Chapter 1

Classical Action Principles

1.1 Classical Lagrange-Hamilton Principle

This is sometimes referred to as the principle of least action. Let the classical system under consideration be described by $N$ generalized coordinates

$$ q = \{ q_i \}_{i=1,..,N}, \tag{1.1a} $$

and the corresponding velocities

$$ \dot{q} = \{ \dot{q}_i \equiv \frac{d}{dt} q_i \}_{i=1,..,N}. \tag{1.1b} $$

The dynamics of the system is specified by giving the Lagrangian, $L = L(q, \dot{q}, t)$. The action $W$ is the time integral of the Lagrangian from some initial time $t_2$ to some final time $t_1$,

$$ W_{12} = \int^{t_1}_{t_2} dt L(q(t), \dot{q}(t), t). \tag{1.2} $$

The action principle states that under infinitesimal variations, the change in the action depends only on the endpoints, that is,

$$ \delta W_{12} = G_1 - G_2, \tag{1.3} $$

where $G_a$ is a function depending only on dynamical variables at time $t_a$. In other words, the action is stationary with respect to variations between 2 and 1. This stationary property picks out the physical trajectory connecting $q_2$, $\dot{q}_2$ and $q_1$, $\dot{q}_1$.

The Lagrangian for a nonrelativistic particle of mass $m$ moving in a potential $V(r)$ is

$$ L = T - V = \frac{1}{2} m \dot{r}^2 - V(r), \tag{1.4} $$

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where the independent variables are \( r \) and \( t \). The possible variations are a change in the path, \( \delta r \), and a change in the time of the endpoints \( \delta t_2, \delta t_1 \). However, for the latter it is more convenient to define a change in the time parameter \( t \to t + \delta t(t) \) where \( \delta t(t_1) = \delta t_1, \delta t(t_2) = \delta t_2 \). Then

\[
dt \to d(t + \delta t) = dt \left( 1 + \frac{d\delta t}{dt} \right),
\]

\[
\frac{d}{dt} \to \left( 1 - \frac{d\delta t}{dt} \right) \frac{d}{dt}.
\]

Because of this change in \( t \), the limits of integration in \( W_{12} \) are not changed.

We are now ready to compute the infinitesimal variation in \( W_{12} \):

\[
\delta W_{12} = \int_1^2 dt \left\{ m \frac{d}{dt} \frac{d}{dr} \delta r - \delta r \cdot \nabla V(r) \right. \\
+ \frac{d\delta t}{dt} \left[ \frac{1}{2} m \left( \frac{d}{dt} r \right)^2 - V(r) \right] - m \left( \frac{d}{dt} r \right)^2 \frac{d\delta t}{dt} \right\}
\]

\[
= \int_1^2 dt \left\{ \frac{d}{dt} \left[ m \dot{r} \cdot \delta r - \delta t \left( \frac{1}{2} m \dot{r}^2 + V \right) \right] \\
+ \delta r \cdot [-m \ddot{r} - \nabla V] + \delta t \frac{d}{dt} \left[ \frac{1}{2} m \dot{r}^2 + V \right] \right\}. \quad (1.6)
\]

Because \( \delta r \) and \( \delta t \) are independent variations, we conclude that

\[
m \ddot{r} = - \nabla V,
\]

which is Newton’s law, and

\[
\frac{dE}{dt} = 0, \quad \text{where} \quad E = \frac{1}{2} m \dot{r}^2 + V(r) \quad \text{is the energy}, \quad (1.7b)
\]

which is the statement of energy conservation. What is left of the variation comes only from the endpoints, so we infer the form of the “generators,”

\[
G = p \cdot \delta r - E \delta t, \quad p = m \dot{r} = \text{momentum}. \quad (1.8)
\]

Let us repeat this analysis for a general Lagrangian, \( L(q_i, \dot{q}_i, t) \). In the following we will adopt a summation convention of summing over repeated indices \( i \). We find

\[
\delta W_{12} = \int_1^2 dt \left\{ \frac{\partial L}{\partial q_i} \frac{d}{dt} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta q_i + \frac{d\delta t}{dt} \left( L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \right\}
\]

\[
= \int_1^2 dt \left\{ \frac{d}{dt} \left[ \frac{\partial L}{\partial q_i} \delta q_i + \delta t \left( L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \right] \\
+ \delta \dot{q}_i \left( - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial L}{\partial q_i} \right) + \delta t \frac{d}{dt} \left( L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \right\}. \quad (1.9)
\]
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From the interior terms we deduce

