Problems in Whittaker and Watson:

Additional problems:

1. Show that the function

\[ f(z) = (z^2 - 1)^{1/2} \]

is single valued when it is defined with the branch line running along the real axis from \(-1\) to \(+1\). [Hint: Consider the net phase change in \(f\) when the branch line is encircled once.]

2. Evaluate

\[ \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2(1 + x^2)^2}} \]

by using a contour which encircles the branch line given in Problem 1, and closed by a circle at infinity. Equivalently, consider a contour of two parts: one that just encloses the branch line from \(z = -1\) to \(z = +1\) and another being a circle about the origin of very large radius. Between these two contours, the function

\[ f(z) = \sqrt{1 - z^2} \]

is analytic.
3. Recall the generating function defining the Bernoulli numbers:

\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.
\]

Show that

\[
B_n = \frac{n!}{2\pi i} \oint_{C_0} \frac{z}{e^z - 1} \frac{dz}{z^{n+1}},
\]

where \(C_0\) is a circle about the origin with radius \(|z| < 2\pi\). From this integral find \(B_0, B_1\) directly. By distorting \(C_0\) into \(C\), an infinite circle about the origin (and hence crossing an infinite number of poles!), show that for \(n\) even, \(n \geq 2\),

\[
B_n = -\frac{(-1)^{n/2} 2^n}{(2\pi)^n} \zeta(n),
\]

where

\[
\zeta(n) = \sum_{p=1}^{\infty} p^{-n}.
\]

4. Use the residue theorem to evaluate the following integrals:

(a)

\[
\int_{0}^{2\pi} \frac{\sin^2 \theta}{a + b \cos \theta} d\theta, \quad a > b > 0.
\]

(b)

\[
\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx, \quad \Re a > \Re b > 0.
\]

What happens if \(a = b\)?

(c)

\[
\int_{0}^{\infty} \frac{x^{2a-1}}{1 + x^2}, \quad 0 < a < 1.
\]

(d)

\[
\int_{0}^{\infty} x \frac{(\log x)^2}{1 + x^2}.
\]