1/2} = (4\pi G \rho_0)^{-1/2}$.

The half Einstein universe, in which $\chi$ varies only from 0 to $\pi/2$, turns out to be an interesting variant of the Einstein static universe idea;
see, for instance, Refs. * and †. As discussed in Ref. ‡, we can consider this universe as a three-dimensional spherical volume, spanned by the “radius” $\chi$ and the angular coordinates $\theta$ and $\phi$, closed by a two-dimensional spherical surface lying at $\chi = \pi/2$. Let us assume that this surface is perfectly conducting. This is the simplest imaginable option in the electromagnetic case, and is analogous to the Dirichlet boundary condition in scalar field theory. The assumption allows the consideration of standing electromagnetic waves in this spherical cavity. We expect to find the same possibility of dividing the possible eigenmodes into independent TE modes and TM modes, as we do when dealing with the conventional standing modes in the Minkowski metric. Explicit calculation of the fundamental electromagnetic field modes was actually given in the cited reference.

Let us assume an orthonormal basis, \( \{\omega^t, \omega^\chi, \omega^\theta, \omega^\phi\} = \{dt, a d\chi, a \sin \chi d\theta, a \sin \chi \sin \theta d\phi\} \), and split off the time factor as \( e^{-i\omega t} \). From Maxwell’s equations in the curvilinear space we find the governing equation for \( E^\chi \) (or \( H^\chi \)). Denoting these field components collectively by \( X \), and writing \( E^\chi(\chi, \theta, \phi) = E^\chi(\chi)Y^m_l(\theta, \phi) \), we obtain

\[
\frac{d^2}{d\chi^2} \left( \sin^2 \chi \, X \right) + (\omega a)^2 \sin^2 \chi \, X - l(l + 1)X = 0.
\]

The solution of this equation is known:

\[
X \propto \sin^{l-1} \chi \, C^{(l+1)}_{n-l}(\cos \chi),
\]

where \( n \) is an integer, \( C^{(l+1)}_{n-1} \) being the Gegenbauer polynomials. The differential equation satisfied by \( C^{(\alpha)}_p(x) \), \( p \geq 1 \) is

\[
(1 - x^2) C^{(\alpha)''}_p(x) - (2\alpha + 1)x C^{(\alpha)'}_p(x) + p(p + 2\alpha) C^{(\alpha)}_p(x) = 0.
\]
Inserting Eq. (81) into Eq. (80) we obtain the eigenfrequencies
\[ \omega_n = \frac{n + 1}{a}, \quad n \geq l \geq 1. \] (83)
For the electromagnetic field, we know that \( l \geq 1 \). Moreover, in order to avoid infinities at the origin \( \chi = 0 \), we must have \( n - l \geq 0 \) in Eq. (81).

As mentioned above, the TE and TM modes are independent and can be considered separately. In the following, we consider the TE modes only.
The TE Modes

The electromagnetic boundary condition at the conducting surface is in this case

\[ H_\chi = 0, \quad \chi = \frac{\pi}{2}. \]  

(84)

Since \( l \geq 1 \) the condition (84), together with the general property that \( C_p^{(\alpha)}(0) = 0 \) when \( p \) is odd, shows that the subscript \( (n - l) \) in Eq. (81) must be odd. The eigenvalues (83) are seen to depend only on \( n \), so that they are degenerate with respect to \( l \) as well as to the magnetic quantum number \( m \) appearing in \( Y_l^m \). We accordingly have to sum \( (2l + 1) \) over all the admissible values of \( l \).

Assume first that \( n \) is even. According to the above, \( l \) can then take all odd values between 1 and \( (n - 1) \), so that the degeneracy becomes

\[
\sum_{l=1,3,5,\ldots(n-1)} (2l+1) = \frac{n}{2}(n+1), \quad n = 2, 4, 6, \ldots.
\]  

(85)
Next, if \( n \) is odd, \( l \) can take all even values between 2 and \((n - 1)\). This leads to the degeneracy

\[
\sum_{l=2,4,6,(n-1)} (2l+1) = \frac{n-1}{2}(n+2), \quad n = 1, 3, 5, \ldots
\]  

(86)

The partition function for the TE modes is accordingly

\[
\ln Z^{\text{TE}} = - \sum_{n=2,4,6}^{\infty} \frac{n}{2} (n+1) \ln \left(1 - e^{-\beta(n+1)/a}\right) \\
- \sum_{n=1,3,5}^{\infty} \frac{n-1}{2} (n+2) \ln \left(1 - e^{-\beta(n+1)/a}\right) \\
= - \sum_{n=1}^{\infty} n(2n+1) \ln \left(1 - e^{-\beta(2n+1)/a}\right) \\
- \sum_{n=1}^{\infty} (n-1)(2n+1) \ln \left(1 - e^{-2\beta n/a}\right). 
\]  

(87)
We write the result for the subtracted energy

\[ E = -\frac{\partial}{\partial \beta} \ln Z^{\mathrm{TE}} = \frac{1}{a} \sum_{n=1}^{\infty} [f(n) + g(n)], \quad (88) \]

where

\[ f(n) = \frac{n(2n + 1)^2}{e^{\beta(2n+1)/a} - 1}, \quad (89) \]
\[ g(n) = \frac{n(2n + 3)2(n + 1)}{e^{2(n+1)\beta/a} - 1}. \quad (90) \]