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \]  
\[ (1.10) \]

which is the Euler-Lagrange equation, and the equation of energy conservation (this assumes that there is no explicit time dependence),

\[ \frac{d}{dt} H = 0, \]
\[ (1.12) \]

where the energy or Hamiltonian is

\[ H = p_i \dot{q}_i - L, \]
\[ (1.13) \]

in terms of the generalized momentum

\[ p_i = \frac{\partial L}{\partial \dot{q}_i}. \]
\[ (1.14) \]

Finally, the generator has the form

\[ G = p_i \delta q_i - H \delta t. \]
\[ (1.15) \]

Let us now return to the simple one-particle system and write the Hamiltonian:

\[ H(p, r) = p \cdot \dot{r} - L = \frac{1}{2} \frac{p^2}{m} + V(r) = T + V. \]
\[ (1.16) \]

We are now to regard \( r, p, \) and \( t \) as independent variables. Then the variation of the action is

\[ \delta W_{12} = \delta \int_2^1 dt \left( p \cdot \frac{dr}{dt} - H \right) \]
\[ = \int_2^1 dt \left[ p \frac{d}{dt} \delta r - \delta r \cdot \frac{\partial H}{\partial r} + \delta p \cdot \frac{dr}{dt} - \delta p \cdot \frac{\partial H}{\partial p} - \frac{d\delta t}{dt} H \right] \]
\[ = \int_2^1 dt \left\{ \frac{d}{dt} \left[ p \cdot \delta r - H \delta t \right] + \delta r \cdot \left[ -\frac{dH}{dt} - \frac{\partial H}{\partial r} \right] \right. \]
\[ + \left. \delta p \cdot \left[ \frac{dr}{dt} - \frac{\partial H}{\partial p} \right] + \delta t \frac{dH}{dt} \right\}. \]
\[ (1.17) \]

From this we infer the three Hamilton’s equations,

\[ \frac{dr}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \text{here,} \]
\[ \frac{dp}{dt} = -\frac{\partial H}{\partial r} = -\nabla V \quad \text{here,} \]
\[ \frac{dH}{dt} = 0. \]
\[ (1.18a/b/c) \]

\[ ^1 \text{If there is explicit time dependence, the equation satisfied by the Hamiltonian is} \]
\[ \frac{d}{dt} H = \frac{\partial}{\partial t} H. \]
\[ (1.11) \]
The generators are
\[ G = \mathbf{p} \cdot \delta \mathbf{r} - H \delta t. \] (1.19)

The generalization to \( H(q_i, p_i) \) is immediate:
\[
\begin{align*}
\frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}, \\
\frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}, \\
\frac{dH}{dt} &= \frac{\partial H}{\partial t},
\end{align*}
\] (1.20a, b, c)

now allowing for explicit time dependence. The generators are
\[ G = p_i \delta q_i - H \delta t. \] (1.21)

Suppose we consider a function of the dynamical variables, \( f(q_i, p_i, t) \). Its time derivative is
\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial f}{\partial p_i} \frac{dp_i}{dt} = \frac{\partial f}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i}
\] (1.22)

We define the Poisson bracket by
\[
\{f, g\} \equiv \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i},
\] (1.23)

so we have
\[
\frac{d}{dt} f = \frac{\partial f}{\partial t} + \{H, f\}. \] (1.24)

Thus, if the following two conditions hold,
1. there is no explicit time dependence of \( f \),
\[
\frac{\partial f}{\partial t} = 0,
\] (1.25a)

and
2. the Poisson bracket of \( f \) and \( H \) vanishes,
\[
\{H, f\} = 0,
\] (1.25b)

then \( f \) is a constant of the motion.

