Physics 5013. Homework 8
Due Tuesday, November 23, 2004

November 9, 2004

1. Let $\rho(x)$ be a positive function on the interval $a \leq x \leq b$. Consider the set of complex-valued functions $\{f\}$ defined on this same interval with the property that

$$\int_a^b dx \rho(x) |f(x)|^2 < \infty.$$  

Prove that this set of functions is an inner product space, with vector addition defined by $(f + g)(x) = f(x) + g(x)$ and the inner product given by

$$\langle f, g \rangle = \int_a^b dx \rho(x) f^*(x) g(x).$$

Using the fact that $L^2_2(a,b)$ is a Hilbert space, prove that this space is one also.

2. Using Green’s theorem,

$$\int_V (d\mathbf{r}) \left[ u \nabla^2 v - v \nabla^2 u \right] = \oint_S d\mathbf{S} \cdot \left[ u \nabla v - v \nabla u \right],$$

where the volume $V$ is bounded by the closed surface $S$, and $d\mathbf{S}$ is the outwardly directed surface element, find the three types of homogeneous boundary conditions which assure that the Laplacian operator $\nabla^2$ is self-adjoint, in terms of the inner product

$$\langle u, v \rangle = \int_V (d\mathbf{r}) u^*(\mathbf{r}) v(\mathbf{r}).$$
3. Find the eigenvalues and eigenfunctions of $\nabla^2$ in two dimensions in the region $0 \leq |\mathbf{r}| \leq a$, subject to the boundary condition that all functions under consideration vanish at $|\mathbf{r}| = a$. (These might describe the normal modes of vibration of a circular drumhead.) [Hint: Solve the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

by separating variables in polar coordinates.]

4. Suppose we have a second-order differential operator of the form

$$L = \frac{1}{f} \frac{d}{dx} \left( f \frac{d}{dx} \right) + q,$$

where $f$ and $q$ are functions of $x$. If $y_1$ and $y_2$ are independent solutions of

$$Ly = 0,$$

the Wronskian

$$\Delta(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

is different from zero. Prove that

$$\frac{d}{dx} \Delta = -\Delta \frac{d}{dx} \ln f,$$

and that

$$y_2(x) = \Delta(x_0) f(x_0) y_1(x) \int_{x_0}^{x} \frac{du}{f(u) y_1^2(u)},$$

where $x_0$ is a point at which

$$y_2(x_0) = 0, \quad y_1(x_0) \neq 0,$$

$$f(x_0) \neq 0, \quad y_1'(x_0) \neq 0.$$

5. Recall that the Bessel functions of integer order are defined by

$$e^{(x/2)(z-1/z)} = \sum_{m=-\infty}^{\infty} z^m J_m(x),$$

or, with $x = kr, \ z = i e^{i \phi}$.

$$e^{ikr \cos \phi} = \sum_{m=-\infty}^{\infty} i^m e^{im \phi} J_m(kr).$$
Use this expression in the two-dimensional completeness statement for the functions
\[ \frac{1}{2\pi} e^{i\mathbf{k} \cdot \mathbf{r}}, \]
that is,
\[ \int \frac{(d\mathbf{k})}{(2\pi)^2} e^{i\mathbf{k} \cdot \mathbf{r}} e^{-i\mathbf{k} \cdot \mathbf{r}'} = \delta(\mathbf{r} - \mathbf{r}'), \]
where the right-hand side is a two-dimensional delta function, which in polar coordinates is
\[ \delta(\mathbf{r} - \mathbf{r}') = \frac{1}{r} \delta(r - r') \delta(\theta - \theta'), \]
and \((d\mathbf{k})\) is the two-dimensional integration element, which is correspondingly given in polar coordinates as
\[ (d\mathbf{k}) = k \, dk \, d\alpha. \]

In this way derive the completeness property of the Bessel functions,
\[ \int_0^\infty k \, dk \, J_m(kr)J_m(kr') = \frac{1}{r} \delta(r - r'). \]