1. One representation for the gamma function is
\[ \frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_C e^{t-z} \, dt \]
where the contour of integration is as shown in Figure 1. Use the method of steepest descents to derive Stirling’s approximation,
\[ \Gamma(x) \sim x^xe^{-x}\sqrt{\frac{2\pi}{x}}, \quad x \to +\infty. \]

2. The Airy function has the asymptotic expansion
\[ \text{Ai}(x) \sim \frac{1}{2} \pi^{-1/2} x^{-1/4} e^{-2x^{3/2}/3} \left[ 1 + O(x^{-3/2}) + O(x^{-3}) + \ldots \right]. \]
Calculate the \(O(x^{-3/2})\) and the \(O(x^{-3})\) terms.

Figure 1: Contour \(C\) used for defining the Hankel representation of the gamma function.
3. Using the integral representation for the Hankel function of the first kind

\[ H^{(1)}_{\nu}(z) = \frac{e^{-i\nu\pi/2}}{\pi} \int_{\pi/2-i\infty}^{\pi/2+i\infty} d\phi e^{i(z \cos \phi + \nu \phi)}, \]

derive Debye’s asymptotic expansion of \( \tan \alpha > 0 \) and \( \nu \) large and positive:

\[ H^{(1)}_{\nu}(\nu \sec \alpha) \sim \sqrt{\frac{2}{\pi \nu \tan \alpha}} e^{-i\pi/4} e^{i\nu(\tan \alpha - \alpha)} \left( 1 + \frac{u_1(\cot \alpha)}{i\nu} + O(1/\nu^2) \right), \]

where

\[ u_1(t) = \frac{3t + 5t^3}{24}. \]