Instructions: This examination consists of three problems. If you get stuck on one part, assume a result and proceed onward. Show all your work. Do not hesitate to ask questions. GOOD LUCK!

1. Consider a configuration of static point charges as shown in the figure.

   (a) Calculate the total charge, the electric dipole moment, and the electric quadrupole moment, defined by the dyadic

   \[
   \mathbf{q} = \int (d\mathbf{r}) \rho(\mathbf{r})[3\mathbf{rr} - 1r^2],
   \]

   where \( \rho \) is the charge density for this configuration of charges. In particular, compute all 9 components of \( \mathbf{q} \). Is there a relationship between the components?

   \[
   \begin{align*}
   z = a & \quad \bullet +e \\
   z = 0 & \quad \bullet -2e \\
   z = -a & \quad \bullet +e
   \end{align*}
   \]
(b) Now suppose the system of charges is exposed to a static electric field, which varies only slightly over the system of charges. That is, let \( a \) be very small compared to the typical scale of variation of \( \mathbf{E} \). Compute the force exerted on the system of charges by this electric field in terms of the total charge, the dipole moment, and the quadrupole moment.

(c) A general formula for the force exerted on an electric quadrupole by a field \( \mathbf{E} \) is
\[
\mathbf{F} = -\nabla U, \quad U = -\frac{1}{6} \nabla \cdot \mathbf{q} \cdot \mathbf{E}.
\]
Show that the result found in part 1b agrees with this formula. [Hint, use the Maxwell’s equations appropriate to electrostatics.]

2. The stress dyadic is
\[
\mathbf{T} = \frac{1}{8\pi} \frac{E^2 + B^2}{4\pi} - \frac{\mathbf{E} \mathbf{E} + \mathbf{B} \mathbf{B}}{4\pi}.
\]
(a) By using Maxwell’s equations, prove directly that
\[
\nabla \cdot \mathbf{T} = -\mathbf{f} - \frac{\partial}{\partial t} \mathbf{G},
\]
and determine the form of the force density \( \mathbf{f} \) and the momentum density of the electromagnetic field \( \mathbf{G} \).

(b) Now consider two point charges of strength \( e \) and \( q \) respectively, fixed in position, and separated by a distance \( R \). For concreteness, assume \( e \) is located at the origin, and \( q \) at \( z = R \). What is the electric field at any point in space?

(c) For the static situation, show that we can calculate the force on charge \( q \) from the surface integral
\[
\mathbf{F}_q = -\int_S d\mathbf{S} \cdot \mathbf{T},
\]
where \( S \) is an arbitrary volume entirely containing the charge \( q \) but not \( e \). By choosing \( S \) to be a very small sphere centered on \( q \), use this formula to calculate all components of \( \mathbf{F} \). Is the result as expected? [Hint: In integrating over solid angles, show that
\[
\int d\Omega \cos \theta = 0, \quad \int d\Omega \cos^2 \theta = \frac{4\pi}{3}.
\]
3. An idealized plane wave propagating in the $z$ direction can be described by the following electric and magnetic fields,

$$E = \hat{x} \text{Re} E_0 e^{i(kz - \omega t)},$$
$$B = \hat{y} \text{Re} B_0 e^{i(kz - \omega t)},$$

in terms of complex amplitudes $E_0$ and $B_0$.

(a) From Maxwell’s equations, what can you say about the relation between $E_0$ and $B_0$, and between $k$ and $\omega$?

(b) Calculate the momentum per unit area carried by this electromagnetic field. [Recall Problem 2a.] Average this over one cycle to get a time-independent result.

(c) Now imagine that this wave is incident upon a perfectly conducting plate perpendicular to the $z$ axis. Describe the reflected wave in a form similar to that given above.

(d) Using the result of part 3b what is the momentum transferred to the plate as a result of the reflection process?