It is sometimes useful to adopt a viewpoint intermediate between that of Lagrange and that of Hamilton. Let us write for the Lagrangian of our single particle system
\[
L = \mathbf{p} \cdot \left( \frac{d\mathbf{r}}{dt} - \mathbf{v} \right) + \frac{1}{2}m\mathbf{v}^2 - V(\mathbf{r})
\]
\[
= \mathbf{p} \cdot \frac{d\mathbf{r}}{dt} - H(\mathbf{r}, \mathbf{p}, \mathbf{v}),
\] (1.26)
where

\[ H(r, p, v) = p \cdot v - \frac{1}{2}mv^2 + V(r). \]  

We are now to regard \( r, p, \) and \( v \) as independent variables. Then the variation of the action is

\[
\delta W_{12} = \int_{t_1}^{t_2} dt \left\{ p \cdot \frac{d}{dt} \delta r - \delta r \cdot \frac{\partial H}{\partial r} 
+ \delta p \cdot \left[ \frac{d}{dt} \frac{\partial H}{\partial p} - \frac{\delta \dot{t}}{dt} \right]
- \delta v \cdot \frac{\partial H}{\partial v}
- \frac{d}{dt} H \right\}.
\]

This implies the four “equations of motion,”

\[
\begin{align*}
\frac{dp}{dt} &= -\frac{\partial H}{\partial r} \quad (= -\nabla V \text{ here}), \\
\frac{dr}{dt} &= \frac{\partial H}{\partial p} \quad (= v \text{ here}), \\
0 &= \frac{\partial H}{\partial v} \quad (= p - mv \text{ here}), \\
\frac{dH}{dt} &= 0,
\end{align*}
\]

and the generator

\[ G = p \cdot \delta r - H \delta t. \]

1.1. Generators

The generators interrelate conservation laws and invariances of the system.

1. Suppose the action is invariant under a rigid displacement (translation) of the coordinate system:

\[
\delta W_{12} = 0 = p_1 \cdot \delta r_1 - p_2 \cdot \delta r_2,
\]

where \( \delta r_1 = \delta r_2 \) for a rigid displacement. Then

\[ p_1 = p_2, \]

that is, momentum is conserved. By our equations of motion, this will be true, of course, only if \( V \) is constant. Conversely, if \( V \) is constant, \( W \) is invariant under a translation of the coordinate system.
2. If $W$ is invariant under a rigid displacement in time (time translation, for which $\delta t_1 = \delta t_2$)

$$\delta W_{12} = 0 = -H_1 \delta t_1 + H_2 \delta t_2,$$

(1.32)

which implies

$$H_1 = H_2,$$

(1.33)

that is, energy is conserved. This is consistent with our equations of motion, unless $H$ has \textit{explicit} time dependence, in which case

$$\frac{dH}{dt} = \frac{\partial H}{\partial t},$$

(1.34)

3. Suppose $W$ is invariant under rigid rotations,

$$\delta \mathbf{r} = \delta \omega \times \mathbf{r}.$$  

(1.35)

Then

$$0 = \delta W_{12} = \mathbf{p}_1 \cdot \delta \mathbf{r}_1 - \mathbf{p}_2 \cdot \delta \mathbf{r}_2$$

$$= \delta \omega \cdot (\mathbf{r}_1 \times \mathbf{p}_1 - \mathbf{r}_2 \times \mathbf{p}_2),$$

(1.36)

which means that $L = \mathbf{r} \times \mathbf{p}$ is conserved. This will be true here provided $V(\mathbf{r}) = V(|\mathbf{r}|)$.

1.2 Classical Field Theory

Let us move on to classical field theory by writing down the appropriate Lagrangian for \textit{relativistic classical electrodynamics}:

$$L = \sum_k \mathbf{p}_k \cdot \left( \frac{d\mathbf{r}_k}{dt} - \mathbf{v}_k \right) - m_{0k} c^2 \sqrt{1 - \frac{v_k^2}{c^2}} + \frac{e_k}{c} v_k^\mu A_\mu (\mathbf{r}_k)$$

$$+ \int (d\mathbf{r}) \left[ -\frac{1}{2} F^{\mu \nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu \nu} F_{\mu \nu} \right],$$

(1.37)

where $m_{0k}$ is the rest mass of the $k$th particle, which has velocity $\mathbf{v}_k$, position $\mathbf{r}_k$, and momentum $\mathbf{p}_k$. Appearing here is the four-velocity

$$v_k^\mu = (c, \mathbf{v}_k), \text{ that is } v_k^0 = c, v_k^i = \mathbf{v}_k^i,$$

