Final Examination
Physics 5013, Mathematical Methods of Physics

December 17, 2004

Instructions: Attempt all parts of this exam. If you get stuck on one part, assume an answer and proceed on. Do not hesitate to ask questions.
Remember this is a closed book, closed notes, exam. Good luck! and have a wonderful holiday!

1. (a) Evaluate the following Principal Value integral

\[ P \int_0^\infty \frac{dx}{1 + x^2} \frac{1}{1 - x} \]

by evaluating the following closed contour integral

\[ \oint_C \frac{dz}{1 + z^2} \frac{\log z}{1 - z} \]

around a suitably-chosen closed path \( C \).

(b) In the same manner, evaluate

\[ \int_0^\infty \frac{dx}{x^2 + 1} \frac{\log x}{1 - x} \]

In the process of working this out, you should reproduce the result found in part 1a.

2. Consider an infinite conducting plane lying at \( x = 0 \). Consider the electrostatic potential in the region to the right of this plane, \( x > 0 \),
due to a point charge in the same region, that is, the Green’s function obeying
\[ -\nabla^2 G(\mathbf{r}, \mathbf{r'}) = 4\pi \delta(\mathbf{r} - \mathbf{r'}). \]

The appropriate boundary conditions are that \( G \) vanishes on the surface \( x = 0 \), and falls off at infinity at least as fast as \( 1/r \), \( r \to \infty \).

(a) Show that by translational invariance in the transverse directions, \( y \) and \( z \), the Green’s function can be written in two-dimensional Fourier transform form, with \( \mathbf{k}_\perp = (k_y, k_z) \),
\[ G(\mathbf{r}, \mathbf{r'}) = -4\pi \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp \cdot (\mathbf{r} - \mathbf{r'})} g(x, x'; k_\perp), \]  
where the reduced Green’s function \( g \) satisfies
\[ \left( \frac{d^2}{dx^2} - k^2_\perp \right) g(x, x'; k_\perp) = \delta(x - x'). \]  

(b) Solve this equation for \( g \) directly by considering two regions,
- \( 0 \leq x < x' \), where \( g(0, x') = 0 \), and
- \( 0 \leq x' < x < \infty \), where \( g \) must tend to zero as \( x \to \infty \).

Determine the arbitrary functions of \( x' \) occurring in the solutions in the two regions by imposing suitable continuity/discontinuity conditions, implied by the differential equation (2), at \( x = x' \). Express \( g \) in closed form in terms of \( x_< \), the lesser of \( x \) and \( x' \), and \( x_> \), the greater of \( x \) and \( x' \).

(c) Alternatively, we can determine \( g \) in terms of the eigenfunctions of the positive Hermitian operator \( -\frac{d^2}{dx^2} \), which eigenfunctions are required to vanish at \( x = 0 \), namely \( \sin \kappa x \). Since the space is unbounded for \( x > 0 \), the eigenvalue \( \kappa^2 \) can take any positive value. Determine the orthonormality property of these functions, that is
\[ \int_0^\infty dx \sin \kappa x \sin \kappa' x, \]  
for \( \kappa, \kappa' \) both positive. Then express the reduced Green’s functions in terms of these eigenfunctions as
\[ g(x, x'; k) = \int_0^\infty dk f(k, \kappa) \sin \kappa x \sin \kappa x', \]  
and give the explicit form of the function \( f \).
(d) Carry out the $\kappa$ integral in (4) by means of the residue theorem, and show that the same result as found in part 2b is obtained.

(e) Alternatively, insert the eigenfunction expansion found in part 2c into (1) to obtain a 3-dimensional integral representation (over $\mathbf{k}_\perp$ and $\kappa$) which is the difference of two empty-space Green's functions, one with the source at $\mathbf{r}' = (x', y', z')$, and one with the source at $\mathbf{r}'' = (-x', y', z')$. Show that the same difference emerges if one uses the closed form of the reduced Green's function found in part 2b. Since the free-space Green's function is

$$G_0 = \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

(verify this), give the image charge interpretation of this result.