Chapter 1

Review of Complex Numbers

Complex numbers are defined in terms of the imaginary unit, \( i \), having the property
\[
i^2 = -1. \tag{1.1}\]

A general complex number has the form
\[
z = x + iy, \tag{1.2}\]
where \( x, y \) are real numbers. We also often write
\[
z = \text{Re } z + i \text{Im } z, \tag{1.3}\]
where Re \( z \) is the “real part of \( z \),” and Im \( z \) is the “imaginary part of \( z \).” Complex numbers are added and multiplied just like real numbers: If
\[
z_1 = x_1 + iy_1, \tag{1.4a}
z_2 = x_2 + iy_2, \tag{1.4b}\]
then
\[
z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \tag{1.5a}
z_1z_2 = x_1x_2 + iy_1x_2 + ix_1y_2 + i^2y_1y_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1). \tag{1.5b}\]

The complex conjugate of a number is obtained by reversing the sign of \( i \): If \( z = x + iy \), we define the complex conjugate of \( z \) by
\[
z^* = x - iy. \tag{1.6}\]

Note that
\[
\text{Re } z = \frac{z + z^*}{2}, \tag{1.7a}
\]
\[
\text{Im } z = \frac{z - z^*}{2i}. \tag{1.7b}\]

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Figure 1.1: Geometrical interpretation of a complex number $z = x + iy$.

Note also that
\[ zz^* = x^2 + y^2 \]  
(1.8)
is purely real and non-negative, so we define the modulus, or magnitude, or absolute value of $z$ by
\[ |z| = \sqrt{zz^*} = \sqrt{(\text{Re} z)^2 + (\text{Im} z)^2}, \]  
(1.9)
where the positive square root is implied.

We give a simple geometrical interpretation to complex numbers, by thinking of them as two-dimensional vectors, as sketched in Fig. 1.1. Here the length of the vector is the magnitude of the complex number,
\[ r = |z|, \]  
(1.10)
and the angle the vector makes with the real axis is $\theta$, where
\[ \tan \theta = y/x; \]  
(1.11)
the quadrant $\theta$ lies in is determined by the sign of $x$ and $y$. We call
\[ \theta = \arg z \]  
(1.12)
the argument or phase of $z$. The above geometrical picture is sometimes called an Argand diagram.

There is an arbitrariness in the choice of the argument $\theta$ of a complex number $z$, for one can always add an arbitrary multiple of $2\pi$ to $\theta$ without changing $z$,
\[ \theta \to \theta + 2\pi n, \quad n \text{ a positive integer}, \quad z \to z. \]  
(1.13)
It is often convenient to define a single-valued argument function $\arg z$. By convention, the principal value of $\arg z$ is that phase angle which satisfies the inequality
\[ -\pi < \arg z \leq \pi. \]  
(1.14)
Figure 1.2: Geometrical interpretation of complex conjugation.

(Note that radian measure is always employed.) For every $z$ there is an unique $\arg z$ lying in this range.

The geometrical significance of complex conjugation is shown in Fig. 1.2. Complex conjugation corresponds to reflection in the $x$-axis.

From the Argand diagram we can write down the “polar representation” of a complex number,

$$ z = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta), \quad (1.15) $$

so if we have two complex numbers,

$$ z_1 = r_1(\cos \theta_1 + i \sin \theta_1), \quad (1.16a) $$
$$ z_2 = r_2(\cos \theta_2 + i \sin \theta_2), \quad (1.16b) $$

the product is

$$ z_1 z_2 = r_1 r_2 \left\{ \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \\
+ i \left[ \cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1 \right] \right\} \\
= r_1 r_2 \left[ \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]. \quad (1.17) $$

That is, the moduli of the complex numbers multiply,

$$ |z_1 z_2| = |z_1||z_2|, \quad (1.18a) $$
while the arguments add,

$$ \arg(z_1 z_2) = \arg z_1 + \arg z_2. \quad (1.18b) $$

The latter statement is to be understood as modulo $2\pi$, i.e., equality up to the addition of an arbitrary integer multiple of $2\pi$. In particular, note that

$$ \left| \frac{1}{z} \right| = \left| \frac{1}{z} \right| |z| = 1, \quad (1.19a) $$
while
\[ 0 = \text{arg} \left( \frac{1}{z} \right) = \text{arg} \left( \frac{1}{z} \right) + \text{arg} z, \]

implying that
\[ \left| \frac{1}{z} \right| = \frac{1}{|z|}, \]
\[ \text{arg} \left( \frac{1}{z} \right) = -\text{arg} z. \]

1.1 De Moivre’s Theorem

From the above, if we choose a unit vector,
\[ z = \cos \theta + i \sin \theta, \]
successive powers follow a simple pattern:
\[ z^n = \cos n\theta + i \sin n\theta, \]

or
\[ (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \]
where \( n \) is a positive integer. This is called De Moivre’s theorem.

1.2 Roots

Suppose we wish to find all the \( n \)th roots of unity, that is, all solutions to the equation
\[ z^n = 1, \]
where \( n \) is a positive integer. If we take the polar form,
\[ z = \rho(\cos \phi + i \sin \phi), \]
this means
\[ \rho^n(\cos n\phi + i \sin n\phi) = 1, \]
which implies
\[ \rho = 1, \]
\[ n\phi = 2\pi k, \]
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where \( k \) is any integer. Thus the \( n \)th root of unity has the form

\[
    z = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}.
\]

These are distinct for \( k = 0, 1, 2, \ldots, n - 1; \)

outside of these values of \( k \), the roots repeat. Thus there are \( n \) distinct \( n \)th roots of unity. For example, for \( n = 8 \), the roots are as shown in Fig. 1.3, in the complex plane.