(1.38)

where we have adopted the usual convention that Greek indices run over four values, $\mu = 0, 1, 2, 3$, while Latin indices take on only the three spatial values, $i = 1, 2, 3$. Note that

$$dt v^\mu = (c \, dt, d\mathbf{r})$$

(1.39)

is a four-vector. The four-gradient is
\[
\partial_\mu = \left( \frac{1}{c} \frac{\partial}{\partial t}, \nabla \right),
\] (1.40)
and the four-vector potential is
\[
A_\mu = (\phi, A),
\] (1.41)
in terms of the usual scalar ($\phi$) and vector ($A$) potentials. $F^{\mu\nu}$ is the electromagnetic field strength tensor, which is antisymmetric,
\[
F^{\mu\nu} = -F^{\nu\mu},
\] (1.42)
and therefore has six distinct nonzero components which are the electric and magnetic field strengths,
\[
F^0_i = E^i, \quad F^{ij} = \epsilon^{ijk} B^k,
\] (1.43)
where the antisymmetric tensor (Levi-Civita symbol) is defined by
\[
\epsilon^{123} = +1, \quad \epsilon^{ijk} = \epsilon^{jki} = \epsilon^{kij} = -\epsilon^{ijk} = -\epsilon^{kji} = -\epsilon^{jik}.
\] (1.44)
Indices are lowered with the metric tensor
\[
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\] (1.45)
so for example
\[
A_\mu = g_{\mu\nu} A^\nu = (-\phi, A),
\] (1.46)
and the summation convention over repeated indices is used.

Let us work out the four independent variations of $L$ with respect to particle variables: ($\nabla_k = \partial/\partial r_k$)
\[
\delta r_k : \quad \delta L = \frac{d}{dt}(\delta r_k \cdot p_k) + \delta r_k \cdot \left( -\frac{d p_k}{dt} + \frac{e_k}{c} v^\nu_k \nabla_k A_\mu(r_k) \right),
\] (1.47a)
\[
\delta p_k : \quad \delta L = \delta p_k \cdot \left( \frac{d r_k}{dt} - v_k \right),
\] (1.47b)
\[
\delta v_k : \quad \delta L = \delta v_k \cdot \left( -p_k + \frac{m_0 v_k}{\sqrt{1 - v_k^2/c^2}} + \frac{e_k}{c} A(r_k) \right),
\] (1.47c)
\[
\delta t : \quad \delta L = \frac{d}{dt}(-H \delta t) + \delta t \frac{dH}{dt},
\] (1.47d)
so the action principle implies
\[
v_k = \frac{d r_k}{d t},
\] (1.48a)
\[ \mathbf{p}_k = \frac{m_0 \mathbf{v}_k}{\sqrt{1 - v_k^2/c^2}} + \frac{e_k}{c} \mathbf{A}(\mathbf{r}_k), \]  
\[ \frac{d\mathbf{p}_k}{dt} = \frac{e_k}{c} \nabla_k v_k^\mu A_\mu(\mathbf{r}_k), \]  
\[ \frac{dH}{dt} = 0, \]

where the Hamiltonian has a particle and a field part,
\[ H = \sum_k H_k + H_f, \]

where
\[ H_k = \mathbf{p}_k \cdot \mathbf{v}_k + m_0 c^2 \sqrt{1 - v_k^2/c^2} - \frac{e_k}{c} v_k^\mu A_\mu(\mathbf{r}_k) \]
\[ = \frac{m_0 c^2}{\sqrt{1 - v_k^2/c^2}} + e_k \phi(\mathbf{r}_k), \]

where Eq. (1.48b) was used to eliminate \( p_k \), and the field part will be given below.

