Physics 6433, Quantum Field Theory
Assignment #1
Due Monday, January 26, 2015

January 12, 2015

1. Consider the description of a single nonrelativistic particle in a central potential, \( V(r) \). First introduce spherical polar coordinates \( r, \theta, \phi \):

\[
x = r \sin \theta \cos \phi, \\
y = r \sin \theta \sin \phi, \\
z = r \cos \theta.
\]

Write the Lagrangian in polar coordinates. What are Lagrange’s equations? Find the canonical momenta \( p_r, p_\theta, p_\phi \). Write the Hamiltonian and the three Hamilton’s equations. What is the physical significance of \( p_\theta \)? Compute \( \{ p_\theta, H \} \).

2. Prove the following identities concerning Poisson brackets:

\[
\{ A, BC \} = \{ A, B \} C + B \{ A, C \}, \\
\{ A, \{ B, C \} \} + \{ B, \{ C, A \} \} + \{ C, \{ A, B \} \} = 0.
\]

3. A vector is an object which transforms in the same way as the position vector \( \mathbf{r} \) under a rotation. Therefore, the infinitesimal change in a vector \( \mathbf{V} \) under an infinitesimal rotation of the coordinate system is \( \delta \mathbf{V} = \delta \omega \times \mathbf{V} \). If \( \mathbf{V} \) is constructed from \( \mathbf{r} \) and \( \mathbf{p} \), show that

\[
\delta \mathbf{V} = \{ \mathbf{L} \cdot \delta \omega, \mathbf{V} \},
\]
where we have used the Poisson bracket notation, and \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \). [Hint: It may help to write \( V \) in the general form

\[
V = ar + bp + cr \times p,
\]
where \( a, b, \) and \( c \) are functions of the scalars \( r, p, \) and \( r \cdot p \).]

4. The generator of dilatations is defined by

\[
D = \sum_i q_i p_i.
\]

If \( A \) is a homogeneous function of the coordinates of degree \( n \), i.e., if

\[
A(\lambda q_i) = \lambda^n A(q_i),
\]
show that

\[
\{D, A\} = nA,
\]
where again \( \{,\} \) represents the Poisson bracket.

5. Verify that when we consider a rigid rotation

\[
\delta \mathbf{r} = \delta \mathbf{\omega} \times \mathbf{r}
\]
in classical particle electrodynamics, the corresponding generator is

\[
G = \delta \mathbf{\omega} \cdot \mathbf{J},
\]
where the angular momentum is

\[
\mathbf{J} = \sum_k \mathbf{r}_k \times m_k \mathbf{v}_k + \frac{1}{c} \int (d\mathbf{r}) \mathbf{r} \times (\mathbf{E} \times \mathbf{B}).
\]

6. Verify that the coordinate transformation law for the field strength tensor,

\[
\delta F_{\mu\nu} = -\delta x^\lambda \partial_\lambda F_{\mu\nu} - (\partial_\mu \delta x^\lambda) F_{\lambda\nu} - (\partial_\nu \delta x^\lambda) F_{\mu\lambda},
\]
is consistent with that for the vector potential

\[
\delta A_\mu = -\delta x^\lambda \partial_\lambda A_\mu - (\partial_\mu \delta x^\lambda) A_\lambda.
\]
7. Using the free Maxwell equations, verify directly the energy-momentum conservation equation,

\[ \partial_\mu t^{\mu \nu} = 0. \]

How is this equation modified if a charge density \( j_\mu \) is present?

8. A scalar particle of mass \( m \) is described by a field \( \phi(x) \) and is governed by the Lagrange density

\[ \mathcal{L} = -\frac{1}{2} \left( \partial_\mu \phi \partial_\mu \phi + m^2 \phi^2 \right). \]

Under an infinitesimal coordinate transformation, \( \phi(x) \) transforms according to

\[ \delta \phi(x) = -\delta x^\nu(x) \partial_\nu \phi(x). \]

As in the lecture, compute the corresponding energy-momentum tensor \( t^{\mu \nu} \).

9. Scale invariance is violated by the energy-momentum tensor computed in the above problem, because, as \( m \to 0 \), \( t^{\mu \mu} \to 0 \). To what limit does \( t \) converge? We can, however, introduce a new energy-momentum tensor, \( \theta^{\mu \nu} \), by adding to \( t^{\mu \nu} \) an identically conserved term,

\[ \theta^{\mu \nu} = t^{\mu \nu} + a \left( \partial_\mu \partial_\nu - g^{\mu \nu} \partial^2 \right) \phi^2. \]

Determine the number \( a \) so that \( \theta = \theta^{\mu \mu} \to 0 \) as \( m \to 0 \). This is the so-called conformal stress tensor. Argue that \( \theta^{\mu \nu} \) and \( t^{\mu \nu} \) are equally as good at describing the energy and momentum of the system. [There is an inherent arbitrariness in defining the energy-momentum tensor.]