October 25, 2004

1. Consider the function \( \frac{1}{z} \log(1 + z) \).

Derive the \([3, 3]\) Padé approximant stated in class

\[
P_3^3(z) = \frac{1 + \frac{17}{14}z + \frac{1}{3}z^2 + \frac{1}{140}z^3}{1 + \frac{12}{7}z + \frac{6}{7}z^2 + \frac{4}{35}z^3}.
\]

Similarly, work out \( P_4^3(z) \), and verify the values given in class for \( z = 0.5, 1, \) and 2.

2. The Stirling series for the Gamma function is

\[
\Gamma(n) = (n-1)! \sim \left( \frac{2\pi}{n} \right)^{1/2} \left( \frac{n}{e} \right)^n \left( 1 + \frac{A_1}{n} + \frac{A_2}{n^2} + \frac{A_3}{n^3} + \frac{A_4}{n^4} + \ldots \right), \quad n \to \infty,
\]

where

\[
A_1 = \frac{1}{12}, \quad A_2 = \frac{1}{288}, \quad A_3 = -\frac{139}{51840}, \quad A_4 = -\frac{571}{2488320}.
\]

Compute the \([1, 1]\) and \([2, 2]\) Padé approximants for \( \Gamma(x)(e/x)^x \sqrt{x/2\pi} \). Compare numerically the values so obtained with the exact function.
for \( x = 0.2, 0.5, \) and \( 1.0, \) which are small values of \( x. \) Can a more accurate approximation be obtained by averaging \( P_1^1 \) and \( P_2^2? \)

3. Consider a continued-fraction representation of the exponential function in the form

\[
e^x = \frac{c_0}{1 + \frac{c_1 x}{1 + \frac{c_2 x^2}{1 + \cdots}}}
\]

Show that

\[
c_0 = -c_1 = 1, \quad c_{2n} = \frac{1}{4n - 2}, \quad c_{2n+1} = -\frac{1}{4n + 2}, \quad n \geq 1.
\]

How many terms must be included to compute \( e \) to 8 significant figures?

4. Using the same form of the continued fraction, compute the first three terms of the continued-fraction representation of the series

(a) \[
\sum_{n=0}^{\infty} \frac{(n!)^2}{(-z)^n},
\]

(b) \[
\sum_{n=0}^{\infty} \frac{(-z)^n}{(2n)!},
\]

(c) \[
\sum_{n=0}^{\infty} \frac{z^n}{n^2 + 1}.
\]