We continue by working out the field variations of Eq. (1.37):
\[ \delta A_\mu : \quad \delta L = \int (d\mathbf{r}) \left( \delta A_\mu \left( j^\mu - \partial_\nu F^{\mu \nu} \right) + \partial_\nu (\delta A_\mu F^{\mu \nu}) \right), \]

where the current density is
\[ j^\mu(\mathbf{r}) = \sum_k \frac{e_k}{c} v_k^\mu \delta(\mathbf{r} - \mathbf{r}_k) = (\rho, \mathbf{j}), \]

so that
\[ \int (d\mathbf{r}) A_\mu(\mathbf{r}) j^\mu(\mathbf{r}) = \sum_k \frac{e_k}{c} v_k^\mu A_\mu(\mathbf{r}_k). \]

The two remaining variations are
\[ \delta F^{\mu \nu} : \quad \delta L = \int (d\mathbf{r}) \delta F^{\mu \nu} \left[ -\frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu \nu} F_\mu^\nu \right], \]
\[ \delta t : \quad \delta W_f = \int dt (d\mathbf{r}) \frac{d\delta t}{dt} \left[ -F^{\mu \nu} \partial_\mu A_\nu + \frac{1}{4} F^{\mu \nu} F_\mu^\nu \right]. \]

The action principle thus implies Maxwell’s equations,
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]
\[ \partial_\nu F^{\mu \nu} = j^\mu, \]

and gives the field part of the Hamiltonian,
\[ H_f = \int (d\mathbf{r}) \left[ F^{\mu \nu} \partial_\mu A_\nu - \frac{1}{4} F^{\mu \nu} F_\mu^\nu \right] = \int (d\mathbf{r}) \left[ \mathbf{E} \cdot \nabla \phi + \mathbf{B} \cdot \nabla \times \mathbf{A} + \frac{1}{2} (E^2 - B^2) \right]. \]
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But using the field equations that give the construction of the field strengths in terms of the potentials,

\[ E_i = -F^0 i - \partial^t A^i + \partial^0 A^i - \nabla_i \phi - \frac{\partial A_i}{\partial t}, \quad (1.58a) \]

\[ B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} = \epsilon_{ijk} \partial_j A_k = (\nabla \times A)_i, \quad (1.58b) \]

the field part of the Hamiltonian becomes

\[ H_f = \int (dr) \left[ \frac{1}{2} (E^2 + B^2) - \rho \phi \right], \quad (1.59) \]

since \( \nabla \cdot E = \rho \). Notice then that the total Hamiltonian, from Eqs. (1.59) and (1.50) is simply, from Eq. (1.52)

\[ H = \sum_k m_0 c^2 (1 - v_k^2/c^2)^{-1/2} + \int (dr) \frac{1}{2} (E^2 + B^2), \quad (1.60) \]

the sum of the free particle Hamiltonians and the pure field part. It appears that the interaction has disappeared. This, of course, is not the case, because \( E, B \) depend on the particle positions and velocities.

What about the generators? From Eqs. (1.47a), (1.47d), and (1.51) we have

\[ G = \sum_k \delta r_k \cdot p_k - \frac{1}{c} \int (dr) \delta A \cdot E - H \delta t. \quad (1.61) \]

This says that just as \( p_k \) is canonically conjugate to \( r_k \), \( -E \) is canonically conjugate to \( A/c \). In fact, if we introduce the Lagrange density according to

\[ L = \int (dr) \mathcal{L}, \quad (1.62) \]

we have from Eq. (1.37),

\[ c \frac{\partial \mathcal{L}}{\partial A_i} = -F^0 i = -E^i. \quad (1.63) \]

[Cf. Eq. (1.14).]

1.2.1 Field Momentum and Angular Momentum

Consider a displacement of the origin of the coordinate system,

\[ \mathbf{r} \rightarrow \mathbf{r} + \delta \mathbf{r}, \quad (1.64) \]

which is sketched in Fig. 1.1. A quantity \( F \) which is coordinate independent is a different function of the old and new coordinates:

\[ F(\mathbf{r}) = \tilde{F}(\mathbf{r} + \delta \mathbf{r}), \quad (1.65) \]
new old

\[ \delta r^* + \delta r \]

\[ P \]

\[ \delta F(r) = F(r - \delta r) - F(r) = -\delta r \cdot \nabla F(r). \] (1.67)

