Physics 5013. Homework 7
Due Friday, December 2, 2011

November 30, 2011

1. An integral representation for the modified Bessel function $K_\nu(x)$ is

$$K_\nu(x) = \frac{1}{2} \int_{-\infty}^{\infty} dt \, e^{-x \cosh t + \nu t}.$$ 

Show that

$$K_{ip}(x) = \sqrt{2\pi (p^2 - x^2)^{-1/4}} e^{-p\pi/2} \sin \phi(x).$$

where

$$\phi(x) - p \cosh^{-1}(p/x) + \sqrt{p^2 - x^2} \sim \frac{\pi}{4}, \quad x \to +\infty, \quad p/x \to +\infty.$$ 

Hint: The contribution comes from the neighborhood of two saddle points satisfying $\sinh t = ip/x$. Explain why it is that although there are an infinite number of saddle points, only two contribute to the leading behavior.

2. An integral representation of the second Airy function $\text{Bi}(x)$ is given by

$$\text{Bi}(x) = \frac{1}{2\pi} \int_{C_+} dt \, e^{xt - t^3/3} + \frac{1}{2\pi} \int_{C_-} dt \, e^{xt - t^3/3},$$

where $C_\pm$ is a contour which originates at $\infty e^{\pm 2\pi i/3}$ and terminates at $+\infty$. Using the method of steepest descents, find the leading asymptotic behavior as $x \to +\infty$. 

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3. One representation for the gamma function is
\[ \frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_C e^{t^{-z}} dt \]
where the contour of integration is as shown in Figure 1. Use the method of steepest descents to derive Stirling’s approximation,
\[ \Gamma(x) \sim x^x e^{-x} \sqrt{\frac{2\pi}{x}}, \quad x \to +\infty. \]

4. The Airy function has the asymptotic expansion
\[ \text{Ai}(x) \sim \frac{1}{2} \pi^{-1/2} x^{-1/4} e^{-2x^{3/2}/3} \left[ 1 + O(x^{-3/2}) + O(x^{-3}) + \ldots \right]. \]
Fill in the steps followed in class, and calculate the \( O(x^{-3/2}) \) and the \( O(x^{-3}) \) terms.

5. Using the integral representation for the Hankel function of the first kind
\[ H^{(1)}_{\nu}(z) = \frac{e^{-i\nu \pi/2}}{\pi} \int_{-\pi/2+i\infty}^{\pi/2-i\infty} d\phi e^{i(z \cos \phi + \nu \phi)}, \]
derive Debye’s asymptotic expansion for \( \tan \alpha > 0 \) and \( \nu \) large and positive:
\[ H^{(1)}_{\nu}(\nu \sec \alpha) \sim \sqrt{\frac{2}{\pi \nu \tan \alpha}} e^{-i\pi/4} e^{i\nu (\tan \alpha - \alpha)} \left( 1 + \frac{u_1(\cot \alpha)}{i\nu} + O(1/\nu^2) \right), \]
where
\[ u_1(t) = \frac{3t + 5t^3}{24}. \]