To get the high-temperature limit, we use the Euler-Maclaurin sum formula. The first integral term is

\[ \int_0^{\infty} dn f(n) = \frac{1}{4} \left( \frac{a}{\beta} \right)^4 \int_{\beta/a}^{\infty} du \frac{u^2(u - \beta/a)}{e^u - 1}. \quad (91) \]

For \( \beta/a \ll 1 \) this is easily seen to be

\[ \int_0^{\infty} dn f(n) = \frac{\pi^4}{60} \left( \frac{a}{\beta} \right)^4 - \frac{\zeta(3)}{2} \left( \frac{a}{\beta} \right)^3 + \frac{1}{24} \frac{a}{\beta} - \frac{1}{96}. \quad (92) \]

In the same way, the integral of the second
function is seen to be

\[
\int_0^\infty dn g(n) = \frac{\pi^4}{60} \left( \frac{a}{\beta} \right)^4 - \frac{\zeta(3)}{2} \left( \frac{a}{\beta} \right)^3 - \frac{\pi^2}{12} \left( \frac{a}{\beta} \right)^2 + \frac{5a}{6\beta} - \frac{1}{3}.
\]

(93)

The contribution of these two integrals to the energy is

\[
E_a = \frac{1}{a} \left[ \frac{\pi^4}{30} \left( \frac{a}{\beta} \right)^4 - \zeta(3) \left( \frac{a}{\beta} \right)^3 - \frac{\pi^2}{12} \left( \frac{a}{\beta} \right)^2 \right.
\]

\[
+ \frac{7}{8} \left( \frac{a}{\beta} \right) - \frac{11}{32} \left. \right].
\]

(94)

The derivative terms in the Euler-Maclaurin formula may be obtained from the small \( \beta \) forms:

\[
f(n) = \frac{a}{\beta} n(2n + 1) - \frac{1}{2} n(2n + 1)^2 + O \left( \frac{\beta}{a} \right),
\]

\[
g(n) = \frac{a}{\beta} n(2n + 3) - n(2n + 3)(n + 1) + O \left( \frac{\beta}{a} \right).
\]

Thus, for high temperature, the only nonzero
derivatives appearing are
\[
\begin{align*}
  f(0) &= 0, \quad f'(0) = \frac{a}{\beta} - \frac{1}{2}, \quad f'''(0) = -12, \\
  g(0) &= 0, \quad g'(0) = 3\frac{a}{\beta} - 3, \quad g'''(0) = -12.
\end{align*}
\]

These give a contribution to the energy of
\[
E_b = \frac{1}{a} \left[ -\frac{B_2}{2} \left( \frac{4a}{\beta} - \frac{7}{2} \right) - \frac{B_4}{24} (-24) \right] = \frac{1}{a} \left( -\frac{a}{3\beta} + \frac{31}{120} \right). \tag{95}
\]

Combining this with \( E_a \), Eq. (94), we obtain
\[
E = \frac{1}{a} \left[ \frac{1}{480\delta^4} - \frac{\zeta(3)}{8\pi^3\delta^3} - \frac{1}{48\delta^2} + \frac{13}{48\pi\delta} - \frac{41}{480} \right]. \tag{96}
\]

This corresponds to the free energy
\[
\beta F^{TE} = -\ln Z^{TE} \tag{97}
\]
\[
= -\frac{\pi}{720\delta^3} + \frac{\zeta(3)}{8\pi^2\delta^2} + \frac{\pi}{24\delta} + \frac{13}{24} \ln \delta - \frac{41\pi}{240} \delta.
\]
Comment on TM Modes

Although we have not considered the TM modes, we expect on physical grounds that they should contribute the same amount to the leading term in the energy as do the TE modes. Accordingly, for the total thermodynamic energy $U = U^{TE} + U^{TM}$ in the half Einstein universe we expect

$$2\pi a U = \frac{\pi}{120 \delta^4} + \text{subleading terms}. \quad (98)$$

This is the same leading term expression as obtained for a conformal scalar field in the $R \times S^3$ geometry considered in Sec. II, and half that for a vector, see Eq. (46). We conclude that in the high-temperature limit, the ratio of entropy to energy is the same as given in Eq. (75),

$$\frac{S}{2\pi a E} \sim \frac{4}{3} \delta, \quad \delta \ll 1. \quad (99)$$
Conclusion

There are two main points in the Verlinde proposal. First, the Friedmann-Robertson-Walker (FRW) equation may be written formally very similarly to the Cardy entropy equation. Even in the presence of higher derivative gravitational terms, or unusual matter content, or in the presence of quantum corrections, one can redefine the energy density to arrive at the standard form of the FRW equations.

Second, the free energy for higher dimensional conformal field theory has formally (at least for high temperature) the same structure as in the two-dimensional Cardy case. Moreover, assuming that (a closed) universe has extensive and subextensive (“Casimir”) contributions to its energy and entropy it was suspected that the “Casimir energy” is less than the Bekenstein-Hawking energy. As a result, the entropy/energy
cosmological bound appears. It is, however, clear that this bound cannot be universal. For example, quantum corrections or the choice of an unusual state of matter would significantly alter such a bound.

Our main qualitative result in this paper is that entropy/energy bounds should be relevant only to the very early universe with very high temperatures. With the decrease of temperature the bound becomes much less reliable, until at low temperature the entropy dominates the energy. This effect occurs already for conformal matter. The situation for non-conformal matter is much more complicated. Hence, it is unclear if entropy/energy bounds should exist at all even for high temperature. This question will be discussed elsewhere.