The field generator corresponding to this coordinate displacement thus is

\[ G_f = -\frac{1}{c} \int (dr) \delta A \cdot E = \frac{1}{c} \int (dr) [(\delta r \cdot \nabla) A] \cdot E \]

\[ = -\frac{1}{c} \int (dr) [\delta r \times (\nabla \times A) - \delta r \cdot (\nabla \cdot A)] \cdot E \]

\[ = \frac{1}{c} \int (dr) [(E \times B) \cdot \delta r - (\nabla \cdot E)(A \cdot \delta r)] \]

\[ = \frac{1}{c} \int (dr)(E \times B) \cdot \delta r - \sum_k \frac{e_k}{c} \varepsilon_k(A(r_k) \cdot \delta r), \] (1.68)

where we used \( \nabla \cdot E = \rho \) and Eq. (1.52). The total generator corresponding to the coordinate displacement is

\[ G = \sum_k G_k + G_f = P \cdot \delta r, \] (1.69)

where the total momentum is, from Eqs. (1.61) and (1.68),

\[ P = \sum_k \left( p_k - \frac{e_k}{c} A(r_k) \right) + \frac{1}{c} \int (dr) E \times B \]

\[ = \sum_k m_k v_k + \frac{1}{c} \int (dr) E \times B, \] (1.70)

where we have used Eq. (1.48b) and introduced the relativistic mass

\[ m_k = m_0 k (1 - v_k^2/c^2)^{-1/2}. \] (1.71)
The corresponding expression for the angular momentum is worked out in the homework:

\[ J = \sum_k r_k \times m_k v_k + \frac{1}{c} \int (\hat{r} \times (E \times B)). \]  

(1.72)

1.3 Energy-Momentum Tensor

Now, let us consider how fields transform under four-dimensional (space-time) coordinate transformations. For a scalar field, the field is the same at the same physical point, so

\[ \tilde{\phi}(\tilde{x}) = \phi(x), \]  

(1.73)

where for an infinitesimal transformation

\[ \tilde{x}^\mu = x^\mu + \delta x^\mu, \]  

(1.74)

so expanding the field,

\[ \tilde{\phi}(x) = \phi(x) + \delta \phi(x) = \phi(x - \delta x) \]

\[ = \phi(x) - \partial_\mu \phi(x) \delta x^\mu, \]  

(1.75)

or

\[ \delta \phi = -\delta x^\mu \partial_\mu \phi. \]  

(1.76)

Take the derivative of this:

\[ \partial_\mu \delta \phi = \delta (\partial_\mu \phi) = -\delta x^\nu \partial_\mu \partial_\nu \phi - (\partial_\mu \delta x^\nu) \partial_\nu \phi. \]  

(1.77)

We will take this to be the rule for how a vector field transforms:

\[ \delta A_\mu = -\delta x^\nu \partial_\nu A_\mu - A_\nu \partial_\mu \delta x^\nu. \]  

(1.78)

A check of this last result is provided by considering a rigid spatial translation,

\[ \delta x_\mu = (0, \delta r), \quad \partial_\nu \delta x_\mu = 0. \]  

(1.79)

Then the rule (1.78) implies correctly [cf. Eq. (1.67)]

\[ \delta A_\mu = -\delta r \cdot \nabla A_\mu. \]  

(1.80)

For a rigid rotation

\[ \delta x_\mu = (0, \delta \omega \times \hat{r}), \]  

(1.81)

so

\[ \partial_1 \delta x_j = \partial_1 \epsilon_{jkl} \delta \omega_k \hat{x}_l = \epsilon_{jkl} \delta \omega_k, \]  

(1.82)

the transformation of a three-vector field is

\[ \delta \mathbf{A} = - (\delta \mathbf{r} \cdot \nabla) \mathbf{A} + \delta \omega \times \mathbf{A}. \]  

(1.83)
The last term here says that $A$, like $r$, is a vector.

A tensor transforms by the obvious generalization of the transformation law for a vector:

$$
\delta F_{\mu\nu} = -\delta x^\lambda \partial_\lambda F_{\mu\nu} - (\partial_\mu \delta x^\lambda) F_{\nu\lambda} - (\partial_\nu \delta x^\lambda) F_{\mu\lambda},
$$

(1.84)

which is consistent with the result found in the homework.

