The second exam will be given on Wednesday, November 8 from 2:45–4:00PM. (Notice, this is postponed one week from the original schedule, due to missed class meetings.) It will cover all material covered in class through Wednesday, November 1, and all homework assignments through this one. It will, of course, be of the closed-book variety.

1. One representation for the gamma function is

\[
\frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_C e^{t-z} dt
\]

where the contour of integration is as shown:

Use the method of steepest descents to derive Stirling’s approximation,

\[
\Gamma(x) \sim x^e e^{-x} \sqrt{\frac{2\pi}{x}}, \quad x \to +\infty.
\]
2. The Airy function has the asymptotic expansion

\[ \text{Ai}(x) \sim \frac{1}{2} \pi^{-1/2} x^{-1/4} e^{-2x^{3/2}/3} [1 + O(x^{-3/2}) + O(x^{-3}) + \ldots]. \]

Calculate the \( O(x^{-3/2}) \) and the \( O(x^{-3}) \) terms. [The first we did in class.]

3. Using the integral representation for the Hankel function of the first kind

\[ H^{(1)}_{\nu}(z) = \frac{e^{-i\nu\pi/2}}{\pi} \int_{\pi/2-i\infty}^{\pi/2+i\infty} d\phi e^{i(z\cos\phi+i\nu\phi)}, \]

derive Debye’s asymptotic expansion of \( \tan \alpha > 0 \) and \( \nu \) large and positive:

\[ H^{(1)}_{\nu}(\nu \sec \alpha) \sim \sqrt{\frac{2}{\pi \nu \tan \alpha}} e^{-i\pi/4} e^{i\nu(\tan \alpha - \alpha)} \left( 1 + \frac{u_1(\cot \alpha)}{i\nu} + O(1/\nu^2) \right), \]

where

\[ u_1(t) = \frac{3t + 5t^3}{24}. \]

4. Consider the function

\[ \frac{1}{z} \log(1 + z). \]

Derive the [3, 3] Padé approximant stated in class

\[ P^3_3(z) = \frac{1 + \frac{17}{144} z + \frac{1}{3} z^2 + \frac{1}{140} z^3}{1 + \frac{12}{7} z + \frac{6}{7} z^2 + \frac{4}{35} z^3}. \]

Similarly, work out \( P^3_4(z) \), and verify the values given in class for \( z = 0.5, 1, \) and 2.

5. The Stirling series for the Gamma function is

\[ \Gamma(n) = (n-1)! \sim \left( \frac{2\pi}{n} \right)^{1/2} \left( \frac{n}{e} \right)^n \left( 1 + \frac{A_1}{n} + \frac{A_2}{n^2} + \frac{A_3}{n^3} + \frac{A_4}{n^4} + \ldots \right), \quad n \to \infty, \]

where

\[ A_1 = \frac{1}{12}. \]
\[ A_2 = \frac{1}{288}, \]
\[ A_3 = -\frac{139}{51840}, \]
\[ A_4 = -\frac{571}{2488320}. \]

Compute the [1, 1] and [2, 2] Padé approximants for \( \Gamma(x)(e/x)^{x} \sqrt{x/2\pi} \). Compare numerically the values so obtained with the exact function for \( x = 0.2, 0.5, \) and 1.0, which are small values of \( x \). Can a more accurate approximation be obtained by averaging \( P_1^1 \) and \( P_2^2 \)?