Problems in Bender and Orzag:
Chapter 8, pp. 412–3: 18, 19, 20, 31

Additional problems:

1. Consider the function \( \frac{1}{z} \log(1 + z) \).

   Derive the [3, 3] Padé approximant stated in class
   \[
   P_3^3(z) = \frac{1 + \frac{17}{14}z + \frac{1}{3}z^2 + \frac{1}{140}z^3}{1 + \frac{12}{7}z + \frac{9}{7}z^2 + \frac{4}{35}z^3}.
   \]

   Similarly, work out \( P_4^3(z) \), and verify the values given in class for \( z = 0.5, 1, \) and 2.

2. The Stirling series for the Gamma function is for \( n \to \infty \),
   \[
   \Gamma(n) = (n - 1)! \sim \left( \frac{2\pi}{n} \right)^{1/2} \left( \frac{n}{e} \right)^n \left( 1 + \frac{A_1}{n} + \frac{A_2}{n^2} + \frac{A_3}{n^3} + \frac{A_4}{n^4} + \ldots \right),
   \]

where

\[
\begin{align*}
A_1 & = \frac{1}{12}, \\
A_2 & = \frac{1}{288}, \\
A_3 & = -\frac{139}{51840}, \\
A_4 & = -\frac{571}{2488320}.
\end{align*}
\]
Compute the $[1,1]$ and $[2,2]$ Padé approximants for $\Gamma(x)(e/x)^x\sqrt{x/2\pi}$. Compare numerically the values so obtained with the exact function for $x = 0.2$, $0.5$, and $1.0$, which are small values of $x$. Can a more accurate approximation be obtained by averaging $P_1^1$ and $P_2^2$?

3. Consider a continued-fraction representation of the exponential function in the form

$$e^x = \frac{c_0}{1 + \frac{c_1}{1 + \frac{c_2}{1 + \cdots}}}.$$ 

Show that

$$c_0 = -c_1 = 1, \quad c_{2n} = \frac{1}{4n-2}, \quad c_{2n+1} = -\frac{1}{4n+2}, \quad n \geq 1.$$ 

How many terms must be included to compute $e$ to $8$ significant figures?