Reflection and Refraction

Using the formalism developed in Chapter 2, it is easy to derive the laws of reflection and refraction by a plane interface separating two dielectric media. Consider the geometry shown in Fig. 4.1. Let us consider an E mode, propagating in a purely dielectric medium. Then from Eqs. (2.32a)–(2.32c) we have

\[
\begin{align*}
E_t &= -\nabla_t \varphi V, \quad (4.1a) \\
H_t &= -\mathbf{e} \times \nabla \varphi I, \quad (4.1b) \\
E_z &= \frac{\gamma^2}{\omega \varepsilon} \varphi I. \quad (4.1c)
\end{align*}
\]

Here, because there are no boundaries in the transverse \((x,y)\) directions, we may take \(\varphi\) to be a transverse plane wave:

\[
\varphi(x, y) = e^{ik_{\perp} \cdot \mathbf{r}_{\perp}}, \quad (4.2a)
\]

and the corresponding eigenvalue is

\[
\gamma^2 = k_{\perp}^2. \quad (4.2b)
\]
Let us consider a plane wave incident on the interface from the left; there will be a wave reflected by the interface, and one refracted into the region on the right. See Fig. 4.2. Thus we take the wave on the left to be described by

\[ I = e^{ik_1 z} + re^{-ik_1 z}, \quad z < 0, \quad \text{(4.3a)} \]

and that on the right to be

\[ I = te^{ik_2 z}. \quad \text{(4.3b)} \]

Here, from (2.36a) the longitudinal wavenumber in each medium is given by

\[ \kappa^2 = k^2 - k_\perp^2, \quad k = \frac{\omega}{c} \sqrt{\tilde{\epsilon}} = \frac{\omega}{c} n, \quad \tilde{\epsilon} = \frac{\tilde{\varepsilon}}{\varepsilon_0}, \quad \text{(4.4)} \]

where we have introduced the index of refraction \( n = \sqrt{\tilde{\epsilon}} \); the subscripts in (4.3a) and (4.3b) refer to the media 1 and 2.

The laws of reflection,

\[ \theta_1 = \theta_3, \quad \text{(4.5)} \]

and of refraction (Snell’s law),

\[ k_1 \sin \theta_1 = k_2 \sin \theta_2 = k_\perp, \quad \text{(4.6)} \]

or

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2, \quad \text{(4.7)} \]

are geometrically obvious from Fig. 4.2.

Now we impose the boundary conditions on the electric field at the interface. From Maxwell’s equations, we may easily derive (see Problem 4.1)

\[ n \times E \quad \text{is continuous}, \quad \text{(4.8a)} \]

\[ n \cdot \varepsilon E \quad \text{is continuous}. \quad \text{(4.8b)} \]

Then from (4.1a) and (4.1c) we learn that both \( I \) and \( V \) must be continuous. From the construction (4.3a) and (4.3b) we learn the continuity condition at \( z = 0 \):
The relation between the $I$ and $V$ functions, (2.32f), which here reads

$$V = \frac{1}{\omega \varepsilon} \frac{d}{dz} I,$$

then implies at $z = 0$

$$\frac{1}{\varepsilon_1} \kappa_1 (1 - r) = \frac{1}{\varepsilon_2} \kappa_2 t,$$

which when multiplied by (4.9) yields

$$\frac{1}{\varepsilon_1} \kappa_1 (1 - r^2) = \frac{1}{\varepsilon_2} \kappa_2 t^2.$$

Equations (4.9) and (4.12) are the usual equations for the reflection and transmission coefficients, as for example given in Ref. [1], (41.96) and (41.97), for the case of $\parallel$ polarization ($E_\perp$ lies in the plane of incidence, defined by $k_\perp$ and $n$, hence the name). These equations may be easily solved:

$$\parallel \text{ polarization: } r = \frac{\kappa_1 / \varepsilon_1 - \kappa_2 / \varepsilon_2}{\kappa_1 / \varepsilon_1 + \kappa_2 / \varepsilon_2}, \quad t = \frac{2 \kappa_1 / \varepsilon_1}{\kappa_1 / \varepsilon_1 + \kappa_2 / \varepsilon_2}. \quad (4.13)$$

Equation (4.12) actually is a statement of energy conservation. If we compute the power (2.76) just to the left, and just to the right, of the interface at $z = 0$, we obtain from (4.10)

$$\frac{1}{2} IV^* = \frac{1}{2 \varepsilon_1} \kappa_1 (1 - r^2) \quad (\text{at } z = 0^-)$$

$$= \frac{1}{2 \varepsilon_2} \kappa_2 t^2 \quad (\text{at } z = 0^+), \quad (4.14)$$

which is the desired equality.

Now we repeat this calculation for the $H$ mode ($\perp$ polarization, since $H_\perp$ lies in the plane of incidence). The fields are now given by

$$H_z = \frac{i}{\omega \mu} k_\perp e^{i k_\perp \cdot r_\perp} V,$$

$$H_\perp = -i k_\perp e^{i k_\perp \cdot r_\perp} I,$$

$$E_\perp = e \times i k_\perp e^{i k_\perp \cdot r_\perp} V,$$

which follow from (2.33a)–(2.33c) when we write

$$\psi = e^{i k_\perp \cdot r_\perp}. \quad (4.16)$$

The continuity of $E_\perp$ implies that of $V$, while the continuity of $H_\perp$ (no surface current) implies that $I$ is continuous. Now we write\footnote{Note that the way we have defined the reflection and transmission coefficients are such that they refer to $B_\perp$ for $\parallel$ polarization, and to $E_\perp$ for $\perp$ polarization.}
\[ V = e^{i\kappa_1 z} + re^{-i\kappa_1 z}, \quad z < 0, \quad \text{(4.17a)} \]
\[ V = te^{i\kappa_2 z}, \quad \text{(4.17b)} \]

so the continuity of \( V \) at \( z = 0 \) implies
\[ 1 + r = t, \quad \text{(4.18)} \]
of the same form as (4.9). Using (2.33g), or
\[ I = \frac{1}{i} \frac{1}{\omega \mu} \frac{d}{dz} V, \quad \text{(4.19)} \]
we see that \( \frac{dV}{dz} \) must be continuous, or
\[ \kappa_1 (1 - r) = \kappa_2 t. \quad \text{(4.20)} \]
Multiplying (4.20) by (4.18) yields the equation of energy conservation,
\[ \kappa_1 (1 - r^2) = \kappa_2 t^2, \quad \text{(4.21)} \]
which follows from the continuity of the complex power, \( \frac{1}{i} IV^* \). This time the solution for the reflection and transmission coefficients is
\[ \perp \text{ polarization: } \quad r = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}, \quad t = \frac{2\kappa_1}{\kappa_1 + \kappa_2}. \quad \text{(4.22)} \]

For further discussion of these phenomena, the reader is referred to standard textbooks, in particular Ref. [1].

### 4.1 Problems for Chapter 4

1. Show from Maxwell’s equations integrated over a small region containing an interface across which the dielectric constant is discontinuous that (\( \mathbf{n} \) is the normal to the interface)
   \[ \mathbf{n} \times \mathbf{E} \quad \text{is continuous,} \quad \text{(4.23a)} \]
   \[ \mathbf{n} \cdot \mathbf{D} \quad \text{is continuous.} \quad \text{(4.23b)} \]

Equation (4.23b) assumes that no free charge resides on the interface.