Now let us calculate the change in the field part of the electrodynamic Lagrangian (1.37)

$$L_f = \int (dr) \left[ -\frac{1}{2} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right],
$$

(1.85)

or better, the change in the corresponding action

$$W = \int dt L,
$$

(1.86)

where now we take the integration to be over all time. Then the Lagrange density $L$ is defined by

$$W = \int (dx) L, \quad (dx) = dt(dr).
$$

(1.87)

If we substitute the field strength construction $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the change of the Lagrange density under a coordinate transformation is

$$\delta L = -\frac{1}{2} \delta F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \delta x^\lambda (\partial_\lambda F_{\mu\nu}) F^{\mu\nu} + (\partial_\mu \delta x^\lambda) F_{\nu\lambda} F^{\mu\nu}
$$

$$= -\delta x^\lambda \partial_\lambda L + F^{\mu\lambda} F^{\nu}_\lambda \partial_\mu \delta x_\nu
$$

$$= -\partial_\lambda (\delta x^\lambda L) + t^{\mu\nu} \partial_\mu \delta x_\nu,
$$

(1.88)

where the electromagnetic energy-momentum or stress tensor is

$$t^{\mu\nu} = F^{\mu\lambda} F^{\nu}_\lambda + g^{\mu\nu} L, \quad L = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}.
$$

(1.89)

Notice that the energy-momentum tensor is symmetric,

$$t^{\mu\nu} = t^{\nu\mu}.
$$

(1.90)

When the region we are considering contains no charges, $\delta W = 0$ by the stationary action principle,

$$0 = \delta W = \int (dx) t^{\mu\nu} \partial_\mu \delta x_\nu
$$

(1.91)

up to a surface term, so since the variation $\delta x_\nu$ is arbitrary at every point in spacetime,

$$\partial_\nu t^{\mu\nu} = 0.
$$

(1.92)
1.3. ENERGY-MOMENTUM TENSOR

This conservation law, which is the local statement of energy-momentum conservation, may be directly verified using Maxwell’s equations. How this is modified when currents are present is also given in the homework.

Let us examine the explicit components of $t^{\mu\nu}$. The time-time component is the energy density:

$$t^{00} = F^{0i}F^i_0 + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}$$

$$= E^2 - \frac{1}{2}(E^2 - B^2) = \frac{1}{2}(E^2 + B^2). \quad (1.93)$$

The time-space components are the momentum density,

$$t^{0i} = t^{i0} = F_{ij} = E^j \epsilon^{ijk}B_k = (E \times B)_i. \quad (1.94)$$

The stress tensor, which measures the flux of the $i$th component of momentum crossing a surface perpendicular to the $j$th direction, is

$$t^{ij} = F^{i0}F^j_0 + F^{ik}F^{jk} + \delta^{jk} \left( -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} \right)$$

$$= -E_iE_j + \epsilon^{ikl} \epsilon^{jkm}B_lB_m + \frac{1}{2}\delta_{ij}(E^2 - B^2). \quad (1.95)$$

If we use the identity

$$\epsilon_{ikl}\epsilon_{jkm} = \delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj}, \quad (1.96)$$

we can write the result in dyadic notation

$$t = -EE - BB + \frac{1}{2}(E^2 + B^2). \quad (1.97)$$

### 1.3.1 Scale Invariance

It is of some significance that the Maxwell stress tensor is traceless:

$$t^\lambda_\lambda = F^{\alpha\beta}F_{\alpha\beta} + 4 \left( -\frac{1}{4} \right) F^{\alpha\beta}F_{\alpha\beta} = 0. \quad (1.98)$$

This reflects the *scale* invariance of the Maxwell theory.

A scale transformation is a particular kind of coordinate transformation,

$$\delta x^\mu = \delta a x^\mu. \quad (1.99)$$

Under such a transformation, the action changes by

$$\delta W = \int (dx)t^{\mu\nu}\partial_\mu \delta x_\nu = \int (dx)t^{\mu\nu} \left( \delta a g_{\mu\nu} + x_\nu \partial_\mu \delta a \right), \quad (1.100)$$

which, because $t = 0$, indeed vanishes if $\delta a$ is constant, and, generally, by the action principle implies

$$\partial_\mu \left( x_\nu t^{\mu\nu} \right) = 0. \quad (1.101)$$

The conserved current here,

$$c^\mu = x_\nu t^{\mu\nu} \quad (1.102)$$

is called the scale current.