1. This problem explores certain properties of the exponential function, and of its inverse, the logarithm.

[5] (a) The exponential function is defined by the power series

\[ e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n. \]

Use the root test to determine the radius of convergence of this series. For what complex values of \(x\) does \(e^x\) possess singularities?

[10] (b) Using the power-series definition of the exponential function, establish the conditions under which

\[ e^{x+y} = e^x e^y. \]

[5] (c) Let \(\pi\) be the smallest positive number for which

\[ e^{2\pi} = 1. \]

Show then that the exponential function is periodic with period \(2\pi i\),

\[ e^{x+2\pi in} = e^x, \quad n \text{ an integer}. \]
(d) Evaluate the derivative of \(e^x\),

\[
\frac{d}{dx} e^x.
\]

(e) If \(x = e^y\), define the inverse function, the logarithm, by \(y = \ln x\). Is the logarithm single-valued? If not, give all the values of \(\ln x\) for a given \(x\). What is \(\ln 1\)?

(f) Evaluate the derivative of the logarithm,

\[
\frac{d}{dx} \ln x.
\]

(g) Compute all the derivatives of \(\ln(1 + x)\), in particular, give a closed-form expression for

\[
\frac{d^n}{dx^n} \ln(1 + x) \bigg|_{x=0}.
\]

(h) Thus determine the power series expansion of \(\ln(1 + x)\) for small \(x\). What is the radius of convergence of this series? Why?

2. Consider the integral

\[
\oint_C dz \, z^\alpha
\]

where \(C\) is a circular contour which is centered on the origin, encircling it once in a counterclockwise sense, and \(\alpha\) is a complex number. Evaluate the integral explicitly for all values of \(\alpha\). From the result, what can you say about the values of \(\alpha\) for which \(z^\alpha\) is analytic? In particular, what can you say about the case when \(\alpha\) is a negative integer?
3. In this problem you may use Cauchy’s theorem to evaluate the integrals.

[10] (a) Consider the integral

$$\oint_{C_0} \frac{dz}{e^z - 1},$$

where $C_0$ is a circle about the origin with radius less than $2\pi$, encircling the origin once in a positive sense. What is the value of this integral?

[10] (b) Now consider the integral

$$\oint_{C_n} \frac{dz}{e^z - 1},$$

where $C_n$ is a positively-oriented circle about the origin with radius $\rho$ in the interval $2\pi n < \rho < 2\pi(n + 1)$. Does this integral have a different value? Why? If it is different, evaluate it.