III. Elementary Transcendental Functions

3.1 Exponential Function

Define, for all complex $z$,

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

By the ratio test,

$$\frac{n!}{(n+1)!} \frac{|z|^n}{n+1} \rightarrow 0 \text{ for all } z,$$

the series converges everywhere. (Uniformly in any finite region.)

Note that

$$\exp(z_1 + z_2) = \sum_{n=0}^{\infty} \frac{1}{n!} (z_1 + z_2)^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{n!}{n! m! (n-m)!} z_1^m z_2^{n-m}$$

$$= \left( \sum_{k=0}^{\infty} \frac{1}{k!} z_1^k \right) \left( \sum_{l=0}^{\infty} \frac{1}{l!} z_2^l \right)$$

$$= \exp(z_1) \cdot \exp(z_2)$$

And, by induction

$$(e^z)^n = e^{nz}.$$
Hyperbolic and trigonometric functions are defined in terms of the exponential function:

\[
\sinh z = \frac{e^z - e^{-z}}{2} \quad \cosh z = \frac{e^z + e^{-z}}{2}
\]

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}
\]

so that

\[i \sin z = \sinh iz\]

\[\cos z = \cosh iz\]

for all complex \( z \).

Example: Note \( e^{iz} = \cos z + i \sin z \).

Therefore, the polar representation of a complex number

\[z = r (\cos \Theta + i \sin \Theta)\]

\[= r e^{i \Theta}\]

becomes a most useful and compact representation.

In particular \( z^n = r^n e^{i n \Theta} \)
implies De Moivre's formula
\[ \cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n \]

There exists a positive number \( \pi \) such that

\[ a) \ e^{\pi i/2} = i \]

\[ b) \ e^{z} = 1 \text{ if and only if } z = 2\pi i n \text{ where } n \text{ is an integer} \]

Hence \( \exp(z) \) is periodic with period \( 2\pi i \)

\[ \exp(z + 2\pi i) = \exp(z) \exp(2\pi i) = \exp(z) \]

The (natural) logarithm

If \( z = re^{i\theta} \), we define

\[ \ln z = \log z = \log r + i\theta \]

where \( \log r \) is defined by

\[ r = e^{\log r} \]
Thus \( z = e^S \) where \( S = \log r + i\Theta \)
\[ = \log z \]

Recall that \( \Theta = \arg z \) is a multivalued function, because \( \Theta \) is only defined up to an arbitrary multiple of \( 2\pi \). (This is just the periodic property mentioned above.) Recall we defined \( \Theta \) the principal value of the argument as that which satisfied

\[-\pi < \arg z \leq \pi \]

Correspondingly, we say that the single valued logarithm function (also denoted \( \log z \)) is defined in the cut plane.

In measuring \( \Theta \) from the \( +x \) axis, one is not allowed to cross the cut along the \( -x \) axis. (Where the cut is placed is an arbitrary convention.) The corresponding defined single valued functions of

\( \arg z \)
and \( \log z = \log |z| + i \arg z \)
are also referred to as the principal values of the argument and logarithm, respectively.

Define
\[
j^z = e^{z \log j}, \quad \log j \text{ defined in the cut plane. Then}
\]
\[
(e^j)^z = e^{z \log e^j} = e^{zj}
\]
when
\[
\arg e^j = \text{Im } j
\]
lies between \(-\pi < \text{Im } j \leq \pi \)

For example
\[
\sqrt{z} = z^{1/2} = e^{1/2 \log z}
\]
is defined only in the cut plane
\(-\pi < \arg z \leq \pi \).

*If this is not so, \( \log e^j = j + 2\pi n i \) so
\(-\pi < \text{Im } (j + 2\pi n i) \leq \pi \), and
\[
(e^j)^z = e^{z(j + 2\pi n i)}
\]
The inverse hyperbolic and trig functions are defined in terms of the logarithms:

\[
\begin{align*}
\text{arc sinh } z &= \log \left[ z + (z^2 + 1)^{\frac{1}{2}} \right] \\
\text{arc cosh } z &= \log \left[ z + (z^2 - 1)^{\frac{1}{2}} \right] \\
\text{arc tanh } z &= \frac{1}{2} \log \frac{1 + z}{1 - z}
\end{align*}
\]

which are defined in the cut planes.

\[
\begin{align*}
\text{arc sinh } z:\quad &\text{Im} \quad +i \\
\text{arc cosh } z:\quad &\text{Re} \quad +1 \\
\text{arc tanh } z:\quad &\text{Re} \quad +1 \\
\text{arc sin } z &= -i \text{ arc sinh } iz \\
&= -i \log \left[ iz + (1 - z^2)^{\frac{1}{2}} \right] \\
\text{arc cos } z &= -i \text{ arc cosh } iz \\
&= -i \log \left[ z + (z^2 - 1)^{\frac{1}{2}} \right] \\
\text{arc tan } z &= -i \text{ arc tanh } iz \\
&= \frac{i}{2} \log \frac{1 - iz}{1 + iz} = \frac{i}{2} \log \frac{i + z}{i - z}
\end{align*}
\]
which are defined in the cut plane:

\[
\begin{align*}
\arcsin z &= i y \\
\arccos z &= \frac{-1}{x} \\
\arctan z &= i
\end{align*}
\]